Strictly according to the latest Syllabus March 2018 prescribed by the Council for the Indian School Certificate Examinations, New Delhi for ISC students taking Board examination in 2018.

ISC MATHEMATICS

BOOK I

FOR CLASS XI

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Preface

We are happy to be able to bring out the thoroughly revised edition of our ISC Mathematics, Book I, for Class XI which has been restructured and redesigned strictly in accordance with the new syllabus. The Council for Indian School Certificate Examinations, New Delhi has made major changes in the syllabus for the ISC examinations to be held in 2018. We have also used this opportunity to incorporate the suggestions and feedback received from learned faculty from time to time.

The book is divided into three sections – **Section A**, **Section B** and **Section C**. Section A is compulsory while the students have a choice in attempting all questions from either Section B or Section C.

All chapters in this book start from the basic level and there is a gradual increase in the difficulty level which ultimately matches the level expected of the students. Most of the topics or sums have been explained in multiple ways, so that the students can understand the concepts without any external help. This will help the students develop the ability to prepare for various competitive examinations by themselves.

Salient features of this book are:

- *This book is designed strictly according to the latest syllabus for ISC (Class XI) prescribed by the Council for Indian School Certificate Examinations.*
- *The chapters have been sequenced in the same order as given in the syllabus released by the Council.*
- *The problems have been carefully selected and well graded keeping in mind their difficulty levels.*
- *A few chapters have been thoroughly revised and rehashed .*
- *In keeping with the modern trend and also the requirements of various entrance examinations, like IIT, AIEEE, etc., effort has been made to lay stress on mathematical accuracy. In addition, certain important questions from past papers of various competitive examinations have been included appropriately.*
- ***Chapter Tests** have been provided at the end of each chapter for reinforcement of the concepts discussed in it.*

The **Model Test Papers** given at the end are based on the guidelines and marking scheme provided by the Council and have been prepared by Mr. S. Ganguly, Senior Maths Teacher, St. Thomas College, Dehradun. We hope these will prove highly useful in understanding the standards expected by the frames of the new syllabus. The authors are deeply grateful to him for providing his valuable assistance.

We would greatly welcome your suggestions and feedback for further improvement.

Authors

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New Syllabus – 2018

Aims:

1. To enable candidates to acquire knowledge and to develop an understanding of the terms, concepts, symbols, definitions, principles, processes and formulae of Mathematics at the Senior Secondary stage.
2. To develop the ability to apply the knowledge and understanding of Mathematics to unfamiliar situations or to new problems.
3. To develop an interest in Mathematics.
4. To enhance ability of analytical and rational thinking in young minds.
5. To develop skills of:
   (a) computation. (b) logical thinking.
   (c) handling abstractions. (d) generalizing patterns.
   (e) solving problems using multiple methods.
   (f) reading tables, charts, graphs, etc.
6. To develop an appreciation of the role of Mathematics in day-to-day life.
7. To develop a scientific attitude through the study of Mathematics.

A knowledge of Arithmetic, Basic Algebra (Formulae, Factorization, etc.), Basic Trigonometry and Pure Geometry is assumed.

As regards to the standard of algebraic manipulation, students should be taught:

(i) To check every step before proceeding to the next particularly where minus signs are involved.

(ii) To attack simplification piecemeal rather than en block.

(iii) To observe and act on any special features of algebraic form that may be obviously present.

CLASS XI

*The syllabus is divided into **three** sections A, B and C.*
*Section A is compulsory for all candidates. Candidates will have a choice of attempting questions from **either** Section B or Section C.*
*There will be one paper of **three** hours duration of 100 marks.*
***Section A (80 marks):** Candidates will be required to attempt **all** questions. Internal choice will be provided in three questions of four marks each and two questions of six marks each.*
***Section B/Section C (20 marks):** Candidates will be required to attempt **all** questions **either** from Section B or Section C. Internal choice will be provided in two questions of four marks each.*

S.No.	*Unit*	*Total Weightage*
	SECTION A: 80 Marks	
1.	Sets and Functions	22 Marks
2.	Algebra	34 Marks
3.	Coordinate Geometry	8 Marks
4.	Calculus	8 Marks
5.	Statistics and Probability	8 Marks

SECTION B: 20 Marks		
6.	Conic Section	12 Marks
7.	Introduction to Three Dimensional Geometry	4 Marks
8.	Mathematical Reasoning	4 Marks
Or **SECTION C: 20 Marks**		
9.	Statistics	6 Marks
10.	Correlation Analysis	6 Marks
11.	Index Numbers and Moving Averages	8 Marks
Total		100 Marks

SECTION A

1. SETS AND FUNCTIONS

(*i*) Sets

Sets and their representations. Empty set. Finite and Infinite sets. Equal sets. Subsets. Subsets of a set of real numbers especially intervals (with notations). Power set. Universal set. Venn diagrams. Union and Intersection of sets. Practical problems on union and intersection of two and three sets. Difference of sets. Complement of a set. Properties of Complement of Sets.

(*ii*) Relations and Functions

Ordered pairs, Cartesian product of sets. Number of elements in the cartesian product of two finite sets. Cartesian product of the set of reals with itself (upto $R \times R \times R$). Definition of relation, pictorial diagrams, domain, co-domain and range of a relation. Function as a special type of relation. Function as a type of mapping. Types of functions (one to one, many to one, onto, into). Domain, co-domain and range of a function. Real valued functions, domain and range of these functions, constant, identity, polynomial, rational, modulus, signum, exponential, logarithmic and greatest integer functions, with their graphs. Sum, difference, product and quotient of functions.

- **Sets:** *Self-explanatory.*
- **Basic concepts of Relations and Functions**
  - Ordered pairs, sets of ordered pairs.
  - Cartesian Product (Cross) of two sets, cardinal number of a cross product.

  Relations as:
  - an association between two sets.
  - a subset of a Cross Product.
  - Domain, Range and Co-domain of a Relation.

  Functions:
  - As special relations, concept of writing "y is a function of x" as $y = f(x)$.
  - Introduction of Types: one to one, many to one, into, onto.
  - Domain and range of a function.
  - Sketches of graphs of exponential function, logarithmic function, modulus function, step function and rational function.

(*iii*) Trigonometry

Positive and negative angles. Measuring angles in radians and in degrees and conversion from one measure to another. Definition of trigonometric functions with the help of unit circle. Truth of the identity $\sin^2 x + \cos^2 x = 1$, for all x. Signs of trigonometric

functions. Domain and range of trigonometric functions and their graphs. Expressing sin $(x \pm y)$ and cos $(x \pm y)$ in terms of sin x, sin y, cos x and cos y and their simple applications. Deducing the identities like the following:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cos(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

$$\sin \alpha \pm \sin \beta = 2 \sin \frac{1}{2}(\alpha \pm \beta) \cos \frac{1}{2}(\alpha \mp \beta)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$$

Identities related to sin $2x$, cos $2x$, tan $2x$, sin $3x$, cos $3x$ and tan $3x$. General solution of trigonometric equations of the type sin y = sin a, cos y = cos a and tan y = tan a. Properties of triangles (proof and simple applications of sine rule, cosine rule and area of triangle).

- **Angles and Arc lengths**
  - *Angles:* Convention of sign of angles.
  - Magnitude of an angle: Measures of Angles; Circular measure.
  - The relation $S = r\theta$ where θ is in radians. Relation between radians and degree.
  - Definition of trigonometric functions with the help of unit circle.
  - Truth of the identity $\sin^2 x + \cos^2 x = 1$.

Note. Questions on the area of a sector of a circle are required to be covered.

- **Trigonometric Functions**
  - Relationship between trigonometric functions.
  - Proving simple identities.
  - Signs of trigonometric functions.
  - Domain and range of the trigonometric functions.
  - Trigonometric functions of all angles.
  - Periods of trigonometric functions.
  - Graphs of simple trigonometric functions (only sketches).

Note. Graphs of sin x, cos x, tan x, sec x, cosec x and cot x are to be included.

- **Compound and Multiple Angles**
  - Addition and subtraction formula: sin$(A \pm B)$; cos$(A \pm B)$; tan$(A \pm B)$; tan $(A + B + C)$, etc. Double angle, triple angle, half angle and one third angle formula as special cases.
  - Sums and differences as products

    $$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right), \text{ etc.}$$

  - Product to sum or difference, *i.e.,* 2 sin A cos B = sin $(A + B)$ + sin $(A - B)$, etc.
- **Trigonometric Equations**
  - Solution of trigonometric equations (General solution and solution in the specified range).
  - Equations expressible in terms of, sin $\theta = 0$, etc.
  - Equations expressible in terms, *i.e.,* sin θ = sin α, etc.

– Equations involving multiple and sub-multiple angles, *i.e.*, $\sin^2 \theta = \sin^2 \alpha$, etc.

– Linear equations of the form $a \cos \theta + b \sin \theta = c$, where $|c| \leq \sqrt{a^2 + b^2}$ and $a, b \neq 0$.

– Properties of Δ

Sine Formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ etc.}$$

Area of triangle: $\Delta = \frac{1}{2} bc \sin A$, etc. Simple applications of the above.

2. ALGEBRA

(*i*) Principle of Mathematical Induction

Process of the proof by induction, motivating the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

Using induction to prove various summations, divisibility and inequalities of algebraic expressions only.

(*ii*) Complex Numbers

Introduction of complex numbers and their representation. Algebraic properties of complex numbers. Argand plane and polar representation of complex numbers. Square root of a complex number. Cube root of unity.

– Conjugate, modulus and argument of complex numbers and their properties.

– Sum, difference, product and quotient of two complex numbers, additive and multiplicative inverse of a complex number.

– Locus questions on complex numbers.

– Triangle inequality.

– Square root of a complex number.

– Cube roots of unity and their properties.

(*iii*) Quadratic Equations

Statement of Fundamental Theorem of Algebra, Solution of quadratic equations (with real coefficients).

- Use of the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In solving quadratic equations.

- Equations reducible to quadratic form.
- Nature of roots

– Product and sum of roots.

– Roots are rational, irrational, equal, reciprocal, one square of the other.

– Complex roots.

– Framing quadratic equations with given roots.

Note. Questions on equations having common roots are to be covered.

- **Quadratic Functions**

  Given α, β as roots then find the equation whose roots are of the form α^3, β^3, etc.

  Case I: $a > 0$ → Real roots; Complex roots; Equal roots

  Case II: $a < 0$ → Real roots; Complex roots; Equal roots

  where 'a' is the coefficient of x^2 in the equations of the form $ax^2 + bx + c = 0$.

  Understanding the fact that a quadratic expression (when plotted on a graph) is a parabola.

  - **Sign of quadratic**

    Sign when the roots are real and when they are complex.
  - **Inequalities**

– *Linear Inequalities*

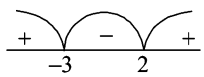
Algebraic solutions of linear inequalities in one variable and their representation on the number line. Graphical representation of linear inequalities in two variables. Graphical method of finding a solution of system of linear inequalities in two variables.

Self-explanatory.

– *Quadratic inequalities*

Using method of intervals for solving problems of the types:

$$x^2 + x - 6 \geq 0$$



A perfect square, *e.g.*, $x^2 - 6x + 9 \geq 0$.

– Inequalities involving rational expression of type $\frac{f(x)}{g(x)} \leq a$, etc., to be covered

(*iv*) Permutations and combinations

Fundamental principle of counting. Factorial n. (n!) Permutations and combinations, derivation of formulae for ${}^nP_r$ and ${}^nC_r$ and their connections, simple application.

- Factorial notation $n!$, $n! = n(n-1)!$
- Fundamental principle of counting.
- Permutations

– ${}^nP_r$.

– **Restricted permutation.**

  – Certain things always occur together.
  – Certain things never occur.
  – Formation of numbers with digits.
  – Word building – repeated letters – no letters repeated.
  – Permutation of alike things.
  – Permutation of repeated things.
  – Circular permutation – Clockwise/counterclockwise – Distinguishable / not distinguishable.

- **Combinations**
  - ${}^nC_r$, ${}^nC_n = 1$, ${}^nC_0 = 1$, ${}^nC_r = {}^nC_{n-r}$, ${}^nC_x = {}^nC_y$, then $x + y = n$ or $x = y$, ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$
  - When all things are different.
  - When all thing are not different.
  - Mixed problems on permutations and combinations.

(*v*) Binomial Theorem

History, statement and proof of the binomial theorem for positive integral indices. Pascal's triangle, General and middle term in binomial expansion, simple applications.

- Significance of Pascal's triangle.
- Binomial theorem (proof using induction) for positive integral powers, *i.e.*, $(x + y)^n = {}^nC_0x^n + {}^nC_1 \cdot x^{n-1}y + \ldots + {}^nC_ny^n$.
  Questions based on the above.

(*vi*) Sequence and Series

Sequence and Series. Arithmetic Progression (A.P.). Arithmetic Mean (A.M.). Geometric Progression (G.P.), general term of a G.P., sum of first n terms of a G.P., infinite G.P. and its sum, geometric mean (G.M.), relation between A.M. and G.M. Formulae for the following special sums Σn, Σn^2, Σn^3.

- **Arithmetic Progression (A.P.)**
  - $T_n = a + (n - 1)d$
  - $S_n = \frac{n}{2}\{2a + (n - 1)d\}$
  - Arithmetic mean: $2b = a + c$
  - Inserting two or more arithmetic means between any two numbers.
  - Three terms in A.P.: $a - d$, a, $a + d$
  - Four terms in A.P.: $a - 3d$, $a - d$, $a + d$, $a + 3d$
- **Geometric Progression (G.P.)**
  - $T_n = ar^{n-1}$, $S_n = \frac{a(r^n - 1)}{r - 1}$, $S_\infty = \frac{a}{1 - r}; |r| < 1$ Geometric Mean, $b = \sqrt{ac}$
  - Inserting two or more Geometric Means between any two numbers.
  - Three numbers are in G.P. ar, a, ar^{-1}
  - Four terms are in G.P. ar^3, ar, ar^{-1}, ar^{-3}
- **Arithmetico Geometric Series**
  Identifying series as A.G.P. (when we substitute $d = 0$ in the series, we get a G.P. and when we substitute $r = 1$ the A.P.)
- Special Sums Σn, Σn^2, Σn^3.
  Using these summations to sum up other related expression.

3. COORDINATE GEOMETRY

(*i*) Straight Lines

Brief recall of two dimensional geometry from earlier classes. Shifting of origin. Slope of a line and angle between two lines. Various forms of equations of a line: parallel to axis, point slope form, slope-intercept form, two-point form, intercept form and normal form. General equation of a line. Equation of family of lines passing through the point of intersection of the lines, Distance of a point from a line.

- **Basic concepts of Points and their coordinates.**
- **The straight line**
  - Slope or gradient of a line.
  - Angle between two lines.

- Condition of perpendicularity and parallelism.
- Various forms of equation of lines.
- Slope intercept form.
- Two point slope form.
- Intercept form.
- Perpendicular/normal form.
- General equation of a line.
- Distance of a point from a line.
- Distance between parallel lines.
- Equation of lines bisecting the angle between two lines.
- Equation of family of lines
- Definition of a locus.
- Equation of a locus.

(*ii*) Circles

- Equations of a circle in:
  - Standard form.
  - Diameter form.
  - General form.
  - Parametric form.
- Given the equation of a circle, to find the centre and the radius.
- Finding the equation of a circle.
  - Given three non collinear points.
  - Given other sufficient data for example centre is (h, k) and it lies on a line and two points on the circle are given, etc.
- Tangents:
  - Condition for tangency.
  - Equation of a tangent to a circle.

4. CALCULUS

Limits and Derivatives

Derivative introduced as rate of change both as that of distance function and geometrically. Intuitive idea of limit. Limits of polynomials and rational functions, trigonometric, exponential and logarithmic functions. Definition of derivative relate it to scope of tangent of the curve, Derivative of sum, difference, product and quotient of functions. Derivatives of polynomial and trigonometric functions.

- **Limits**
  - Notion and meaning of limits.
  - Fundamental theorems on limits (statement only).
  - Limits of algebraic and trigonometric functions.
  - Limits involving exponential and logarithmic functions.

Note. Indeterminate forms are to be introduced while calculating limits.

- **Differentiation**
  - Meaning and geometrical interpretation of derivative.
  - Derivatives of simple algebraic and trigonometric functions and their formulae.
  - Differentiation using first principles.
  - Derivatives of sum/difference.
  - Derivatives of product of functions.
  - Derivatives of quotient of functions.

5. STATISTICS AND PROBABILITY

(*i*) Statistics

Measures of dispersion: range, mean deviation, variance and standard deviation of ungrouped/grouped data. Analysis of frequency distributions with equal means but difference variances.

- Mean deviation about mean and median.
- Standard deviation – by direct method, short cut method and step deviation method.

Note: Mean, Median and Mode of grouped and ungrouped data and required to be covered.

(*ii*) Probability

Random experiments; outcomes, sample spaces (set representation). Events; occurrence of events, 'not', 'and' and 'or' events, exhaustive events, mutually exclusive events, Axiomatic (set theoretic) probability, connections with other theories studied in earlier classes. Probability of an event, probability of 'not', 'and' and 'or' events.

- Random experiments and their outcomes.
- Events: sure events, impossible events, mutually exclusive and exhaustive events.
  - Definition of probability of an event.
  - Laws of probability addition theorem.

SECTION B

6. CONIC SECTION

Sections of a cone, ellipse, parabola, hyperbola, a point, a straight line and a pair of intersecting lines as a degenerated case of a conic section. Standard equations and simple properties of parabola, ellipse and hyperbola.

- Conics as a section of a cone.
  - Definition of Foci, Directrix, Latus Rectum.
  - $PS = ePL$ where P is a point on the conics, S is the focus, PL is the perpendicular distance of the point from the directrix.

(*i*) Parabola

$e = 1$, $y^2 = \pm 4ax$, $x^2 = 4ay$, $y^2 = -4ax$,

$x^2 = -4ay$, $(y - \beta)^2 = \pm 4a\,(x - \alpha)$,

$(x - \alpha)^2 = \pm 4a\,(y - \beta)$.

- Rough sketch of the above.
- The latus rectum; quadrants they lie in; coordinates of focus and vertex; and equations of directrix and the axis.
- Finding equation of parabola when foci and directrix are given, etc.
- Application questions based on the above.

(*ii*) Ellipse

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ e < 1,\ b^2 = a^2\,(1 - e^2)$
- $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$

- Cases when $a > b$ and $a < b$.
- Rough sketch of the above.
- Major axis, minor axis, latus rectum; coordinates of vertices, focus and centre; and equations of directrices and the axes.
- Finding equation of ellipse when focus and directrix are given.
- Simple and direct questions based on the above.
- Focal property, *i.e.*, $SP + SP' = 2a$.

(*iii*) Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ e > 1,\ b^2 = a^2(e^2 - 1)$$

- $\dfrac{(x-\alpha)^2}{a^2} - \dfrac{(y-\beta)^2}{b^2} = 1$
- Cases when coefficient of y^2 is negative and coefficient of x^2 is negative.
- Rough sketch of the above.
- Focal property, *i.e.*, $SP - S'P = 2a$.
- Transverse and Conjugate axes, Latus rectum, coordinates of vertices, foci and centre; and equations of the directrices and the axes.
- General second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
  - Case 1: Pair of straight line if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
  - Case 2: $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, then represents a parabola if $h^2 = ab$, ellipse if $h^2 < ab$, and hyperbola if $h^2 > ab$.
- Condition that $y = mx + c$ is a tangent to the conics, general equation of tangents, point of contact and locus problems.

7. INTRODUCTION TO 3-DIMENSIONAL GEOMETRY

Coordinate axes and coordinate planes in three dimensions. Coordinates of a point. Distance between two points and section formula.

- As an extension of 2-D.
- Distance formula.
- Section and midpoint form.

8. MATHEMATICAL REASONING

Mathematically acceptable statements. Connecting words/phrases – consolidating the understanding of "if and only if (necessary and sufficient) condition", "implies", "and/or", "implied by", "and", "or", "there exists" and their use through variety of examples related to the Mathematics and real life. Validating the statements involving the connecting words, Difference between contradiction, converse and contrapositive.

Self-explanatory

SECTION C

9. STATISTICS

- Combined mean and standard deviation.
- The Median, Quartiles, Deciles, Percentiles and Mode of grouped and ungrouped data.

10. CORRELATION ANALYSIS

- Definition and meaning of covariance.
- Coefficient of Correlation by Karl Pearson. If $x - \bar{x}$, $y - \bar{y}$ are small non-fractional numbers, we use

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{\Sigma(y-\bar{y})^2}}$$

If x and y are small numbers, we use

$$r = \frac{\Sigma xy - \frac{1}{N}\Sigma x \Sigma y}{\sqrt{\Sigma x^2 - \frac{1}{N}(\Sigma x)^2}\sqrt{\Sigma y^2 - \frac{1}{N}(\Sigma y)^2}}$$

Otherwise, we use assumed means

A and B, where $u = x - A$, $v = y - B$

$$r = \frac{\Sigma uv - \frac{1}{N}(\Sigma u)(\Sigma v)}{\sqrt{\Sigma u^2 - \frac{1}{N}(\Sigma u)^2}\sqrt{\Sigma v^2 - \frac{1}{N}(\Sigma v)^2}}$$

- Rank correlation by Spearman's (Correction included).

11. INDEX NUMBERS AND MOVING AVERAGES

- **Index Numbers**
  - Price index or price relative.
  - Simple aggregate method.
  - Weighted aggregate method.
  - Simple average of price relatives.
  - Weighted average of price relatives (cost of living index, consumer price index).
- **Moving Averages**
  - Meaning and purpose of the moving averages.
  - Calculation of moving averages with the given periodicity and plotting them on a graph.
  - If the period is even, then the centred moving average is to be found out and plotted.

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CHAPTER 1

SETS

1.01. What is a set?

A set is a **well defined** *collection of objects.* By 'well defined' we mean that it must be possible to tell beyond doubt whether or not a given object belongs to the collection that we are considering.

For example, the following are well defined collections and so are example of sets:

(*i*) The set of numbers 1, 3, 5, 7 and 10.

(*ii*) The set of pupils of your class.

The following do not describe a well defined collection and so are not sets:

(*i*) The vegetables which taste good to all. People may have different tastes.

(*ii*) All good movies. People may have different likings.

1.02. Members of a set and symbol for 'belongs to'

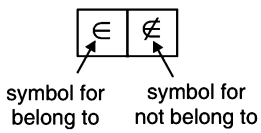
The objects that belong to a set are called **members or elements** of the set. The Greek letter epsilon '$\in$' is used for '***belong to***'. For example,

$a \in \{a, b, c\}$, $b \in \{a, b, c\}$, $c \in \{a, b, c\}$,

We also say that 'a', 'b', 'c' are included' in the set.

The symbol $\notin$ is used to mean '***is not a member of.***'

Thus $1 \notin \{0, 2, 4, 6\}$.



Note. The braces { } are used to enclose the members of a set.

1.03. Representation of a set

Capital letters, e.g., A, B, S etc., are used to name sets. There are three ways of representing a set.

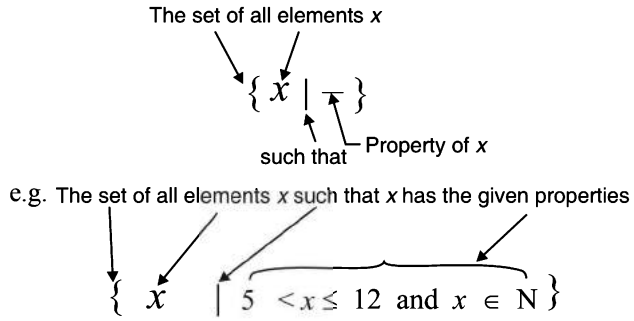
1. **In words.** We can use words to describe a set, e.g., the set of multiples of whole numbers less than 10.
2. **Roster or Tabulation method**
3. **Rule method or Set builder method**

Tabulation form. In this form we define a set by actually listing its elements. For example, the elements in the set A of letters of the English alphabet can be listed as $A = \{a, b, c, \ldots z\}$. The set of whole numbers is $\{0, 1, 2, 3, \ldots\}$

Note. We do not list an element more than once in a given set. Thus $\{b, o, k\}$ and not $\{b, o, o, k\}$ denotes the set of different letters in the word 'book'. Also $\{k, b, o\}$ defines the same set as $\{b, o, k\}$. ***When listing the members of a set, the name of a member is listed only once, and the members can be written in any order.***

Set builder method. In this method a set is defined by stating properties which the statement of the set must satisfy. We use braces { } to write set in this form. The brace on the left is followed by a lower case italic letter that represents any element of the given set. This letter is followed by a vertical bar and the brace on the right. Symbolically, it is of the form $\{(x \mid —\}$. Here, we write the condition which x satisfies, or more briefly, $\{x \mid P(x)\}$, where $P(x)$ is a proposition stating a condition

for x. The x is a sort of place holder, for all possible elements x that have the given property. The vertical line is a symbol for '*such that*' and the symbolic form $A = \{x \mid x \text{ is even}\}$ reads "A is the set of numbers x such that x is even. Sometimes a colon (:) or a semicolon (;) is also used in place of the vertical bar."



Ex. 1. Write the following sets in the set builder form.

(*i*) the number 1, 3, 5 ... **(*ii*) the solutions of the equation $x^2 + 5x - 24 = 0$**

(*iii*) $F = \left\{\frac{3}{8}, \frac{4}{9}, \frac{5}{10}, \frac{6}{11}, \frac{7}{12}, \frac{8}{13}\right\}$ **(*iv*) {0, 1, 8, 27, 64, 125, 216}**

Sol. (*i*) $\{x \mid x \text{ is an odd number}\}$ (*ii*) $\{x : x^2 + 5x - 24 = 0\}$

(*iii*) We observe that in the given set, the denominator of each fraction is 5 more than the numerator. Hence, the set builder form of the set is

$$F = \left\{x \mid x = \frac{n}{n+5}, n \in N \text{ and } 3 \le n \le 8\right\}$$

(*iv*) We observe that the elements of the given set are the cubes of the first seven whole numbers.
$\therefore$ In the set builder form $C = \{x \mid x = n^3, n \in W \text{ and } n \le 6\}$

TYPES OF SETS

1.04. Finite and Infinite sets

Intuitively, a set is finite if it consists of a definite number of different elements, i.e., if in counting the different numbers of the set, the counting can come to an end. A set is infinite if the process of counting its elements never comes to an end.

For example, the following are **finite** sets.

1. The set of teachers in your school.
2. The set of continents.
3. The set of prime numbers between 1 and 50.
4. $\{x : x = 5n + 3, n \in Z, -2 \le n \le 2\}$.

The following are **infinite** sets.

1. The set of square numbers.
2. The set of all points on a line between two distinct points A and B on it.
3. $\{x : x \in Z, x < 0\}$.

1.05. The empty set or the null set

A set that contains no members is called the empty set or the null set.

For example, each one of the following is the empty set.

(1) The set of the months of a year that have fewer than 15 days.

(2) The set of students in your class who are more than 3 metre tall.

(3) $\{x : x \in R, x \neq x\}$

(4) $\{x \mid x \in R, x^2 = -1\}$

ϕ symbol for the Empty Set

because a real number is always equal to itself and because the square of a real number, cannot be negative, so no such real number exists.

- The emplty set is written as $\{\ \}$ or ϕ.

Maths Alert!

1. There is only one empty set and it is called '*the empty set*' there is nothing like 'an empty set' or '*empty sets*'
2. $\{0\}$ and $\{\phi\}$ are not empty sets because each of these sets contains one element.

1.06. Singleton set

A set containing only one element is called a ***singleton set***, e.g., $A = \{5\}$ is a singleton set.

1.07. Equal sets

Two sets A and B are called ***identical or equal*** *sets, written as A = B, if they have exactly the same elements.*

For example

(1) If $A = \{p, q, r\}$, and $B = \{q, r, p\}$, then $A = B$.

(2) If $P = \{x \mid x^2 - 7x + 6 = 0\}$, $Q = \{1, 6\}$, $R = \{6, 1\}$, then $P = Q = R$.

1.08. Cardinal number

The number of elements in a finite set is called the cardinal number of the set. The cardinal number of a set A is abbreviated as $\boldsymbol{n(A)}$.

n (A) symbol for cardinal number of set A

For example

1. If $A = \{a\}$, $B = \{a, b\}$, $C = \{p, q, r\}$, then $n(A) = 1$, $n(B) = 2$, $n(C) = 3$.
2. Let P be the set of letters in the word 'MATHEMATICS', then A = [M, A, T, H, E, I, C, S] and $n(P) = 8$.

Note

1. Cardinal number of an infinite set is not defined.
2. Cardinal number of the empty set is zero, *i.e.*, $n(\phi) = 0$.

1.09. Equivalent sets

Two finite sets A and B are said to be equivalent, if they have the same number of elements, i.e., if $n(A) = n(B)$.

The equivalence of two sets A and B is expressed by writing $A \leftrightarrow B$.

Ex. 2. Let $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, then since each set contains 4 elements, *i.e.*, $n(A) = n(B) = 4$, therefore, they are equivalent sets.

For example

1. Let A set of the letters of the word '*PAN*', B = set of the letters of the word '*NAP*', C = set of letters of the word '*PEN*', then

$A = B$, $A \neq C$, $B \neq C$.

Also, since $n(A) = n(B) = n(C)$, we have $A \leftrightarrow B$, $A \leftrightarrow C$, $B \leftrightarrow C$.

Maths Alert ! It is obvious from the definition of equal and equivalent sets that equal sets are always equivalent but equivalent sets may or may not be equal.

1.10. Overlapping sets

*Two sets are called **overlapping sets** if, they have at least one element in common.*

For example, the sets $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 9, 11\}$ are overlapping sets, since the element 3 is common in them.

1.11. Disjoint sets

Two sets are called **disjoint sets** *if they have no element in common.*

For example, the set $R = \{1, 3, 5, 7\}$, $S = \{2, 4, 6, 8\}$ are disjoint sets, since no element is common to them.

1.12. Sets of numbers

You are familiar with the following infinite sets of numbers.

1. Set **N** of **natural numbers,** *i.e.*, $\mathbf{N} = \{1, 2, 3, ...\}$
2. Set **W** of **whole numbers**, *i.e.*, $\mathbf{W} = \{0, 1, 2, 3, ...\}$
3. Set **Z** of **integers**, i.e., $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
4. Set **Q** of **rational numbers**, i.e., numbers that can be expressed as a quotient (or ratio) of two integers. The decimal form of a rational number either terminates, such as $\frac{1}{4} = 0.25$, or repeats, such as $\frac{-4}{3} = -1.\overline{3} = -1.3333...$

The following numbers are all rational numbers.

$$5,\ 0.7,\ -8,\ 3\frac{5}{11},\ 0,\ 5.3\overline{12}$$

5. The set of **irrational numbers**, such as $\sqrt{3}$ and π, which cannot be expressed as a quotient of two integers, and their decimal forms do not terminate or repeat. However, we can approximate these numbers using terminating decimals, e.g., $\sqrt{2} = 1.414$, $\pi = 3.14$, $\sqrt{3} = 1.732$.

The following numbers are all irrational numbers.

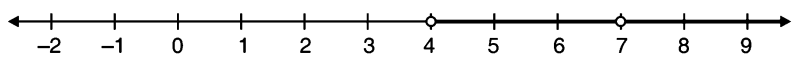
$$-7\sqrt{5},\ \pi,\ e,\ \sqrt{8},\ \frac{\sqrt{3}}{2},\ \sqrt[3]{11}$$

6. The set of **real numbers R** includes all the above sets of numbers.

1.13. Intervals

Many infinite sets, such as the real numbers, cannot be represented in roster form. There are other methods of representing these sets. For example, the number line represents the set of all real numbers.

The set of real numbers between 4 and 7, which is also an infinite set, can be represented on a number line or by an inequality.



Number line

$4 < x < 7$

Inequality

An interval is the set of all numbers between two endpoints, such as 4 and 7.

In **Interval notation** the symbols **[and]** are used to include an endpoint from an interval, and the symbols **(and)** are used to exclude an endpoint from an interval, e.g.,

(4, 7) The set of real numbers betweens 4 and 7 but not including 4 and 7.

The notation [ ] indicates **closed interval** as it includes the end points also while the notation ( ) is called an **open interval** as it does not include the endpoints of the interval.

- A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The end-points are also called **boundary points;** they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together comprise the interval's **interior**. Infinite intervals are closed if they contain a finite endpoint, and open otherwise. The entire real line R is an infinite interval that is both open and closed.
- An interval that extends forever in the positive direction goes to infinity (∞), and an interval that extends forever in the negative direction goes to negative infinity (– ∞).

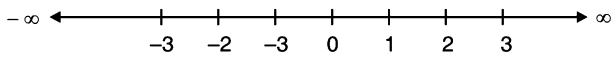


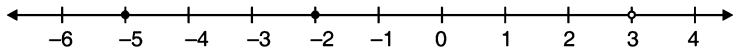
Table: Types of intervals

	Notation	Set description	Type	Picture
Finite	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	R (set of all real numbers)	Both open and closed	

Ex. 3. Use interval notation to represent each set of numbers.

(i) $5 \le x < 7$

(ii)



Number line

Sol. (*i*) [5, 7); *5 is included but 7 is not.*

(*ii*) There are two intervals graphed on the number line

[–5, –2] *Both – 5 and – 2 are included.*

(3, ∞) 3 is not included and the interval continues forever in the positive direction.

The representation is [–5, – 2] or [–3, ∞]; The word 'or' is used to indicate that a set includes more than one interval.

EXERCISE 1 (*a*)

Type I. (*Question related to 'definitions and description of sets*)

1. Which of the collections are sets?

(*i*) The collection of all months of a year, beginning with letter J

(*ii*) The collection of most talented writers of India.

(*iii*) The collection of all natural numbers less than 100.

(*iv*) A collection of most dangerous animals of the world. **(*NCERT*)**

2. Let $A = \{1, 2, 3, 4, 5, 6\}$. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces. ***(NCERT)***

(*i*) 5 ... *A*, (*ii*) 8 ... *A*, (*iii*) 0 ... *A* (*iv*) 4 ... *A*

3. Write down a description of each of the following sets. (There could be different suitable descriptions.)

(*i*) {2, 4, 6, 8} (*ii*) {7, 14, 21, 28, 35} (*iii*) {1, 2, 3, 4, 6, 12}

4. List the following sets in roster form.

(*i*) The set of square numbers less than 40.

(*ii*) The set of colours of the rainbow.

(*iii*) (*a*) The set of factors of 144.

(*b*) The set of prime factors of 144.

(*iv*) The set of natural numbers less than 50.

(*v*) The set of consonants before i in the English alphabet.

(*vi*) The set of letters in the word '***Satellite***'

5. Rewrite the following sets in the indicated notation.

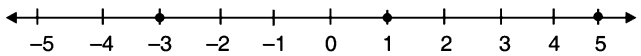
(*i*) {–2, – 4, – 6, – 8}; (*a*) *Words* (*b*) *Set-builder notation*

(*ii*) Positive multiples of 11; *Roster form*

(*iii*) {– 9, – 7, – 5, – 3, – 1}; *Set-builder notation*

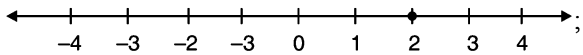
(*iv*) Even numbers between 27 and 39 (*a*) *Roster form* (*b*) *Set-builder notation*

(*v*) $\{x \mid 0 < x < 1\}$; (*a*) Words (*b*) *Roster form*

(*vi*) 

Roster form

(*vii*) Negative multiples of 3 **Set-builder form**

(*viii*) 

; **Set-builder form**

(*ix*) Numbers more than 2 units from 8, **Set-builder form.**

(*x*) $x \neq 5$ and $x \leq 10$; **Set-builder form.**

(*xi*) $\{y \mid y = 5x - 2, y \in N\}$; **Roster form**

(*xii*) $\left\{x \mid x = \dfrac{3p+1}{2p-1}, p \in W \text{ and } p \leq 5\right\}$; **Roster form**

Type II. (*Question related to types of sets*)

6. State whether each of the following sets is finite or infinite:

(*i*) The set of lines which are parallel to the *x*-axis.

(*ii*) The set of letters in the English alphabet.

(*iii*) The set of numbers which are multiple of 5.

(*iv*) The set of animals living on earth.

(*v*) The set of circles through the origin (0, 0). ***(NCERT)***

(*vi*) The set of whole numbers greater than 5.

(*vii*) The set of natural numbers less than one billion.

(*viii*) The set of integers between – 4 and 4.

(*ix*) The set of rational numbers between 0 and 1.

7. Which of the following sets are empty sets?

(*i*) $A = \{x : x \text{ is a human being living on Mars}\}$

(*ii*) $B = \{x : x \text{ is an odd number divisible by 2}\}$

(*iii*) $C = \{x : x \text{ is a point common to any two parallel lines.}\}$

(*iv*) $D = \{0\}$

(*v*) $E = \{x : x$ is a natural number, $x < 5$ and simultaneously $x > 7\}$ **(*NCERT*)**

8. Are the following sets equal? Give reasons.

(*i*) $A = \{2, 3\}$; $B = \{x : x$ is a solution of $x^2 + 5x + 6 = 0\}$

(*ii*) $A = \{x : x$ is a letter in the word FOLLOW$\}$

$B = \{y : y$ is a letter in the word WOLF$\}$ **(*NCERT*)**

9. Which of the following are singleton sets?

(*i*) $A = \{x : |x| = 5, x \in N\}$ (*ii*) $B = \{x : x^2 - 11x + 24 = 0, x \in N\}$

(*iii*) $C = \{x : x^3 = -125, x \in Z\}$

10. State the value of $n(A)$ for each of the following sets.

(*i*) $A =$ {Months of the year} (*ii*) $A =$ {Planets of our solar system}

(*iii*) $A = \{x : x$ is an integer and $-8 \le x \le 3\}$

(*iv*) $A = \{x : x$ is an even number$\}$

Type III. Questions related to interval notation

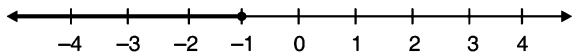
Use interval notation to represent each set of numbers.

11. (*i*) $-17 < x < 0$ (*ii*) $6 \le x \le 12$ (*iii*) $-1 < x \le 4$ (*iv*) $-4 \le x < 7$

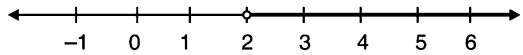
(*v*) $x \le 3$ or $5 < x \le 9$ (*vi*) $\{x \mid x \ge 99\}$ (*vii*) $\{x \mid x \ne 1\}$ (*viii*) $\{1, 3, 5, 7, ...\}$

(*ix*) $x \ne 3$

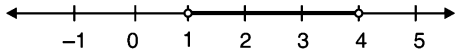
12. (*i*)



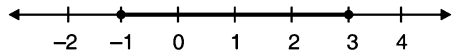
(*ii*)



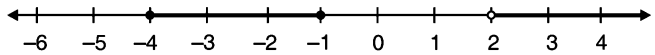
(*iii*)



(*iv*)



(*v*)



ANSWERS

1. (*i*) and (*iii*) are sets **2.** (*i*) $\in$ (*ii*) $\notin$ (*iii*) $\notin$ (*iv*) $\in$

3. (*i*) the set of even numbers between 1 and 9, or simply {even numbers between 1 and 9}

(*ii*) The set of multiples of 7 (*iii*) The set of factors of 12.

4. (*i*) {1, 4, 9, 16, 25, 36} (*ii*) {red, orange, yellow, green, blue, indigo, violet}

(*iii*) (*a*) {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 144}

(*b*) {2, 3} (*iv*) {1, 2, 3, 4, 5, 6, ..., 49}

(*v*) {*b*, *c*, *d*, *f*, *g*, *h*} (*vi*) {*s*, *a*, *t*, *e*, *l*, *i*}

5. (*i*) (*a*) negative even numbers; (*b*) $\{x \mid x < 0$ and x is even$\}$

(*ii*) {11, 22, 33, 44, 55, 66, 77, ...} (*iii*) $\{x \mid -9 \le x \le -1$ and x is odd$\}$

(*iv*) (*a*) {28, 30, 32, 34, 36, 38} (*b*) $\{x \mid 27 < x < 39$ and x is even$\}$

(*v*) (*a*) Numbers greater than 0 and less than 1

(*b*) cannot be expressed in roster form

(*vi*) $\{-3, 1, 5\}$ (*vii*) $\{x \mid x = -3n$ and n is a natural number$\}$

(*viii*) $\{x \mid x = 2\}$ (*ix*) $\{x \mid x = 8 + 2\}$

(*x*) $\{x \mid x < 5$ or $5 < x \le 10\}$ (*xi*) {3, 8, 13, 18, ...}

(*xii*) $\left\{-1, 4, \frac{7}{3}, 2, \frac{13}{7}, \frac{16}{9}\right\}$

6. (*i*) Infinite (*ii*) Finite (*iii*) Infinite (*iv*) Finite (*v*) Infinite
(*vi*) Infinite (*vii*) Finite (*viii*) Finite (*ix*) Infinite

7. (*i*), (*ii*), (*iii*) and (*iv*) are empty set. (*v*) is not the empty set because it contains the element '0'.

8. (*i*) They are not equal because they do not contain exactly. $A = \{2, 3\}$ and $B = \{-2, -3\}$ they are however equivalent sets.

(*ii*) Equal because $A = \{F, O, L, W\}$ and $B = \{W, O, L, F\}$. Thus they contain exactly the same elements.

9. (*i*) and (*iii*) are singleton sets.

10. (*i*) 12 (*ii*) 8 (*iii*) 12
(*iv*) It is an infinite set therefore, its cardinal number cannot be determined.

11. (*i*) $(-17, 0)$ (*ii*) $[6, 12]$ (*iii*) $(-1, 4]$
(*iv*) $[-4, 7)$ (*v*) $(-\infty, 3]$ or $(5, 9]$ (*vi*) $[99, \infty)$
(*vii*) $(-\infty, 1)$ or $(1, \infty)$ (*viii*) cannot represented using interval notation
(*ix*) $(-\infty, -1)$ or $(3, \infty)$

12. (*i*) $(-\infty, -1)$ (*ii*) $(2, \infty)$ (*iii*) $(1, 4)$
(*iv*) $[-1, 3]$ (*v*) $[-4, -1]$ or $(2, \infty)$

1.14. Subsets

Sets whose all elements are contained in another set X_s are called subsets of the set X. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{1, 2\}$, than B is a subset of A.

Definition: ***If A and B are sets such that every member of set A is a member of set B, then set A is called a subset of set B.***

Since, by definition, if $A = \{p, q, r\}$ and $B = \{p, q, r\}$ then A is a subset of B. Also since A is equal to B, we use the symbol $\subseteq$ for 'is a subset of '. Thus, $A \subseteq B$, the symbol $\subseteq$ means contained in or possibly equal to.

● **Proper subset.** If a set Q is a subset of set P and set Q is not equal to set P, then set Q is called a **proper subset** of set P and is expressed as $Q \subset P$. The set P is called the **superset** of set Q. It is expressed as $P \supset Q$.

$P = \{a, b, c, d\}$
$Q = \{a, c\}$

The symbol $\not\subseteq$ indicates is not a subset of and the symbol $\not\subset$ indicates is not a proper subset of.

$\subseteq$	$\subset$	$\not\subset$	$\supset$
is a subset of	is a proper subset of	is not a proper subset of	is a superset of

● **Some theorems on subsets**

Theorem 1. ***Every set is a subset of itself.***

Theorem 2. ***The empty set is a subset of all sets,***

For example, if $A = \{1, 4, 9\}$, $B = \{1, 4, 9\}$, then $B \subseteq A$ and $\phi \subseteq A$.

Remark: If A is a subset of the empty set, *i.e.*, if $A \subseteq \phi$, then $A = \phi$, *i.e.*, A itself is the empty set. Thus, $\phi \subseteq \phi$,

Theorem 3. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. Let $x \in A$. Then since $A \subseteq B$, $x \in B$.

Now, $x \in B \Rightarrow x \in C$ because $B \subseteq C$

$x \in A$ and $x \in C \Rightarrow A \subseteq C$.

Theorem 4. *For any two sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$ and vice versa.*

Let $x \in A$, then $x \in B$ since $A \subseteq$ B ... (1)

Also, $x \in B \Rightarrow x \in A$ since $B \subseteq A$

(1) and (2) $\Rightarrow A = B$

Conversely, Let $A \subseteq$ B. then,

$x \in A \Rightarrow x \in B$, since $A = B \therefore A \subseteq B$

Similarly, $x \in B \Rightarrow x \in A \quad (\because A = B) \therefore B \subseteq A$

For example, let A = {Ashok, Kavita, John}, then all the possible subsets of A are

B = {Ashok, Kavita, John}, C = {Ashok, Kavita}, D = {Ashok, John}, E = {Kavita, John}, F = (Ashok), G = {Kavita}, H = {John}, $I = \phi$

Note that the sets B and I above are considered as subsets of A, *i.e.*, $B \subseteq A$ and $I \subseteq A$ but sets C, D, E, F, G and H are considered ***proper subsets*** of A, *i.e.*, $C \subset A$, $D \subset A$, $E \subset A$, $F \subset A$, $G \subset A$ and $H \subset A$

Math Alert! while writing subsets of a given set don't forget the given set itself and the empty set.

1.15. Number of subsets of a given set

A set having n element has 2^n subsets

For example

1. The subsets of the set {1, 2} are {1, 2}, {1}, {2}, ϕ

The number of subsets is 4, i.e., 2^2

2. The subsets of the set {a, b, c} are {a, b, c}, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, ϕ

The number of subsets is 8, *i.e.*, 2^3.

Ex. 4. The sets of numbers N = the set of natural numbers 1, 2, 3, ..., W = the set of whole numbers, 0, 1, 2, 3, ..., I or Z = the set of integers ..., – 2, – 1, 0, 1, 2, ..., Q = the set of rational numbers, R = the set of real numbers.

The above sets are related as N $\subset$ W $\subset$ Z $\subset$ Q $\subset$ R.

Ex. 5. If $A = \{x : x = 2n, n \in N\}$ and $B = \{x : x = 2^n, n \in N\}$. Is $A \subseteq B$ or $B \subseteq A$?

Sol. $A = \{x : x = 2n, n \in N\} = \{2, 4, 6, 8, 10, ...\}$ i.e., A contains all positive integral multiplies of 2.

$B = (x = 2^n, n \in N) = \{2, 2^2, 2^3, 2^4, ..\}$ *i.e.*, B contains all positive integral power of 2.

Since a power of 2 is a multiple of 2, therefore, every element of B is in $A \therefore B \subseteq A$.

Ex. 6. Prove that $\phi \subseteq A$ for any set A. Is it true that $\phi \subset$ A.

Sol. Let, if possible, ϕ be not a subset of A. Then there should be an element in ϕ which is not in A. Since there is no element in ϕ, this is not possible. Therefore, our assumption that ϕ is not a subset of A is wrong.

$\therefore \phi \subseteq A$

$\phi \subset A$ is not true since, if $A = \phi$, then we cannot find an element of A which is not in ϕ.

Ex. 7. Prove that $A \subseteq \phi \Rightarrow A = \phi$

Sol. We know that two sets **A** and **B** are equal if $A \subseteq B$ and $B \subseteq A$.

Here, we know that $\phi \subseteq A$. Also, $A \subseteq \phi$ (Given)

$\therefore A = \phi$.

1.16. Family of sets and power set

If we have a set whose elements are sets themselves, then the set formed is called the family of ***sets***, or the ***class of sets*** or the ***set of sets***.

For example, if A = {1, 2, 3}, B = {p, q}, C = {5, 6} then the set S whose elements are all or some of these sets is a family of sets. Then S = {{1, 2, 3}, {p, q}, {5, 6}}.

The family of all the subsets of a given set is called the **power set**. Power set of a given set A is denoted by $\boldsymbol{P(A)}$.

Thus, if $A = \{1, 2, 3\}$, then its subsets are ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, $\{1, 2, 3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$

Remark. It may be noted that $P(A)$ is a set. Therefore, we have $\{1\} \in P(A)$ true; $1 \in P(A)$ false; and $\{1\} \subset P(A)$ false. $\{(1)\} \subset P(A)$ true.

Notes:

1. ϕ and A are both elements of $P(A)$.

2. If $A = \phi$, then $P(\phi) = \{\phi\}$.

3. If $A = \{a\}$, then $P(A) = \{\phi, \{a\}\}$

Ex. 8. Prove that for a null set, ϕ, $n\{P(P(P(\phi))\} = 4$ where P indicates power set.

Sol. $P(\phi) = \{\phi\}$, $P(P(\phi)) = \{\phi, \{\phi\}\}$, $P(P(P(\phi)) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

This contains 4 elements $\therefore n\{P(P(P(\phi))\} = 4$.

EXERCISE 1(*b*)

1. Find the subsets of (*i*) $\{a\}$ (*ii*) {Reena, Sonu} (*iii*) ϕ (*iv*) $\{5, \{7\}\}$

2. Let $A = \{p, q, r\}$

(*i*) List all the subsets of A. (*ii*) List all the proper subsets of A.

3. Let P = {whole numbers less than 30}

(*i*) List the subset Q {even numbers} (*ii*) List the subset R {odd numbers}

(*iii*) List the subset S {prime numbers} (*iv*) List the subset T {square numbers}

(*iv*) List the subset U {triangle numbers}

4. Tell in each of the following, whether first set is a subset of the second set or not.

(*i*) A = Set of letters in the word 'LATE'
B = Set of letters in the word 'PLATE'

(*ii*) P = Set of even prime numbers.
$Q = \{x | x = 2p, p \in N \text{ and } 1 \le p \le 3\}$

(*iii*) L = Set of digits in the number 1590
M = Set of digits in the number 178902

(*iv*) E = Set of all triangles having 4 sides.
F = Set of digits in the number '100'.

5. Write the proper subsets of the following sets:

(*i*) $\{7\}$ (*ii*) $\{1, 3\}$ (*iii*) $\{c, a, b\}$ (*iv*) ϕ

6. How many subsets do the following sets have?

(*i*) A set having 5 elements.

(*ii*) The set of letters of the word 'CENTENARY'

7. How many proper subsets do the following sets have?

(*i*) The set of factors of 12.

(*ii*) The set A $\{x | x$ is a prime number, $x < 20\}$

8. Answer true or false

(*i*) $3 \subseteq \{3, 0\}$ (*ii*) $\{3\} \subseteq \{3, 0\}$ (*iii*) $\phi \in \{3, 0\}$ (*iv*) $0 \in \{3, 0\}$ (*v*) $\phi \subset \{3, 0\}$

(*vi*) $\phi \subseteq \{\phi\}$ (*vii*) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$.

(*viii*) Every set has a proper subset. (*ix*) Every subset of a finite set is finite.

(*x*) Every subset of an infinite set is infinite.

9. Find the power set of each of the following sets:

(*i*) A = {digits in the number, 98} (*ii*) B = {letters in the word 'KID'}

(*iii*) $S = \{2, 3\}$ (*iv*) $T = \{4, 7, 9\}$

ANSWERS

1. (*i*) $\{a\}$, ϕ (*ii*) {Reena, Sonu}, {Reena}, {Sonu}, ϕ (*iii*) ϕ (*iv*) $\{5, \{7\}\}$, $\{5\}$, $\{\{7\}\}$, ϕ.

2. (*i*) $\{p, q, r\}$, $\{p, q\}$, $\{p, r\}$, $\{q, r\}$, $\{p\}$, $\{q\}$, $\{r\}$, ϕ

(*ii*) $\{p, q\}$, $\{p, r\}$, $\{q, r\}$, $\{p\}$, $\{q\}$, $\{r\}$

3. (*i*) $Q = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\}$
(*ii*) $R = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29\}$
(*iii*) $S = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
(*iv*) $T = \{1, 4, 9, 16, 25\}$ (*v*) $U = \{1, 3, 6, 10, 15, 21, 28\}$

4. (*i*) yes (*ii*) yes (*iii*) no (*iv*) yes, because the empty set is a subset of every set.

5. (*i*) ϕ (*ii*) $\{1\}, \{3\}, \phi$ (*iii*) $\{c, a\}, \{c, b\}, \{a, b\}, \{c\}, \{a\}, \{b\}, \phi$ (*iv*) None

6. (*i*) 32 (*b*) 128

7. (*i*) 62 excluding $\{12\}$ and ϕ (*ii*) 254 excluding A and ϕ.

8. (*i*) False (*ii*) True (*iii*) False (*iv*) True (*v*) False (*vi*) True (*vii*) False (*viii*) False (*ix*) True (*x*) False

9. (*i*) $P(A) = \{\{9, 8\}, \{9\}, \{8\}, \phi\}$
(*ii*) $P(B) = \{\{K, I, D\}, \{K, I\}, \{K, D\}, \{I, D\}, \{K\}, \{I\}, \{D\}, \phi\}$
(*iii*) $P(S) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$
(*iv*) $P(T) = \{\phi, \{4\}, \{7\}, \{9\}, \{4, 7\}, \{7, 9\}, \{4, 9\}, \{4, 7, 9\}\}$

1.17. Universal set

A universal set is the set of elements from which elements may be chosen to form sets for a particular discussion. Thus, the set of even numbers is a subset of the universal set of whole numbers. We denote a universal set by the symbol ξ or U or E. In this book as per notation specified in the syllabus, the universal set will be denoted by the symbol ξ. The universal set may change from problem to problem. **For example,** the set of letters of the alphabet is the universal set from which the letters of any word may be chosen to form a set.

ξ or U or E
Universal Set

For example,

(*i*) In plane geometry, the universal set consists of all the points in a plane.

(*ii*) For the set of prime numbers, composite numbers, positive odd numbers and positive even numbers the universal set can be taken as the set of natural numbers.

(*iii*) For the set of players of football eleven of your school, the universal set may be taken as the set of pupils of your school.

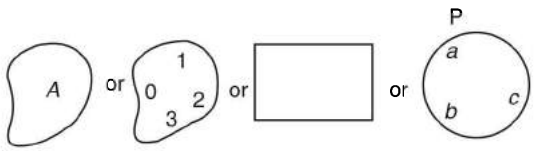
(*iv*) Suppose we have to solve the inequation $\{x \mid x < 5, x \in W\}$. The solution will be $\{0, 1, 2, 3, 4\}$. If universal set were the set of integers, *i.e.,* $\{x \mid x < 5, x \in Z\}$ then the solution would be $\{\ldots -3, -2, 0, 1, 2, 3, 4\}$. It will be an infinite set.

1.18. Venn diagrams

Venn diagrams are set diagrams drawn to represent sets. A closed curve or a rectangle is drawn and the ***points of the interior*** of the closed curve represent the elements of the set under consideration. The diagrams can be identified:

(*a*) By using the letter name of the set illustrated, or

(*b*) By writing the members of the set within a curve as shown in Fig. 1.01.



Venn Diagrams
Fig. 1.01

It may be noted that only the points in the interior of the closed curve belong to the set. A point on the boundary of the region does not belong to the set.

The universal set is generally represented by the set of points of a rectangular region as shown in the figure.

After having got the universal set represented by the interior region of the rectangle, smaller closed geometrical figures are used to represent various subsets of the universal set.

Illustrations

Fig. 1.02 illustrates two intersecting sets, A and B as subsets of universal set ξ. Fig. 1.03 shows two non intersecting sets, A and B as subsets of a universal set ξ. Fig. 1.04. shows set B as a proper subset of set A.

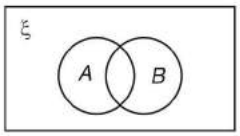


Fig. 1.02

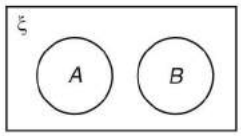


Fig. 1.03

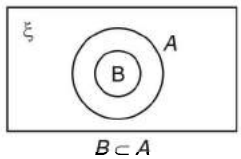


Fig. 1.04

For example,

In Fig. 1.05, the universal set is represented by a rectangle and the set A which is contained in the universal set by a circle 'a' is an element of set A, *i.e.*, $a \in A$.

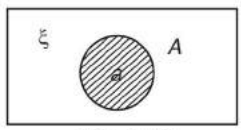


Fig. 1.05

Ex. 9. What universal sets would you propose for each of the following:

(*i*) The set of right triangles (*ii*) The set of isosceles triangles **(*NCERT*)**

Sol. (*i*) Set of triangles (*ii*) set of equilateral triangles or set of triangles.

1.19. OPERATIONS ON SETS

1.19 (A). Union of sets

The union of two sets *A and B is a set C formed by combining the elements of A and B. It contains all elements in either of the sets A or B.*

Elements that are common to both A and B need to be listed only once in set C. The symbol $A \cup B$ means the union of A and B and is read "A union B".

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{both } A \text{ and } B\}$$

For example,

(*i*) If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(*ii*) If $A = \{a, b, c, d\}$, $B = \{a, b, x, y, z\}$, then $A \cup B = \{a, b, c, d, x, y, z\}$

(*iii*) If $A = \{\text{factors of } 12\}$, $B = \{\text{factors of } 16\}$ then

$A = \{1, 2, 3, 4, 6, 12\}$, $B = \{1, 2, 4, 8, 16\}$ and $A \cup B = \{1, 2, 3, 4, 6, 8, 12, 16\}$

(*iv*) If $A = \{x \mid x = 3p, p \in N\}$, $B = \{y \mid y = 4n, n \in N\}$ then

$A = \{3, 6, 9, 12, 15, 18, 21, 24, ...\}$, $B = \{4, 8, 12, 16, 20, 24, 28, ...\}$

So $A \cup B = \{3, 4, 6, 8, 9, 12, 15, 16, 18, 20, 21, 24, 28, ..,\} = \{x \mid x \in N, x \text{ is divisible by 3 or by 4}\}$

(*v*) If $A = \{2, 3, 5, 7, 11\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{0, 1, 2, 3\}$ then,

$A \cup B = \{1, 2, 3, 5, 7, 9, 11\}$.

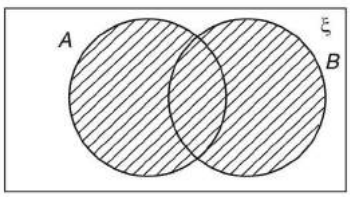
$(A \cup B) \cup C = \{1, 2, 3, 5, 7, 9, 11\} \cup \{0, 1, 2, 3\} = \{0, 1, 2, 3, 5, 7, 9, 11\}$

$B \cup C = \{1, 3, 5, 7, 9\} \cup \{0, 1, 2, 3\} = \{0, 1, 2, 3, 5, 7, 9\}$

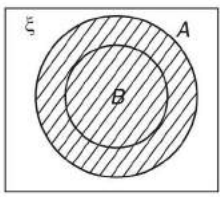
$A \cup (B \cup C) = \{2, 3, 5, 7, 11\} \cup \{0, 1, 2, 3, 5, 7, 9\} = \{0, 1, 2, 3, 5, 7, 9, 11\}$

Do you observe from (1) and (2) that $(A \cup B) \cup C = A \cup (B \cup C)$?

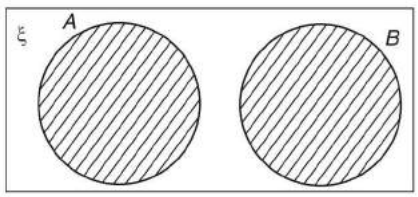
The shaded portions in the following diagrams illustrate the union of two sets A and B.



$A \cup B$
(Overlapping sets)



$A \cup B$
(B is a subset of A)



$A \cup B$
(Disjoint sets)

Fig. 1.06

Ex. 10. Is it true that for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.

Sol. To prove, let us take an example suppose, $A = \{p\}$, $B = \{q\}$. Then, $A \cup B = \{p, q\}$.

$\therefore$ $P(A)$ = set of subsets of $A = \{\phi, \{p\}\}$

$P(B)$ = set of subsets of $B = \{\phi, \{q\}\}$

$\Rightarrow$ $P(A) \cup P(B) = \{\phi, \{p\}, \{q\}\}$ ... (1)

$A \cup B = \{p, q\} \Rightarrow P(A \cup B)$ = set of subsets of $P(A \cup B)$

$\Rightarrow$ $P(A \cup B) = \{\phi, \{p\}, \{q\}, \{p, q\}\}$ ... (2)

From (1) and (2) is clear that $P(A) \cup P(B) \neq P(A \cup B)$

Properties of Union of sets

Let A, B, C be any sets, then

1. $A \cup B = B \cup A$ (Commutative law) **2. $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)**

3. $A \subseteq A \cup B$ and $B \subseteq A \cup B$ **4. If $A \subseteq B$, then $A \cup B = B$**

5. $A \cup \phi = A$

1.19 (B). Intersection of sets

The intersection of two sets A and B is a set that contains elements that are both in A and B. The symbol $A \cap B$ means the intersection of A and B.

$$A \cap B = \{x \mid x \in A, x \in B\}$$

For example:

(*i*) If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \cap B = \{1, 3, 5\}$. Do you observe that $A \cap B \subseteq A$ and $A \cap B \subseteq B$?

(*ii*) If $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$, then these two sets do not have any common members. They are **disjoint sets**. Their intersection is the null set, that is, $A \cap B = \phi$.

Two sets A and B are said to be **disjoint** or **mutually** exclusive if and only if there is no element common to A and B, i.e., if their intersection is the empty set, *i.e.*, $A \cap B = \phi$.

(*iii*) Let $A = \{2, 4, 6, 8, 9, 10, 12\}$,

$B = \{1, 6, 7, 10, 11, 12\}$,

$C = \{2, 5, 8, 12, 15\}$ then

$A \cap B = \{2, 4, 6, 8, 9, 10, 12\} \cap \{1, 6, 7, 10, 11, 12\} = \{6, 10, 12\}$

$B \cap A = \{1, 6, 7, 10, 11, 12\} \cap \{2, 4, 6, 8, 9, 10, 12\} = \{6, 10, 12\}$

$\therefore$ $A \cap B = B \cap A$.

Now, $(A \cap B) \cap C = \{6, 10, 12\} \cap \{2, 5, 8, 12, 15\} = \{12\}$.

Again $B \cap C = \{1, 6, 7, 10, 11, 12\} \cap \{2, 5, 8, 12, 15\} = \{12\}$

$A \cap (B \cap C) = \{2, 4, 6, 8, 9, 10, 12\} \cap \{12\} = \{12\}$

$\therefore$ $(A \cap B) \cap C = A \cap (B \cap C)$

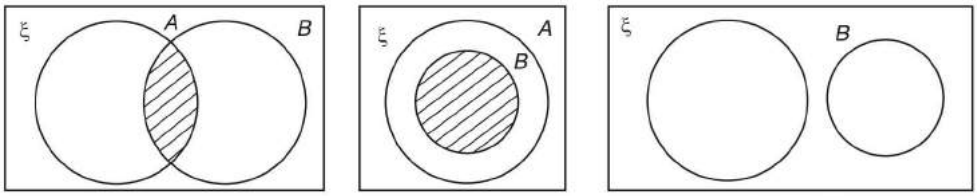
(*iv*) Let $A = \{x : 2x + 9 = 0, x \in N\}$, $B = \{1, 2, 3, 4\}$

$2x + 9 = 0$ gives $x = -\frac{9}{2}$ which is not a natural number.

$\therefore$ $A = \phi$

$\therefore$ $A \cap B = \phi \cap \{1, 2, 3, 4\} = \phi$.

The following Venn diagrams show the intersection of two sets A and B.



$A \cap B$ (overlapping sets) | $A \cap B = B$ (B is a proper subset of A) | $A \cap B = \phi$ (Disjoint sets)

Fig. 1.07

The shaded portions in the above diagrams represent the intersection of A and B.

Properties of Intersection of Sets

Let A , B, C be any sets, then

1. $A \cap B = B \cap A$ (Commutative law) 2. $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

3. $A \subseteq B$ then $A \cap B = A$

4. For any sets A and B, we have $A \cap B \subseteq A$ and $A \cap B \subseteq B$

5. $A \cap \phi = \phi$

6. **Distributive law: If A, B and C are any three sets then**

(*i*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (*ii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1.20. Difference of sets

1. Let A and B be two sets. Then the set of all elements of A which belong to A but do not belong to B is called the difference of sets A and B and is denoted by $A - B$.

2. The set of all elements which belong to B but do not belong to A is called the difference of sets B and A and is denoted by $B - A$. Thus

$$\boldsymbol{A - B = \{x | x \in A, x \notin B\}}$$
$$\boldsymbol{B - A = \{x | x \in B, x \notin A\}}$$

For example:

(*i*) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$ then $A - B = \{1, 2\}$, $B - A = \{6, 7\}$

Note that $A - B \neq B - A$.

(*ii*) Let $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$, $B = \{3, 5, 7, 9\}$ then $A - B = \{1, 11, 13, 15\}$, $B - A = \phi$.

Note that $A - B \neq B - A$.

Remark. The difference of sets A and B is also denoted by $A \backslash B$.

Ex. 11. If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $D = \{5, 10, 15, 20\}$ find:

(*i*) $A - B$ **(*ii*) $A - C$** **(*iii*) $A - D$** **(*iv*) $B - A$**

(*v*) $C - A$ **(*vi*) $D - A$** **(*vii*) $B - C$** **(*viii*) $B - D$**

(*ix*) $C - B$ **(*x*) $D - B$** **(*xi*) $C - D$** **(*xii*) $D - C$** **(*NCERT*)**

Sol.

(*i*) $A - B$ = {the element which are in A but not in B}

= $\{3, 6, 9, 12, 15, 18, 21\} - \{4, 8, 12, 16, 20\} = \{3, 6, 9, 15, 18, 21\}$

(*ii*) $A - C$ = {elements which are in A but not in C}

= $\{3, 6, 9, 12, 15, 18, 21\} - \{2, 4, 6, 8, 10, 12, 14, 16\} = \{3, 9, 15, 18, 21\}$

(*iii*) $A - D$ = {the elements which are in A but not in D}

$= \{3, 6, 9, 12, 15, 18, 21\} - \{5, 10, 15, 20\}$

$= \{3, 6, 9, 12, 18, 21\}$

(*iv*) $B - A$ = {the elements which are in B but not in A}

$= \{4, 8, 12, 16, 20\} - \{3, 6, 9, 12, 15, 18, 21\}$

$= \{4, 8, 16, 20\}$

(*v*) $C - A$ = {the elements which are in C but not in A}

$= \{2, 4, 6, 8, 10, 12, 14, 16\} - \{3, 6, 9, 12, 15, 18, 21\}$

$= \{2, 4, 8, 10, 14, 16\}$

(*vi*) $D - A$ = {the elments which are in D but not in A}

$= \{5, 10, 15, 20\} - \{3, 6, 9, 12, 15, 18, 21\}$

$= \{5, 10, 20\}$

(*vii*) $B - C$ = {the elements which are in B but not in C},

$= \{4, 8, 12, 16, 20\} - \{2, 4, 6, 8, 10, 12, 14, 16\} = \{20\}$

(*viii*) $B - D$ = {the elements are in B but not in D}

$= \{4, 8, 12, 16, 20\} - \{5, 10, 15, 20\} = \{4, 8, 12, 16\}$

(*ix*) $C - B$ = {the elements which are in C but not in B}

$= \{2, 4, 6, 8, 10, 12, 14, 16\} - \{4, 8, 12, 16, 20\}$

$= \{2, 6, 10, 14\}$

(*x*) $D - B$ = {the elements which are in D but not in B}

$= \{5, 10, 15, 20\} - \{4, 8, 12, 16, 20\}$

$= \{5, 10, 15\}$

(*xi*) $C - D$ = {the elements which are in C but not in D}

$= \{2, 4, 6, 8, 10, 12, 14, 16\} - \{5, 10, 15, 20\}$

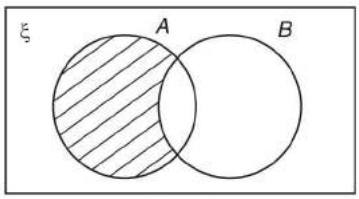
$= \{2, 4, 6, 8, 12, 14, 16\}$

(*xii*) $D - C$ = {the elements which are in D but not in C}

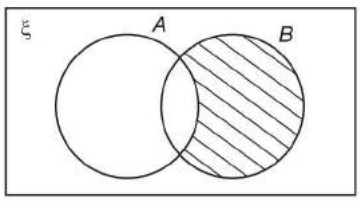
$= \{5, 10, 15, 20\} - \{2, 4, 6, 8, 10, 12, 14, 16\}$

$= \{5, 15, 20\}$

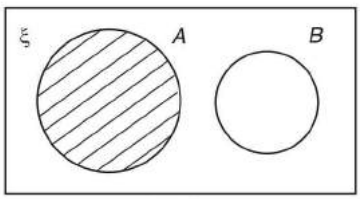
The following Venn diagrams will help you to understand the concept of difference of two sets.



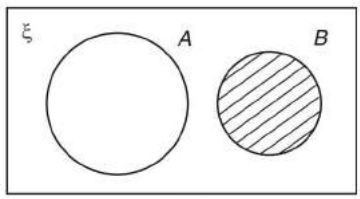
$A - B$
(Overlapping sets)



$B - A$
(Overlapping sets)

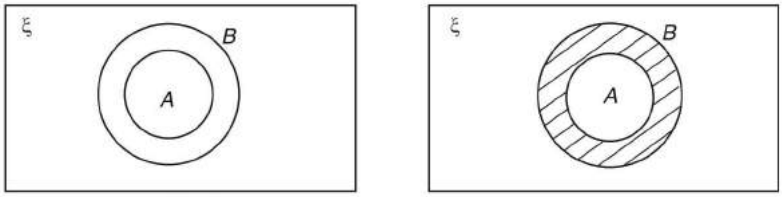


$A - B$



$B - A$

Disjoint sets



When A is a proper subsets of B

Fig. 1.08

Notes

1. If $A \subset B$, then $A - B = \phi$ and $B - A$ is not ϕ.
2. If $A = \phi$, then $A - B = \phi$ and $B - A = B$.
3. If $A \cap B = \phi$, then $A - B = A$ and $B - A = B$.
4. If $A = B$, then $A - B$ and $B - A$ are both null sets.
5. Set $A - B$ is a subset of A, i.e., $A - B \subseteq A$.
6. The sets $A - B$, $A \cap B$ and $B - A$ are mutually disjoint.
7. $A - B = (A \cup B) - B = A - A \cap B$
8. $A - B \neq B - A$.

1.21. Symmetric difference of two sets

Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.

Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

The shaded part in Fig. 1.09 represents $A \Delta B$.

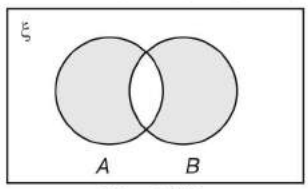


Fig. 1.09

Notes:

1. $A \Delta B = (A \cup B) - (B \cap A)$
2. $A \Delta B = B \Delta A$.
3. $A \Delta A = \phi$, $A \Delta \phi = A$, $A \Delta B = \phi \Leftrightarrow A = B$.

Ex. 12. If $A = \{5, 6, 7, 8, 9, 10\}$, $B = \{5, 7, 9, 11\}$, then find $A \Delta B$.

Sol. $A \Delta B = \{6, 8, 10, 11\}$

Ex. 13. If $A = \{x \in R : 0 < x < 4\}$ and $B = \{x \in R : 1 \leq x \leq 7\}$ then find $A \Delta B$.

Sol. $A - B = \{x \in R : 0 < x < 1)$, $B - A = \{x \in R : 4 \leq x \leq 7\}$

$A \Delta B = (A - B) \cup (B - A) = \{x \in R : 0 < x < 1\} \cup \{x \in R : 4 \leq x \leq 7\}$

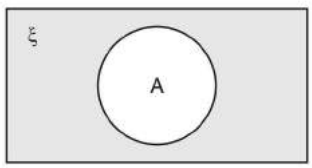
$= \{x \in R, 0 < x < 1 \text{ or } 4 \leq x \leq 7\}$

1.22. Complement of a set

If A is a subset of a universal ξ then the set of all those elements of ξ which do not belong to A is the ***complement*** *of A and is denoted by A' or $\overline{A}$ or A^c.*

Thus, $A' = \xi - A = \{x \mid x \in \xi, x \notin A\}$

We may also simply say $A' = \{x \mid x \notin A\}$. Since it is understood that $x \in \xi$



This shaded points represents $A' = \xi - A$

Fig. 1.10

For example,

(i) Let ξ be the set of all pupils of a class and A the set of all girls in that class, then A' is the of all boys in that class.

(*ii*) If ξ = the set of counting numbers and A = set of counting numbers less than 100, then $A' = \{100, 101, 102, ...\}$

(*iii*) If ξ = set of natural numbers, A = set of even natural numbers, then A' = set of odd numbers.

(*iv*) Let ξ = the set of the English alphabet and let $V = \{x \mid x$ is a vowel of the English alphabet$\}$, then $V' = \{x \mid x$ is a consonant of the English alphabet.$\}\}$

(*v*) (*a*) Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{1, 3, 5, 7\}$ then $A' = \{2, 4, 6, 8\}$.

Also, $n(\xi) = 8$, $n(A) = 4$, $n(A') = 4$

(*b*) If $n(\xi) = 30$, $n(A) = 16$, $n(B) = 21$, then $n(A') = n(\xi) - n(A) = 30 - 16 = 14$

$n(B') = n(\xi) - n(B) = 30 - 21 = 9$.

It is obvious that the complement of A' is A, that is, $(A')' = A$. The **complement of the universal set is the empty set,** that is $\xi' = \phi$ and the complement of the empty set is the universal set, *i.e.*, $\phi' = \xi$.

1.23. Laws of the complement of a set

Law 1. $(A')' = A$, *i.e.*, the complement of the complement of a set is the set itself.

Law 2. $\xi' = \phi$ and $\phi' = \xi$, *i.e.*, the complement of the universal set is the empty set and vice-versa.

Law 3. (*i*) $A \cup A' = \xi$ and (*ii*) $A \cap A' = \phi$.

Law 4. De Morgan's laws

(*a*) $(A \cup B)' = A' \cap B'$ (*b*) $(A \cap B)' = A' \cup B'$

The following Venn diagrams show $(A \cup B)'$ and $(A \cap B)'$.

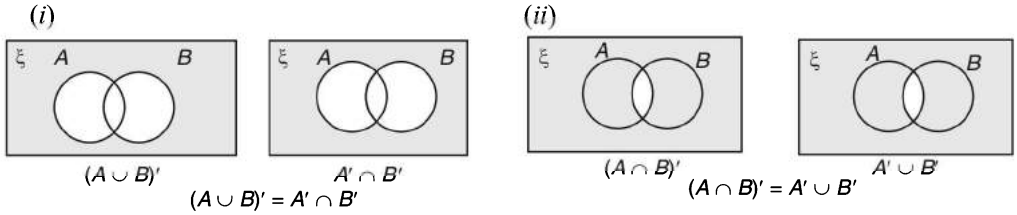


Fig. 1.11

EXERCISE 1(*c*)

1. Suggest a universal set for each of the following:

(*i*) {Jaipur, Chennai, Bangalore, Itanagar} (*ii*) {Narmada, Cauvery, Mahanadi, Jhelum}

(*iii*) {Asia, Europe, Antarctica} (*iv*) {Earth, Mars, Venus}

(*v*) {0, 5, 10, 15, 20, 25}

2. What universal set may be proposed for the following sets?

(*i*) The set of parallelograms

(*ii*) The set of irrational numbers

(*iii*) The set of positive even numbers

3. Solve the following equations:

(*i*) $\{x \mid 2x + 6 = 0, x \in Z\}$ (*ii*) $\{x \mid 5x + 16 = 1, x \in N\}$ (*iii*) $\{x \mid 2x - 3 < 7, x \in W\}$

(*iv*) $\{x \mid 4x - 25 > 13, x \in Z\}$ (*v*) $\{y \mid \frac{5y}{3} - 7 \le 13,$ y is a prime number$\}$

4. $\xi = \{-2\frac{1}{2}, -1, \sqrt{2}, 3.5, \sqrt{30}, \sqrt{36}\}$

$X = \{$integers$\}$, $Y = \{$irrational numbers$\}$

List the members of (*i*) X (*ii*) Y ***(C.U.O Level 2003)***

5. $\xi = \{40, 41, 42, 43, 44, 45, 46, 47, 48, 49\}$
$A = \{\text{prime numbers}\}$, $B = \{\text{odd numbers}\}$

(*i*) Place the ten numbers in the correct places on the diagram.

(*ii*) Write the set $B \cap A'$. **(C.U)**

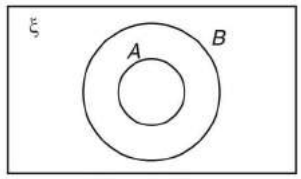
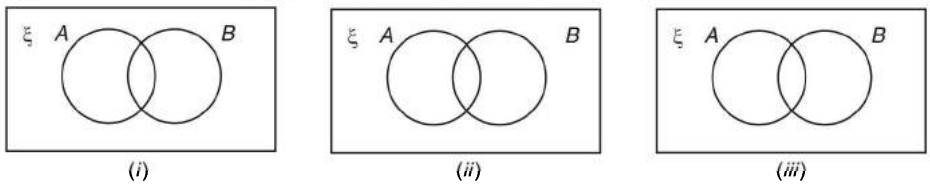


Fig. 1.12

6. Shade the regions as directed? **(C.U)**

(*i*) $A \cap B$ (*ii*) $(A \cup B)'$ (*iii*) Complementary set B

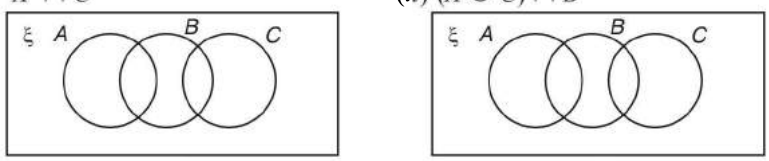


(*i*) (*ii*) (*iii*) **(C.U)**

Fig. 1.13

7. On the Venn diagrams shade the regions:

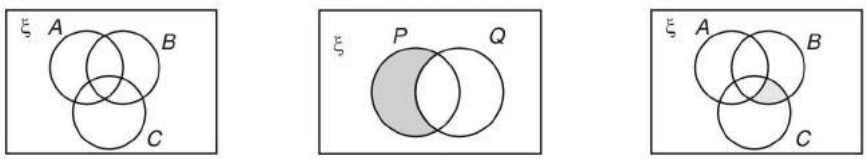
(*i*) $A' \cap C'$ (*ii*) $(A \cup C) \cap B$



(*i*) (*ii*) **(C.U)**

Fig. 1.14

8. (*i*) On a copy of the Venn diagram, shade the set $A \cup (B \cap C)$.



(*i*) (*ii*) (*iii*) **(C.U)**

Fig. 1.15

(*ii*) Express in set notation the subset shaded in the Venn diagram.

(*iii*) Express in set notation, as simply as possible, the subset shaded in the Venn diagram.

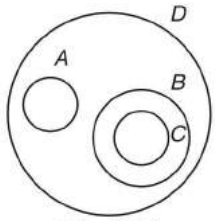
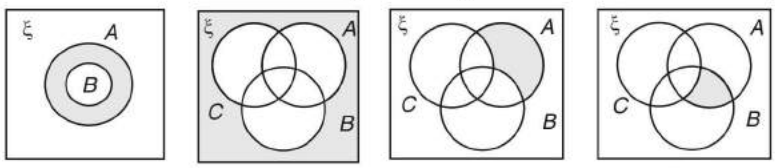


Fig. 1.16

9. Answer true or false. Refer to the adjoining figure.

(*i*) $A \subset D$ (*ii*) $B \subset C$ (*iii*) $A \not\subset B$ (*iv*) $C \subset D$

10. Describe the shaded areas in the following Venn diagrams.



(*i*) (*ii*) (*iii*) (*iv*)

Fig. 1.17

11. Use the given diagram to shade the following regions.

(*i*) $A' \cap B \cap C$ (*ii*) $A' \cap B \cap C'$ (*iii*) $(A \cap B \cap C)'$

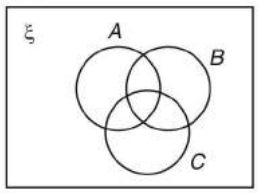


Fig. 1.18

12. X = {all triangles}, P = {isosceles triangles}, Z = {equilateral triangles}, Draw a Venn diagram to illustrate the relationship between these sets.

13. Draw a Venn diagram to show the relationship between the following sets:
ξ = {quadrilaterals}, A = {parallelograms}, B = {rectangles}, C = {rhombuses}
Show in your diagram the region that represents the set of squares.

14. Using ξ = {books}, N = {novels} and D = {detective novels}, represent the following statement as a Venn diagram, 'Some novels are not detective novels'

15. Illustrate by a Venn diagram the relationship between the sets A, B and C given that $B \subset A$, $C \subset B'$ and $A \cap C = \phi$.

16. If ξ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 3, 6, 10}, B = {1, 3, 5, 7, 9} and C = {1, 4, 9} then list the elements of the following sets.

(*i*) $A \cap B$ (*ii*) $B \cup C$ (*iii*) C' (*iv*) $n(B)$ (*v*) $B \cap C$ (*vi*) $A \cap B$
(*vii*) $A \cup C$ (*viii*) $A \cup B \cup C$ (*ix*) $(A \cap B)'$ (*x*) $A - B$ (*xi*) $B - A$
(*xii*) $(A \cup B \cup C)'$ (*xiii*) $(A \cap B \cap C)'$ (*xiv*) $A \Delta B$

17. Let L = (letters of CRICKET), M = {letters of CATERPILLAR} and N = {letters of CARE TAKER} find:

(*i*) $L \cup M$ (*ii*) $M \cup N$ (*iii*) $L \cup N$

18. Taking the set of natural numbers as the universal set, write down the complements of the following sets:

(*i*) $\{x : x$ is a positive multiple of 3$\}$ (*ii*) $\{x : x$ is a prime number$\}$
(*iii*) $\{x : x$ is a natural number divisible by 3 and 5$\}$
(*iv*) $\{x : x + 5 = 8\}$ (*v*) $\{x : 2x + 5 = 9\}$
(*vi*) $\{x : x \geq 7\}$ (*vii*) $\{x : x \in N$ and $2x + 1 > 10\}$ ***(NCERT)***

19. Refer to the Venn diagram. List the elements of the following sets.

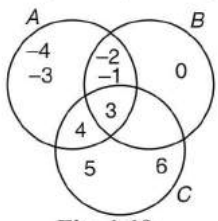


Fig. 1.19

(*i*) $(A \cap B) \cup C$ (*ii*) $A \cap (B \cup C)$ (*iii*) $(A \cap C) \cap (B \cap C)$
(*iv*) $(A \cup B) \cap (B \cap C)$ (*v*) $B - C$ (*vi*) $C - B$
(*vii*) $B \Delta C$

20. If ξ = {1, 2, 3, 4}, A = {1, 4}, B = {1, 3}, then list the elements of
(*i*) A' (*ii*) B' (*iii*) $(A \cap B)'$ (*iv*) $(A \cup B)'$ (*v*) $A' \cap B'$ (*f*) $A' \cup B'$
Also show that $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$

ANSWERS

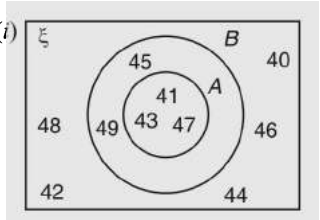
1. (*i*) {State capitals of India} (*ii*) {Rivers of India} (*iii*) {Continents} (*iv*) {Planets of our solar system}
(*v*) $\{x | x = 5p, p \in W, 0 \leq p \leq 5$ or {Multiples of 5 greater than or equal to 0}

2. (*i*) the set of quadrilaterals (*ii*) the set of real numbers (*iii*) the set of natural numbers

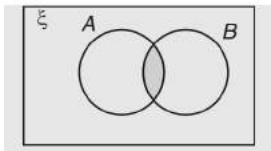
3. (*i*) {–3} (*ii*) ϕ (*iii*) {0, 1, 2, 3, 4} (*iv*) {10, 11, 12, ...} (*v*) {2, 3, 5, 7, 11}

4. (*i*) $-1, \sqrt{36}$ (*ii*) $\sqrt{2}, \sqrt{30}$

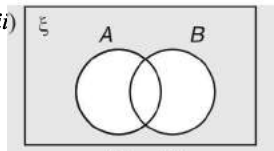
5. (*i*)



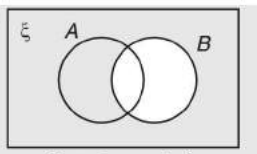
(*ii*) {45, 49}

6. (*i*) 

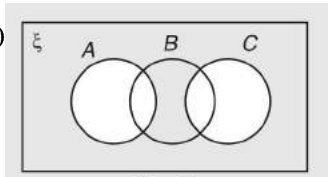
$A \cap B$

(*ii*) 

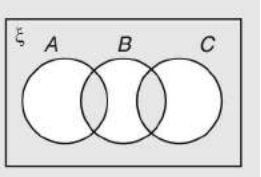
$(A \cup B)'$

(*iii*) 

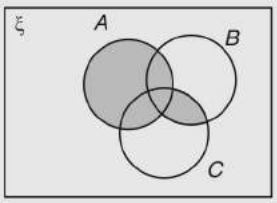
Complement of set B

7. (*i*) 

$A' \cap C'$

(*ii*) 

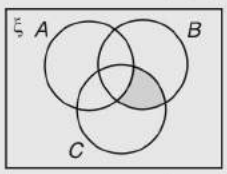
$(A \cup C) \cap B$

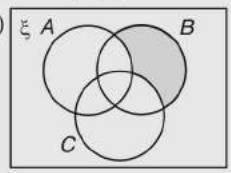
8. (*i*) 

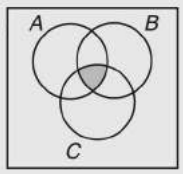
(*ii*) $P \cap Q'$ (*iii*) $B \cap C \cap A'$

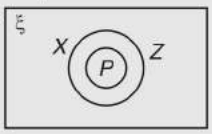
9. (*i*) True (*ii*) False (*iii*) True (*iv*) True

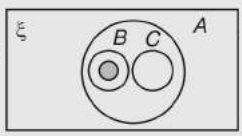
10. (*i*) $A \cap B'$ (*ii*) $(A \cup B \cup C)'$ (*iii*) $(A \cap B' \cap C')$ (*iv*) $(A \cap B \cap C')$

11. (*i*) 

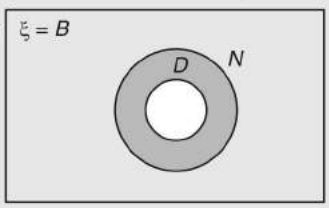
(*ii*) 

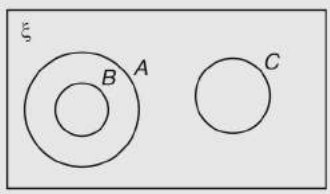
(*iii*) 

12. 

13. 

The shaded region represents the set of squares.

14. 

15. 

B = {books}, N = {novels}
D = {detective novels}

16. (*i*) {1, 3} (*ii*) {1, 2, 3, 4, 5, 7, 9} (*iii*) {2, 3, 5, 6, 7, 8, 10}
(*iv*) 5 (*v*) {1, 9} (*vi*) {1} (*vii*) {1, 3, 4, 6, 9, 10} (*viii*) {1, 3, 4, 5, 6, 7, 9, 10}
(*ix*) {2, 4, 5, 6, 7, 8, 9, 10} (*x*) {6, 10} (*xi*) {5, 7, 9} (*xii*) {2, 8} (*xiii*) {2, 3, 4, 5, 6, 7, 8, 9, 10}
(*xiv*) {5, 6, 7, 9, 10}

17. (*i*) {*c*, *r*, *i*, *k*, *e*, *t*, *a*, *p*, *l* } (*ii*) {*c*, *a*, *t*, *e*, *r*, *i*, *p*, *l*, *k*} (*iii*) {*c*, *r*, *i*, *k*, *e*, *t*, *a*}

18. (*i*) $\{x : x \in N$ and x is not a multiple of 3$\}$
(*ii*) $\{x : x$ is a positive composite number and $x = 1\}$
(*iii*) $\{x : x$ is a positive integer which is not divisible by 15$\}$
(*iv*) $\{x : x \in N$, and $x \neq 3\}$
(*v*) $\{x : x \in N$ and $x \neq 2\}$
(*vi*) $\{x : x \in N$ and $x < 7\}$, *i.e.*, $\{1, 2, 3, 4, 5, 6\}$
(*vii*) $\{x : x \in \text{N}$ and $x \leq \frac{9}{2}\}$, *i.e.*, $\{1, 2, 3, 4\}$

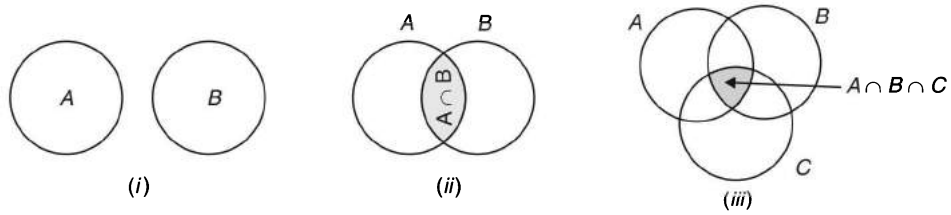
19. (*i*) $\{-2, -1, 3, 4, 5, 6\}$ (*ii*) $\{-2, -1, 3, 4\}$ (*iii*) $\{3\}$ (*iv*) $\{3\}$
(*v*) $\{0, -1, -2\}$ (*vi*) $\{4, 5, 6\}$ (*vii*) $\{0, -1, -2, 4, 5, 6\}$

20. (*i*) $\{2, 3\}$ (*ii*) $\{2, 4\}$ (*iii*) $\{2, 3, 4\}$ (*iv*) $\{2\}$ (*v*) $\{2\}$ (*vi*) $\{2, 3, 4\}$

1.24. Cardinal number

The number of elements in a set is often referred to as the *cardinal number* of the set. The cardinal number of a set A is abbreviated as $n(A)$. Thus, we have

$n(\phi) = 0$, $n\{a\} = 1$, $n\{a, b\} = 2$, $n\{a, b, c\} = 3$, and so on.



(*i*) (*ii*) (*iii*)

Fig. 1.20

Suppose we wish to find the number of elements in the union of two sets. Let A and B be any two sets. There are two possibilities.

Case (*i*). The sets are disjoint, that is $A \cap B = \phi$. [Fig. 1.20 (*i*)]. In this case, obviously

$$n(A \cup B) = n(A) + n(B)$$

Case (*ii*). The sets are not disjoint, that is, $A \cap B \neq \phi$. [Fig. 1.20 (*ii*)]. In this case, $n(A) + n(B)$ counts the common elements in both sets twice and so exceeds $n(A \cup B)$ by $n(A \cap B)$. Therefore

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

It is possible to develop a formula similar to the above, for three or more sets, but it is easier to rely on the use of Venn diagrams to solve such problems.

If *A*, *B*, *C* are any three sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Also from the two intersecting sets A and B shown in the adjoining Venn diagram, it is obvious that

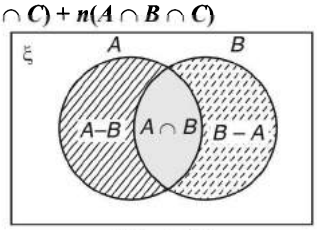


Fig. 1.21

(1) $n(A - B) = n(A) - n(A \cap B) \Rightarrow n(A) = n(A - B) + n(A \cap B)$
(2) $n(B - A) = n(B) - n(A \cap B) \Rightarrow n(B) = n(B - A) + n(A \cap B)$
(3) $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
(4) $n(\xi) = n(A\,) + n(A') = n(B) + n(B')$, where ξ is the universal set.

Ex. 14. If $n(\xi) = 600$ $n(A) = 460$, $n(B) = 390$, and $n(A \cap B) = 325$, draw a Venn diagram to find:

(*i*) $n(A \cup B)$ (*ii*) $n(A \cup B)'$ (*iii*) $n(A - B)$

Sol. Given $n(\xi) = 600$, $n(A) = 460$, $n(B) = 390$, $n(A \cap B) = 325$

(*i*) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 460 + 390 - 325 =$ **525**

The given sets are intersecting sets. The Venn diagram is as shown.

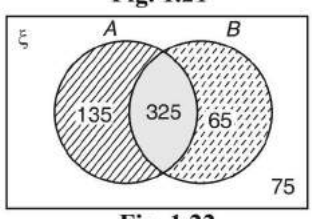


Fig. 1.22

(*ii*) $n(A \cup B)' = n(\xi) - n(A \cup B) = 600 - 525 =$ **75**

(*iii*) $n(A - B) = n(A) - n(A \cap B) = 460 - 325 =$ **135**

Ex. 15. The Venn diagram shows
ξ = **{pupils in a school},**
A = **{those who have a brother} and**
B = **{those who have a sister}.**
Write down the number who:

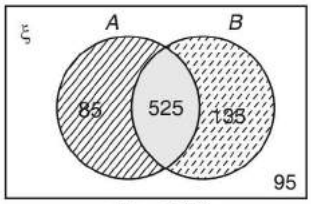


Fig. 1.23

(*i*) have a brother (*ii*) have a sister
(*iii*) have both a brother and a sister
(*iv*) have neither brother nor sister (*v*) are in the school altogether

Sol. (*i*) Number of pupils who have a brother = $n(A) = 85 + 525 =$ **610**
(*ii*) Number of pupils who have a sister = $n(B) = 525 + 135 =$ **660**
(*iii*) Number of pupils who have both a brother and a sister = $n(A \cap B) =$ **525**
(*iv*) Number of pupils who have neither brother nor sister = $n(A \cup B)' =$ **95**
(*v*) Total number of pupils in the school = $n(\xi) = 85 + 525 + 135 + 95 =$ **840**.

Ex. 16. In a college 500 students study Mathematics and 400 study Economics. If 200 students study both the subjects, find the total number of students enrolled in these two subjects.

Sol. Let
A = the set of students studying Mathematics.
B = the set of students studying Economics, then it is given that
$n(A) = 500$, $n(B) = 400$, $n(A \cap B) = 200$
We have to find $n(A \cup B)$

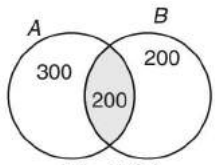


Fig. 1.24

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 500 + 400 - 200 = 700.$$

Therefore the total number of students studying the two subjects is 700.
Alternatively, See Fig. 1.24.

Ex. 17. In a class of 32 students, each of whom must take English or Hindi; 20 take English, 9 take both English and Hindi. How many take Hindi?

Sol. Let E = {students taking English}. H = {students taking Hindi}
The shaded region represents the set of 9 students who take both English and Hindi. Therefore, the number of students taking only Hindi

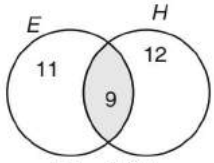


Fig. 1.25

$$= 32 - 20 = 12.$$

The total number of students taking Hindi = 12 + 9 = 21.

Ex. 18. In a group of 70 people, 48 speak Tamil, 36 speak English and all the people speak at least one language. Find:

(*i*) how many speak both the languages.
(*ii*) how many speak only Tamil;
(*iii*) how many speak only English.

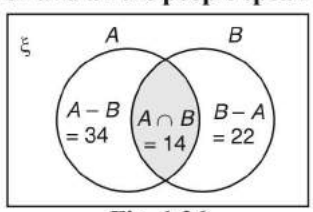


Fig. 1.26

Sol. Let A denote the set of people who speak Tamil and B the set of people who speak English. Then $n(A) = 48$, $n(B) = 36$.
'$A - B$' represents the set of people who speak only Tamil. '$B - A$' represents the set of people who speak only English and $A \cap B$ the set of people who speak both languages. We have

$n(A \cup B) + n(A \cap B) = n(A) + n(B)$
$\therefore\ n(A \cap B) = n(A) + n(B) - n(A \cup B) = 48 + 36 - 70 = 84 - 70 = 14$
$\therefore$ numbers of people who speak both languages= **14**
number of people who speak only Tamil = $n(A - B) = n(A) - n(A \cap B) = 48 - 14 =$ **34**
number of people who speak only English = $n(B - A) = n(B) - n(A \cap B) = 36 - 14 =$ **22**.

Ex. 19. Out of 100 candidates who appeared in an entrance test in English and Mathematics, 80 passed in at least one subject. If 60 passed in English and 65 in Mathematics, find:

(*i*) how many passed in both the subjects;

(*ii*) how many passed in English only;

(*iii*) how many failed in Mathematics.

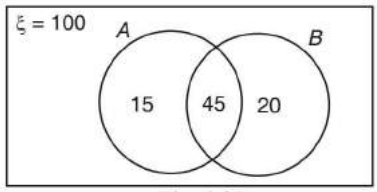


Fig. 1.27

Sol. Let ξ be the set of all the candidates who took the test, A the set of candidates who passed in English and B of those who passed in Mathematics. Then

Given: $n(\xi) = 100$, $n(A \cup B) = 80$, $n(A) = 60$, $n(B) = 65$

(i) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 60 + 65 - 80 = 45$

$\therefore$ Number of candidates who passed in both the subjects = **45**.

(ii) Number of candidates who passed only in English = $n(A - B) = n(A) - n(A \cap B) = 60 - 45 =$ **15**.

(iii) Number of candidates who failed in Mathematics = $n(B') = n(\xi) - n(B) = 100 - 65 =$ **35**.

The Venn diagram is shown above.

Ex. 20. 100 of the 120 students at a college take at least one of the languages French, German and Russian, Suppose 65 study French, 45 study German, 42 study Russian, 20 study French and German, 25 study French and Russian and 15 study German and Russian. Find the number of students who study all the three languages.

Sol. Let F, G and R denote the sets of studying French. German and Russian respectively. Then

$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$

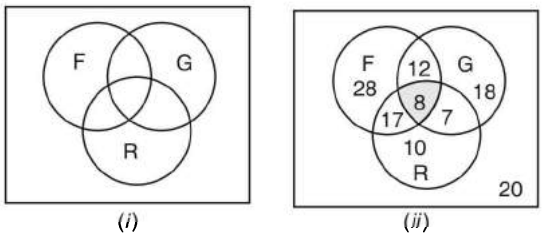


Fig. 1.28

Now, $n(F \cup G \cup R) = 100$, because 100 of the students study at least one of the languages. Substituting

$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(F \cap G \cap R)$, giving $n(F \cap G \cap R) = 8$, *i.e.*, 8 students study all three languages.

Note. We may use this result to fill in the Venn diagrams, we have:

8 study all the three languages.

$20 - 8 = 12$ study French and German but not Russian.

$25 - 8 = 17$ study French and Russian but not German.

$15 - 8 = 7$ study German and Russian but not French.

$65 - 12 - 8 - 17 = 28$ study only French;

$45 - 12 - 8 - 7 = 18$ study only German;

$42 - 17 - 8 - 7 = 10$ study only Russian;

$120 - 100 = 20$ do not study any of the languages.

Thus, the completed diagram is Fig. 1.28 (ii). Observe that $28 + 18 + 10 = 56$ students study exactly one of the three languages.

EXERCISE 1(*D*)

1. If $n(\xi) = 80$, $n(A) = 48$, $n(B) = 40$ and $n(A \cap B) = 25$, draw a Venn diagram to find:

(i) $n(A \cup B)$ (ii) $n(A \cup B)'$ (iii) $n(A - B)$ (iv) $n(B - A)$

(v) $n(A \cap B')$ (vi) $n(A' \cap B)$

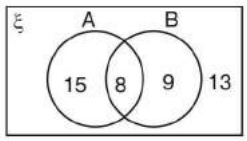
2. If $\xi = \{x \mid x \in N, x < 10\}$,

$A = \{x \mid x$ is a prime number, $x < 10\}$, $B = \{x \mid x$ is an even number, $x < 10\}$ draw a Venn diagram to find:

(*i*) $n(A \cup B)$ (*ii*) $n(A \cap B)$ (*iii*) $n(A \cup B)'$ (*iv*) $n(A \cap B')$ (*v*) $n(A' \cap B)$

3. Given $n(\xi) = 40$, $n(A') = 12$, $n(B) = 15$ and $B \subset A$. Draw a Venn diagram to illustrate this information. Hence find $n(A - B)$.

4. The Venn diagram shows:



ξ = {pupils in class 8}

A = {pupils who play cricket}

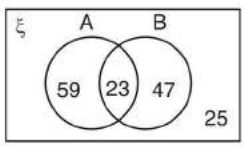
B = {pupils who play basketball}

How many pupils:

(*i*) are in class 8? (*ii*) play cricket (*iii*) play basketball?

(*iv*) play both cricket and basketball (*v*) play neither cricket nor basketball?

5. In the Venn diagram



ξ = {people at a function}

A = {those who asked for tea}

B = {those who watched the ballet}

Write down the number who:

(*i*) asked for tea (*ii*) asked for tea and watched the ballet

(*iii*) neither asked for tea nor watched the ballet (*iv*) attended the function

6. In a group of 30 people, 18 play squash and 19 play tennis. How many play both games, provided everyone plays at least one game?

7. In a class of 50 students, 22 like History, 25 like Geography and 10 like both subjects. Draw a Venn diagram and find the number of students who

(*i*) do not like History (*ii*) do not like Geography

(*iii*) like neither History nor Geography

8. 2000 candidates appear in a written test in Mathematics and Gerneral Awareness for a Government job. 1800 passed in at least one subject. If 1200 passed in Mathematics and 1500 in General Awareness find:

(*i*) how many passed in both the subjects?

(*ii*) how many passed in Mathematics only?

(*iii*) how many failed in General Awareness?

9. In a group of 80 people, 40 like Indian food, 36 like Chinese food and 27 do not like any kind of these foods. Draw Venn diagram to find:

(*i*) how many like both kind of food?

(*ii*) how many like only the Indian food?

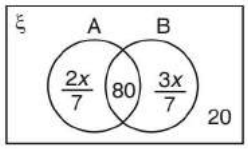
(*iii*) how many like only the Chinese food?

[**Hint:** Let A and B be the sets of those people who like Indian and Chinese food respectively. Then, we have $n(A) = 40$, $n(B) = 36$, $n(A \cup B) = 80 - 27 = 53$]

10. In a group of people, two-seventh speak Bengali only and three-seventh speak Hindi only. If 20 people speak none of these languages and 80 speak both, find using Venn diagram the total number of people in the group.

[**Hint:** Let A and B be the sets of people speaking Bengali and Hindi respectively. Let the number of people in the group be x. Then $n(\xi) = x$]

$$\left[\therefore \frac{2x}{7} + \frac{3x}{7} + 80 + 20 = x\right]$$



11. In a class of 150 students, the following results were obtained in a certain examination; 45 students failed in Maths, 50 students failed in Physics, 48 students failed in Chemistry, 30 students failed in both Maths and Physics, 32 failed in Physics and Chemistry. 35 failed in both Maths and Chemistry, 25 failed in all the three subjects. Draw a Venn diagram corresponding to this data and find the number of students who have failed in at least one subject.

12. A firm has 40 workers working in the factory premises, 30 working in its office and 20 working in both the places. How many workers are there in the firm? How many are working in the (*i*) factory (*ii*) office alone?

ANSWERS

1. (*i*) 63 (*ii*) 17 (*iii*) 23 (*iv*) 15 (*v*) 23 (*vi*) 15

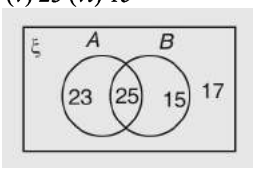


Fig. A1

2. (*i*) 7 (*ii*) 1 (*iii*) 2 (*iv*) 3 (*v*) 3

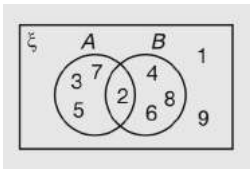


Fig. A2

3. See Fig. A3

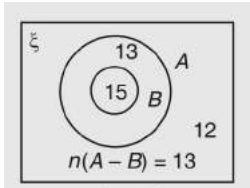


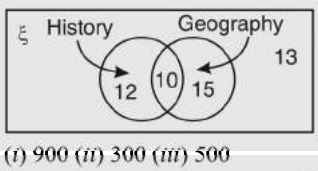
Fig. A3

4. (*i*) 45 (*ii*) 23 (*iii*) 17 (*iv*) 8 (*v*) 13

5. (*i*) 82 (*ii*) 70 (*iii*) 25 (*iv*) 154

6. 7

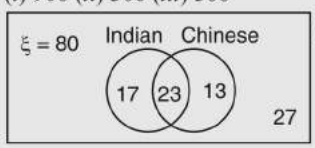
7.



(*i*) 28 (*ii*) 25 (*iii*) 13

8. (*i*) 900 (*ii*) 300 (*iii*) 500

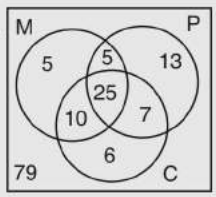
9.



(*i*) 23 (*ii*) 17 (*iii*) 13

10. 350

11.



, 71

12. 50; (*i*) 20 (*ii*) 10

1.25. Algebra of sets (summary of laws)

Sets under the operations of union, intersection, and complement, satisfy the laws listed below. You can verify these laws by drawing Venn diagrams.

If A, B, C are any sets and ξ is the universal set, then

Laws of operations with Sets

(1) $(A')' = A$

(2) $\phi' = \xi$ (2A) $\xi' = \phi$

(3) $A \cup \phi = A$ (3A) $A \cap \xi = A$

(4) $A \cup \xi = \xi$ Indempotent Laws (4A) $A \cap \phi = \phi$

(5) $A \cup A = A$ (5A) $A \cap A = A$

(6) $A \cup A' = \xi$ (6A) $A \cap A' = \phi$

Associative Laws

(7) $(A \cup B) \cup C = A \cup (B \cup C)$ (7A) $(A \cap B) \cap C = A \cap (B \cap C)$

Commutative Laws

(8) $A \cup B = B \cup A$ (8A) $A \cap B = B \cap A$

Distribution Laws

(9) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (9A) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e., Union and intersection are distributive over intersection and union respectively.

De Morgan's Laws

(10) $(A \cup B)' = A' \cap B'$ (10A) $(A \cap B)' = A' \cup B'$

Proofs of some of the Above Laws

1. **$(A \cup B) \cup C = A \cup (B \cup C)$**

Let x be an arbitrary elements of $(A \cup B) \cup C$.

Then $x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C \Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C) \Rightarrow x \in A$ or $x \in (B \cup C) \Rightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$. Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$

2. **$(A \cap B) \cap C = A \cap (B \cap C)$**

Let x be an arbitrary element of $A \cap (B \cap C)$. Then

$x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in (B \cap C) \Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C \Rightarrow x \in (A \cap B)$ and $x \in C \Rightarrow x \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$

3. **$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$**

Let x be an arbitrary element of $A \cup (B \cap C)$. Then

$x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in (B \cap C) \Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ ('or' is distributive over 'and')

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C) \Rightarrow x \in ((A \cup B) \cap (A \cup C))$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. **$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$**

Let x be an arbitrary element of $A \cap (B \cup C)$. Then

$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C) \Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

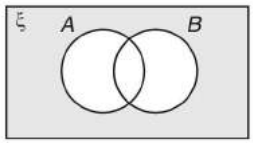
$\Rightarrow x \in (A \cap B) \cup (A \cap C)$ $\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

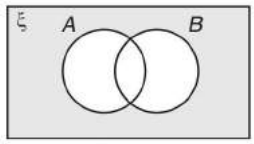
Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. **(*i*)** $(A \cup B)' = A' \cap B'$ **(*ii*)** $(A \cap B)' = A' \cup B'$

(*i*)



$(A \cup B)'$



$A' \cap B'$

Let x be any arbitrary element of $(A \cup B)'$, then

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \notin B' \Rightarrow x \in A' \cap B'$

$\therefore (A \cap B)' \subseteq A' \cap B'$ ...(1)

Again, let y be any arbitrary element of A' and B' then

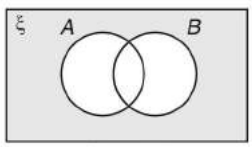
$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B' \Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B) \Rightarrow y \in (A \cup B)'$

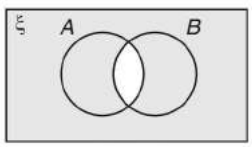
$\Rightarrow A' \cap B' \subseteq (A \cup B)'$ ... (2)

(1) and (2) $\Rightarrow (A \cup B)' = A' \cup B'$

(*ii*)



$(A \cap B)'$



$A' \cup B'$

Prove similarly as above.

6. $(A')' = A$

Let x be any arbitrary element of $(A')'$. Then

$x \in (A')' \Rightarrow x \notin A' \Rightarrow x \in A$

$\Rightarrow (A')' \subseteq A$ ... (1)

Now, let y be any arbitrary element of A, then

$y \in A \Rightarrow y \notin A' \Rightarrow y \in (A')'$

$\Rightarrow A \subseteq (A')'$ ... (2)

(1) and (2) $\Rightarrow (A')' = A$

CHAPTER SUMMARY

1. A **set** is a well defined collection of objects.
2. Each thing in a set is called a **member** or an **element** of the set.
3. The symbol '$\in$' is used to denote '**is a member of**' or '**belongs to**'. Thus, if x is a member of set A, we write $x \in A$.
4. If x does not belong to set A, we write $x \notin A$.
5. Sets are specified by any of the following methods:
   (*i*) **Description method:** By writing the description in braces, viz. A = {months of year}, B = {primary colours}

(*ii*) **Roster method:** By listing the names of the elements in braces.

For example, the set of letters in the word 'NAINITAL' is written as A = {N, A, I, T, L}

Repetition is not done while listing the elements

(*iii*) **Set builder form.** A set is specified in the form $\{x|$ statement of property which x satisfies$\}$.

For example, $\{x|x$ is a multiple of 4, $x \leq 20\}$

6. A set containing only one element is called a **singleton set**.

7. Two sets are called **overlapping sets,** if they have at least one element in common. Thus, A and B are overlapping sets if $A = \{0, 2, 3, 5, 7, 9\}$, $B = \{0, 3, 6, 9, 12, 15\}$

8. If two sets have no elements in common, they are called **disjoint sets.** For example, if $A = \{$letters of 'CAT'$\}$, $B = \{$letters of 'SUN'$\}$, then A and B are disjoint sets.

9. A set which contains a definite number of objects *i.e.*, in which the counting comes to an end is called **finite set**, *e.g.*, $A = \{$people on Earth$\}$ is a finite set.

If the process of counting the elements of a set cannot come to an end, it is an **infinite set**. The set of integers is an infinite set.

10. A set that has no members is called the **empty set** or the **null set**. It is denoted by symbol { } or ϕ.

11. The number of members in a set is called the **cardinal number** of the set. The cardinal number of a set A is denoted by $n(A)$.

For example, if $A = \{p, q, r\}$ then $n(A) = 3$

The cardinal number of an infinite set is not defined.

The cardinal number of the empty set is zero, *i.e.*, $n(\phi) = 0$.

12. Sets containing equal number of elements are called **equivalent sets.** Sets containing the same elements are called **equal sets.**

For example, if $A = \{$letter of 'TALE'$\}$, $B = \{$letters of 'MAIL'$\}$ and $C = \{$letters of '*LATE*'$\}$, then A and C are equal sets, while A and B, B and C are equivalent sets.

All equal sets are equivalent sets while all equivalent sets are not equal sets.

13. If every member of a set A is also a member of a second set B, then the set A is called a **subset** of B. This member expressed by writing $A \subseteq B$. If a set E is a part of a set F, then the set E is called a **proper subset** of set F. This is expressed by writing $A \subset B$.

For example, if $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 3, 5, 7\}$, $C = \{3, 5\}$, $D = \{x|x \in N, x \leq 7\}$, then $B \subset A$, $C \subset A$, $C \subset B$, $A \subseteq D$, and $D \subseteq A$. A and D are equal sets.

The symbol $\not\subseteq$ is used to denote '**is not a subset of** ' and $\not\subset$ to denote '**not a proper subset of**'.

1. Every set is a subset of itself, *i.e.*, $A \subseteq A$

2. The null set is a subset of every set, i.e., $\phi \subseteq A$, where A is any set.

14. The set of all possible subsets of a set A is called the **power set** of A. It is denoted by $P(A)$.

For example, if $A = \{m, a, t\}$ then $P\{A\} = \{\phi, \{m\}, \{a\}, \{t\}, \{m, a\}, \{a, t\}, \{m, t\}, \{m, a, t\}\}$.

15. The total number of subsets of a set containing n elements is 2^n. The total number of proper subsets is 2^n–1.

For example, if $A = \{x|x \in Z, -3 \leq x < 4\}$, then $A = \{-3, -2, -1, 0, 1, 2, 3\}$ and A has 2^7, i.e., 128 subsets. The number of proper subsets is $2^7 - 1$, i.e., 128 – 1 i.e., 127.

16. A **universal set** is the set of elements from which elements may be chosen for a particular discussion. It is denoted by the symbol 'ξ'.

17. If A is a subset of a universal set ξ, then the set of all those elements of ξ which do not belong to A is called the **complement** of A and is denoted by A' or $\overline{A}$ or A^C.

For example, if ξ = {digits of the whole number system}

A = set of digits in the number '203786' then

A' = {1, 4, 5, 9} Also, $n(\xi) = 10$, $n(A) = 6$, $n(A') = 4 = n(\xi) - n(A)$

$(A')' = A$

$\xi' = \phi$

$\phi' = \xi$

18. The **union of two sets** A and B denoted by $A \cup B$, is a set C formed by combining the elements of A and B. If A and B are overlapping sets, then the elements that are common to both A and B are listed only once in set C.

$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}$.

For example, if $A = \{2, 4, 6, 8, 10\}$, $B = \{5, 7, 9\}$,

$C = \{0, 1, 2, 3, 4\}$ then

$A \cup B = \{2, 4, 5, 6, 7, 8, 9, 10\}$, $A \cup C = \{0, 1, 2, 3, 4, 6, 8, 10\}$,

$B \cup C = \{0, 1, 2, 3, 4, 5, 7, 9\}$

$A \cup \phi = A$

$A \cup \xi = \xi$

$A \cup A = A$

$A \cup A' = \xi$

If $A \subseteq B$, then $A \cup B = B$

$A \subseteq A \cup B$, $B \subseteq A \cup B$

19. The **Intersection of two sets** A and B, denoted by $A \cap B$, is the set of element common to A and B.

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

For example, (*i*) If $A = \{x \mid x \in N, x \leq 10\}$,

$B = \{x \mid x$ is an even number, $x \leq 10\}$ then

$A \cap B = \{2, 4, 6, 8, 10\}$

(*ii*) If A = {letters of 'MOUSE'}

B = {letters of 'CALL'}, then $A \cap B = \phi$

$A \cap \xi = A, A \cap A = A$

$A \cap \phi = \phi, A \cap A' = \phi$

$A \cap B \subseteq A, A \cap B\ A$ B

20. Difference of two sets. Let A and B be two sets, then '$A - B$' is the set of elements which belong to A but do not belong to B.

Thus, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x \mid x \in B \text{ and } x \notin A\}$

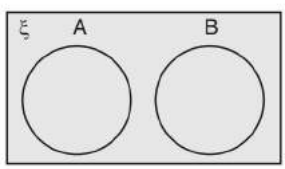
For example, (*i*) If $A = \{m, o, u, s, e\}$, $B = \{h, o, p, e\}$, then

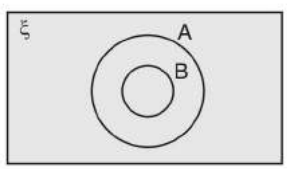
$A - B = \{m, u, s\}$, $B - A = \{h, p\}$

(*ii*) $A = \{a, b, c, d, e, f\}$, $B = \{b, c, e\}$, then

$A - B = \{a, d, f\}$, $B - A = \phi$.

21. Venn diagrams are diagrams to express various ideas about sets and relationships between them. Usually a universal set is represented by a rectangle and its subsets by closed figures such as circles or ovals.





CHAPTER TEST

1. Answer true or false:

(*i*) If $A = \{1, 2, 3, 4, 5, 6\}$ and B = {whole numbers less than 6}, then $A = B$.

(*ii*) The empty set has no subsets.

(*iii*) The empty set may be represented by (ϕ)

(*iv*) $\{1, 2\} = \{2, 1\}$

(*v*) $\phi = \{0\}$

(*vi*) $\phi \subset \{1, 2\}$

(*vii*) Given that $A = \{x | x \text{ is a square}\}$ and $B = \{x | x \text{ is a rectangle}\}$. State which of the following is true: $A = B$ or $A \neq B$.

(*viii*) The total number of subsets of a finite set which contains n elements is 2^n.

(*ix*) If $A \subset B$ and $C \subset B$, then $A \subset C$.

(*x*) The cardinal number of the set of the letters in the word 'INDIA' is 4.

(*xi*) There are as many numbers in the set of natural numbers as in the set of natural numbers divisible by 17.

(*xii*) If $n(A) = n(B)$, then $A \leftrightarrow B$ *i.e.*, A and B are equivalent sets.

(*xiii*) If $x \in A'$, then $x \notin (A')'$

(*xiv*) If $A = \{0, 2, 4\}$, and $B = \{0, 3, 5, 7\}$, then $A \cap B = \phi$.

(*xv*) $\xi \cap \phi' = \xi$

(*xvi*) The union of two overlapping (intersecting) sets is either of the two sets.

(*xvii*) $A \cap B = \phi$ then $A \cap \phi = B$.

(*xviii*) For any sets X and Y, $(X \cup Y)' = X' \cap Y'$.

(*xix*) For any sets A and B, $A \cup B = B \cup A$.

2. A is the set of all integers from 20 to 70, both inclusive.

$B = \{x : x \in A, x \text{ is a perfect square}\}$,

$C = \{x : x \in A, x \text{ is a prime number}\}$

$D = \{x : x \in A, x \text{ is the first digit of } x > \text{its second digit}\}$.

List the following sets.

(*i*) $B \cap C$ (*ii*) $B \cap D$ (*iii*) $B \cap C \cap D$

3. If $X = \{a, b, c, d\}$ and $Y = \{f, b, d, g\}$ find

(*i*) $X - Y$ (*ii*) $Y - X$ (*iii*) $X \cap Y$ (*iv*) $X \cup Y$

4. Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$, find

(*i*) A' (*ii*) B' (*iii*) $(A \cup C)'$ (*iv*) $(A \cup B)'$ (*v*) $(A \cap C)'$ (*vi*) $(A')'$

(*vii*) $(B - C)'$ **(*NCERT*)**

5. If $\xi = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 5\}$, and $B = \{6, 7\}$, verify that $A - B = A \cap B' = B' - A'$.

6. (*i*) In the Venn diagram, shade the set $A \cup (B \cap C)$.

(*ii*) Express in set notation the subset shaded in the Venn diagram.

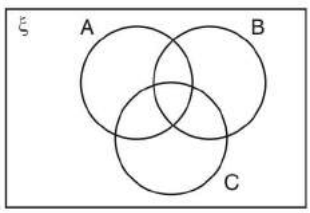


Fig. Q. 6 (*i*)

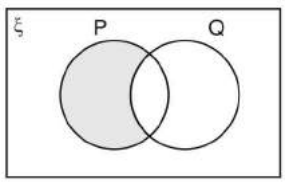


Fig. Q. 6 (*ii*)

(*iii*) In a class of 36 students, 25 study History, 20 study Geography and 4 study neither History nor Geography. Find how many students study both History and Geography? **(*C.U.*)**

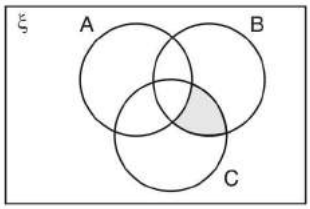
7. $\xi = \{x : x \text{ is an integer and } 1 \leq x \leq 8\}$

$P = \{x : x > 5\}$, $Q = \{x : x \leq 3\}$

(*i*) (*a*) Find the value of $n(P \cup Q)$

(*b*) List the elements of $P' \cap Q'$

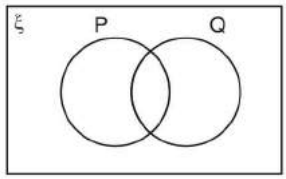
(*ii*) Express, in set notation, as simply as possible, the subset shaded in the Venn diagram.



(*iii*) It is given that $n(\xi) = 40$, $n(P) = 18$, $n(Q) = 20$ and $n(P \cap Q) = 7$. Find (*a*) $n(P \cup Q)$, (*b*) $n(P' \cup Q')$.

(*iv*) There are 27 children in a class. Of these children, 19 own a bicycle, 15 own a scooter and 3 own neither a bicycle nor a scooter. Using a Venn diagram, or otherwise, find the number of children who own a bicycle but not a scooter. **(*C.U.*)**

8. (*i*) In the Venn diagram, shade $P \cup Q'$.



(*ii*) A group of 60 children attend an often school club. Of these, 35 children play football and 29 play hockey. 3 children do not play either football or hockey. By drawing a Venn diagram or otherwise, find the number of children who play only hockey. **(*C.U.*)**

9. (*i*) A, B and C are subsets of universal set ξ. If $A = \{2, 4, 6, 8, 12, 20\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20\}$ and ξ is the set of whole numbers, draw a Venn diagram showing the relation of A, B and C. **(*NCERT, Examplar*)**

(*ii*) If a set A has n elements, then the number of elements in power set of A is

(*a*) n (*b*) 2^n (*c*) n^2 (*d*) none of these **(*Roorkee*)**

(*iii*) If a finite set S contains n elements, then the number of non-empty proper subsets of S is

(*i*) 2.2^{n-1} (*ii*) $2(2^n - 1)$ (*iii*) $(2^{n-1} - 1)$ (*iv*) $2(2^{n-1} - 1)$

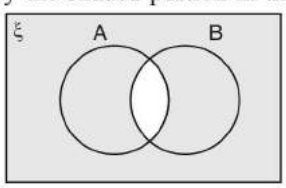
10. If $A = \{3, 6, 8, 15, 19\}$ and $B = \{1, 2, 6, 8, 14, 15\}$, then verify and $A \Delta B = (A \cup B) - (A \cap B)$.

11. (*i*) If A, B, C are three non-zero sets, then $(A \cap B) \cap (B \cap C) \cap (C \cap A)$ is equal to

(*a*) $A \cap B \cap C$ (*b*) $A \cup B \cup C$ (*c*) ϕ (*d*) none of these

(*UPSEAT 2002*)

(*ii*) Consider the given Venn diagram if $n(\xi) = 42$, $n(A) = 15$, $n(B) = 12$ and $n(A \cup B) = 22$, then the area represented by the shaded portion in the Venn diagram is



(*a*) 25 (*b*) 27 (*c*) 32 (*d*) 37 **(*NDA*)**

(*iii*) In a class of 100 students, 55 students have passed in mathematics and 67 students have passed in physics. Then, the number of students who have passed in physics only is

(*a*) 22 (*b*) 33 (*c*) 10 (*d*) 45 **(*ISM Dhanbad*)**

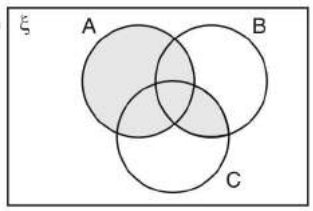
ANSWERS

1. (*i*) False (*ii*) False (*iii*) False (*iv*) True (*v*) False (*vi*) False (Empty set is considered a subset of every set and not a proper subset) (*vii*) $A \neq B$, (*viii*) True (*ix*) False (*x*) True (*xi*) True (*xii*) True (*xiii*) True (*xiv*) False (*xv*) True (*xvi*) False (*xvii*) False (*xviii*) True (*xix*) True

2. (*i*) ϕ (*ii*) {64} (*iii*) ϕ

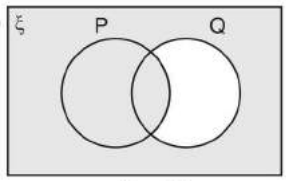
3. (*i*) {*a*, *c*}(*ii*) {*f*, *g*} (*iii*) {*b*, *d*} (*iv*) {*a*, *b*, *c*, *d*, *f*, *g*}

4. (*i*) $A' = \{5, 6, 7, 8, 9\}$ (*ii*) $B' = \{1, 3, 5, 7, 9\}$ (*iii*) $(A \cup C)' = \{7, 8, 9\}$ (*iv*) $(A \cup B)' = \{5, 7, 9\}$
(*v*) $(A \cap C)' = \{1, 2, 5, 6, 7, 8, 9\}$ (*vi*) $(A')' = \{1, 2, 3, 4\} = A$ (*vii*) $(B - C)' = \{1, 3, 4, 5, 6, 7, 9\}$

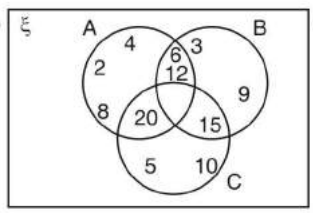
6. (*i*) 

(*ii*) $P \cap Q'$ (*iii*) 13

7. (*i*) (*a*) 6 (*b*) 4, 5 (*ii*) $B \cap C \cap A'$ (*iii*) (*a*) 31 (*b*) 9 (*iv*) 9

8. (*i*) 

$P \cup Q'$ (*ii*) 22

9. (*i*) 

(*ii*) (*b*) (*iii*) *d*

11. (*i*) (*a*) Hint check by Venn diagram (*ii*) *d* (*iii*) *d*

CHAPTER 2

RELATIONS AND FUNCTIONS

SECTION A
CARTESIAN PRODUCT

2.01. Ordered pair

In daily life, there are situations in which order is important and in others it is not. For example, when we take off our shoes, it does not matter whether we remove the shoe on our right first. We will ultimately get both shoes off. The end result is not affected by the order in which we perform the steps. On the other hand, if we have a sock and a shoe, the order in which they are put on is not only important but essential also.

Study the following examples:

(*i*) The table below shows how three countries are paired with their capitals.

Country	→	India	England	China
Capital	→	New Delhi	London	Beijing

Instead of using the above table, we can show this by means of ordered pairs.

(India, New Delhi), (England, London), (China, Beijing). In the ordered pair, (India, New Delhi), India is the **first component** and New Delhi is the **second component.**

India ↗ **first component** — New Delhi ↗ **second component**

Definition. ***An ordered pair is a pair of objects whose components occur in a special order. It is written by listing the two components in the specified order, separating them by a comma and enclosing the pair in parentheses. In the ordered pair (a, b), a is called the first component and b, the second component.***

Note 1. It may be noted that (a, b) is not the same as $\{a, b\}$. The former denotes an ordered pair, whereas the latter denotes a set.

Note 2. Ordered pairs may have the same first and second components such as (2, 2), (5, 5).

2.02. Equality of ordered pairs

Two ordered pairs of numbers are defined to be equal when both the first components are equal and also their second components are equal. Thus, $(-3, 8) = (-\frac{15}{5}, \frac{24}{3})$ but $(2, 3) \neq (3, 2)$, $(5, 7) \neq (5, 8)$ and $(1, 9) \neq (4, 9)$.

Ex. 1. If $A = \{a, b\}$, then all possible ordered pairs are (a, a) (b, b) (a, b) (b, a).

Ex. 2. If $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then all possible ordered pairs such that in each ordered pair the first component is an element of set A and second component is an element of set B are obtained by pairing each element from set A with each element from set B. This can be done by the following scheme:

$(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1), (c, 2), (c, 3), (c, 4)$.

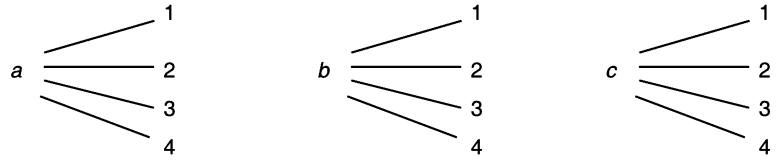


Fig. 2.01

Ex. 3. Express $\{(x, y) : x^2 + y^2 = 25$, where $x, y \in W\}$ as a set of ordered pairs.

Sol. Let us find possible values of whole numbers x, y which make $x^2 + y^2 = 25$.

$$x = 0,\ y = 5 \Rightarrow x^2 + y^2 = 0^2 + 5^2 = 25$$
$$x = 3,\ y = 4 \Rightarrow x^2 + y^2 = 3^2 + 4^2 = 25$$
$$x = 4,\ y = 3 \Rightarrow x^2 + y^2 = 4^2 + 3^2 = 25$$
$$x = 5,\ y = 0 \Rightarrow x^2 + y^2 = 5^2 + 0^2 = 25$$

$\therefore$ The required set of ordered pairs is $\{(0, 5), (3, 4), (4, 3), (5, 0)\}$.

Ex. 4. The number of elements in the set $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$, where Z is the set of all integers is

(*a*) 2 **(*b*) 4** **(*c*) 8** **(*d*) 12** ***(Kerala Eng. 2005, 2007)***

Sol. Given set is $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$

By trial, we see that $2(\pm 2)^2 + 3(\pm 3)^2 = 35$ and $2(\pm 4)^2 + 3(\pm 1)^2 = 35$

$\therefore$ 8 elements of the set are

$\{(2, 3), (2, -3), (-2, 3), (-2, -3), (4, 1), (4, -1), (-4, 1), (-4, -1)\}$

2.03. Cartesian product

Consider a set B of two boys and another set G of four girls.

$B = \{\text{Suresh, Amitabh}\}$, $G = \{\text{Vidya, Hema, Rekha, Saroj}\}$

The possible number of ways of forming partnership for a mixed doubles badminton team from these two sets is equal to the possible number of ordered pairs that can be formed from the set of two boys and the set of 4 girls. The set of all these possible ordered pairs is called the ***Cartesian product*** of the given sets.

In the above mentioned illustration, the possible partnerships are (Suresh, Vidya), (Suresh, Hema), (Suresh, Rekha), (Suresh, Saroj), (Amitabh, Vidya), (Amitabh, Hema), (Amitabh, Rekha), (Amitabh, Saroj).

The cartesian product of sets B and G, denoted by $B \times G$, is $B \times G = \{$(Suresh, Vidya), (Suresh, Hema), (Suresh, Rekha), (Suresh, Saroj), (Amitabh, Vidya), (Amitabh, Hema), (Amitabh, Rekha), (Amitabh, Saroj)$\}$.

Definition. ***Let A and B be two non-empty sets. Then the set of all possible ordered pairs (x, y) such that the first component x of the ordered pairs is an element of A, and the second component y is an element of B, is called the Cartesian product of sets A and B. It is denoted by $A \times B$ which reads "A cross B".***

Thus, $A \times B = \{(a, b) \mid a \in A$ and $b \in B\}$

Also, $n(A \times B) = n(A) \cdot m(B)$, i.e., if set A has n elements and set B has m elements then the product set $A \times B$ has nm elements.

Note 1. The cartesian product $A \times B$ is not the same as $B \times A$. In $A \times B$, the set A is named first and so its elements will appear as the first components of the ordered pairs.

In $B \times A$, the set B is named first so in this case its elements will appear as the first components of the ordered pairs.

Note 2. If either A or B is the null set, then we define $A \times B$ to be the null set. For example, if $A = \{a, b\}$, and $B = \phi$, then $A \times B = \phi$.

Note 3. If either A or B is an infinite set and other is a non-empty set, then $A \times B$ is also an infinite set.

Note 4. $A \times \phi$ is the empty set where A is any set.

Note 5. If A and B are two non-empty sets having n elements in common, then $A \times B$, $B \times A$ have n^2 elements in common.

Note 6. If $A = B$, then $A \times B$ becomes $A \times A$ and is denoted by A^2.

Ex. 5. If $A = \{a, b\}$, and $B = \{1\}$, then $A \times B = \{(a, 1), (b, 1)\}$ and $B \times A = \{(1, a), (1, b)\}$.

Ex. 6. If $A = \{2, 4, 6\}$, then

$$A^2 = A \times A = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}.$$

Help line: $2 \to 2, 4, 6$; $4 \to 2, 4, 6$; $6 \to 2, 4, 6$

Ex. 7. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B. **[N.C.E.R.T]**

Sol. Given: $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$

$$= (a, b) \times (x, y)$$

$$\Rightarrow \quad A = \{a, b\}, B = \{x, y\}$$

Ex. 8. Let $A = \{x \in N : x \leq 4\}$ and $B = \{y \in N : 3 < y \leq 5\}$. Find out

(i) $A \times A$ **(ii)** $A \times B$ **(iii)** $B \times A$ **(iv)** $(B \times B)$

(v) $n(A \times A)$ **(vi)** $n(A \times B)$ **(vii)** $n(B \times A)$ **(viii)** $n(B \times B)$

Sol. Here $A = \{1, 2, 3, 4\}$, $B = \{4, 5\}$

(*i*) $A \times A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$.

(*ii*) $A \times B = \{1, 2, 3, 4\} \times \{4, 5\}$.

$= \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$.

(*iii*) $B \times A = \{(4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)\}$.

(*iv*) $B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$.

(*v*) $n(A \times A) = 4 \times 4 = 16$ (*vi*) $n(A \times B) = 4 \times 2 = 8$

(*vii*) $n(B \times A) = 2 \times 4 = 8$ (*viii*) $n(B \times B) = 2 \times 2 = 4$.

Ex. 9. If $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{1, 2, 3, 4, 5\}$ find (*i*) $A \times B$ (*ii*) $C \times B$ (*iii*) $B \times B$. Hence prove that $(C \times B) - (A \times B) = B \times B$.

Sol. $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

$C \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5), (5, 4), (5, 5)\}$

$B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$

$(C \times B) - (A \times B) = \{(4, 4), (4, 5), (5, 4), (5, 5)\} = B \times B$.

Ex. 10. If each one of the two sets A and B has 3 elements, how many elements are there in $A \times B$? If three of the elements in $A \times B$ be (2, 3), (4, 4) and (6, 6), find the remaining elements.

Sol. It is given that $n(A) = 3$, $n(B) = 3$

$\therefore \quad n(A \times B) = n(A) \cdot n(B) = 3 \times 3 = 9.$

Now (2, 3), (4, 4) and (6, 6) $\in A \times B$

$\therefore$ The first components 2, 4, 6 must be in A.

$\therefore$ $A = \{2, 4, 6\}$, because A has exactly 3 elements.

Similarly, $B = \{3, 4, 6\}$

$\therefore \quad A \times B = \{(a, b) : a \in A, b \in B\}$

$= \{(2, 3), (2, 4), (2, 6), (4, 3), (4, 4), (4, 6), (6, 3), (6, 4), (6, 6)\}.$

$\therefore$ The remaining elements of $A \times B$ are

$(2, 4), (2, 6), (4, 3), (4, 6), (6, 3), (6, 4).$

Ex. 11. Is $A \times \phi$ the empty set or not, where ϕ denotes the empty set and A is any set?

Sol. ϕ is the empty set. It contains no element. Therefore, no ordered pair is possible in $A \times \phi$. Hence, it is clear that $A \times \phi$ contains no elements, *i.e.*, $A \times \phi$ is the empty set.

Ex. 12. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$, verify that

(*i*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(*ii*) $A \times C$ is a subset of $B \times D$. **[*N.C.E.R.T*]**

Sol. (*i*) $(B \cap C) = \{1, 2, 3, 4\} \cap \{5, 6\} = \phi$

$\Rightarrow \quad A \times (B \cap C) = \{1, 2\} \times \phi = \phi$ ... (1)

$(A \times B) = \{1, 2\} \times \{1, 2, 3, 4\} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\}$

$(A \times C) = \{1, 2\} \times \{5, 6\} = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$\Rightarrow \quad (A \times B) \cap (A \times C) = \phi$ ... (2)

(1) and (2) $\Rightarrow A \times (B \cap C) = (A \times B) \cap (A \times C)$

(*ii*) $(A \times C) = \{1, 2\} \times \{5, 6\} = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$ ... (1)

$(B \times D) = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$

$= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$ ... (2)

From (1) and (2) it is clear that all the elements of $(A \times C)$ are contained in $(B \times D)$. Hence, $A \times C$ is a subset of $(B \times D)$.

Ex. 13. If $A = \{1, 4\}$ $B = \{2, 3, 6\}$ and $C = \{2, 3, 7\}$, verify that $A \times (B - C) = (A \times B) - (A \times C)$.

Sol. $B - C = \{2, 3, 6\} - \{2, 3, 7\} = \{6\}$

$\therefore \quad A \times (B - C) = \{1, 4\} \times \{6\} = \{(1, 6), (4, 6)\}$ ... (1)

Also, $A \times B = \{1, 4\} \times \{2, 3, 6\} = \{(1, 2), (1, 3), (1, 6), (4, 2), (4, 3), (4, 6)\}$

$A \times C = \{1, 4\} \times \{2, 3, 7\} = \{(1, 2), (1, 3), (1, 7), (4, 2), (4, 3), (4, 7)\}$

$\therefore \quad (A \times B) - (A \times C) = \{(1, 6), (4, 6)\}$ ... (2)

From (1) and (2) it follows that $A \times (B - C) = (A \times B) - (A \times C)$.

Ex. 14. $A \subseteq B$ and $C \subseteq D$, prove that $A \times C \subseteq B \times D$.

Sol. Let (a, b) be any arbitrary element of $A \times C$. ... (1)

Then, $(a, b) \in A \times C \Rightarrow a \in A$ and $b \in C$

$\Rightarrow$ $a \in B$ and $b \in D$ [Since $A \subseteq B$ and $C \subseteq D$]

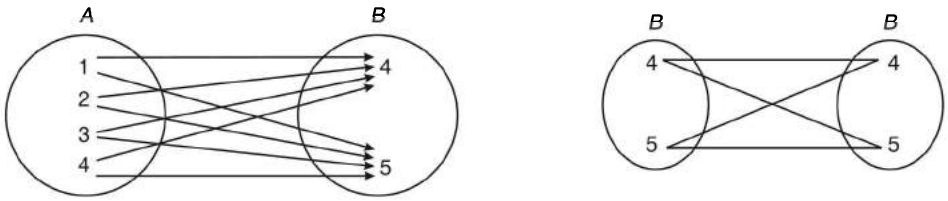
$\Rightarrow$ $(a, b) \in B \times D$ ... (2)

From (1) and (2) it follows that $A \times C \subseteq B \times D$.

Graphical Representation

Ex. 15. Taking $A = \{1, 2, 3, 4\}$, $B = \{4, 5\}$ in Ex. 8, represent $A \times B$ and $B \times B$ pictorially, using arrows.

Sol.

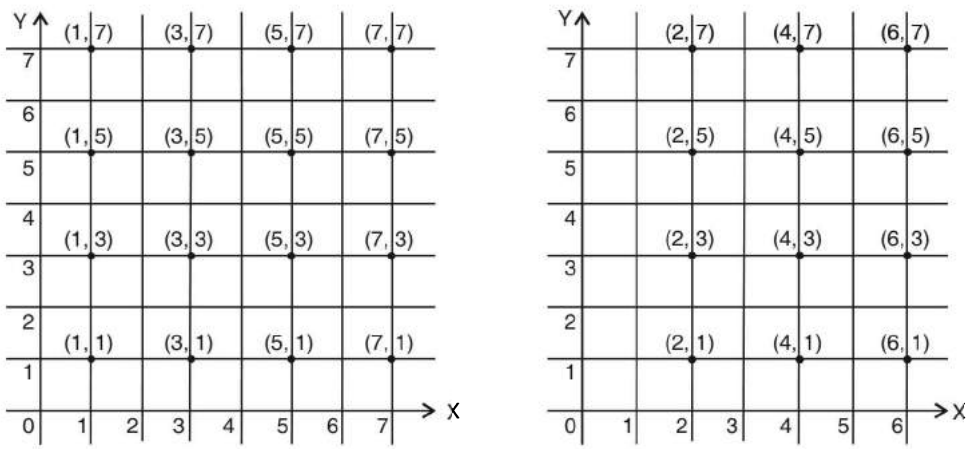


Representation of $A \times B$ Representation of $B \times B$

Fig. 2.02

Ex. 16. Taking $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$, draw the graphs of (*i*) $A \times A$ (*ii*) $B \times A$.

Sol. $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$



Graph of $A \times A$ Graph of $B \times A$

Fig. 2.03

$A \times A = \{1, 3, 5, 7\} \times \{1, 3, 5, 7\}$

$= \{(1, 1), (1, 3), (1, 5), (1, 7), (3, 1), (3, 3), (3, 5), (3, 7), (5, 1), (5, 3), (5, 5), (5, 7), (7, 1), (7, 3), (7, 5), (7, 7)\}$

$B \times A = \{2, 4, 6\} \times \{1, 3, 5, 7\}$

$= \{(2, 1), (2, 3), (2, 5), (2, 7), (4, 1), (4, 3), (4, 5), (4, 7), (6, 1), (6, 3), (6, 5), (6, 7)\}$

Their graphs are shown in Fig. 2.03

GENERALISATION: CARTESIAN PRODUCT OF MORE THAN TWO SETS

2.04. Ordered triplet

If there are 3 sets A, B, C, then choosing 3 elements a, b, c such that $a \in A$, $b \in B$, $c \in C$ we form an **ordered triplet** (a, b, c). The set of all such ordered triplets is called the cartesian product of the three sets A, B, C, and is denoted by $\boldsymbol{A \times B \times C}$.

Definition. For any non-empty set A, we define

$$(A \times A \times A) = \{(a, b, c) : a, b, c \in A\}$$

The three numbers a, b, c, are listed in a specific order and enclosed in parentheses. Then,

$$(3, 4, 5) \neq (4, 3, 5) \neq (5, 4, 3), \text{ etc.}$$

Similarly, if there are n sets $A_1, A_2, \ldots A_n$ then choosing n elements $a_1, a_2, \ldots a_n$ such that $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$ $a_n \in A_n$, we may form an ordered n-triple. The set of all such ordered n-triplets is called the cartesian product of n sets $A_1, A_2, \ldots A_n$ and is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or briefly by $\prod_{i=1}^{n} \boldsymbol{A_i}$.

Ex. 17. If $A = \{-1, 1\}$, find $A \times A \times A$. **[N.C.E.R.T]**

Sol. $A \times A = \{-1, 1\} \times (-1, 1) = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$

$A \times A \times A = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\} \times (-1, 1)$

$= \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1 -1, 1), (1, 1, -1), (1, 1, 1)\}$

Cartesian product of the set of reals with itself.

Ex. 18. Let R be the set of real numbers. What does $(R \times R \times R)$ represent?

Sol. $R \times R \times R = \{(a, b, c) : a, b, c \in R\}$

Thus, $(R \times R \times R)$ represents the set of all coordinates of points in three-dimensional plane.

Important results on Cartesian Product of sets.

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. If A and B are any two non-empty sets and $A \times B = B \times A$, then $A = B$.
5. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
6. If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.
7. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$
8. For any sets A, B, C, D
   $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
9. If A and B are two non-empty sets and $A \cap B$ has $\boldsymbol{n}$ elements, then $A \times B$ and $B \times A$ have n^2 elements in common.

EXERCISE 2 (*a*)

1. $x \in \{2, 4, 6, 9\}$ and $y \in \{4, 6, 18, 27, 54\}$. Form all ordered pairs (x, y) such that x is a factor of y and $x < y$.
2. Find the numbers x and y if $(x + 3, y - 5) = (5, 0)$.
3. If $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$, find (*i*) $A \times A$ (*ii*) $A \times B$ (*iii*) $B \times A$ (*iv*) $B \times B$ (*v*) $n(A \times A)$ (*vi*) $n(A \times B)$ (*vii*) $n(B \times A)$ (*viii*) $n(B \times B)$.

4. Answer true or false:

(*i*) If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$

(*ii*) $\{(a, x), (a, y), (b, x), (b, y)\}$ is a product set.

(*iii*) If $n(A) = x$ and $n(B) = y$ and $A \cap B = \phi$, then $n(A \times B) = xy$.

(*iv*) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$.

5. Given $A = \{1, 2\}$, $B = \{3\}$, $C = \{4, 5\}$, test whether the following are true:

(*i*) $A \times (B \cup C) = (A \cup B) \times (A \cup C)$.

(*ii*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

6. If $A = \{1, 2, 3, 4\}$, $B = \{5, 7, 9\}$, $C = \{2, 4, 6\}$, find

(*i*) $A \times B$ (*ii*) $(B \times C)$ (*iii*) $C \times A$ and draw their graphs.

7. Some elements of $A \times B$ are (a, x), (c, y), (d, z). If $A = \{a, b, c, d\}$, find the remaining elements of $A \times B$ such that $n(A \times B)$ is least.

[**Hint.** Set B must be $\{x, y, z\}$]

8. The ordered pairs (1, 1), (2, 2), (3, 3) are among the elements in the set $A \times B$. If A and B have 3 elements each, how many elements in all does the set $A \times B$ have? Also find the remaining elements.

[**Hint.** Type Solved Ex. 10]

9. If $A = \{1, 4\}$, $B = \{2, 3, 6\}$ and $C = \{2, 3, 7\}$, then verify that

(*i*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (*ii*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

10. If $A = \{2, 3\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 4\}$ show that $A \times A = (B \times B) \cap (C \times C)$.

11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.

12. The Cartesian product $A \times A$ has 9 elements among which are found (–1, 0) and (0, 1). Find the set A and the remaining elements of $A \times A$.

ANSWERS

1. $\{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54), (9, 18), (9, 27), (9, 54)\}$.

2. $x = 2, y = 5$.

3. (*i*) $A \times A = \{(1, 1), (1, 3), (1, 5), (1, 7), (3, 1), (3, 3), (3, 5),$
$(3, 7), (5, 1), (5, 3), (5, 5), (5, 7), (7, 1), (7, 3), (7, 5), (7, 7)\}$.

(*ii*) $A \times B = \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6), (7, 2), (7, 4), (7, 6)\}$.

(*iii*) $B \times A = \{(2, 1), (2, 3), (2, 5), (2, 7), (4, 1), (4, 3), (4, 5), (4, 7), (6, 1), (6, 3), (6, 5), (6, 7)\}$.

(*iv*) $B \times B = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}$.

(*v*) $n(A \times A) = 16$, (*vi*) $n(A \times B) = 12$, (*vii*) $n(B \times A) = 12$, (*viii*) $n(B \times B) = 9$.

4. (*i*) False (*ii*) True (*iii*) True (*iv*) False

5. (*i*) False (*ii*) True

7. $(a, y), (a, z), (b, x), (b, y), (b, z), (c, x), (c, z), (d, x), (d, y)$

8. 9; (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)

11. $A = \{x, y, z\}$, $B = \{1, 2\}$

12. $A = \{–1, 0, 1\}$. The remaining elements of $A \times A$ are (–1, –1), (–1, 1), (0, –1), (0, 0), (1, – 1), (1, 0), (1, 1).

HINTS TO SELECTED QUESTIONS

11. $n(A) = 3$ and $n(B) = 3 \Rightarrow A$ contains 3 elements and B contains 2 elements.
Also, $(x, 1), (y, 2), (z, 1)$ are in $A \times B$
$\therefore$ obviously, $A = \{x, y, z\}$ and $B = \{1, 2\}$

12. $(-1, 0) \in A \times A \Rightarrow -1, 0 \in A$
Also, $(0, 1) \in A \times A \Rightarrow 0, 1 \in A$
$\Rightarrow -1, 0, 1 \in A$. It is given that A has exactly 3 elements. Hence $A = \{-1, 0, 1\}$.

SECTION B
RELATIONS

2.05. Relations (Intuitive or non-mathematical concept)

Study the following sentences:

1. Dashratha **was father of** Rama.
2. New Delhi is **the capital of** India.
3. 52 is **greater than** 43.
4. The line AB is **parallel to the** line CD.
5. $\{a, b\}$ **is a subset of** $\{a, b, c\}$.

All these sentences express **relationship** between two objects. In mathematics, symbols replace phrases. Thus, the phrases "*is larger than", "is parallel to", "is a subset of",* are respectively replaced by the symbols, '$>$', '$\|$' , '$\subset$'.

Intuitively, we can say that the word "relation" implies an association of two objects (people, numbers, ideas, etc.) according to some property possessed by them.

2.06. Relation as an association between two sets (Mathematical concept)

Consider the sentence; 'x is the capital of y'.

A few ordered pairs satisfying the sentence are (Colombo, Sri Lanka), (London, England), (Ottawa, Canada) and (Tokyo, Japan). Similarly, the ordered pairs.

$\{(8, 6), (7, 3), (2, 1), (1, 0)\}$ satisfy the sentence 'x is *greater than* y' while

$(7, 8), (3, 7), (1, 2), (0, 1)$ do not.

From the above we can conclude that we can always form a set of ordered pairs from the given relation.

Definition. ***A relation is a set of ordered pairs obtained by virtue of an association between two sets. Any set of ordered pairs is, therefore, a relation. The set of first components of the ordered pairs is called the*** **domain** ***and the set of second components is called the*** **range.**

For example, in the relation $\{(6, 8), (3, 7), (1, 2), (0, 1)\}$ the domain is $\{6, 3, 1, 0\}$ and the range is $\{8, 7, 2, 1\}$.

2.07. Notation for relation

We use the letter 'R' to designate a relation. For example,

(*i*) $R = \{(1, 5), (4, 8), (2, 13), (16, 20)\}$.

(*ii*) $R = \{(p, q) : p \in B, q \in G, q \text{ is the sister of } p\}$.

(*iii*) $R = \{(x, y) : x, y \in W, y > x\}$.

2.08. The notation xRy

If (x, y) is a member of a relation R, then we also use the symbolism 'xRy' and read 'x is the relation R to y'.

i.e., $(x, y) \in R \Leftrightarrow xRy$. If $(x, y) \notin R$, then $x \not R\ y$.

For example, for the relation '*is the capital of* ', we have 'Patna R Bihar.', 'Kolkata R West Bengal, 'Bhopal R M.P.' We give below more examples of relations which can be expressed via the 'xRy' symbolism.

(*i*) Let 'R' mean 'is *equal to',* then 'xRy' means $x = y$.

(*ii*) Let $A = \{2, 3, 4, 5, \ldots 12\}$ and R mean '*is one-third of* ' in $A \times A$, then

$2R6, 3R9, 4R12 \quad \Rightarrow \quad R = \{(2, 6), (3, 9), (4, 12)\}$

Note 1. We emphasize that relations are sets and therefore any statements about sets or any operation defined on sets are appropriately applied to relations. One can thus speak of the intersection or union of two relations or speak of one relation being a subset of another.

Note 2. In the Art. 2.03, we have seen that the product set $A \times B$ is a set of ordered pairs $(a \in A, b \in B)$. Since a relation is also a set of ordered pairs, therefore a relation is the subset of a product set.

2.09. Relation as a subset of the cartesian product

Definition. ***If A and B are any two non-empty sets, then any subset of A × B is defined as a relation from A to B.***

For example, suppose $A = \{1, 2, 3\}$, and $B = \{1, 2, 3, 4\}$. Then $\{(2, 3), (2, 4), (1, 3)\}$, is a relation in $A \times B$. Many more relations (subsets) can be selected similarly at random from our product set $A \times B$.

2.10. Domain, codomain and range of a relation

Let R be a relation from set A to set B. Then the set of first entries of the ordered pairs in R is called the ***domain***.

Thus domain $= \{a \mid (a, b) \in R\}$

Similarly the set of second entries of the ordered pairs in R is called the ***range***.

Thus range $= \{ b \mid (a, b) \in R\}$

The second set B is called the ***codomain*** of R.

For example. If $A = \{16, 25, 36, 49\}$, $B = \{1, 4, 5, 6\}$ and R be the relation "*is square of*" from A to B then

$R = \{(a, b), \mid a = b^2, a \in A, b \in B\}$

$\therefore \quad R = \{(16, 4), (25, 5), (36, 6)\}$, Domain of $R = \{16, 25, 36\}$

Range of $R = \{4, 5, 6\}$, codomain of $R = \{1, 4, 5, 6\}$.

Note 1. The number of relations that are possible from a set *A* of *m* elements to another set *B* of *n* elements.

$\because$ Number of elements in $A = m$, Number of elements in $B = n$

$\therefore$ Number of elements in $A \times B = mn$. $\therefore$ Number of subsets of $A \times B = \mathbf{2^{mn}}$

Since every subset of $A \times B$ is a relation from A to B, therefore $\mathbf{2^{mn}}$ relations are possible from A to B.

Note 2. If R_1 and R_2 are two relations from A to B, then $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ are also relations from A to B.

Note 3. Since $\phi \subseteq A \times B$, therefore, ϕ is a relation from A to B. Also Dom $\phi = \phi$ and Range $\phi = \phi$.

2.11. The inverse of a relation

For any binary relation R, a second relation can be constructed by merely interchanging first and second components in every ordered pair. The relation thus obtained is called the *inverse* of the first one and is designed by R^{-1}. Symbolically,

$$R^{-1} = \{(y, x), (x, y) \in R\}$$

Thus, Domain (R^{-1}) = Range (R) and Range (R^{-1}) = Domain (R)

Illustrations. (1) The inverse of the husband-wife relation is the wife-husband relation.

(2) Let $R = \{(2, 1), (3, 2), (4, 3), (4, 5)\}$.

Domain (R) = (2, 3, 4), Range (R) = {1, 2, 3, 5}

$\Rightarrow$ Domain (R^{-1}) = Range (R) = {1, 2, 3, 5} and

Range (R^{-1}) = Domain (R) = {2, 3, 4}

Hence, $R^{-1} = \{(1, 2), (2, 3), (3, 4), (5, 4)\}$

(3) If $R = \{(x, y) \mid x = y + 2\}$ and $S = \{(x, y) \mid y = x^2\}$ then

$R^{-1} = \{(x, y) \mid y = x + 2\}$ and $S^{-1} = \{(x, y) \mid x = y^2\}$.

Note. It can be easily shown that $(R^{-1})^{-1} = R$.

Ex. 19. Let R be the relation from $A = \{1, 2, 3, 4, 5, 6\}$ to $B = \{1, 3, 5\}$ which is defined by the open sentence "x is less than y".

(*i*) Find the solution set of R, that is, write R as a set of ordered pairs.

(*ii*) Plot R on a co-ordinate diagram of $A \times B$.

(*iii*) State the domain, range and codomain of R.

Sol. (*i*) R consists of those ordered pairs $(a, b) \in A \times B$ for which $a < b$; hence

$R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

(*ii*) R is sketched on the co-ordinate diagram of $A \times B$ as shown in the figure.

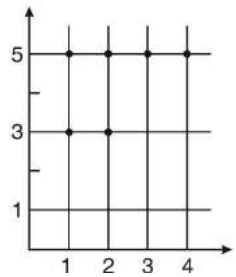


Fig. 2.04

(*iii*) Domain of R = {1, 2, 3, 4}

Range of R = {3, 5}.

Codomain of R = {1, 3, 5}.

Ex. 20. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. Which of the following are relations from A to B?

(*i*) $\{(a, y), (a, z), (c, x), (d, y)\}$ **(*ii*) $\{(a, x), (b, y), (c, x), (a, d)\}$**

(*iii*) $\{(a, x), (y, d), (x, c)\}$ **(*iv*) $\{(y, a), (z, a), (z, c), (y, d)\}$**

(*v*) $\{(a, x), (x, a), (b, y), (y, b)\}$ **(*vi*) $\{(a, x), (b, y), (c, z), z\}$**

(*vii*) $\{a, b, x, y, z\}$

Sol. (*i*) Yes.

(*ii*) No, because in the ordered pair (a, d), $a \in A$ and $d \notin B$.

(*iii*) No, because in (y, d), $y \in B$.

(*iv*) No. because here the first entries in all the ordered pairs are in the set B.

(*v*) No.

(*vi*) No, because the element z is not an ordered pair.

(*vii*) No, because the elements of the set are not ordered pairs.

Ex. 21. Determine the domain and range of the following relations:

(*i*) $\{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$

(*ii*) $\{(x, y) : x$ is a multiple of 3 and y is a multiple of 5$\}$

(*iii*) $\{(x, x^2) : x$ is a prime number less than 15$\}$

Sol. (*i*) Domain = {–3, –1, 1, 3}, Range = {0, 1}

(*ii*) Domain = $\{x : x \text{ is a multiple of } 3\} = \{3n : n \in Z\}$

Range = $\{y : y \text{ is a multiple of } 5\} = \{5n : n \in Z\}$

(*iii*) Relation = $\{(x, x^2) : x \text{ is a prime number less than } 15\}$

= {(2, 4), (3, 9), (5, 25), (7, 49), (11, 121), (13, 169)}

Domain = {2, 3, 5, 7, 11, 13}, Range = {4, 9, 25, 49, 121, 169}

Ex. 22. Let *N* be the set of natural numbers. Describe the following relation in words giving its domain and the range.

{(1, 1), (16, 2), (81, 3), (256, 4)}

Sol. The given relation stated in words is

$R = \{(x, y) : x \text{ is the fourth power of } y;\ x \in N, y \in \{1, 2, 3, 4\}\}$.

Ex. 23. *I* is the set of integers. Describe the following relations in words, giving its domain and range.

{(0, 0), (1, – 1), (2, – 2), (3, –3) ....}

Sol. $R = \{(0, 0), (1, -1), (2, -2), (3, -3) ...\} = \{(x, y) : y = -x, x \in W\}$

Domain = {0, 1, 2, 3, ... } = *W*, Range = { ..., – 3, – 2, – 1, 0}

Ex. 24. Let $A = \{1, 2, 3, 4, 6\}$. Let *R* be the relation on A defined by $\{(a, b) = a \in A, b \in A, a$ divides $b\}$. Find

(*i*) *R **(*ii*) domain of *R*** **(*iii*) range of *R*** **(*N.C.E.R.T.*)**

Sol. Given, set $A = \{1, 2, 3, 4, 6\}$. As per definitions of *R* in the question 1 divides each of the elements 1, 2, 3, 4, 6; 2 divides each of the elements 2, 4, 6; 3 divides each of the elements 3, 6 and 4 divides 4; 6 divides 6.

$\therefore$ $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

Also, Domain of $R = \{a : (a, b) \in R\} = \{1, 2, 3, 4, 6\} = A$

Range of $R = \{b : (a, b) \in R\} = \{1, 2, 3, 4, 6\} = A$

Ex. 25. If $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$, then which of the following are relations from *A* to *B*? Give reasons for your answer.

(*i*) $R_1 = \{(1, 5), (2, 7), (3, 8)\}$ (*ii*) $R_2 = \{(6, 2), (3, 7), (4, 7)\}$

Sol. (*i*) Since $R_1 \subseteq A \times B$, therefore, R_1 is a relation from *A* to *B*.

(*ii*) $(6, 2) \in R_2$ but $(6, 2) \notin A \times B$, therefore,

$R_2 \not\subseteq A \times B \Rightarrow R_2$ is not a relation from *A* to *B*.

Ex. 26. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from *A* into *B*.

Sol. $n(A) = 3$, $n(B) = 2 \Rightarrow n(A \times B) = 3 \times 2 = 6$

$\therefore$ The number of subsets of $A \times B = 2^6 = 64$

$\Rightarrow$ The number of relations from *A* into *B* = 64.

EXERCISE 2 (*b*)

1. Write down the relation shown by the arrow diagram, by listing the ordered pairs. State the domain, co-domain, and the range of the relation.

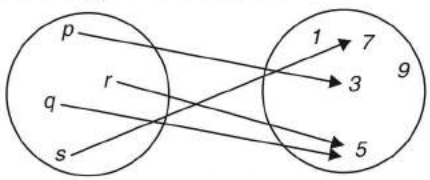


Fig. 2.05

2. Which of the following are relations from B to A, where $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$?

(i) $\{(z, x), (z, y), (x, a)\}$ (ii) $\{(z, a), (z, b), (z, c), (z, d)\}$

(iii) $\{(x, b), (y, a)\}$ (iv) $\{(b, y), (c, z), (a, x)\}$

(v) $\{(x, d), (y, c), (z, b)\}$

3. In each of the following, state which of the ordered pairs belong to the given relations?

(i) $\{(x, y) : x > y + 5\}$; (1, 0), (8, 2), (0, 1), (2, 8), (9, 3), (10, 7), (123, 4)

(ii) $\{(x, y) : xy = 12\}$; (3, 4), (4, 3), (12, 0), (0, 12), (12, 1), (6, 2), (7, 5)

(iii) $\{(x, y) : y = \dfrac{x+3}{x-1},\ x \neq 1\}$; (0, 1), (2, 5), (5, 2), (3, 3), (7, 5), $(7, \dfrac{5}{3})$

4. Let N be the set of natural numbers. Describe the following relations in words, giving their domain and the range.

(a) $\{(2, 1), (4, 2), (10, 5), (18, 9), (20, 10)\}$ (b) $\{(3, 1), (6, 2), (15, 5)\}$

(c) $\{(1, 4), (5, 16), (7, 22), (12, 37)\}$.

5. Z is the set of integers. Describe the following relation in set builder form, given its domain and range.

$$\{(0, -7), (2, -5), (4, -3), (-13, -20), ...\}$$

6. Write down the domain and range of the relation $(x, y) : x = 3y$ and x and y are natural numbers less than 10.

7. Determine the domain and range of the relation R.

(a) $R = \{(x + 1, x + 5) \mid x \in \{0, 1, 2, 3, 4, 5\}\}$. Draw the graph of R.

(b) $R = \{(x, x^3) \mid x$ is a prime number less than $10\}$. **[*N.C.E.R.T*]**

8. Given $A = \{-2, -1, 0, 1, 2\}$, list the ordered pairs determined by each of the following relations applied on A:

(i) R_1 = "is less than" (ii) R_2 = "is the square of"

(iii) R_3 = "is the additive inverse of" (iv) R_4 = "is equal to".

9. Given $A = \{2, 3, 4, 5, 6\}$. List the elements of each of the following relations:

(i) $\{(x, y) \in A \times A : x = y\}$

(ii) $\{(x, y) \in A \times A : x > y,\ \dfrac{x}{y} \notin W\}$

(iii) $\{(x, y) \in A \times A : x$ is a divisor of y and $x \neq y\}$.

10. (i) If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is **[*N.D.A 2003*]**

(a) 2^9 (b) 9^2 (c) 3^2 (d) $2^9 - 1$

(ii) Let A be a finite set. The number of relations on A where A has 3 elements are:

(a) 9 (b) 0.81 (c) 24.3 (d) 512 **[*A.M.U 2001*]**

(iii) Let $n(A) = p$. Then the number of all relations on A is

(a) 2^p (b) $2^{p!}$ (c) 2^{p^2} (d) None of these

ANSWERS

1. $\{(p, 3), (q, 5), (r, 5), (s, 7)\}$, Domain is $\{p, q, r, s\}$,
Co-domain is $\{1, 3, 5, 7, 9\}$; Range is $\{3, 5, 7\}$.

2. (*i*) No, (*ii*) Yes, (*iii*) Yes, (*iv*) No, (*v*) Yes

3. (*i*) (8, 2), (9, 3), (123, 4) (*ii*) (3, 4), (4, 3), (12, 1), (6, 2)

(*iii*) (2, 5), (3, 3), $(7, \frac{5}{3})$

4. (*a*) Relation = $\{(x, y), x = 2y, y \in \{1, 2, 5, 9, 10\}\}$
Domain = $\{2, 4, 10, 18, 20\}$, Range = $\{1, 2, 5, 9, 10\}$
(*b*) Relation = $\{(x, y), x = 3y, y \in \{1, 2, 5\}\}$
Domain = $\{3, 6, 15\}$, Range = $\{1, 2, 5\}$
(*c*) Relation = $\{(x, y): y = 3x + 1, x \in \{1, 5, 7, 12\}\}$
Domain = $\{1, 5, 7, 12\}$ Range = $\{4, 16, 22, 37\}$

5. Relation = $\{(x, y): x - y = 7, x \in Z\}$
Domain = $\{0, 2, 4, -13, ...\}$,
Range = $\{-7, -5, -3, -20, ...\}$

6. Domain = $\{3, 6, 9\}$,
Range = $\{1, 2, 3\}$

7. (*a*) Domain is $\{1, 2, 3, 4, 5, 6\}$
Range is $\{5, 6, 7, 8, 9, 10\}$
Graph is shown in Fig. 2.06
(*b*) Domain is $\{2, 3, 5, 7\}$
Range is $\{8, 27, 125, 343\}$.

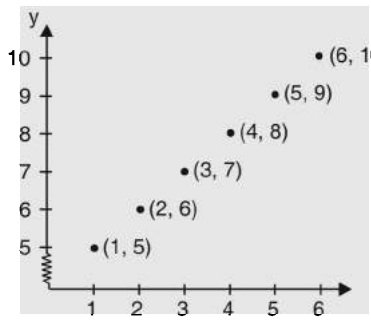


Fig. 2.06

8. (*i*) $R_1 = \{(-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$.
(*ii*) $R_2 = \{(1, -1), (0, 0), (1, 1)\}$.
(*iii*) $R_3 = \{(-2, 2), (2, -2), (-1, 1), (1, -1), (0, 0)\}$.
(*iv*) $R_4 = \{(-2, -2), (-1, -1), (0, 0) (1, 1), (2, 2)\}$.

9. (*i*) $\{(2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$
(*ii*) $\{(3, 2), (4, 3), (5, 2), (5, 3), (5, 4), (6, 4), (6, 5)\}$.
(*iii*) $\{(2, 4), (2, 6), (3, 6)\}$

10. (*i*) (*a*) (*ii*) (*d*) (*iii*) (*c*)

SECTION C
FUNCTIONS

2.12. Concept and definition of a function

A. Function as a special type of relation

Consider the following examples:

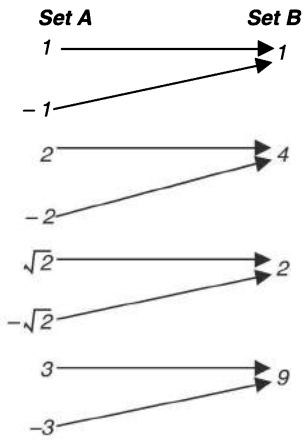


Fig. 2.07

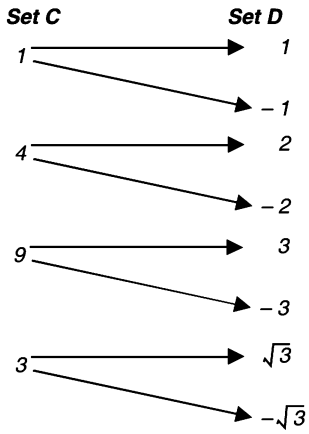


Fig. 2.08

In Fig. 2.07, a number belonging to set A is associated to its square number in set B. As a result of this association, we obtain the set of ordered pairs $P = \{(1, 1), (-1, 1), (\sqrt{2}, 2), (-\sqrt{2}, 2), (2, 4), (-2, 4), (3, 9), (-3, 9)\}$.

In Fig. 2.08, a number belonging to set C is associated to its square root number in set D. As a result, we obtain the set of ordered pairs $Q = \{(1, 1), (1, -1) (4, 2), (4, -2), (9, 3), (9, -3), (3, \sqrt{3}), (3, -\sqrt{3})\}$.

Observe the distinction between the diagrams of the two examples. In Fig. 2.07, only one arrow starts from every element thus, yielding a set of ordered pair in which no two or more ordered pairs have the same first component. In Fig. 2.08, more than one arrow starts from an element of set C, thus, yielding a set of ordered pairs in which more than one ordered pair have the same first component such as (1, 1), (1, –1). Set of ordered pairs such as obtained in Fig. 2.07, in which no two ordered pairs have the same first component is called a **function**. The set of ordered pairs obtained in Fig. 2.08 is not a function. Thus, **functions are special types of relations.**

Definition. ***A function is a special type of relation in which no two different ordered pairs have the same first component.***

Stated directly in terms of a set of ordered pairs, the definition is as given below.

Definition. ***A function is a set of ordered pairs, no two of which have the same first component.*** The set {(0, 1), (1, 1)} is a function but the set {(1, 0), (1, 1)} is not a function as the first component is repeated.

A function is usually denoted by f, g, h, F, G, H.

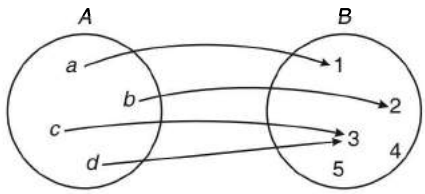
Maths Alert! By convention, the empty set is not considered a function.

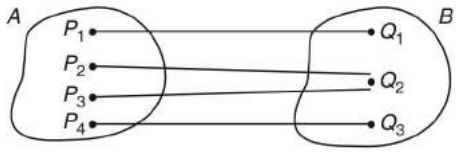
A function is usually denoted by letters f, g, F, G. Thus, here, $f = \{(a, 1), (b, 2), (c, 3), (d, 3)\}$.

Note. Range $(f) \subseteq$ co-domain (f).

B. Function as a type of mapping

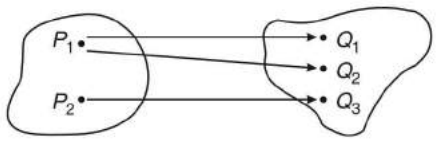
A function is also thought of as a **mapping** of its domain into its range.





Function as a mapping

Fig. 2.09



This mapping is not a function

Fig. 2.10

For the definition of a function to be satisfied, it is essential that each element of the domain is mapped into a unique element of the co-domain. If this condition is not met, the mapping is not a function. In terms of the pictures, for a mapping to be function, each arrow should emanate from a different point in the domain. Whether they terminate at the same point in the co-domain is immaterial in the definition.

Remark. Consider the following mapping.

From the figure displaying the association of a player to the game he or she plays, we observe that the player Ronaldo of set A is not associated with any game in set B.

This association or mapping is **not a function**.

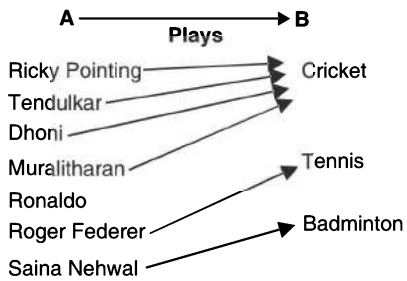


Fig. 2.11

2.13. Essential requirements for the definition of a function

A function $f : A \to B$ is defined under the following conditions:

(1) *Every $x \in A$ is associated with some y in B, i.e., a function is defined only when the domain is entirely "used up".*

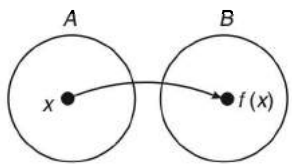
The set B may not be entirely "used up" by the function.

(2) *The function may associate more than one x to the same y.*

(3) *No element in A should have more than one image in B.*

2.14. The terms value and image of a function

If f is a function and if x is any element of the domain, we use the symbol $f(x)$ to denote the object which f associates with x. $f(x)$ is the functional value at x. The symbol $f(x)$ is read as "the value of f at x", or "f at x", or "f of x". $f(x)$ is called the **image** of x and x is called the **pre-image** of $f(x)$. We could also designate the function as $f : A \to B$ or $f : x \to f(x)$ [read "f takes x into $f(x)$".]



For example, (*i*) Let f be $\{(0, 1), (1, 2), (2, 3),\}$. It is a function whose domain is $\{0, 1, 2\}$ and range is $\{1, 2, 3\}$. Here $f(0) = 1$, $f(1) = 2$ and $f(2) = 3$.

Euler invented a symbolic way to write $f : \to f(x)$ as $\boldsymbol{y = f(x)}$ while is read "y equals f of x".

(*ii*) Consider the two sets for the function $f(x) = x^2$.

$$A=\left\{-2, 2, \frac{3}{4}, 0\right\}, B=\left\{4, \frac{9}{16}, 0\right\}$$

The set $\left\{-2, 2, \frac{3}{4}, 0\right\}$ is the domain and the set $\left\{4, \frac{9}{16}, 0\right\}$ is the range, 4 is the image of each of –2 and 2, $\frac{9}{16}$ is the image of $\frac{3}{4}$ and 0 is the image of 0. If we denote the function by f, then

$$f(-2)=4, f(2)=4, f\left(\frac{3}{4}\right)=\frac{9}{16}, f(0)=0.$$

If $f(x)$ is a given function of x and if a is in its domain then by $f(a)$ is meant the number obtained by replacing x by a in $f(x)$ or the value assumed by $f(x)$ when $x = a$.

Illustration: If $f(x) = x^2 - 5x + 3$, then $f(1) = (1)^2 - 5(1) + 3 = 1 - 5 + 3 = -1$

$f(-3) = (-3)^2 - 5(-3) + 3 = 9 + 15 + 3 = 27$; $f(a) = a^2 - 5a + 3$.

Note. The domain of a function may be restricted by context. For example, the domain of the area function given by $A = \pi r^2$ only allows the radius r to be positive. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation, the range is $\{x^2 \mid x \geq 2\}$ or $\{y \mid y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real valued**. The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals.The intervals may be open, closed or half open, and may be finite or infinite.

Identifying Domain and Range

Illustration:

Functions	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4-x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$

Solution. The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is non-negative and every non-negative number y is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = \frac{1}{x}$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero.* The range of $y = \frac{1}{x}$, the set of reciprocals of all non-zero real numbers, is the set of all non-zero real numbers, since $y = 1/(1/y)$

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every non-negative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4-x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4-x}$ is $[0, \infty)$, the set of all non-negative numbers.

The formula $y = \sqrt{1-x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1-x^2}$ is $[0, 1]$.

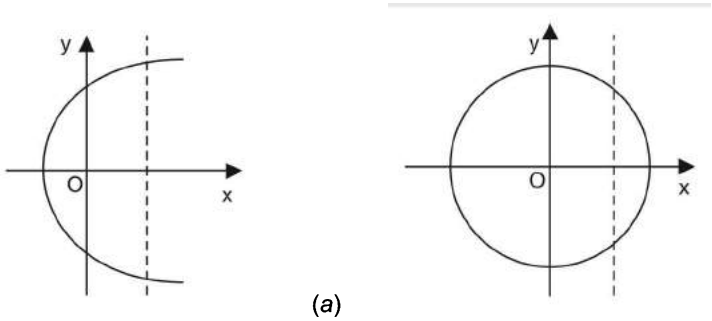
2.15. Vertical line test for a function

You can test whether a given relation is a function or not by applying the following tests:

(1) Examine whether the first set, *i.e.,* the domain is fully used up or not.

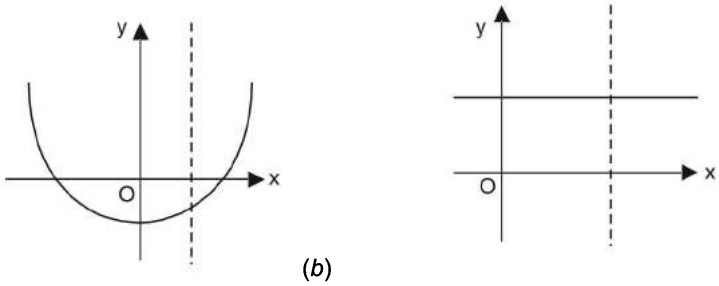
(2) Examine whether the first members of all the ordered pairs are different or not. In an arrow diagram, examine whether each element of the first set has only one image in the second set or not.

(3) The vertical line test. *If it is possible to draw a vertical line that intersects the graph of a relation in more than one point, then the relation is not a function. If it is not possible to draw such a vertical line, then the relation is a function.*



(*a*)

These are not functions because the vertical line cuts the graph in more than one point.



(*b*)

These are functions because the vertical line cuts the graph in only one point.

Fig. 2.12

It is not difficult to read the theory behind this. If there are two points of the graph of a relation on a certain vertical line, then the two corresponding ordered pairs in the relation have the same first component but different second components. This means that the relation is not a function by the definition of a function.

Ex. 27. Which of the following relations are functions? Give reasons. In case of a function, determine its domain and range.

(i) **{(1, – 2), (3, 7), (4, – 6), (8, 11)}**

(ii) **{(1, 0), (1, – 1), (2, 3), (4, 10)}**

(iii) **{(*a*, *b*), (*b*, *c*), (*c*, *b*), (*d*, *c*)}**

Sol. (*i*) and (*iii*) are functions because no two of their ordered pairs have the same first component; (*ii*) is not a function since the two ordered pairs (1, 0) and (1, – 1) have the same first component.

Domain of (*i*) is {1, 3, 4, 8}. Range is {– 2, 7, – 6, 11}.

Domain of (*iii*) is {*a*, *b*, *c*, *d*}. Range is {*b*, *c*}.

Ex. 28. State whether or not each of the following diagrams defines a function of $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

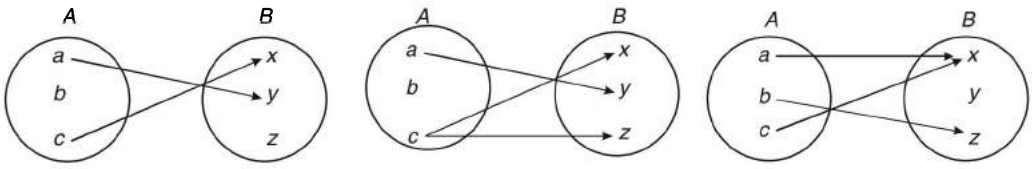


Fig. 2.13

Sol. (*i*) No, the domain set A is not entirely used up.

(*ii*) No, Two elements x and z are assigned to the same element c in the domain set A. This will yield ordered pairs (c, x), (c, z) having the same first component.

(*iii*) Yes, because a given association is a function even if (*a*) one element of the second set B is assigned to more than one element of the domain, (*b*) the second set B is not entirely used up.

Ex. 29. Which of the following relations are functions?

(i) $y = 3x + 2$.

(ii) ***a* is the capital of *b* where $b \in B$ and B is the set of all countries, $a \in A$ and A is the set of capital cities of countries.**

(iii) $y < x + 3$.

(iv) ***y* is the Maths teacher of *x* where *x* represents any pupil taking up Maths in a school.**

(v) ***y* is *a* Maths pupil of *x*, where *x* represents any Maths teacher in a school.**

Sol. (*i*) *Function.* Each replacement of x gives one and only one replacement of y which makes the sentence true.

(*ii*) *Function.* Because normally each country has exactly one capital city.

(*iii*) *Not a function.* For each replacement of x there are many replacements of y which makes the sentence true.

(*iv*) *Function.* If we assume that every pupil has only one Maths teacher.

(*v*) *Not a function.* Each Maths teacher ordinarily has a number of pupils.

Ex. 30. State the domain of these functions:

(i) $f : x \to 5x$

(ii) $g : x \to 5x, x \in Z$

(iii) $h : x \to \dfrac{2}{x-7}$

(iv) $F : x \to 5x, x \in \{0, 1, 2\}$

(v) $f : x \to \dfrac{x}{5}$

(vi) $F : x \to \dfrac{6}{x}$

(*vii*) $H : x \to x^2 + 5x - 6$ (*viii*) $g : x \to \dfrac{x-4}{(x-3)(x+6)}$

(*ix*) $g : x \to \dfrac{x}{1}, x \in \{2,4,6\}$ (*x*) $g : x \to \dfrac{1}{x}, x \in R$

Sol. (*i*) R, (*ii*) Z, (*iii*) The domain of function h is the set of all real numbers except, 7, since division by 0 is not defined. (*iv*) {0, 1, 2} (*v*) R (*vi*) $R, x \neq 0$, (*vii*) R, (*viii*) $R, x \neq 3, x \neq -6$ (*ix*) {2, 4, 6} (*x*) $R, x \neq 0$.

Ex. 31. Does the relation $\{(x, y) \mid y = |x|, x \in R\}$ define a function? Write the range and draw the graph.

Sol. Since in $y = |x|$, for every values of x, there exists a unique value of y, therefore, the relation defines a function. The domain of the function is the set of real numbers and the range is the set of positive real numbers.

To draw its graph, we first find a few ordered pairs in it. For the convenience of plotting the points we take only integral values of x.

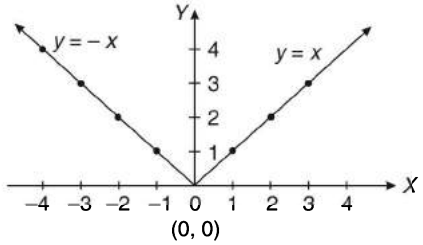


Fig. 2.14

x	0	1	2	3	...	...	− 1	− 2	− 3	...	...
$y = \|x\|$	0	1	2	3	...	...	1	2	3	...	...

Plotting the ordered pairs and joining the points so obtained we get the graph of the function $y = |x|$ as shown in Fig. 2.14.

Ex. 32. Does the relation 'a square root of' in the set of real numbers represent a function?

Sol. Here every positive real number say "a" in R, is associated with $\pm\sqrt{a}$, *i.e.*, every element of the domain does not have a unique image in R. Moreover, negative numbers in the domain do not have any image in the second set, *i.e.*, the domain set is not entirely used up. Hence this relation is not a function.

EXERCISE 2 (*c*)

1. Let $X = \{1, 2, 3, 4\}$. Determine whether or not each relation is a function from X into X.

(*i*) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$

(*ii*) $g = \{(3, 1), (4, 2), (1, 1)\}$

(*iii*) $h = \{(2, 1), (3, 4), (1, 4), (4, 4)\}$

2. State for each of the following relations whether it is function or not. (Write Yes or No)

(*i*) $\{(1, 2), (2, 2), (3, 2), (4, 2)\}$.

(*ii*) $\{(x, y) : x \in A, y \in B$ is surname of $x\}$ where A is the set of people in India and B is the set of surnames.

(*iii*) $\{(x, y) : x \in A, y \in B, y$ is the area of a square of side $x\}$ where A is the set of measurements of length.

(*iv*) $\{(x, y) : x \in B, y \in P, y$ is a passenger on $x\}$ where B is the set of buses of a school and P is the set of pupils of some schools.

(*v*) $\{(x, y) : x \in A, y \in B, y$ is sewn onto $y\}$, where A is the set of buttons and B is the set of shirts.

3. The ordered pairs are represented by the points shown. For each diagram, state whether it represents a relation or a function. Justify your answer.

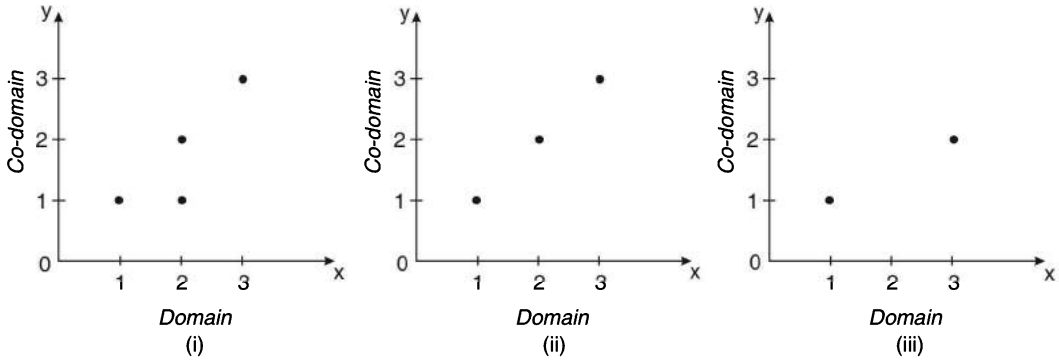


Fig. 2.15

4. The range and domain of a function $f(x) = \frac{3}{x} + 1$ are subsets of A and B respectively, where $A = \left\{-\frac{1}{2}, 0, \frac{2}{3}, \frac{6}{7}, 1\right\}$ and $B = \left\{-5, 0, 4\frac{1}{2}, 5, 5\frac{1}{2}\right\}$. List the elements of the function as ordered pairs.

5. Which of the four statements given below is different from others?

(a) $f: A \to B$ (b) $f: x \to f(x)$

(c) f is a mapping of A into B (d) f is a function of A into B **[*UPSEAT 2000*]**

6. $A = \{-2, -1, 1, 2\}$ and $f = \left\{\left(x, \frac{1}{x}\right), x \in A\right\}$

(i) List the domain of f (ii) List the range of f (iii) Is f a function?

7. $f: x \to$ highest prime factor of x.

(i) Find the range of f when the domain is $\{12, 13, 14, 15, 16, 17\}$.

(ii) State a domain of five integers for which the range is (3).

(iii) A set of positive integers is called S. What can be said about these integers if $f(S) = S$?

8. (i) For $x > 3$, $f(x) = 3x - 2$ and for $-2 \le x \le 2$, $f(x) = x^2 - 2$, find $f(0) + f(4)$.

(ii) If $f: R \to R$ defined by $f(x) = \begin{cases} 4x - 1 \text{ for } x > 4 \\ x^2 - 2 \text{ for } -2 \le x < 4, \\ 3x + 4 \text{ for } x < -2 \end{cases}$

find $f(5) + f(0) + f(-5)$.

9. What is the fundamental difference between a function and a relation? Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$. Determine which of the following sets are:

(i) relation (ii) function (iii) neither

(a) $f_1 = \{(x, y) : y = x^2, x \in X, y \in Y\}$ (b) $f_2 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$

(c) $f_3 = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ (d) $f_4 = \{(1, 1), (2, 7), (3, 5)\}$ **[*Roorkee*]**

10. A certain jet plane has an average speed of 500 km per hour. It can carry sufficient fuel for a 5 hour flight.

(i) Define the relation, as a set, between the distance d (in km) and time t (in hours) for this plane.

(*ii*) State the domain of the relation.

(*iii*) State the range of this relation.

(*iv*) Is this relation a function?

11. The domain of a function is the set of positive integers less than 12. If $y = f(x) = |x - 4|$, find all ordered pairs satisfying the function. Graph the function.

12. Let $X = \{2, 3\}$ and $Y = \{1, 3, 5\}$. How many different functions are there from X into Y ?

ANSWERS

1. (*i*) No, (*ii*) Yes, (*iii*) Yes,

2. (*i*) Yes, (*ii*) No, (*iii*) No, (*iv*) No, (*v*) Yes.

4. $\left\{\left(-\frac{1}{2}, 5\right), \left(\frac{2}{3}, 5\frac{1}{2}\right), \left(\frac{6}{7}, 4\frac{1}{2}\right)\right\}$ **5.** All the four statements are the same

6. (*i*) $\{-2, -1, 1, 2\}$, (*ii*) $\left\{-\frac{1}{2}, -1, 1, \frac{1}{2}\right\}$, (*iii*) Yes,

7. (*i*) $\{2, 3, 5, 7, 13, 17\}$, (*ii*) $\{3, 6, 9, 12, 18\}$, (*iii*) S is the set of prime numbers.

8. (*i*) 8 (*ii*) 6

10. (*i*) $\{d : d = 500\,t, t \in R \text{ and } \leq 5\}$ (*ii*) $\{0 < t \leq 5, t \in R\}$

(*iii*) $\{d : 0 < d \leq 2500, d \in R\}$, (*iv*) Yes

11. $\{(1, 3), (2, 2), (3, 1), (4, 0), (5, 1), (6, 2), (7, 3), (8, 4), (9, 5), (10, 6), (11, 7)\}$

Graph consists of points corresponding to these ordered pairs.

12. Nine

HINTS TO SELECTED QUESTIONS

3. (*i*) The set of ordered pairs is $\{(1, 1), (2, 1), (2, 2), (3, 3)\}$.

This is a relation and not a function because two ordered pairs, (2, 1) and (2, 2) have the same first component.

(*ii*) The set of ordered pairs is $\{(1, 1), (2, 2), (3, 3)\}$. This is a function as all members of the domain are used up and no two ordered pairs have the same first component.

(*iii*) The set of ordered pairs is $\{(1, 1), (3, 2)\}$. This is neither a relation nor a function, because all members of the domain have not been used up.

9. Every subset of $X \times Y$ is a relation from X to Y but only that subset of $X \times Y$ is a function from X to Y whose domain is X and in which no two different ordered pairs have the same first component. Now

(*i*) $f_1 = \{(1, 1), (3, 9), (4, 16)\}$, which is clearly a relation from X to Y but domain $f_1 \neq X$, so f_1 is not a function.

(*ii*) f_2 is clearly a function from X to Y.

(*iii*) f_3 being a subset of $X \times Y$ is a relation from X to Y, but two different ordered pairs, namely, (2, 9) and (2, 11) in f_3 have the same first component therefore, f_3 is not a function.

(*iv*) $f_4 \not\subseteq X \times Y$, therefore, f_4 is neither a relation nor a function from X to Y.

12. If $n(A) = p$ and $n(B) = q$, then the number of possible mappings from A to B is q^p.

Here, $n(X) = 2$ and $n(Y) = 3$, so the number of functions $= 3^2 = 9$.

TYPES OF FUNCTIONS

2.16. One-one function (Injective function)

1. A function $f : A \to B$ is called a **one-one** or **injective function** if **distinct elements** of A have distinct images in B, *i.e.*, if $a_1, a_2 \in A$ and $a_1 \neq a_2$

$\Rightarrow f(a_1) \neq f(a_2)$.

Equivalently, we say $f : A \to B$ is one-one if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

In terms of pictures, we say that in one : one function, no more than one arrow terminates at any given element of B.

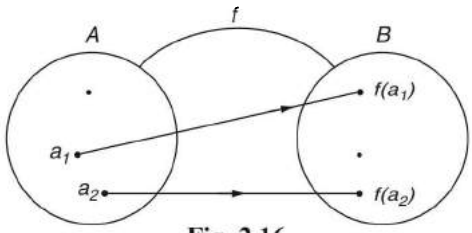


Fig. 2.16

Illustrations

1. If $A = \{4, 5, 6\}$ and $B = \{a, b, c, d\}$ and if $f : A \to B$ such that $f = \{(4, a), (5, b), (6, c)\}$, then f is 1 – 1.

2. The mapping $f : R \to R$ such that $f(x) = x^2$ is not a **one-one** function since $f(-2) = 4$ and $f(2) = 4$, that is, two distinct elements – 2 and 2 have the same image 4.

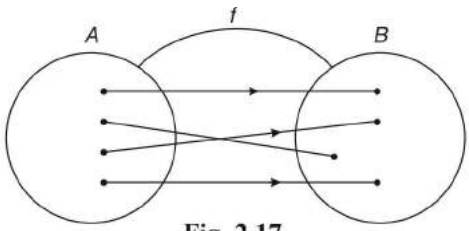


Fig. 2.17

Ex. 33. Find whether the following functions are one-one or not.

(*i*) $f : R \to R$, defined by $f(x) = x^3$, $x \in R$.

(*ii*) $f : Z \to Z$, defined by $f(x) = x^2 + 5$ for all $x \in Z$.

(*iii*) $f : R - \{3\} \to R$, defined by $f(x) = \dfrac{5x+7}{x-3}$, $x \in R - \{3\}$.

[Method to check whether a function is one-one or not]

Step 1. *Take two arbitrary elements a_1, a_2 in the domain of f.*

Step 2. *Put $f(a_1) = f(a_2)$ and solve the equation.*

Step 3. *If it yields $a_1 = a_2$, only then $f : A \to B$ is a one-one function otherwise not.*

Remark. Let $f : A \to B$ and let $a_1, a_2 \in A$. Then $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$ is always true from the definition, but, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ is true only when f is one-one.

Sol. (*i*) Let a_1, a_2 be two arbitrary elements of domain f, $(a_1, a_2 \in R)$ such that $f(a_1) = f(a_2)$. Then $f(a_1) = f(a_2) \Rightarrow a_1^3 = a_2^3 \Rightarrow a_1 = a_2$

Hence, f is a one-one function.

(*ii*) Let a_1, a_2 be two arbitrary elements of Z such that $f(a_1) = f(a_2)$. Then

$$f(a_1) = f(a_2) \Rightarrow a_1^2 + 5 = a_2^2 + 5 \Rightarrow a_1^2 = a_2^2 \Rightarrow a_1 = \pm a_2$$

Since $f(a_1) = f(a_2)$ does not yield a unique answer $a_1 = a_2$ but gives $a_1 = \pm a_2$, so f is not a one-one function.

In fact, we have $f(2) = 2^2 + 5 = 9$ and $f(-2) = (-2)^2 + 5 = 9$. Thus, two distinct elements 2 and – 2 have the same image.

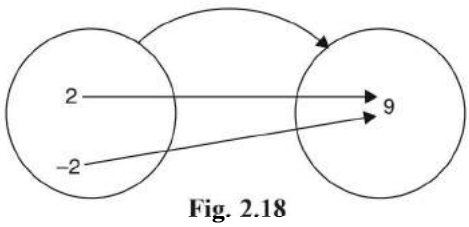


Fig. 2.18

(*iii*) We have $f(a_1) = \dfrac{5a_1 + 7}{a_1 - 3}$, $x \in R - \{3\}$

Let a_1, a_2 be two arbitrary elements of domain $f(a_1, a_2) \in R - \{3\}$ such that $f(a_1) = f(a_2)$.

Then $f(a_1) = f(a_2) \Rightarrow \dfrac{5a_1 + 7}{a_1 - 3} = \dfrac{5a_2 + 7}{a_2 - 3}$

$\Rightarrow \quad 5a_1a_2 - 15a_1 + 7a_2 - 21 = 5a_1a_2 - 15a_2 + 7a_1 - 21$

$\Rightarrow \quad -22a_1 = -22a_2 \Rightarrow a_1 = a_2$

$\therefore$ f is one-one.

Ex. 34. Show that the modulus function $f : R \to R$, given by $f(x) = |x|$, is not one-one function.

Sol. Given : $f(x) = |x|$

$\Rightarrow \quad f(1) = |1| = 1$ and $f(-1) = |-1| = 1$

Thus, $-1 \neq 1$ but $f(-1) = f(1)$

Hence, $f : A \to B$ is not one-one function.

It is a many-one function.

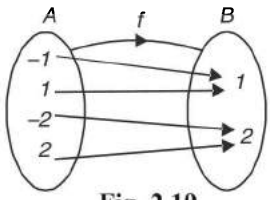


Fig. 2.19

2.17. Many-one function

If the function $f : A \to B$ is such that two or more elements a_1, a_2, ... of A have the same f-image in B, then the function is called many-one function.

Note that if the function $f : A \to B$ is not one-one, then f is many-one.

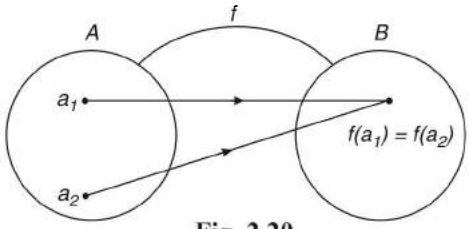


Fig. 2.20

Illustrations

1. Let $f : A \to B$ be a function represented by the diagram shown in Fig. 2.21.

Clearly, $f(a) = 1, f(b) = 1, f(c) = 1, f(d) = 2$, $f(e) = 2$ and $a \neq b \neq c$, $d \neq e$ so f is a many-one function.

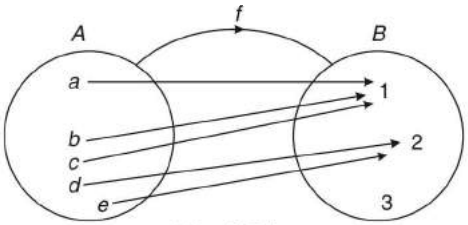


Fig. 2.21

2. $f : R \to R$, defined by $f(x) = x^2 + 5$ is a many-one function, since $f(-1) = (-1)^2 + 5 = 6$ and $f(1) = (1)^2 + 5 = 6$, that is, two distinct elements -1 and 1 have the same image is 6.

Ex. 35. Prove that the greatest integer function $f: R \to R$, given by $f(x) = [x]$ is a many-one function.

Sol. The diagram shows that even though $1.3 \neq 1.4$ yet $f(1.3) = f(1.4) = 1$

Thus, two elements 1.3 and 1.4 in set A have the same image 1 in set B. Hence, $f: A \to B$ is not a one-one function. It is a many-one function.

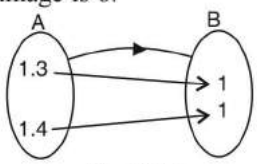


Fig. 2.22

Method to check whether the given function is many-one.

Step 1. *Consider any two arbitrary elements a_1, $a_2 \in A$.*

Step 2. *Put $f(a_1) = f(a_2)$ and solve the equation.*

Step 3. *If we get $a_1 \neq a_2$, then f is many-one.*

Note. Recall that if we get $a_1 = a_2$, then f is a one-one function.

2.18. Onto function (Surjective function)

The mapping $f : A \to B$ is called an **Onto** function if the set B is entirely used up, *i.e.*, if every element of B is the image of at least one element of A.

$\Rightarrow$ For every $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$.

$\Rightarrow$ Range of f is the co-domain of f (Range = Co-domain)

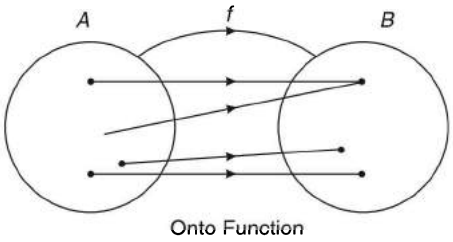


Fig. 2.23. Onto Function

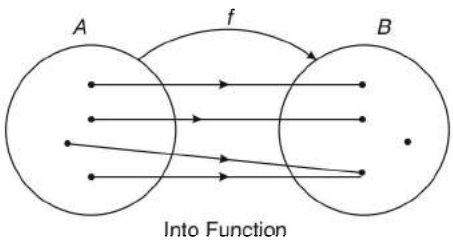


Fig. 2.24. Into Function

If function $f: A \to B$ is not onto, that is, some of the elements of B remain unmated, then 'f' is called an **into function.**

Remarks 1. An onto function is also called a **surjective** function or a **surjection.**

2. A one-one function may be both onto and into.

Illustrations

1. Let $A = \{-2, 2, -3, 3\}$, $B = \{4, 9\}$ and $f: A \to B$ be a function defined by $f(x) = x^2$, then f is onto, because $f(-2) = 4, f(2) = 4, f(-3) = 9, f(3) = 9$, *i.e.*, $f(A) = \{4, 9\} = B$.

Here, range = $\{4, 9\} = B$ = co-domain.

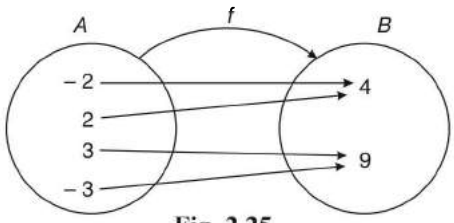


Fig. 2.25

2. A function $f : N \to N$ defined by $f(x) = 7x$ is not an onto function, because $f(N) = \{7, 14, 21, ...\} \neq N$.

$\Rightarrow$ Range $\neq$ co-domain (N).

3. The function $f : R \to R$ defined by $f(x) = |x|, x \in R$ is not onto since $|x|$ is never negative and therefore no negative real number can be the image of any real number under 'f'.

Here, also range = set of non-negative real numbers $\neq$ co-domain (R).

4. Consider the function $f : R \to R^+$ defined by $f(x) = \sqrt{x^2}$, $x \in R$.

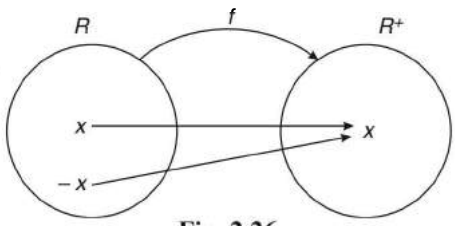


Fig. 2.26

Let $x \in R^+$, then x is non-negative real number.

$\therefore \quad \sqrt{x^2} = x \Rightarrow f(x) = x,$

Also, $\sqrt{(-x)^2} = \sqrt{x^2} = x \Rightarrow f(-x) = x$

Thus, all numbers in R (domain) have an image in R^+ (co-domain), *i.e.*, range (all numbers in R) = co-domain (R^+).

$\therefore$ f is onto.

5. Let the function $f : R \to R$ be defined by the formula $f(x) = x^2$. The f is not an onto function since the negative numbers do not appear in the range of f, that is, no negative number is the square of a real number. Here, range of $f \neq R$ (co-domain). It is a subset of R.

Ex. 36. Find whether the following are onto functions (surjections) or not.

(*i*) $f : R \to R$ defined by $f(x) = x^3 + 5$ for all $x \in R$.

(*ii*) $f : R \to R$ defined by $f(x) = x^2 + 3$ for all $x \in R$.

(*iii*) $f : Z \to Z$ defined by $f(x) = 5x - 9$ for all $x \in Z$.

Method :

Suppose f : *$A \to B$ is a given function.*

Step 1. *Choose any arbitrary element y in B.*

Step 2. *Put $f(x) = y$.*

Step 3. *Solve the equation $f(x) = y$ for x and obtain x in terms of y. Let $x = g(y)$.*

Step 4. *If for all values of $y \in B$, the values of x obtained from $x = g(y)$ are in A, then f is onto. If there are some $y \in B$ for which x, given by $x = g(y)$, is not in A, then f is not onto.*

Sol. (*i*) Let there be an arbitrary element y in R. Then $f(x) = y = x^3 + 5 \Rightarrow x = (y-5)^{\frac{1}{3}}$.

Now, for all $y \in R$, $(y-5)^{\frac{1}{3}}$ is a real number. So, for all $y \in R$ (co-domain), there exists $x = (y-5)^{\frac{1}{3}}$ in R (domain) such that $f(x) = x^3 + 5$.

Here, $f : R \to R$ is an onto function.

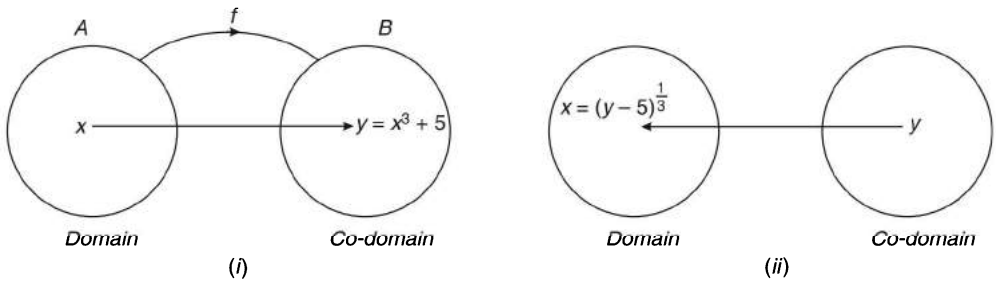


Fig. 2.27

(*ii*) $f(x) = x^2 + 3 \geq 3$ for all $x \in R$. If we take $f(x)$ = a real number < 3, say $= 2$, then $x^2 + 3 = 2$

$\Rightarrow x^2 = -1$ which is not true for any $x \in R$ (domain).

$\therefore$ 2 is not the image of any element in the domain.

$\therefore$ f is not onto.

(*iii*) Let y be an arbitrary element of Z (co-domain). Then

$$f(x) = y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$$

Now, if $y = 0$, then $x = \frac{-2}{3} \notin Z$. Thus, $y = 0$ in Z (co-domain) does not have its pre-image in Z (domain).

Hence, f is not an onto function.

2.19. One-one onto function (Bijection)

If the function f is both one-one and onto, then it is called a one-one onto function. It is also called a **bijection.**

In other words, a function $f: A \to B$ is one-one onto if

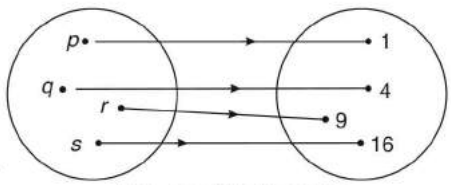
(i) it is one-one, i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) it is onto, i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Note. If A and B are finite sets and $f: A \to B$ is a one-one onto function, then $n(A) \leq n(B)$ and $n(B) \leq n(A) \Rightarrow n(A) = n(B)$.

Illustrations

1. If $A = \{p, q, r, s\}$ and $B = \{1, 4, 9, 16\}$ then $f = \{(p, 1), (q, 4), (r, 9), (s, 16)\}$ is a one-one onto function.



One-one Onto Function

Fig. 2.28

2. Let $A = \{a, b, c, d, e\}$ and $f: A \to A$, defined by $f(a) = e, f(b) = d, f(c) = b, f(d) = a, f(e) = c$, then f is a bijection from A to A.

3. If R is the set of real numbers and Z the set of integers, then $f: R \to R$ defined by $f(x) = x^3$ is a one-one onto function while $f: Z \to Z, f(x) = x^3$ is not a bijection. It is one-one but not onto since an element like-3 in co-domain Z has no pre-image in the domain Z.

Ex. 37. If $A = R - \{3\}$ and $B = R - \{1\}$ and $f: A \to B$ is a mapping defined by $f(x) = \dfrac{x-2}{x-3}$ show that f is one-one onto function. ***[CBSE DB Comptt. 2006]***

Sol. One-one onto function : Let x, y be any two elements of A. Then,

$$f(x) = f(y) \Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3} \Rightarrow (x-2)(y-3) = (x-3)(y-2)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6 \Rightarrow x = y$$

Since, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$, so f is one-one.

Onto : Let y be any arbitrary element of B. Then

$f(x) = y \Rightarrow \dfrac{x-2}{x-3} = y \Rightarrow (x-2) = y(x-3) \Rightarrow x = \dfrac{2-3y}{1-y}$. Clearly, $\dfrac{2-3y}{1-y}$ is a real number for all $y \neq 1$. Also, $\dfrac{2-3y}{1-y} \neq 3$ for any y, for, we take $\dfrac{2-3y}{1-y} = 3$, then we get $2 = 1$, which is wrong.

Thus, every element y in B has its pre-image x in A given by $x = \dfrac{2-3y}{1-y}$. So, f is onto.

Thus, f is both one-one and onto.

Ex. 38. Let $f = N \to N$ be defined by $f(x) = x^2 + x + 1, x \in N$, then prove that f is one-one onto. ***[AMU 2000]***

Sol. Let $x, y \in N$ such that $f(x) = f(y)$. Then $f(x) = f(y) \Rightarrow x^2 + x + 1 = y^2 + y + 1$.

$\Rightarrow (x - y)(x + y + 1) = 0 \Rightarrow x = y$ or $x = (-y - 1) \notin N$. $\therefore$ f is one-one.

Again, since for each $y \in N$, there exists $x \in N$ $\therefore$ f is onto.

Ex. 39. Let $f : R \to R$ be a function defined by $f(x) = \dfrac{x-m}{x-n}$, where $m \neq n$. Then show that f is one-one but not onto. [*UPSEAT 2001*]

Sol. For any $x, y \in R$, we have $f(x) = f(y) \Rightarrow \dfrac{x-m}{x-n} = \dfrac{y-m}{y-n} \Rightarrow x = y$. $\therefore$ f is one-one.

Let $\alpha \in R$ such that $f(x) = \alpha \Rightarrow \dfrac{x-m}{x-n} = \alpha \Rightarrow x = \dfrac{m-n\alpha}{1-\alpha}$

Clearly $x \notin R$ for $\alpha = 1$, so f is not onto.

Ex. 40. The function $f : R \to R$ defined by $f(x) = e^x$ is

(*a*) Onto (*b*) Many-one (*c*) One-one and into (*d*) Many one and onto.

[*Karnataka CET 2002, UPSEAT 2002*]

Sol. Ans. (*c*) Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ or $e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$ therefore, f is one-one.

Now, let $f(x) = e^x = y \Rightarrow x = \log y$. Since negative real numbers have no pre-image, the function is not onto. Also since zero is not the image of any real number, f is into.

Ex. 41. If $f : R \to R$ be defined as $f(x) = x^4$. Choose the correct answer:

(*a*) f is one-one onto (*b*) f is many-one onto

(*c*) f is one-one but not onto (*d*) f is neither one-one nor onto

Sol. Given $f(x) = x^4$

Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$

Then, $a_1^4 = a_2^4 \Rightarrow a_1^4 - a_2^4 = 0$

$\Rightarrow (a_1 + a_2)(a_1 - a_2)(a_1^2 + a_2^2) = 0$

$\Rightarrow$ $a_1 = -a_2$, *i.e.*, $a_1 \neq a_2 \Rightarrow f$ is many-one function ...(1)

Now, let $y \in B$ so $y = x^4 \Rightarrow x = y^{1/4}$

Since, $y \in R$ so $y^{1/4} \in R$

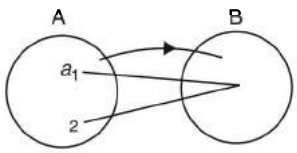


Fig. 2.29

Consider $y = -1$ in B. Its pre-image $(-1)^{1/4} \notin R$ and so does not exist in A.

Hence, $f : R \to R$ is not an onto function. ...(2)

From (1) and (2), we conclude that f is neither one-one nor onto.

Hence, the correct answer is (*d*).

Ex. 42. Let $f : N \to N$ be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N.$$

Find whether the function is bijective. Justify your answer. [*AI CBSE 2009*]

Sol. We have $f(1) = \dfrac{1+1}{2} = \dfrac{2}{2} = 1$ and $f(2) = \dfrac{2}{2} = 1$. Thus, $f(1) = f(2)$ while $1 \neq 2$.

$\therefore$ f is not one-one $\Rightarrow$ It is many-one.

Now, in order to show that f is onto, consider any $n \in N$.

Case 1. If n is odd, then $(2n - 1)$ is odd, and $f(2n - 1) = \dfrac{(2n-1)+1}{2} = \dfrac{2n}{2} = n$

Case 2. If n is even, then $2n$ is even, and $f(2n) = \dfrac{2n}{2} = n$

Thus, for every $n \in N$. (Whether even or odd), there exists its pre-image in N.

$\therefore$ f is onto.

Hence, f is many-one onto, *i.e.*, f is not bijective.

EXERCISE 2 (*d*)

1. A function f given as f: {(2, 7), (3, 4), (7, 9), (– 1, 6), (0, 2), (5, 3)}.
Is this function one-one onto?
Interchange the order of the elements in the ordered pairs and form the new relation. Is this relation a function? If it is a function, is it one-one onto.

2. Determine if each function is one-one.
(*i*) To each person on the earth assign the number which corresponds to his age.
(*ii*) To each country in the world assign the latitude and longitude of its capital.
(*iii*) To each book written by only one author assign the author.
(*iv*) To each country in the world which has a prime minister assign its prime minister.

3. Let $f: A \to B$. Find $f(A)$, *i.e.*, the range of f, if f is an onto function.

4. Show that the function $f: R \to R$ given by $f(x) = \cos x$ for all $x \in R$, is neither one-one nor onto.

5. Let $A = \{-1, 1\}$. Let functions f, g and h of A onto A be defined by:
(1) $f(x) = x$, (2) $g(x) = x^3$, (3) $h(x) = \sin x$.
Which function, if any, is onto?

6. Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$, construct an example of each of the following.
(*i*) A one-one mapping from A to B
(*ii*) A mapping from A to B which is not one-one
(*iii*) A mapping from B to A.

7. Are the following sets of ordered pairs functions? If so, examine whether the mapping is onto or one-one.
(*i*) $\{(x, y) : x$ is a person, y is the mother of $x\}$
(*ii*) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$. **[*IIT*]**

8. Is the function $f: N \to N$ (N is the set of natural numbers) defined by
$f(n) = 2n + 3$ for all $n \in N$ onto?

9. Let $A = \{x = 0 \le x \le 2\}$ and $B = \{1\}$. Give an example of a function from A to B. Can you define a function from B and A which is onto? Give reasons for your answer.

10. Prove that the function $f: R \to R, f(x) = x^2 + x$ is a many- one into function. **[*Raj. PET 1999*]**

11. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4) (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one but not onto.

12. Show that the function $f: R \to R : f(x) = 3 - 4x$ is one-one onto and hence bijective.

ANSWERS

1. Yes, Yes, Yes

2. (*i*) No, Many people in the world have the same age.
(*ii*) Yes, (*iii*) No, There are different books by the same author.
(*iv*) Yes, Different countries in the world have different Prime Ministers.

3. $f(A) = B$. **5.** (1) and (2) are onto functions.

6. (*i*) $f = \{(2, 5), (3, 7), (4, 6)\}$, (*ii*) $g = \{(2, 2), (3, 5), (4, 2)\}$
(*iii*) $h = \{(2, 2), (5, 3), (6, 4), (7, 4)\}$.

8. No.

HINTS TO SELECTED QUESTIONS

4. **One-one:** $f(0) = \cos 0 = 1$ and $f(2\pi) = \cos 2\pi = 1$. So, different elements in R (domain) may have the same image. Hence, f is not one-one.

Onto : Since the values of $\cos x$ lie between – 1 and 1, therefore, the entire co-domain is not used up, *i.e.*, the range of $f(x)$ is not equal to co-domain. So, f is not one-one.

7. (*i*) The given set of ordered pairs is a function since each person has one and only one mother. This function is onto but not a one-one function or one-one because more than one person may have the same mother.

(*ii*) The given set of ordered pairs is not a function since a person has many ancestors (*e.g.,* father, mother, grandfather, great grandmother and so on). Thus in this case the image $f(a)$ of an element a of the domain is not unique.

9. A function $f: A \to B$ is defined by $f(x) = 1 \forall x \in A$.

Since B contains only one element whereas A contains infinite number of elements (*i.e.*, all numbers from 0 to 2), it is not possible to define a function from B to A which is onto.

10. Type Solved Ex. 38. There is no pre-image of –1. So it is an into function.

12. Show $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one. Now, put $y = 3 - 4x$ and proceed as in Ex. 37.

2.20. Algebraic operations on functions

If f and g are real-valued functions of x with domain set A, B respectively, then both f and g are defined in $A \cap B$. Now we define $(f + g)$, $(f - g)$, $f(g)$, $\left(\frac{f}{g}\right)$ as follows:

(*i*) $(f + g)(x) = f(x) + g(x)$; Domain $A \cap B$

(*ii*) $(f - g)(x) = f(x) - g(x)$; Domain $A \cap B$

(*iii*) $(fg)(x) = f(x) \cdot g(x)$; Domain $A \cap B$

(*iv*) $\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)}$; Domain = $\{x \mid \in A \cap B\}$

(*v*) $(f + k)x = f(x) + k$; k is a constant. Domain A

(*vi*) $(kf)\,x = kf(x)$

(*vii*) $f^n(x) = \{f(x)\}^n$.

Ex. 43. If $f(x) = \frac{x - |x|}{|x|}$, then find $f(-1)$.

Sol. $f(-1) = \frac{-1 - |-1|}{|-1|} = \frac{-1-1}{1} = -2.$

Ex. 44. If $f(x) = \frac{x}{x-1}$, then prove that $\frac{f(a)}{f(a+1)} = f(a^2)$

Sol. $\frac{f(a)}{f(a+1)} = \frac{\frac{a}{a-1}}{\frac{(a+1)}{(a+1-1)}} = \frac{a}{a-1} \cdot \frac{a}{a+1} = \frac{a^2}{a^2-1} = f(a^2)$

Ex. 45. If $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$ for $x > 2$ then find $f(11)$.

[*EAMCET 2003*]

Sol. $f(11) = \dfrac{1}{\sqrt{11+2\sqrt{22-4}}} + \dfrac{1}{\sqrt{11-2\sqrt{22-4}}}$

$= \dfrac{1}{\sqrt{11+2\sqrt{18}}} + \dfrac{1}{\sqrt{11-2\sqrt{18}}} = \dfrac{1}{\sqrt{(3+\sqrt{2})^2}} + \dfrac{1}{\sqrt{(3-\sqrt{2})^2}}$

$= \dfrac{1}{3+\sqrt{2}} + \dfrac{1}{3-\sqrt{2}} = \dfrac{3-\sqrt{2}+3+\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})}$

$= \dfrac{6}{3^2-(\sqrt{2})^2} = \dfrac{6}{9-2} = \dfrac{6}{7}.$

Ex. 46. If $f(x) = \dfrac{1+x}{1-x}$, show that $f[f(\tan\theta)] = -\cot\theta$.

Sol. $f(x) = \dfrac{1+x}{1-x}, \quad \therefore f(\tan\theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$

$$f[f(\tan\theta)] = \frac{1+f(\tan\theta)}{1-f(\tan\theta)} = \frac{1+\dfrac{1+\tan\theta}{1-\tan\theta}}{1-\dfrac{1+\tan\theta}{1-\tan\theta}}$$

$$= \frac{1-\tan\theta+1+\tan\theta}{1-\tan\theta-1-\tan\theta} = \frac{2}{-2\tan\theta} = -\cot\theta.$$

Ex. 47. If $f(x) = \cos(\log x)$, then $f(x^2)\,.f(y^2) - \dfrac{1}{2}\left[f(x^2\,y^2) + f\left(\dfrac{x^2}{y^2}\right)\right] =$

(*a*) –2 (*b*) –1 (*c*) $\dfrac{1}{2}$ (*d*) 0

Sol. $f(x) = \cos(\log x) \Rightarrow f(x^2) = \cos(\log x^2) = \cos(2\log x)$, $f(y^2) = \cos(2\log y)$

$$f\left(\frac{x^2}{y^2}\right) = \cos\left(\log\frac{x^2}{y^2}\right) = \cos(2\log x - 2\log y),\ f(x^2\,y^2) = \cos(2\log x + 2\log y)$$

$$f(x^2)f(y^2) - \frac{1}{2}\left[f\left(\frac{x^2}{y^2}\right) + f(x^2\,y^2)\right]$$

$$= \cos(2\log x)\,.\,\cos(2\log y) - \frac{1}{2}\left[\cos(2\log x - 2\log y) + \cos(2\log x + 2\log y)\right]$$

$$= \cos(2\log x)\,.\,\cos(2\log y) - \frac{1}{2}\times 2\cos(2\log x)\,.\,\cos(2\log y) = 0.$$ **Ans : (*d*)**

[Applying C, D formula of Trigonometry]

2.21. Real valued functions

If R be the set of real numbers and A, B are subsets of R, then the function f: $A \to B$ is called a real function or **real valued function.**

2.22. Piece function

Although functions are frequently described by single formulae, it is not necessary that it should always be so. Thus sometimes a function is defined in two or more parts, such as the function g defined by

$g(x) = x$, for each number x such that $x \geq 0$,

$g(x) = -x$, for each number x such that $x < 0$.

This is a single rule as it defines one function, even though it involves two equations. It is customary to abbreviate this rule in the form $g(x) = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

The domain of this function is the set of all real numbers and the range is the set of all non-negative real numbers. The above function is just the absolute value function $g(x) = |x|, x \in R$.

Suppose we have a piece function h defined as

$$h(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

We notice that h assigns a number to every real number, hence the domain of h is the set of all real numbers. Now, since h (every negative real number) $= -1$, $h(0) = 0$, h (every positive real number) $= 1$, therefore, the range of the function h defined as above, is the set $\{-1, 0, 1\}$.

Ex. 48. If $f(x) = \begin{cases} 2x - 1, & \text{when } x \leq 0 \\ x^2, & \text{when } x > 0 \end{cases}$, **then find** $f\left(\frac{1}{2}\right)$ **and** $f\left(\frac{-1}{2}\right)$.

Sol. When $x \leq 0, f(x) = 2x - 1 \quad \therefore f\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right) - 1 = -2$

when $x > 0, f(x) = x^2 \ \therefore \ f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Ex. 49. If $f(x) = \begin{cases} 1 + x, & -1 \leq x < 0 \\ x^2 - 1, & 0 < x < 2 \\ 2x, & 2 \leq x \end{cases}$

Find $f(3), f(-2), f(0), f\left(\frac{1}{2}\right) f(2-h), f(-1+h)$, **where** $h > 0$ **is very small.**

Sol. $f(3) = 2 \times 3 = 6$ $(\because f(x) = 2x \text{ for } x \geq 2)$

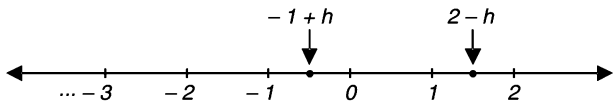


Fig. 2.30

$f(-2)$ and $f(0)$ are not defined.

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4}$ (Since $f(x) = x^2 - 1$ for $0 < x < 2$)

$f(-1+h) = 1 + (-1+h) = h$ $(\because f(x) = 1 + x, \text{ for } -1 \leq x < 0)$

$f(2-h) = (2-h)^2 - 1 = h^2 - 4h + 3$ $(\because f(x) = x^2 - 1 \text{ for } 0 < x < 2)$

Ex. 50. Graph the following piece wise-defined function:

$$\begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

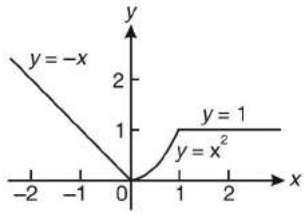


Fig. 2.31: To graph the function $y = f(x)$ shown here, we apply different formulae to different parts of its domain.

Sol. The given piece wise function is defined on the entire real line but has values given by different formulae depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. The function, however, is just one function whose domain is the entire set of real numbers (Figure).

2.23. Classification of functions

Functions can be classified into two groups:

I. ALGEBRAIC FUNCTIONS

A function which consists of a finite number of terms involving powers and roots of the independent variable x and the four fundamental operations of addition, subtraction, multiplication and division is called an algebraic functions.

Thus, $5x^3 - 7x^2 + 6x + 2$, $(5x - 3)^3$, $3\sqrt{6x-11}$, $\dfrac{\sqrt{2x+3}}{(x-1)^2}$ are all algebraic functions.

Three particular cases of algebraic functions are:

(*i*) **Polynomials or rational integral functions.** A function of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots a_n$ where $f(x)$ is a positive integer and $a_0, a_1, a_2, \ldots a_n$ are real constants is called a polynomial or rational integral function of x, *e.g.,* $5x^3 - 6x^2 + 4x - 2$ is a polynomial.

(*ii*) **Rational functions**. A function of the form $\dfrac{f(x)}{g(x)}$ $g(x) \neq 0$, where $f(x)$ and $g(x)$ are polynomials in x is called a rational function, *e.g.,* $\dfrac{5x^2 + 6x - 2}{x^3 + x^2 + 3}$ is a rational functions.

(*iii*) **Irrational functions.** The functions involving radicals are called irrational functions.

e.g., $\sqrt{5x^2 - 7x + 2}$, $\dfrac{(x+1)^{\frac{1}{2}} - (x-3)^{\frac{1}{2}}}{(5+3x)^{\frac{1}{2}} + (7-2x)^{\frac{2}{3}}}$.

II. TRANSCENDENTAL FUNCTIONS

A function which is not algebraic is called transcendental.

In other words, transcendental functions are not capable of being produced by the algebraical operations of addition, multiplication, and involution, or the inverse operations.

The following are different types of transcendental functions.

(*i*) **Trigonometric functions,** such as $\cos x$, $\tan 3x$, etc.

(*ii*) **Inverse trigonometric functions**, such as $\sin^{-1} x$, $\sec^{-1} x$, etc.

(*iii*) **Exponential functions.** A function, in which the domain (independent) variable is an exponent, is an exponential function. A simple exponential function is $y = 2^x$.

(*iv*) **Logarithmic function**. The correspondence $x \to \log x$ is a logarithmic function.

III. SPECIAL FUNCTIONS

1. Constant function. If k is a fixed real number, then, the function $f : R \to R$: $f(x) = k$ for all $x \in R$ is called the **constant function**.

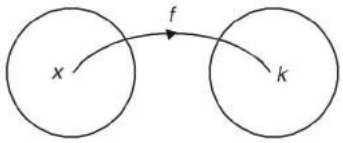
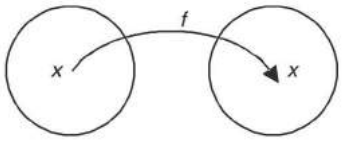


Fig. 2.32

For the constant function $f(x) = k \; \forall \; x \in R$ we have domain of $f = R$ and range of $f = \{k\}$.

2. Identity function. The identity function maps each member of the domain onto itself. That is, if I is the identity function, then for any x in the domain, $I(x) = x$.



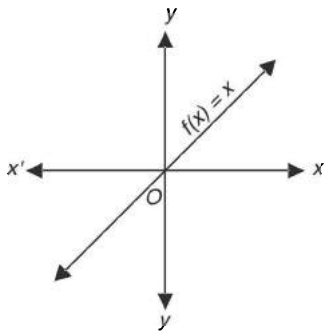


Fig. 2.33

If we let y stand for $I(x)$, then $y = (x)$. It is easy to see that if the domain of I is the set of real numbers, then the graph of the identity function is the line $y = x$.

3. Modulus function. The function defined by

$$f(x) = |x| = \begin{cases} x; \text{ when } x \geq 0 \\ -x; \text{ when } x < 0 \end{cases}$$

is called the modulus function.

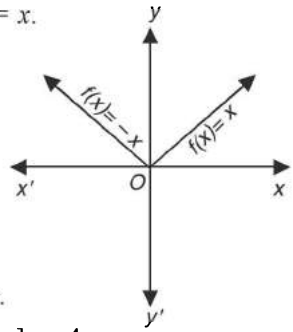


Fig. 2.34

Note. For any real number x, we have $\sqrt{x^2} = |x|$.

4. Greatest integer function.

The symbol $[x]$ denotes the greatest integer less than or equal to x.

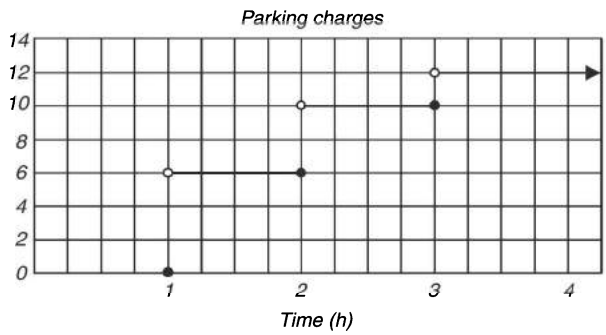
For example, $[4] = 4$, $[4.1] = 4$, $[0] = 0$; $[-1.3] = -2$, $[\pi] = 3$, $[-\pi] = -4$

The function with equation $y = [x]$ is called the greatest integer functions. This is an example of **step function**.

The graph of a step function can be drawn as shown below. It is known as **step** graph because it increases (or decreases) in *steps* rather than being a continuous line.

Illustrations

1. This step graph shows the charges at a parking station.



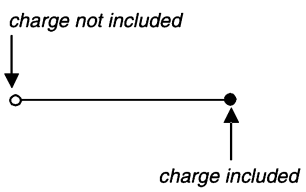


Fig. 2.35

(*i*) What is the charge for parking for:

(*a*) 2 h (*b*) $2\frac{1}{2}$h (*c*) 3 h (*d*) 4 h

(*ii*) How long can you park for free?

Sol. (*i*) (*a*) Cost for 2 h parking = ₹ 6 (*b*) Cost for $2\frac{1}{2}$h parking = ₹ 10

(*c*) Cost for 3 h parking = ₹ 10 (*d*) Cost for 4 h parking = ₹ 12

(*ii*) You can park for up to 1 h free.

3. Fig. 2.36 shows the partial step graph of the function $G = x \to [x]$, where $x \in R$.

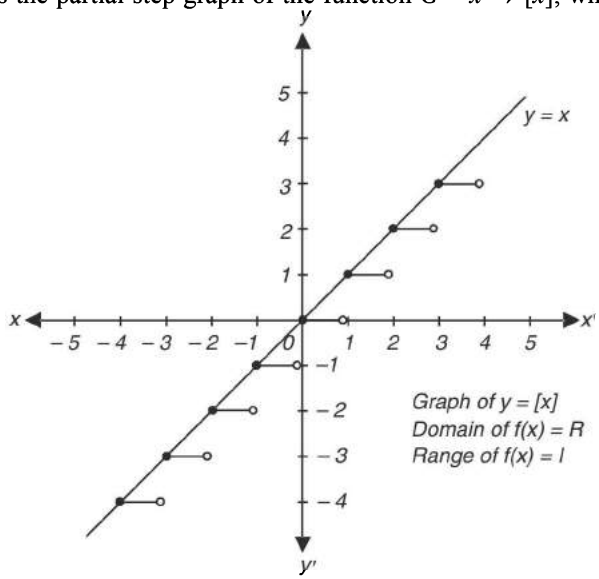


Fig. 2.36

Note that the function G maps each real number to x into the greatest integer that is less than or equal to x. For instance, G maps 2 into 2 because the greatest integer less than or equal to 2 is 2 itself. But G also maps $2\frac{1}{2}$ into 2, since the greatest integer less than or equal to $2\frac{1}{2}$ is 2. We thus have

$$[2] = 2,\ \left[2\frac{1}{2}\right] = 2,\ [2.69] = 2, \text{ but } \left[-2\frac{1}{2}\right] = -3,$$

$$[0] = 0,\ \left[1\frac{1}{3}\right] = 1,\ \left[-\frac{1}{5}\right] = -1,\ [2.99] = 2,\ [3.1] = 3,$$

$$[-1] = -1,\ \left[4\frac{3}{4}\right] = 4,\ \left[-3\frac{1}{10}\right] = -4,\ [-5.1] = -6, \text{ etc.}$$

Definition: ***Let $x \in R$, then, A function $f(x) = [x]$ where $f(x) = n$ (an integer) such that $n \le x < n + 1$, is called a* step *or* greatest integer function.***

Thus, for all x such that $0 \le x < 1$, $f(x) = 0$,

$1 \le x < 2 \Rightarrow f(x) = 1$, $2 \le x < 3 \Rightarrow f(x) = 2$,

$-1 \le x < 0 \Rightarrow f(x) = -1$, $-2 \le x < -1 \Rightarrow f(x) = -2$

Domain of $f = R$, Range of f = set of integers

5. Signum function.

The function $f : R \to R$ defined as,

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}, \quad i.e., \ \text{Sig } x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

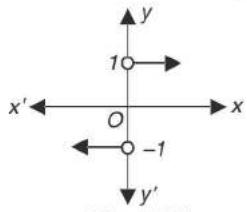


Fig. 2.37

Its domain is R and range is the set $\{-1, 0, 1\}$. The graph is as shown.

6. Even and odd functions.

If $f(x)$ is a function of x such that $f(-x) = f(x)$ for every x of its domain, then it is called an *even* function of x. For example,

If, $f(x) = \cos(x)$ then $f(-x) = \cos(-x) = \cos x$ so that $f(x) = f(-x)$. Thus it is an even function. Similarly, $y = x^2 + 7$, $y = \cos 2x$ are also even functions.

If $f(x)$ is a function of x such that $f(-x) = -(x)$ for every x of its domain, then it is called an odd function of x. For example,

If $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x$ so that $f(-x) = -f(x)$. Hence, $f(x)$ is an odd function. Similarly, $y = x^2 + 6x$, $y = \tan x$ are also odd functions.

Remark. There are functions which are neither even nor odd, *e.g.*, $y = x^2 - 3x + 1$.

2.24. Undefined functions

If $f(a)$ is any of the forms $\dfrac{0}{0}, \dfrac{\infty}{\infty}, (0 \times \infty), (\infty \times -\infty). 0^0$, 1^∞ and ∞^0, then we say that $f(x)$ is not defined at $x = a$. These forms are called **indeterminate forms.** Thus,

if $f(x) = \dfrac{x^2 - 4}{x - 2}$, then $f(2) = \dfrac{0}{0}$, and, therefore, $f(x)$ is not defined at $x = 2$.

Similarly, $f(x) = x^x$ is not defined at $x = 0$, since $f(0) = 0^0$, which is indeterminate.

$f(x) = \dfrac{\log(\cos x)}{\sec x}$ is not defined at $x = \dfrac{\pi}{2}$, since $f\left(\dfrac{\pi}{2}\right) = \dfrac{\infty}{\infty}$, which is indeterminate.

[**Note.** You will study indeterminate functions in class XII.]

Ex. 51. Examine whether $x\left(\dfrac{a^x + 1}{a^x - 1}\right)$ is an odd or even function.

Sol. Let $f(x) = x\left(\dfrac{a^x + 1}{a^x - 1}\right)$. Then $f(-x) = (-x)\dfrac{a^{-x} + 1}{a^{-x} - 1} = (-x)\dfrac{\dfrac{1}{a^x} + 1}{\dfrac{1}{a^x} - 1}$

$$= (-x)\frac{1 + a^x}{1 - a^x} = x \cdot \frac{a^x + 1}{a^x - 1} = f(x).$$

Since $f(x) = f(-x)$, so it is an even function.

Ex. 52. Examine whether the function $f(x) = \sin[\log(x + \sqrt{x^2 + 1})]$ is an odd or even function. [*Orissa JEE 2002*]

Sol. $f(-x) = \sin[\log(-x + \sqrt{1 + x^2})] = \sin \log[(\sqrt{1 + x^2} - x)\dfrac{(\sqrt{1 + x^2} + x)}{(\sqrt{1 + x^2} + x)}]$

$$= \sin \log \left(\frac{1}{x + \sqrt{1+x^2}} \right) = \sin \log [(x + \sqrt{1+x^2})^{-1}]$$

$$= \sin [-\log (x + \sqrt{1+x^2})] = -\sin [\log (x + \sqrt{1+x^2})] = -f(x)$$

Since $f(-x) = -f(x)$, the given function is an odd function.

Ex. 53. (*i*) Determine whether the function is even, odd, or neither even nor odd.

(*a*) $f(x) = 5 - x^2$ (*b*) $f(x) = |-x|$ (*c*) $f(x) = [x]$ (*d*) $f(x) = |x - 2|$ (*e*) $f(x) = -x|x|$.

(*ii*) If f is an odd function and $f(0)$ is defined, must $f(0) = 0$?

(*iii*) If $f(x) = x^2 + kx + 1$ for all x and f is an even function, find k, $k \in R$.

Sol. (*i*) (*a*) f is even, since $f(-x) = f(x)$.

(*b*) Since $|-x| = |x|$, this function is even.

(*c*) $f\left[\frac{1}{2}\right] = 0$ and $f\left[-\frac{1}{2}\right] = -1$. Hence, this function is neither even nor odd.

(*d*) $f(2) = 0$ and $f(-2) = |-2-2| = |-4| = 4$, so $f(x)$ is neither even nor odd.

(*e*) $f(-x) = -(-x)\,|-x| = x|x| = -f(x)$, so this is an odd function.

(*ii*) Yes, $f(0) = f(-0) = -f(0)$. Hence, $f(0) = 0$.

(*iii*) $f(1) = 2 + k$ and $f(-1) = 2 - k$. By the evenness of f, $f(-1) = f(1)$. Hence,

$2 + k = 2 - k \Rightarrow k = 0$.

Ex. 54. The function $f(x) = \sin x + \cos x$ will be

(*a*) an even function **(*b*) an odd function**

(*c*) a constant function **(*d*) None of these** **[*Chhattisgarh PET 2009*]**

Sol. $f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x$

$\therefore$ It is neither odd nor even function.

Also, it is not a constant function.

Hence, option (*d*) is correct.

EXERCISE 2 (*e*)

1. If the function $f: N \to N$ is defined by $f(x) = \sqrt{x}$, then find $\frac{f(25)}{f(16) + f(1)}$. ***[MP PET 2009]***

2. If $f(x) = \frac{x^3}{2} - \frac{x^2}{2} + x - 16$, find $f\left(\frac{1}{2}\right)$.

3. If $f(x) = 7x^4 - 2x^3 - 8x - 5$, find $f(-1)$.

4. If $f(x) = \begin{cases} 3x - 2 \text{ when } x \le 0 \\ x + 1 \text{ when } x > 0 \end{cases}$, find $f(-1)$ and $f(0)$.

5. If $f(x) = \log\left(\frac{1-x}{1+x}\right)$, show that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$.

6. If $f(x) = 2x\sqrt{1-x^2}$, then show that $f\left(\sin\frac{x}{2}\right) = \sin x$.

7. If $f(x) = \cos(\log x)$, then prove that

$$f\left(\frac{1}{x}\right).f\left(\frac{1}{y}\right) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right] = 0$$

8. If $y = f(x) = \dfrac{5x+3}{4x-5}$, then show that $f(y) = x$.

9. If $f(x) = x^2 + kx + 1$, for all x and if it is an even function, find k.

10. If $f(x) = x^3 - (k-2)x^2 + 2x$, for all x and if it is an odd function, find k.

11. Is there a function f which is both even and odd?

12. The function $f(x) = \log(x + \sqrt{x^2+1})$, is (*a*) an even function (*b*) an odd function (*c*) a periodic function (*d*) Neither an even nor an odd function.

[*AIEEE 2003, MPPET 2003, UPSEAT 2003*]

13. Prove that $f(x) = (1/x)\log\sqrt{x+\sqrt{x^2+1}}$ is an even function.

[**Hint.** Show that $f(-x) = f(x)$]

ANSWERS

1. 1 **2.** $\dfrac{-249}{16}$ **3.** 12 **4.** –5, –2 **9.** $k = 0$ **10.** $k = 2$

11. The zero function **12.** (*b*) Odd function [**Hint.** $f(-x)$ will come out to be equal to $-f(x)$.}

2.25. Further discussion on the domain of a function

Consider the following functions.

1. The function $\dfrac{1}{x}$ has no meaning when $x = 0$, so this function is defined for all real numbers except 0.

2. The function $\sqrt{x}$ is defined for only the non-negative real numbers.

3. Since a negative number does not have a real square root, therefore, the function $\sqrt{x(8-x)}$ is defined only when $x(8-x) \geq 0$.

This will be true only when both the factors x and $8 - x$ are greater than or equal to 0, or both less than or equal to 0.

If $x \geq 0$ and $8 - x \geq 0$, then $0 \leq x \leq 8$.

The alternative $x \leq 0$ and $8 - x \leq 0$, is impossible because $8 - x \leq 0 \Rightarrow 8 \leq x \Rightarrow x \geq 8$.

We cannot have $x \leq 0$ and $x \geq 8$ at the same time. ***So the function $\sqrt{x(8-x)}$ is defined only if $0 \leq x \leq 8$.***

Definition. *The set of numbers x for which function f(x) is defined is called the domain of the function.*

Thus, the domain of the function $f(x) = \sqrt{x(8-x)}$ is the set of real numbers such that $0 \leq x \leq 8$. These are any numbers for which $\sqrt{x(8-x)}$ exists.

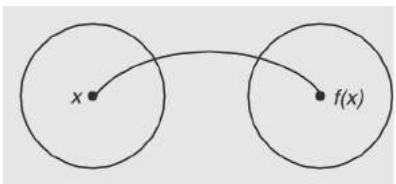


Fig. 2.38

2.26. Tips for finding the domain of a function

1. **Algebraic functions**
   (*i*) Denominator should be non-zero.
   (*ii*) Expression under the even root should be non-negative.
2. **Trigonometric functions**
   (*i*) sin x and cos x are defined for all real values of x.
   (*ii*) tan x and sec x are defined for all real values of x except $x = (2n+1)\frac{\pi}{2}$, where $n \in I$.
   (*iii*) cot x and cosec x are defined for all real values of x except $x = n\pi$, where $n \in I$.
3. **Logarithmic functions**
   $\log_b a$ is defined when $a > 0$, $b > 0$ and $b \neq 1$.
4. **Exponential functions**
   a^x is defined for all real values of x, where $a > 0$.

Ex. 55. If f is a real function, find the domain of

(*i*) $f(x) = \sqrt{(a^2 - x^2)}$ **(*ii*)** $f(x) = \dfrac{1}{3x+2}$

(*iii*) $f(x) = \dfrac{1}{\log|x|}$ **(*iv*)** $f(x) = 10^{-x}$.

Sol. (*i*) Since f is real, $a^2 - x^2 \geq 0 \Rightarrow (a + x)(a - x) \geq 0$

$\Rightarrow \quad -a \leq x \leq a \quad \therefore$ Domain of $f(x) = \sqrt{a^2 - x^2}$ is $\{x \mid -a \leq x \leq a\}$.

(*ii*) Since f is real, x can take all real values except the value for $3x + 2 = 0 \Rightarrow x \neq -\frac{2}{3}$.

$\therefore$ Domain of $f(x) = \dfrac{1}{3x+2}$ is $R - \left\{-\frac{2}{3}\right\}$.

(*iii*) Since f is real, x can take all real values except 0, 1, – 1.

$\therefore$ Domain of $f(x) = \dfrac{1}{\log|x|}$ is $R - \{0, 1, -1\}$

(*iv*) $f(x) = \dfrac{1}{10^x} \quad \therefore$ Domain of f is R.

Ex. 56. Find the domain of (*i*) $\sqrt{x} + \sqrt{2x-1}$ **(*ii*)** $\log(x-2) - \sqrt{(3-x)}$.

Sol. (*i*) Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2x-1}$,

Let domain of $f(x) = A$ and domain of $g(x) = B$.

Thus, $A = [0, \infty)$ and $B = \left[\frac{1}{2}, \infty\right)$

$\therefore$ Domain of $\sqrt{x} + \sqrt{2x-1} = A \cap B = \left[\frac{1}{2}, \infty\right)$.

(*ii*) Let $f(x) = \log(x - 2)$ and $g(x) = \sqrt{(3-x)}$

For $f(x)$ to be defined, $x - 2 > 0 \Rightarrow x > 2 \Rightarrow x \in (2, \infty)$ and $g(x)$ to be defined, $3 - x \geq 0$ $\Rightarrow 3 \geq x \Rightarrow x \leq 3 \Rightarrow x \in (-\infty, 3]$.

Thus $A = (2, \infty)$ and $B = (-\infty, 3]$

$\therefore$ Domain of $\log (x - 2) - \sqrt{(3-x)} = A \cap B - \{x \mid g(x) = 0\}$.

$= A \cap B - \{3\} = (2, \infty) \cap (-\infty, 3) - \{3\} = (2, 3)$.

Ex. 57. Find the domain of the following functions.

(i) $f(x) = \log_{3+x} (x^2 - 1)$. **[*Orissa JEE 2003*]**

(ii) $f(x) = \dfrac{\sin^{-1} (3 - x)}{\ln (|x| - 2)}$ **[*Orissa JEE 2002*]**

(iii) $f(x) = \sin^{-1} \log_2 \dfrac{x}{3}$ **[*AMU 1996*]**

(iv) $f(x) = \dfrac{\log_2 (x + 3)}{(x^2 + 3x + 2)}$ **[*IIT (S) 2001, UPSEAT 2001*]**

(v) $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9 - x^2}}$

(vi) $f(x) = \log_e \dfrac{2+x}{2-x}$ **[*WBJEE*]**

Sol. (*i*) $f(x)$ is defined when $x^2 - 1 > 0 \Rightarrow x^2 > 1$ and $3 + x > 0$

$\Rightarrow x < -1$ or $x > 1$ and $x > -3$ and $x \neq -2$, since base $3 + x \neq 1$)

$\therefore$ Domain is $(-3, -2) \cup (-2, -1) \cup (1, \infty)$.

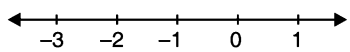


Fig. 2.39

[Recall that in the definition of a logarithm, $\log_a n = x$, the a, n are positive real numbers, x is any real number and $a \neq 1$.]

(*ii*) Let $g(x) = \sin^{-1} (3 - x) \Rightarrow -1 \leq 3 - x \leq 1$ $\therefore$ Domain of $g(x)$ is $[2, 4]$.

Let $h(x) = \log [|x| - 2] \Rightarrow |x| - 2 > 0 \Rightarrow |x| > 2 \Rightarrow x < -2$ or $x > 2$

$\Rightarrow$ Domain $h(x) = (-\infty, -2) \cup (2, \infty)$

Now, we know that $\left(\dfrac{f}{g}\right) x = \dfrac{f(x)}{g(x)} \forall x \in D_1 \cap D_2 - \{x \in R : g(x) = 0\}$

$\therefore$ Domain of $f(x) = (2, 4] - \{3\} = (3, 4]$.

(*iii*) Let $f(x) = \sin^{-1} z$ where, $z = \log_2 \dfrac{x}{3}$. Since, $\sin^{-1} z$ is defined only for $-1 \leq z \leq 1$, therefore, the domain is given by $-1 \leq \log_2 \dfrac{x}{3} \leq 1 \Rightarrow 2^{-1} \leq \dfrac{x}{3} \leq 2^1$

$\Rightarrow \dfrac{1}{2} \leq \dfrac{x}{3} \leq 2 \Rightarrow \dfrac{3}{2} \leq x \leq 6 \Rightarrow$ Domain is $\left[\dfrac{3}{2}, 6\right]$. (Since $\log_a x = k \Rightarrow x = a^k$)

(*iv*) For the function to be defined $x + 3 > 0$ and $x^2 + 3x + 2 \neq 0$

$\Rightarrow x > -3$ and $(x + 1)(x + 2) \neq 0$, *i.e.*, $x \neq -1, -2$.

$\therefore$ Domain is $(-3, \infty) - \{-1, -2\}$.

(*v*) For $f(x)$ to be defined

(*i*) $9 - x^2 > 0 \Rightarrow -3 < x < 3$; (*ii*) $-1 \leq x - 3 \leq 1$ [$\sin\theta$ is defined for $-1 \leq \theta \leq 1$] $\Rightarrow 2 \leq x \leq 4$

From (*i*) and (*ii*), $2 \leq x < 3$, *i.e.*, the domain is $[2, 3)$.

(*vi*) $f(x)$ will be defined when $\frac{2+x}{2-x} > 0$.

i.e., when $2 + x > 0$ and $2 - x > 0$ or $2 + x < 0$ and $2 - x < 0$

i.e., when $x > -2$ and $x < 2$ or $x < -2$ and $x > 2$

i.e., when $-2 < x < 2$ or $x < -2$ and $x > 2$

As the second option is absurd, the domain is $\mathbf{-2 < x < 2}$.

Ex. 58. The domain of the definition of the function $f(x) = \sqrt{1+\log_e(1-x)}$ is

(a) $\mathbf{-\infty < x \le 0}$ ***(b)*** $\mathbf{-\infty < x \le \frac{e-1}{e}}$

(c) $\mathbf{-\infty < x \le 1}$ ***(d)*** $\mathbf{x \ge 1-e}$. ***[WB JEE 2009]***

Sol. For domain, $(1 - x) > 0$ and $\log_e (1 - x) \ge -1$

$\Rightarrow x < 1$ and $1 - x \ge e^{-1} \Rightarrow x < 1$ and $x \le 1 - \frac{1}{e}$

$\Rightarrow -\infty < x \le \frac{e-1}{e}$

Ans. (*b*)

Ex. 59. Domain of definition of the function $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is

(a) **(1, 2)** ***(b)*** **(–1, 0) ∪ (1, 2)** ***(c)*** **(–1, 2) ∪ (2, ∞)**

(d) **(–1, 0) ∪ (1, 2) ∪ (2, ∞)**

Sol. For domain of $f(x)$

(*i*) $x^3 - x > 0 \Rightarrow x(x-1)(x+1) > 0$

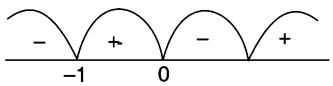


Fig. 2.40

$\therefore$ Region is $(-1, 0) \cup (1, \infty)$

(*ii*) $4 - x^2 \ne 0 \Rightarrow x \ne 2$ $\therefore$ Region is $(-\infty, 2) \cup (2, \infty)$.

$\therefore$ Domain is the common region of (*i*) and (*ii*), *i.e.*, $(-1, 0) \cup (1, 2) \cup (2, \infty)$ **Ans.** (*d*)

2.27. Further discussion on the range of a function

Once we know the domain of the function, we can determine the values $f(x)$ can take. The set of all these values is called the **range** of the function.

For example,

1. The function $f(x) = x(8 - x)$ can be evaluated for any real number x, so the domain is the complete set of real numbers.

   Now, let $y = f(x)$ $\Rightarrow y = x(8 - x)$ $\Rightarrow x^2 - 8x + y = 0$

   $\Rightarrow$ $x = -\frac{(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times y}}{2 \times 1}$ $\Rightarrow x = \frac{8 \pm \sqrt{64 - 4y}}{2}$

   x will be real if $64 - 4y \ge 0 \Rightarrow 64 \ge 4y \Rightarrow y \le 16$, so the range consists of all real numbers less than or equal to 16.

2. Let $y = f(x) = \sqrt{x(8-x)}$

   Its domain is the set of real numbers $0 \le x \le 8$.

Now, $y = \sqrt{8x - x^2} \Rightarrow y^2 = 8x - x^2 \Rightarrow x^2 - 8x + y^2 = 0 \Rightarrow x = \dfrac{8 \pm \sqrt{64 - 4y^2}}{2}$

x will be real if $64 - 4y^2 \geq 0 \Rightarrow y^2 - 16 \leq 0 \Rightarrow (y-4)(y+4) \leq 0$

The above inequality is true when $-4 \leq y \leq 4 \Rightarrow y \in [-4, 4]$

Now, $y = \sqrt{16 - x^2} \geq 0$ for all $x \in [-4, 4]$

$\therefore$ $y \in [0, 4]$ to all $x \in [-4, 4]$.

Hence, range (f) = [0, 4].

Method for finding the range of a function $y = f(x)$.

Step 1. Find the domain of the function $y = f(x)$.

Step 2. Solve the equation $y = f(x)$ to find x in term of y. Find the real values of y for which x is real.

The set of values of y so obtained makes up the range of f.

Ex. 60. Find the range of the following functions.

(*i*) $y = \dfrac{x^2}{1+x^2}$ **(*ii*)** $f(x) = \sqrt{3x^2 - 4x + 5}$

(*iii*) $f(x) = \log_e(3x^2 - 4x + 5)$ **(*iv*)** $f(x) = \dfrac{5}{3 - x^2}$

Sol. (*i*) Since, $y = \dfrac{x^2}{1+x^2}$ is defined for all real x, therefore, domain of $y = (-\infty, \infty)$

Now, $y = \dfrac{x^2}{1+x^2} \Rightarrow x^2y + y = x^2 \Rightarrow x^2 = \dfrac{y}{1-y} \Rightarrow x = \sqrt{\dfrac{y}{1-y}} \Rightarrow x = \dfrac{\sqrt{y(1-y)}}{(1-y)}$

For x to be real, $1 - y \neq 0$, *i.e.*, $y \neq 1$ and $y(1-y) \geq 0$

$\Rightarrow$ $y(y-1) \leq 0$ and $y \neq 1 \Rightarrow 0 \leq y < 1$.

$\therefore$ Range of $y = [0, 1)$*.

(*ii*) $f(x)$ is defined if $3x^2 - 4x + 5 \geq 0$

$\Rightarrow 3\left(x^2 - \dfrac{4}{3}x + \dfrac{5}{3}\right) \geq 0 \Rightarrow 3\left[\left(x - \dfrac{2}{3}\right)^2 + \dfrac{11}{9}\right] \geq 0$

which is true for all real x.

$\therefore$ Domain (f) = $(-\infty, \infty)$

Let $y = \sqrt{3x^2 - 4x + 5} \Rightarrow y^2 = 3x^2 - 4x + 5 \Rightarrow 3x^2 - 4x + (5 - y^2) = 0$

For x to be real, $(-4)^2 - 4 \times 3 \times (5 - y^2) \geq 0 \Rightarrow y \geq \sqrt{\dfrac{11}{3}}$

$\therefore$ Range of $y = \left[\sqrt{\dfrac{11}{3}}, \infty\right)$

(*iii*) $f(x)$ is defined if $3x^2 - 4x + 5 > 0 \Rightarrow 3\left[x^2 - \dfrac{4}{3}x + \dfrac{5}{3}\right] > 0$

*$y = [0, 1)$ is another way of writing $0 \leq y < 1$

$\Rightarrow \quad 3\left[\left(x-\frac{2}{3}\right)^2+\frac{11}{9}\right]>0$ which is the true for all real x.

$\therefore$ Domain of $f = (-\infty, \infty)$.

Let $y = \log_e(3x^2-4x+5) \quad \Rightarrow \quad e^y = 3x^2-4x+5 \quad \Rightarrow \quad 3x^2-4x+(5-e^y)=0$

For x to be real,

$$(-4)^2-12(5-e^y)\geq 0 \quad \Rightarrow \quad 12e^y \geq 44 \quad \Rightarrow \quad e^y \geq \frac{11}{3} \quad \Rightarrow \quad y \geq \log_e \frac{11}{3}$$

$\therefore$ Range of $f = \left[\log_e \frac{11}{3}, \infty\right)$.

(*iv*) For $f(x)$ to be defined, $3 - x^2 \neq 0$, *i.e.*, $x \neq \pm\sqrt{3}$. Domain of $f = R - \{-\sqrt{3}, \sqrt{3}\}$

Now, let $y=\frac{5}{3-x^2} \quad \Rightarrow x^2=\frac{3y-5}{y} \quad \Rightarrow \quad x=\pm\sqrt{\frac{3y-5}{y}}$

Now, x will take real values other than $-\sqrt{3}$ and $\sqrt{3}$, if

$\frac{3y-5}{y}\geq 0 \quad \Rightarrow y<0$ and $y \geq \frac{5}{3}$. Also, $y \neq 0$

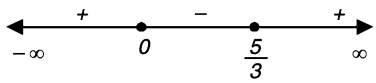


Fig. 2.41

Hence, range $(f) = (-\infty, 0) \cup \left[\frac{5}{3}, \infty\right)$.

Ex. 61. Find the domain and range of the function

(*i*) $f(x)=\frac{4-x}{x-4}$, **(*ii*)** $f(x) = [\sin x]$.

Sol. (*i*) $f(x)=\frac{4-x}{x-4}$.

Domain of *f*: $f(x)$ is defined for all x except at $x = 4$.

$\therefore$ Domain of $f = R - \{4\}$.

Range of *f*: For any $x \in$ Domain (f), we have

$$f(x) = \frac{4-x}{x-4} = \frac{-(x-4)}{x-4} = -1 \qquad \therefore \text{ Range } (f) = \{-1\}.$$

(*ii*) Since $\sin x$ is defined for all $x \in R$, therefore, $f(x) = [\sin x]$ is defined for all $x \in R$.

$\therefore$ Domain $(f) = R$.

Now, $-1 \leq \sin x \leq 1 \quad \Rightarrow \quad [\sin x] = -1, 0, 1$

Hence, Range of $f = \{-1, 0, 1\}$.

EXERCISE 2 (*f*)

1. The range and domain of function $f(x)=\frac{3}{x}+1$ are subsets of A and B respectively, where $A=\left\{-\frac{1}{2},0,\frac{2}{3},\frac{6}{7},1\right\}$ and $B=\left\{-5,0,4\frac{1}{2},5,5\frac{1}{2}\right\}$. List the elements of the function as ordered pairs.

2. $A=\{-2,-1,1,2\}$ and $f=\left\{\left(x,\frac{1}{x}\right),x\in A\right\}$.

(*i*) List the domain of f (*ii*) List the range of f (*iii*) Is f a function?

3. $f: x \to$ highest prime factor of x.

(*i*) Find the range of f when the domain is {12, 13, 14, 15, 16, 17}.

(*ii*) State a domain of five integers for which the range is (3).

(*iii*) A set of positive integers is called S. What can be said about these integers if $f(S)=S$?

4. A function f is defined on the set of integers as follows:

$$f(x)=\begin{Bmatrix}1+x;\ 1\le x<2\\ 2x-1;\ 2\le x<4\\ 3x-10;\ 4\le x<6\end{Bmatrix}$$

(*i*) Find the domain of the function

(*ii*) Find the range of the function

(*iii*) Find the value of $f(4)$.

5. Let f be a function whose domain is the set of all real number. If $f(x)=|x|-x$, what is range of f?

6. Write the domain of the following real functions

(*i*) $\sqrt{9-x^2}$ (*ii*) $\sqrt{1-2x-3x^2}$ (*iii*) 10^x

(*iv*) $\frac{1}{\sqrt{x^2-7}}$ (*v*) $\log(2-3x)$ (*vi*) $\log\left(\sqrt{x-4}+\sqrt{6-x}\right)$ **[*Raj. PET 2001*]**

(*vii*) $\left[\log_{10}\left(\frac{5x-x^2}{4}\right)\right]^{\frac{1}{2}}$ **[*Chhattisgarh PET 2009, AIEEE 2002, UPSEAT 2001*]**

(*viii*) $\sin^{-1}\left[\log_2\left(\frac{x}{2}\right)\right]$ **[*Raj. PET 2002*]**

7. Find the range of each of the following functions

(*i*) $|x-3|$ (*ii*) $\sqrt{x-5}$ (*iii*) $\cos\left(\frac{x}{3}\right)$ **[*Raj PET 2002*]**

(*iv*) $\frac{x+2}{|x+2|}$ **[*IIT*]** (*v*) $\sec\left(\frac{\pi}{4}\cos^2 x\right), -\infty<x<\infty$ **[*Orissa JEE 2002*]**

(*vi*) $\frac{x^2+x+2}{x^2+x+1}; x\in R$ **[*IIT (S) 2003*]** (*vii*) $\frac{x}{1+x^2}$ **[*DCE 2009*]**

(*viii*) $f(x)=\frac{3}{2-x^2}$

8. Find the domain and range of the function $\frac{x^2-4}{x-2}$.

9. If the domain of the function $\frac{|x|}{x}$ be [3, 7], then its range is

(*a*) [–1, 1] (*b*) [–1, 1] (*c*) {1} (*d*) {–1}

ANSWERS

1. $\left\{\left(-\frac{1}{2},5\right),\left(\frac{2}{5},5\frac{1}{2}\right),\left(\frac{6}{7},4\frac{1}{2}\right)\right\}$

2. (*i*) {–2, –1, 1, 2} (*ii*) $\left\{-\frac{1}{2},-1,1,\frac{1}{2}\right\}$, (*iii*) yes

3. (*i*) {2, 3, 5, 7, 13, 17} (*ii*) {3, 6, 9, 12, 18} (*iii*) S is the set of prime numbers

4. (*i*) {1, 2, 3, 4, 5} (*ii*) {2,3,5} (*iii*) 2 **5.** $\{x : x \geq 0\}$

6. (*i*) $\{-3 \leq x \leq 3\}$ (*ii*) $\{-1 \leq x \leq \frac{1}{3}\}$ (*iii*) R (*iv*) $(-\infty,-\sqrt{7})\cup(\sqrt{7},\infty)$

(*v*) $\{x : x \in \left(-\infty,\frac{2}{3}\right)$ and $x \in R\}$ (*vi*) [4, 6] (*vii*) $1 \leq x \leq 4$ (*viii*) [1, 4]

7. (*i*) $[0,\infty)$ (*ii*) $[0,\infty)$ (*iii*) (–1, 1) (*iv*) {–1, 1} (*v*) $[1,\sqrt{2}]$

(*vi*) $[1,\frac{7}{3}]$ (*vii*) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (*viii*) $(-\infty,0]\cup\left[\frac{3}{2},\infty\right)$

8. Domain = $R - \{2\}$, Range is $R - \{4\}$ **9.** (*c*)

HINTS TO SELECTED QUESTIONS

6. (*vii*) $\log_{10}\left(\frac{5x-x^2}{4}\right) \geq 0 \Rightarrow \frac{5x-x^2}{4} \geq 10^0 \Rightarrow \frac{5x-x^2}{4} \geq 1 \Rightarrow (x-1)(x-4) \geq 0$

(*viii*) Type Solved Ex. 57 (*c*).

7. (*i*) Since $f(x) = |x-3|$ is defined for all $x \in R$, therefore, domain of $f(x) = R$

Now, $|x-3| \geq 0$ for all $x \in R$ ...(1)

$\Rightarrow 0 \leq |x-3| < \infty$ for all $x \in R$ ...(2)

(1) and (2) $\Rightarrow f(x) \in [0,\infty)$ for all $x \in R$

(*v*) $0 \leq \cos^2 x \leq 1$. At $\cos x = 0, f(x) = 1$ and at $\cos x = 1, f(x) = \sqrt{2}$ $\therefore$ $1 \leq x \leq 2$.

(*vi*) Let $y = \frac{x^2+x+2}{x^2+x+1} \Rightarrow (y-1)x^2 + (y-1)x + y - 2 = 0$

$\Rightarrow$ $(y-1)^2 - 4(y-1)(y-2) \geq 0 \Rightarrow (y-1)(3y-7) \leq 0 \Rightarrow y \in \left(1,\frac{7}{3}\right]$.

(*vii*) Clearly, domain (f) = R



Let $f(x) = y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2}$

x will have real values if $1 - 4y^2 \geq 0$, *i.e.*, if $4y^2 - 1 \leq 0$

$$\Rightarrow y^2 - \frac{1}{4} \leq 0 \Rightarrow \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2} \Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$\therefore$ Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(*viii*) Type Solved Ex. 60 (*iv*).

8. Let $y = \dfrac{x^2 - 4}{x - 2}$ $\therefore$ $y = x + 2$ if $x \neq 2$.

Since $\dfrac{(x^2 - 4)}{(x - 2)}$ is not defined only when $x = 2$, domain of $\dfrac{(x^2 - 4)}{(x - 2)}$ is $R - \{2\}$. But y will be equal to 4 when $x = 2$ which is not possible. Hence y takes all real values except 4.

$\therefore$ Range of $\dfrac{(x^2 - 4)}{(x - 2)}$ is $R - \{4\}$.

9. Since the domain is the set of positive real numbers x such that $3 \leq x \leq 7$, therefore, $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

2.28. Graphs of real functions

The graph of a function f consists of all points (x, y) such that $y = f(x)$. To graph a function we carry out three steps:

Step 1. *Make a table of pairs from the function.*

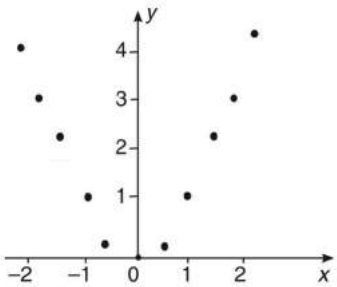
Step 2. *Plot enough of the corresponding points to learn the shape of the graph. Add more pairs to the table if necessary.*

Step 3. *Complete the sketch by joining the points.*

Illustration. Suppose we have to draw the graph of function $y = x^2$ over the interval $-2 < x < 2$.

Sol.

x	$y = x^2$
–2.0	4.0
–1.75	3.06
–1.5	2.25
–1.0	1.0
–0.5	0.25
0	0
0.5	0.25
1.0	1.0
1.5	2.25
1.75	3.06
2.0	4.0



The Graph of $y = x^2$, $-2 \leqslant x \leqslant 2$

Fig. 2.42

Note: We have graphed the function $y = x^2$ over the interval $-2 \leq x \leq 2$. What about the rest of the graph?

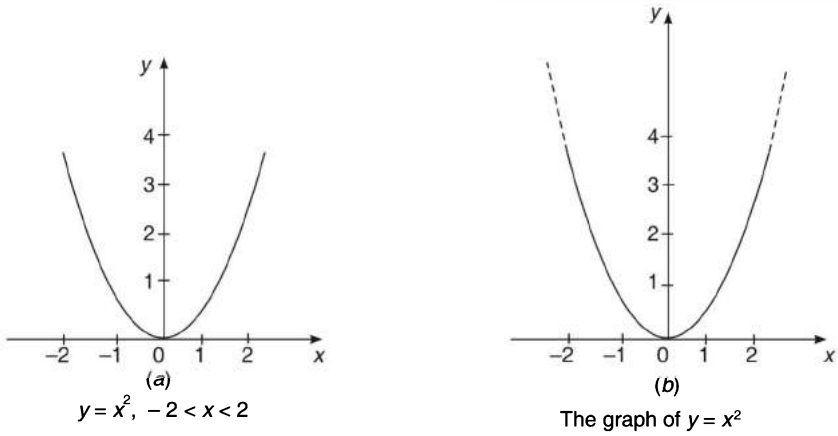


Fig. 2.43

The domain and range of $y = x^2$ are both infinite, so we cannot hope to draw the entire graph. But we can imagine what the graph looks like by examining the formula $y = x^2$ and by looking at the picture we already have. As x moves away from the interval $-2 \leq x \leq 2$ in either direction $y = x^2$ increases rapidly. When x is 5, y is 25. When x is 10, y is 100. Continuing the pattern we see that the graph goes up as shown in Fig. 2.43.

The basic idea for graphing curves that are not straight lines is to plot points until we see the curve's shape, and then to use the formula for y to find out how y changes as x moves between or away from the plotted values. But what points should we plot? Here are some guidelines about choosing good points to plot.

Guidelines for choosing points for graphing $y = f(x)$

1. Plot any points where the graph crosses or touches the axes. These points are often easy to find by setting $y = 0$ and $x = 0$ in the equation $y = f(x)$.

2. Plot a few points near the origin. When the values of x are small, the values of y are often easy to compute or estimate.

3. Graph the function at or near any end points of its domain.

2.29. Absolute function

Ex. 62. The absolute value function $y = f(x) = |x|$.

Sol. The domain consists of all real numbers, but the range is made up of all non-negative real numbers. The graph is shown in Fig. 2.44. When $x \geq 0$, $y = |x|$ is equivalent to $y = x$, the equation of the straight line through the origin with slope 1. When $x < 0$, $y = |x|$ is equivalent to $y = -x$, the equation of the straight line through the origin with slope – 1.

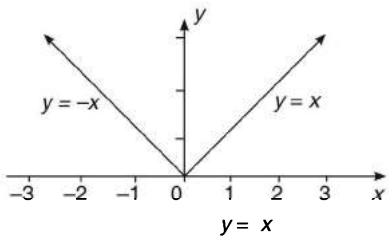


Fig. 2.44

Ex. 63. Draw the graph of the function $f(x) = |x - 1|$.

Sol. The domain is the set of all real numbers. The range is the set of all non-negative real numbers. The graph is the graph of $y = |x|$ shifted one unit to the right.

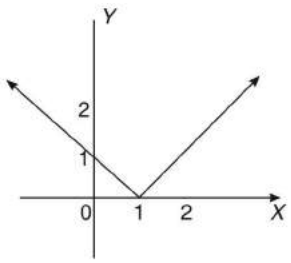


Fig. 2.45. The graph of $f(x) = |x - 1|$

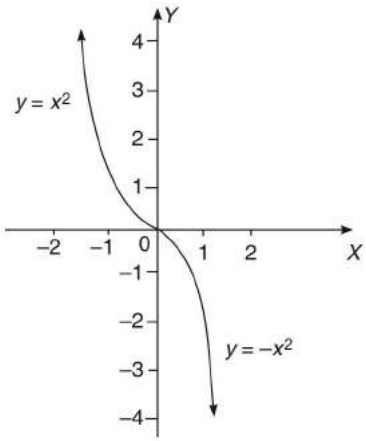


Fig. 2.46. Graph of $f(x) = -x\,|x|$

Ex. 64. Draw the graph of the function $f(x) = -x\,|x|$.

Sol. If $x > 0$, then $y = f(x) = -x.x = -x^2$, *i.e.*, $y = -x^2$, for $x > 0$ ... (1)

If $x = 0$, then $y = f(x) = 0$, *i.e.*, $y = f(x) = 0$, for $x = 0$ ... (2)

If $x < 0$, then $y = f(x) = (-x)(-x) = x^2$, *i.e.*, $y = f(x) = x^2$ for $x < 0$ ... (3)

From (*i*), as $x \to \infty, y = -\infty$ for $x > 0$;

From (*iii*) as $x \to -\infty, y \to \infty$ for $x < 0$; From (*ii*), $y = f(x) = 0$

The figure is drawn as Fig. 2.46.

Ex. 65. Draw the graph of the following functions.

(*i*) $f(x) = \dfrac{|x|}{x}$ **(*ii*)** $f(x) = |x| + |x - 1|$

Sol. (*i*) The domain is the set of all non-zero real numbers. The graph is the right half of the line $y = 1$ for $x > 0$, plus the left half of the line $y = -1$ for $x < 0$. The range is $\{1, -1\}$. The graph is not defined for $x = 0$

(*ii*) **Case 1.** $x \geq 1$. Then $f(x) = x + x - 1 = 2x - 1$.

Case 2. $0 \leq x < 1$. Then $f(x) = x - (x - 1) = 1$.

Case 3. $x < 0$. Then $f(x) = -x - (x - 1) = -2x + 1$.

So, the graph consists of a horizontal line segment and two half-lines.

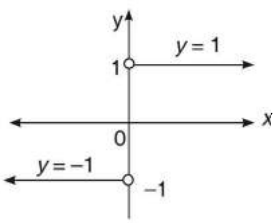


Fig. 2.47. The graph of $f(x) = \dfrac{|x|}{x}$

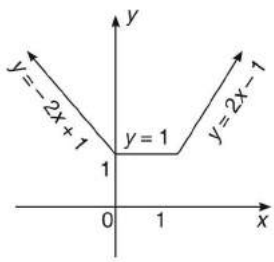


Fig 2.48. The graph of $f(x) = |x| + |x - 1|$

Ex. 66. Draw the graph of the function

$$y = |x - 2| + |x - 3|.$$

Sol. $y = |x - 2| + |x - 3|$

Here, we consider separate intervals.

When $x > 3$, $x - 2$, and $x - 3$ both will be positive, therefore,

$y = x - 2 + x - 3 \Rightarrow \mathbf{y = 2x - 5}$. Some points of the graph are

x	3	4	6
y	1	3	7

When $x = 3$, $y = |3 - 2| + |3 - 3| = 1$

When $x = 2$, $y = |2 - 2| + |2 - 3| = 1$

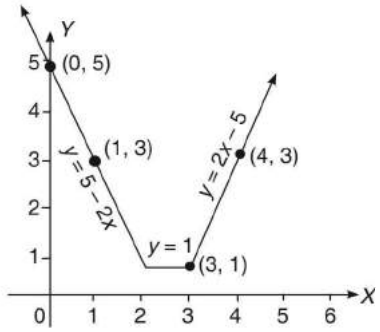


Fig. 2.49

and if $2 < x < 3$ then $x - 2$ will be positive and $x - 3$ will be negative, thus,

$$y = (x - 2) - (x - 3) = 1$$

$\therefore$ **For $2 \le x \le 3$, $y = 1$** (a line parallel to the x-axis)

Now, if $x < 2$, in this case both $(x - 2)$ and $(x - 3)$ will be negative and, therefore,

$$y = -(x - 2) - (x - 3) = 5 - 2x$$

Some points on the line **$y = 5 - 2x$ for $x < 2$** are

x	–1	0	1	2
y	7	5	3	1

The graph is as shown in Fig. 2.49.

2.30. The greatest integer function (or step function)

Ex. 67. Draw the graph of the function $y = [x]$ in $-2 < x < 4$.

Sol. $y = [x]$. Here, the value of y is the integer value of x. Some points of the graph are

x	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x < 2$	$2 \le x < 3$	$3 \le x < 4$
y	–2	–1	0	1	2	3

i.e.,
$$f(x) = \begin{cases} -2, \text{ when } x \in [-2, -1) \\ -1, \text{ when } x \in [-1, 0) \\ 0, \text{ when } x \in [0, 1) \\ 1, \text{ when } x \in [1, 2) \\ \text{and so on} \end{cases}$$

The required graph is shown in Fig. 2.50. Clearly, the function has jumps at the points $(-1, -2)$, $(0, -1)$, $(1, 0)$, $(2, 1)$ etc. In other words, the given function is discontinuous at each integral value of x.

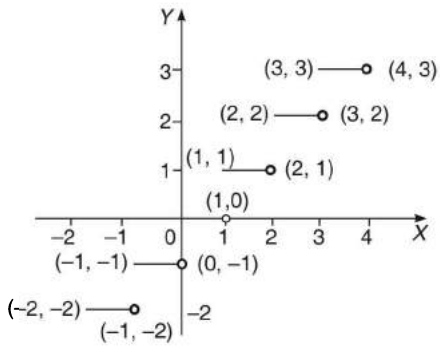


Fig. 2.50. The graph of $y = [x]$

2.31. Exponential function

Ex. 68. Draw the graph of $y = 2^x$.

Sol. Here, we see that as $x \to \infty$, $y \to \infty$ and when $x \to -\infty$, $y \to 0$.

Some points of the graph are

x	–2	–1	0	1	2	3	4
y	0.25	0.5	1	2	4	8	16

The required graph is as shown.

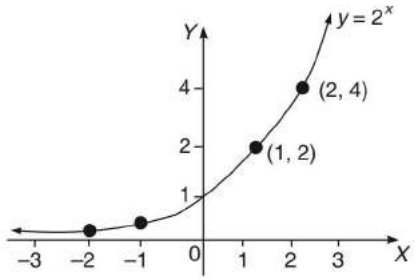


Fig. 2.51. The graph of $y = 2^x$.

Note: You can round $\frac{1}{8} = 0.125$ as 0.1 and $\frac{1}{4} = 0.25$ as 0.3 and draw the graph of $y = 2^x$ for $-3 \le x \le 4$ as shown below.

x	–3	–2	–1	0	1	2	3	4
2^x	0.1	0.3	0.5	1	2	4	8	16

Any function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an exponential function with base a. The graph of the function has equation $y = a^x$.

2.32. To draw the graph of $y = a^x$

Before we draw the graph, we should keep in mind the following facts.

(a) $\log_a x$ is defined for positive values of x only and base a is always positive.

(b) For all values of a, $\log_a 1 = 0$ since $a^0 = 1$.

(c) When $a > 1$, $\log_a \infty = \infty$, since $a^\infty = \infty$; and $\log_a 0 = -\infty$, since $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$.

(d) When $0 < a < 1$, $\log_a \infty = -\infty$, since $a^{-\infty} = \left(\frac{1}{a}\right)^\infty = \infty$ as $\frac{1}{a} > 1$ and $\log_a 0 = \infty$, since

$$a^\infty = \left(\frac{1}{a}\right)^{-\infty} = 0 \text{ as } \frac{1}{a} > 1.$$

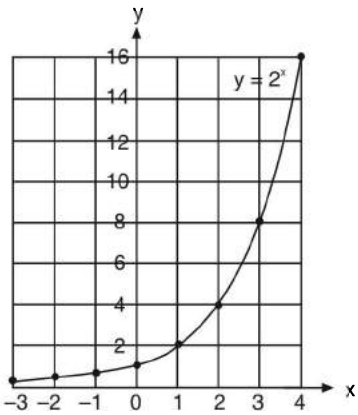


Fig. 2.52

(e) For all positive values of a, a^x is positive, whether x is positive or negative, and a^x never vanishes for finite values of x.

Now we shall draw the graph of a^x, *i.e.*, of $y = a^x$.

Case I. Let $a > 1$. $y = a^x$.

(i) The curve does not pass through the origin.

(ii) It cuts the x-axis when $y = 0$ *i.e.*, $0 = a^x$ only when $x = -\infty$.

$\therefore$ The curve meets the negative direction of the x-axis at infinity,

i.e., $y \to 0$ as $x \to -\infty$

(iii) It cuts the y-axis when $x = 0$.

$\therefore$ $y = a^0 = 1$.

$\therefore$ The curve meets the y-axis at (0, 1).

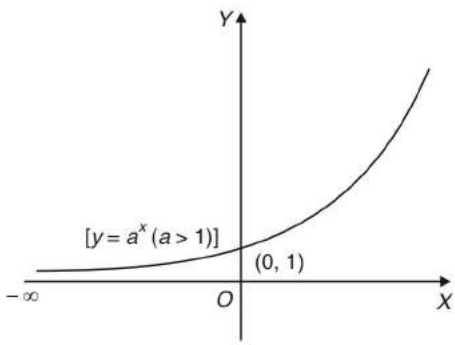


Fig. 2.53

(*iv*) a^x, *i.e.*, y is positive whether x is positive or negative. In other words, the curve lies only in the first and the second quadrants, *i.e.*, above the x-axis.

(*v*) As x increases, y increases, *i.e.*, y is an increasing function of x,

i.e., $y \to \infty$ as $x \to \infty$

Hence, the curve is drawn in Fig. 2.53.

Cor. The above holds good for the function e^x also as $e > 1$.

Case II. Let $0 < a < 1$. $y = a^x$.

(*i*) The curve does not pass through the origin.

(*ii*) It cuts the x-axis when $y = 0$, *i.e.*, $x = \infty$.

$\therefore$ The curve meets the positive direction of the x-axis at infinity, *i.e.*,

$y \to 0$ as $x \to \infty$,

(*iii*) It cuts the y-axis when $x = 0$, *i.e.*, $y = 1$

$\therefore$ The curve meets the y-axis at (0, 1).

(*iv*) a^x is positive, whether x is positive or negative. In other words, the curve lies only in the first and the second quadrants, *i.e.*, above the x-axis.

(*v*) As x increases, y decreases, *i.e.*, y is a decreasing function of x, *i.e.*, $y \to 0$ as $x \to \infty$,

i.e., $y \to \infty$ as $x \to -\infty$

Hence, the curve is drawn as shown in Fig. 2.54.

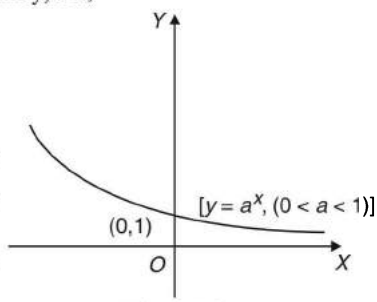
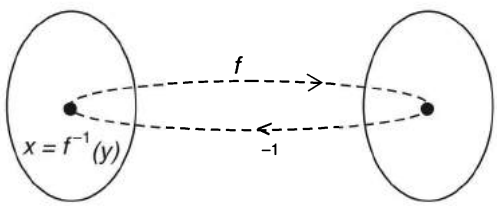


Fig. 2.54

Note. In the case of the function a^x it is the exponent 'x' which is a variable and the base 'a' is a constant. This is the justification of calling a^x as "exponential function". For the function x^n, the base is a variable and the exponent 'n' is a constant. This is called algebraic function.



2.33. Logarithmic function

Look at the graph $y = a^x$ of the exponential function $f(x) = a^x, a > 1$, on the right.

For each real number x, y is a positive real number. Conversely, for each positive real number y there is exactly the real number x such that $y = a^x$.

$f(x)$ sets up a one-to-one correspondence between the set of real numbers on the x-axis (the domain) and the set of positive real numbers on OY (the range).

You know that

$$\boxed{y = a^x \Leftrightarrow x = \log_a y}$$

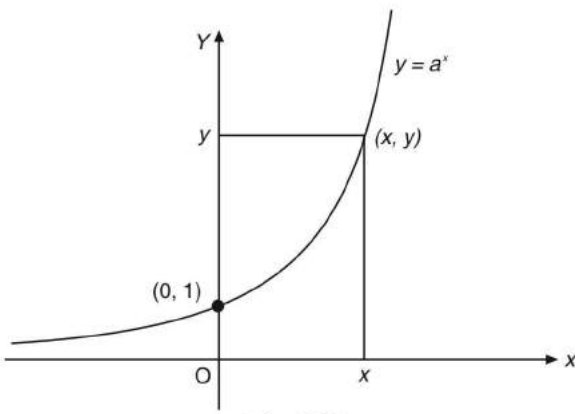


Fig. 2.55

Notice that the graph $y = \log_a x$ is the mirror image *of* $y = a^x$ in the line $y = x$, as shown.

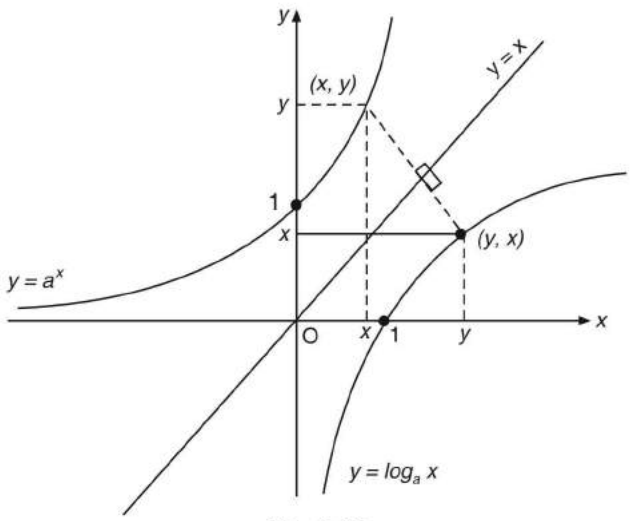


Fig. 2.56

Two special logarithms

Using the form

$y = a^x \Rightarrow \log_a y = x$

(*i*) $1 = a^0 \Rightarrow \log_a 1 = 0$

(log 1 to any base = 0)

(*ii*) $a = a^1 \Rightarrow \log_a a = 1$, (log of a number to that base = 1).

Using $\log_a 1 = 0$ and $\log_a a = 1$, we can obtain two points on the graph of $y = \log_a x$.

If $x = 1, y = \log_a 1 = 0$, giving (1, 0).

If $x = a, y = \log_a a = 1$, giving $(a, 1)$.

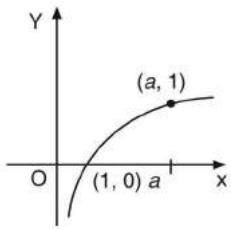


Fig. 2.57

Note. The **domain of the logarithmic function** is the set of all positive real numbers and **the range** is the set R of all real numbers. Graph of this function is as shown.

If $x > 1$, as $x \to \infty, f(x) \to \infty$

If $0 < x < 1$, as $x \to \infty$, $f(x) \to -\infty$

If $x = 1, f(x) = 0$

$\therefore$ Curve meets the x-axis at (1, 0).

$\therefore$ Range of f is $(-\infty, \infty)$.

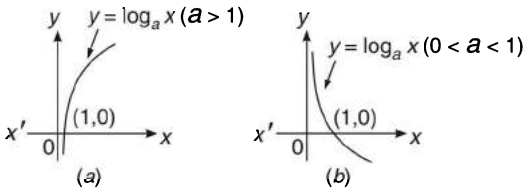


Fig. 2.58

Ex. 69. Draw the graph of $y = \log_2 x$.

Sol. We know that,

$y = \log_2 x \Rightarrow x = 2^y$

Therefore, for different values of y, we can get the corresponding values of x as follows.

x	1	2	4	8	16	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$	$\frac{1}{8} = 0.125$
y	0	1	2	3	4	–1	–2	–3

Plotting these points we can draw the required graph

Also, as $x \to \infty, y \to \infty$ and as $x \to 0, y \to -\infty$

Note. You may round off 0.25 to 0.3 and 0.125 to 0.1 for plotting.

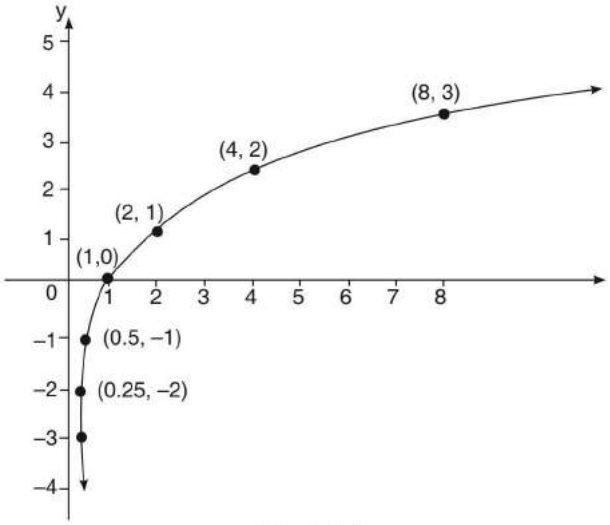


Fig. 2.59

2.34. Sketching of (i) e^x (ii) e^{-x} (iii) e^{2x}

(*i*) We know that $2 < e < 3$. Thus this is a particular case of a^x where $a > 1$. We note the following points:

1. The function e^x is always positive, whether x is positive or negative, *i.e.,* the graph lies above the x-axis.

2. As x increase, e^x increases, therefore e^x is an increasing function of x.

As $x \to \infty, y \to \infty$; as $x \to -\infty, y \to 0$

3. If $x = 0, y = 1$, therefore the graph cuts the y-axis at (0, 1). Hence, the graph is drawn as shown in Fig. 2.60.

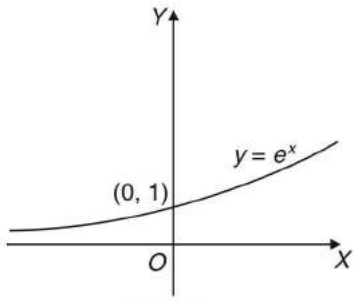


Fig. 2.60

(*ii*) Sketching of e^{-x}, $y = e^{-x} = \dfrac{1}{e^x}$. We note the following points :

1. y or $\dfrac{1}{e^x}$ is always positive, whether x is positive or negative, *i.e.,* the graph lies above the x-axis.

2. As x increases, e^x increases, but $\dfrac{1}{e^x}$ decreases, therefore $\dfrac{1}{e^x}$ is a decreasing function of x.

   As $x \to \infty, \dfrac{1}{e^x} \to 0$; as $x \to -\infty, \dfrac{1}{e^x} \to \infty$, *i.e.*, $y \to \infty$.

3. If $x = 0, y = 1$, therefore the graph cuts the y-axis at (0, 1). Hence, the graph is drawn as shown in Fig. 2.61.

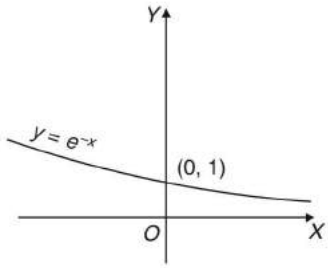


Fig. 2.61

(*iii*) The graph of $y = e^{2x}$ is also similar to the graph of $y = e^x$. This graph also cuts the y-axis at (0,1). For positive values of x, the value of e^{2x} is greater than the values of e^x. For negative values of x, the value of e^{2x} is less than the value of e^x. Therefore in the first quadrant the graph of e^{2x} is above the graph of e^x, and in the second quadrant the graph of e^{2x} is below the graph of e^x.

Hence, all the three graphs, with the same axes, are drawn in Fig. 2.62.

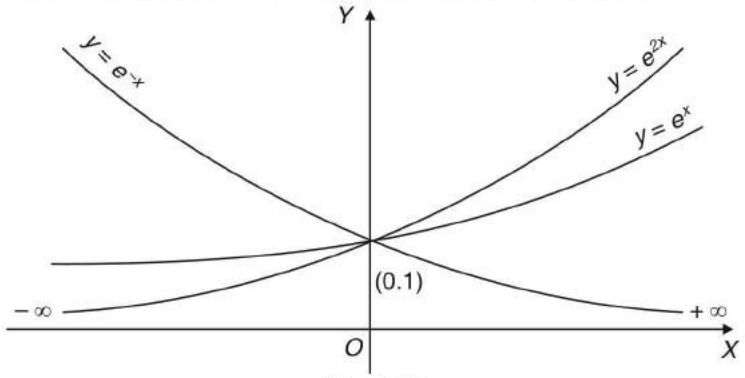


Fig. 2.62

2.35. Rational functions

A function that can be expressed as a ratio of two polynomials is called a *rational function*. If $P(x)$ and $Q(x)$ are polynomials, then the ratio $\frac{P(x)}{Q(x)}$ defines a rational function. Although a polynomial function is defined for every real number x, a rational function is defined only when the denominator is non-zero, thus, the domain of $f(x) = \frac{P(x)}{Q(x)}$ consists of all values of x such that $Q(x) \neq 0$.

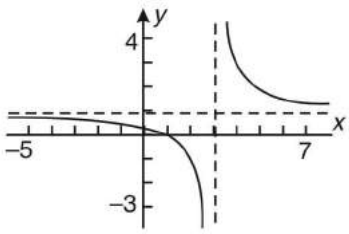
For example, consider the rational functions

(*i*) $g(x) = \frac{x^2 - 1}{x^2 - 2x - 3}$ (*ii*) $h(x) = \frac{3x + 2}{x^2 + 1}$

(*i*) The domain of $g(x) = \frac{x^2 - 1}{x^2 - 2x - 3}$ is the set of all real numbers except when $x^2 - 2x - 3 = 0$ which gives $x = -1$ and $x = 3$. Therefore, the domain of g is the set of all real numbers except $x = -1$ and $x = 3$.

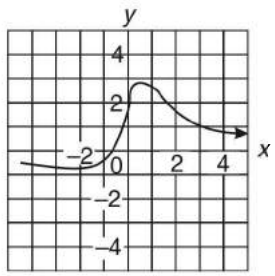
(*ii*) The domain must exclude values of x for which $x^2 + 1 = 0$. It is not possible for $x^2 + 1 = 0$, because $x^2 \geq 0$, and so $x^2 + 1$ cannot be equal to zero. Therefore, there are no real numbers that must be excluded from the domain of f. Hence, the domain is $\{x | x \in$ real numbers$\}$.

The graphs of the above two functions are shown below:



$y = \frac{x^2 - 1}{x^2 - 2x - 3}$

Fig. 2.63



$y = \frac{3x + 2}{x^2 + 1}$

Fig. 2.64

Special features of rational graphs

- Unlike graphs of polynomials which are continuous (unbroken) curves, there are breaks in the graph of a rational function wherever the function is not defined, that is, at the zeros of the denominator. Except for breaks, the graph is a continuous unbroken curve. Additionally, the graph has no sharp corners.
- **Asymptote.** A line that a function gets increasingly close to but never meets is called an asymptote. thus, for graph of a^x, the x-axis is an asymptote. (Fig. 2.55 Art. (2.33) and for graph of $y = \log_a x$, y-axis is an asymptote (Fig. 2.56 and 2.57 Art 2.33). We may also say that an asymptote is a line associated with a curve such that as a point moves along a branch of the curve, the distance between the line and the curve approaches zero.
- One very common feature of graphs of rational functions is asymptote. The graph may have *vertical asymptotes* and *horizontal asymptotes*.

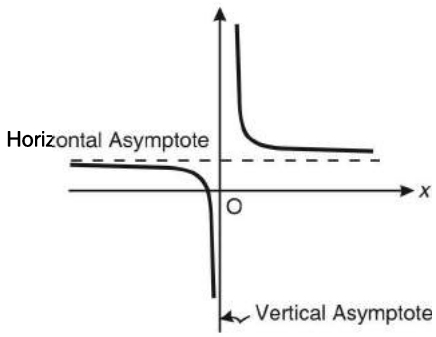


Fig. 2.65

Ex. 70. Draw the graph of the function $y = \frac{1}{x}$ for x: $-4 \le x \le 4$.

Sol. We first form the following table for values in the interval $-4 \le x \le 4$.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Using a scale of 1 cm to represent 1 unit on both axes we draw the graph as under:

In the graph of $y = \frac{1}{x}$, which is called a **reciprocal function,** we see the curve getting closer and closer to the y-axis, but since the function is not defined at $x = 0$ (dividing by 0 being not permissible), the graph would never touch the y-axis.

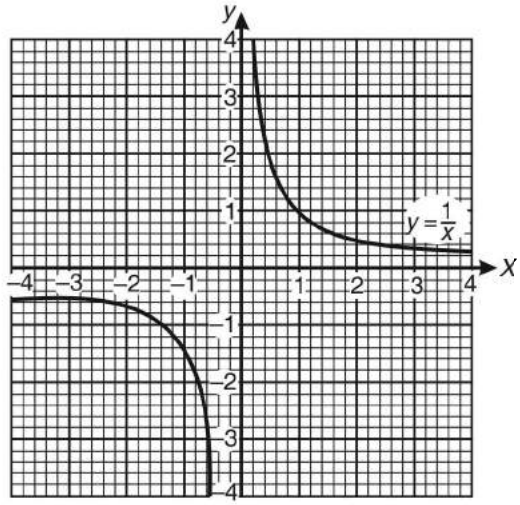
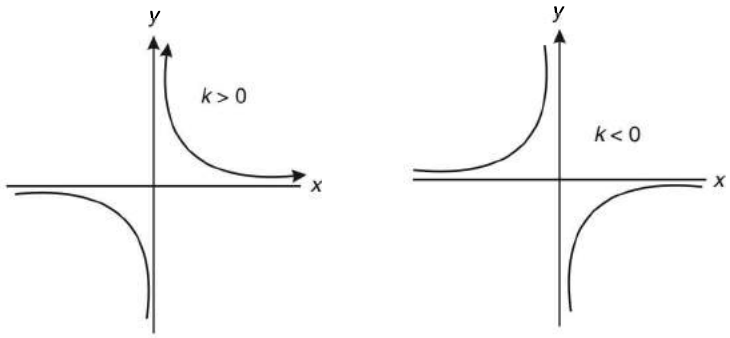
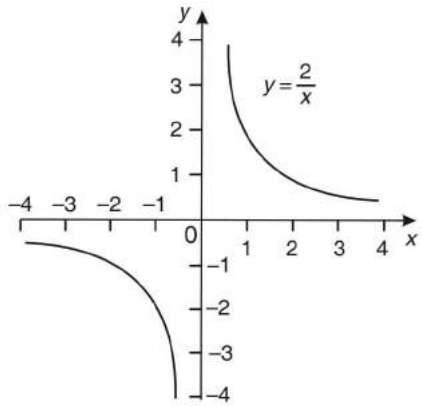


Fig. 2.66

A reciprocal function has the form $f(x) = \frac{k}{x}$. The domain of f is $x \neq 0$ and the range is $y \neq 0$. The graph of $f(x)$ is a hyperbola.



Reciprocal graphs can be written in the form $y = \frac{a}{x}$ or $xy = a$ (where a is constant). They all have approximately the same basic shape as shown in the example below.



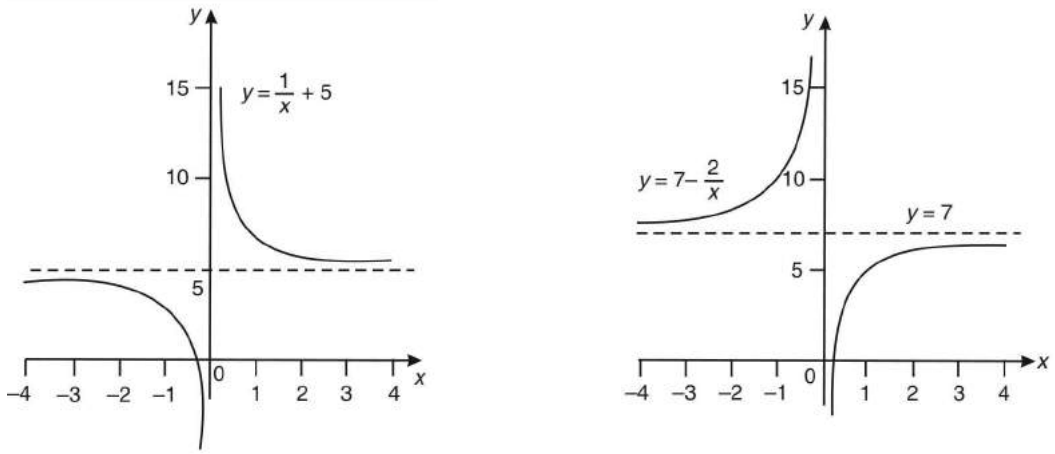


Fig. 2.67

Ex. 71. The graph of $xy = 12$ or the same as $y = \frac{12}{x}$ or $\left(x = \frac{12}{y}\right)$ is drawn below. We need to use non-integer values between 0 and 1 and 0 and – 1 to find out sufficient detail as to be able to draw its graph.

Table of values

x	– 4	– 3	– 2	– 1	1	2	3	4
$y=\frac{12}{x}$	– 3	– 4	– 6	– 12	12	6	4	3

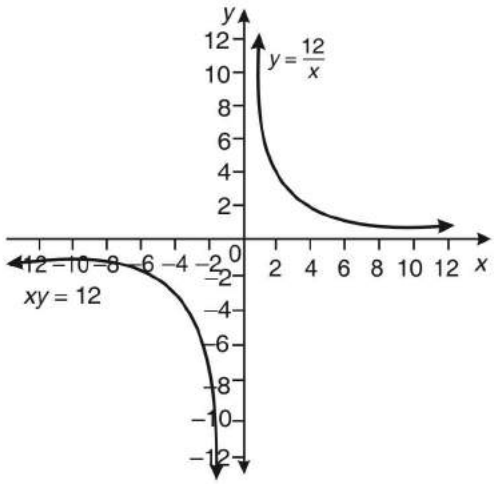


Fig. 2.68

Ex. 72. Graph the function $xy – y – x – 2 = 0$.

Sol. $xy – y – x – 2 = 0 \Rightarrow y = \frac{x+2}{x-1}$, $x = \frac{y+2}{y-1}$. The function exists for $x \neq 1$ and for $y \neq 1$. The

asymptotes are $x = 1$ (vertical asymptote) and $y = 1$ (horizontal asymptote). The x-intercept is -2 and the y-intercept is -2. The table of values is

x	-10	-4	-3	-2	-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{3}{2}$	2	10
y	$\frac{8}{11}$	$\frac{2}{5}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	-2	-5	-11	13	7	4	$\frac{4}{3}$

To draw the graph, first draw in the asymptotes $x = 1$ and $y = 1$ (dotted lines). Though the asymptotes are not a part of the graph, they serve as very convenient guide lines. Since the curve does not exist for $x = 1$ and $y = 1$, it does not cross the asymptotes. Since there is one value of y for each value of $x \neq 1$, that is, since y is single valued, the graph appears in but two of the form regions into which the plane is separated by the asymptotes. From the table it is evident that the curve lies in the region to the right of the vertical asymptote and above the horizontal asymptote refer to the portion of the table to the right of the double line and in the region to the left of the vertical asymptote and below the horizontal asymptote (refer to the portion of the table to the left of the double line). It is seen in the graph, it is symmetric with respect to (1, 1) the point of intersection of the asymptotes.

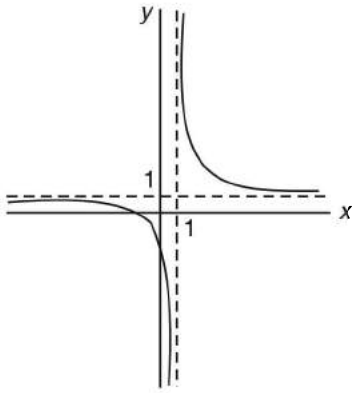


Fig. 2.69

EXERCISE 2 (*g*)

Graph each function.

1. $y = -|x|$ **2.** $y = \frac{|x| - x}{2}$ **3.** $y = \frac{1}{|x|}$ **4.** $y = |4 - x^2|, -3 \leq x \leq 3.$

5. $y = |x| + x, -2 \leq x \leq 2$ **6.** $y = |x + 2| + x$

7. (*i*) Copy and complete this table of values :

x	-2	-1	0	1	2	3
3^x	0.1	0.3	1			

(*ii*) Draw the graph $y = 3^x$ on squared paper, for $-2 \leq x \leq 3$.

(*iii*) What features do the graphs of $y = 2^x$ (drawn on page Ch 2-51) and $y = 3^x$ have in common?

8. (*i*) Copy and complete this table.

x	-3	-2	-1	0	1	2	3
$\left(\frac{1}{2}\right)^x$	8			1		0.3	0.1

(*ii*) Draw the graphs $y = 2^x$ (See Solved Ex. 68) and $y = \left(\frac{1}{2}\right)^x$, on the same diagram, for $-3 \leq x \leq 3$.

(*iii*) Which line is the axis of symmetry in the diagram?

9. A sketch of the graph $y = a \log_4 (x + b)$ is shown in Fig. 2.70. Find the values of a and b.

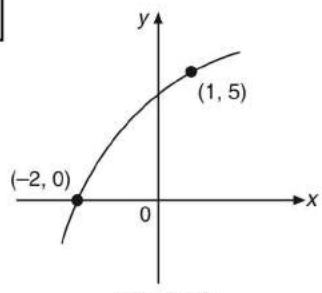


Fig. 2.70

10. Diagram (*i*) shows the curve $y = \log_a x$. What is the value of a?

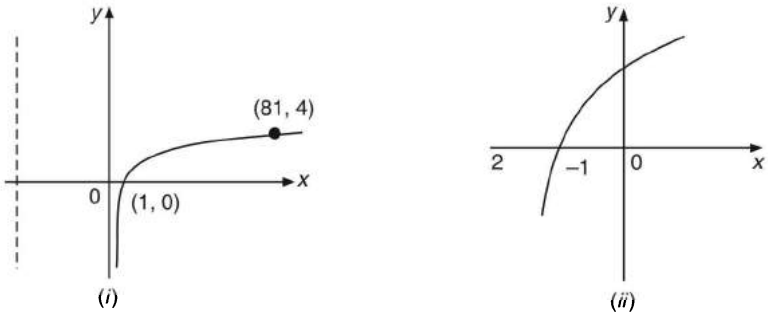


Fig. 2.71

11. Diagram 2.71 (*ii*) shows the curve $y = \log_{10}(x + p)$. What is the value of p?

12. (*i*) Sketch the graphs $y = 2$ and $y = \log_{10} 2x$ on the same diagram.

(*ii*) Find the point of intersection of the graphs by solving the equation $\log_{10} 2x = 2$.

13. The sketch shows part of the graph $y = a \log_2 (x - b)$. [Fig. 2.72].

Find the values of a and b.

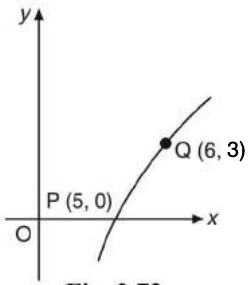


Fig. 2.72

14. (*i*) Sketch the graphs $y = 4 - x$ and $y = \log_{10} x$ on the same diagram.

(*ii*) Write down an equation to find the x-coordinate of the point of intersection of the graphs.

(*iii*) Show that x satisfies $3.4 < x < 3.5$.

(*iv*) From your graph find x, correct to 2 decimal places.

15. Sketch the graphs.

(*i*) $y = \log_2 x$ (*ii*) $y = \log_2 x + 1$ (*iii*) $y = \log_2 (x + 1)$

16. Sketch the graphs.

(*i*) $\log_4 x$ (*ii*) $2 \log_4 x$ (*iii*) $3 \log_4 x$

17. For $-2 < x < 1$, draw the graph of $y = 2^x$. (use 1 cm = 1 unit on both axes).

Use this graph to solve $2^x = 2x$.

18. (*i*) Complete the following table for $y = 4^x$. Enter the values of x and y correct to 1 decimal place.

x	–2	–1	...	0.5	0.75
y	...	...	1	...	...

Copy the table on your answer book and enter the values there.

(*ii*) Taking 4 cm = 1 unit on both axes, draw the graph of $y = 4^x$ for $0.75 \geq x \geq -2$

(*iii*) From your graph estimate $\log_4 1.25$. **[*ISC 1991*]**

19. Copy and complete the table, for the function $y = \frac{3}{x}$, giving your answer correct to 1 d.p. Then draw the graph.

x	–3	–2.5	–2	–1.5	–1	–0.5	0.5	1	1.5	2	2.5	3
y	–1	–1.2	...	...	–3	...	6	...	...	1.5	...	...

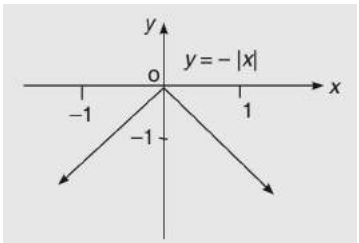
20. Sketch the graphs of the following rational functions

$(i)\ y = \dfrac{x+3}{x-2}$ $(ii)\ y = \dfrac{6}{x-6}$ $(iii)\ y = \dfrac{5}{2x+1}$

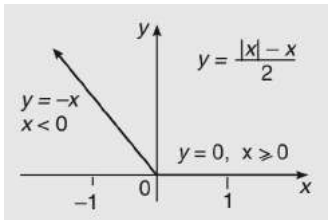
$(iv)\ y = \dfrac{2x+1}{x-3}$ $(v)\ y = \dfrac{7-2x}{3x+5}$

ANSWERS

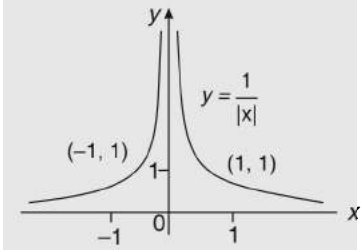
1.



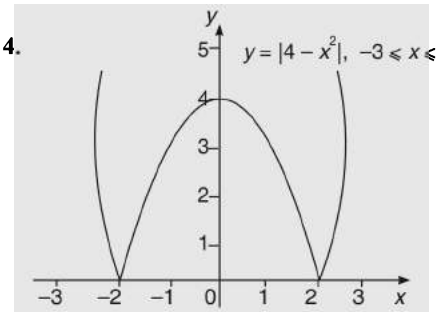
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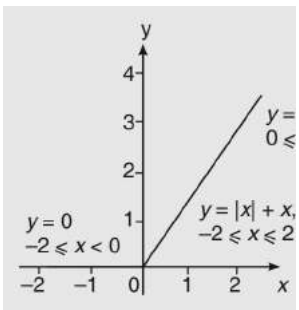
3.



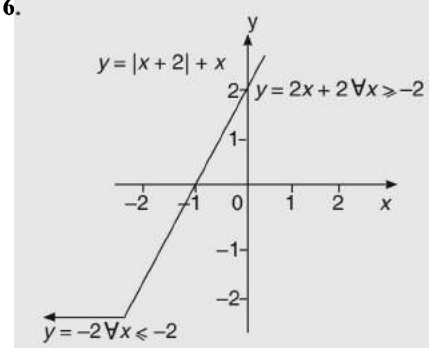
4.



5.



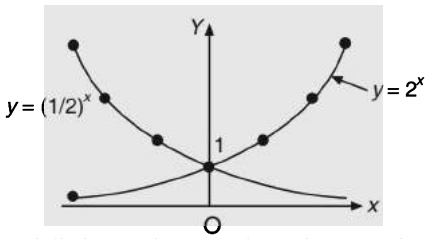
6.



7. (i) 3, 9, 27

(ii) They pass through (0, 1), rise steadily from left to right, have the x-axis as the tangent at infinity, and $y \to \infty$ as $x \to \infty$.

8. (i)



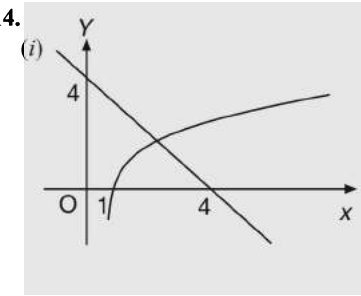
9. (i) $a = 5, b = 3$

(ii) Both lie in Quadrant I and Quadrant II only and both pass through the point (0,1).

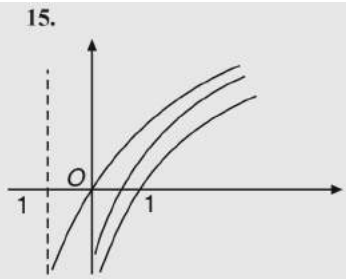
(iii) the y-axis.

10. 3 **11.** 2 **13.** $a = 3, b = 4$

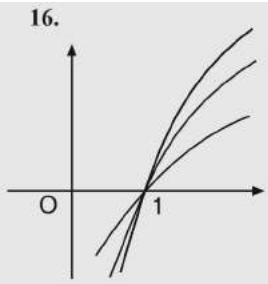
14.



15.



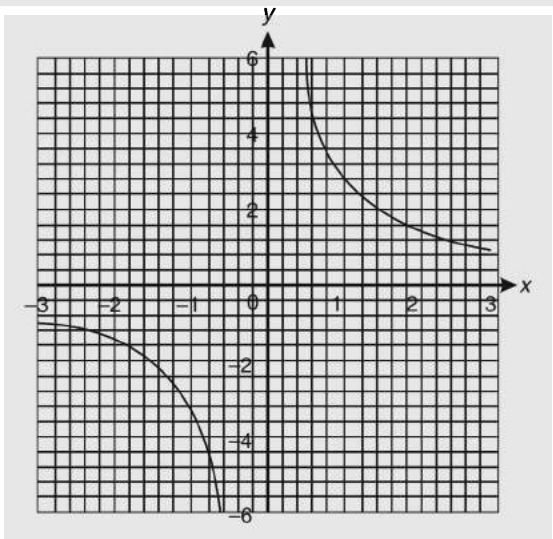
16.



(*ii*) $4 - x = \log_{10} x$ (*iv*) 3.46

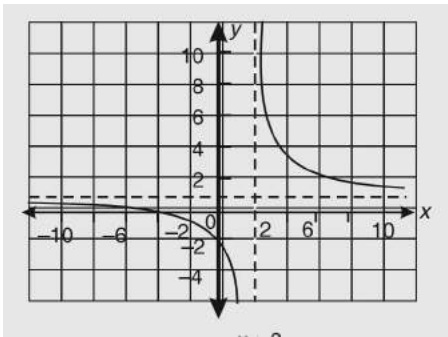
19. Table of values for the function $y = \dfrac{3}{x}$

x	-3	-2.5	-2	-1.5	-1	-0.5	0.5	1	1.5	2	2.5	3
y	-1	-1.2	-1.5	-2	-3	-6	6	3	2	1.5	1.2	1



Graph of $y = \dfrac{3}{x}$

20. (*i*) The y-intercept is when $x = 0 \Rightarrow y = -\dfrac{3}{2}$. Also, the x-intercept is when $y = 0 \Rightarrow k = -3$. The vertical asymptote is $x = 2$.

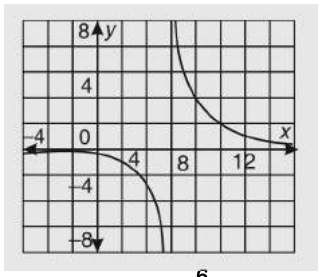


Graph of $y = \dfrac{x+3}{x-2}$

(*ii*) **Hint:** Table of values is

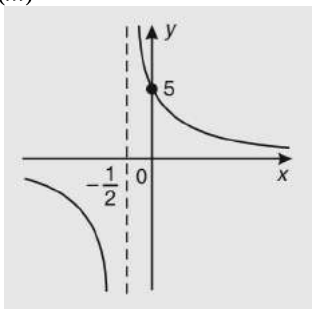
x	-6	0	5	6	7	12
y	-0.5	-1	-6	undefined	6	1

Plot the points and connect them with smooth curves.



Graph of $y = \frac{6}{x-6}$

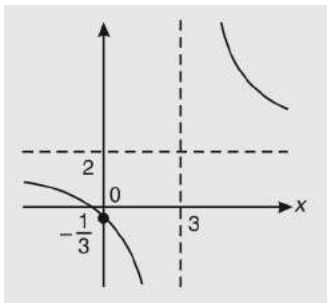
(*iii*)



Graph of $y = \frac{5}{2x+1}$

Vertical asymptote is $y = -\frac{1}{2}$ and y-intercept = 5

(*iv*)



Graph of $y = \frac{2x+1}{x-3}$

Graph of $y = \frac{5}{2x+1}$

Horizontal asymptote is at $y = 2$ and vertical asymptote is $x = 3$.

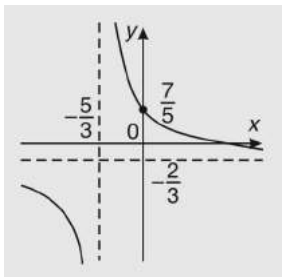
The y-intercept is $y = -\frac{1}{3}$

(*v*) Graph of $y = \frac{7-2x}{3x+5}$

Vertical asymptote at $x = -\frac{5}{3}$

Horizontal asymptote at $y = -\frac{2}{3}$

y-intercept is $y = \frac{7}{5}$.



CHAPTER 3

ANGLES AND ARC LENGTHS

3.01. What is trigonometry?

The word trigonometry is derived from two Greek words "trigonon", meaning a triangle and "metron" meaning measurement. Thus, the word literally means the science, which deals with the measurement of triangles. The early applications of Trigonometry were to astronomy, surveying, navigation and engineering. In modern times, it finds wide application in warfare, all sorts of vibratory phenomena, sound, light, electricity, etc.

Historical Note.

Hipparchus is known as the founder of trigonometry. He was a Greek astronomer born about 180 B.C. in northern Asia Minor. The subject was developed by him as a tool to help in his astronomical work.

3.02. Angles

In geometry, an angle is defined as the inclination of two straight lines, which meet but are not in the same straight line. Thus, in geometry, an angle is always less than two right angles. In trigonometry, we need a broader definition of an angle. **An angle is thought of as traced out by the rotation of a line (called the revolving line) from the initial position OX to a terminal position OP.** Then O, is the vertex, OX is called the initial side, and OP is called the terminal side of the angle. [Fig. 3.01]

3.03. Convention of signs of angles

It is customary to regard the angle traced out by a *counterclockwise* rotation as positive and one traced out by a *clockwise* rotation as negative (Fig. 3.02). The angle is indicated by using an arrow, starting from the initial line and ending at the terminal line. If the angles are not marked in this manner, it will not be possible to distinguish such angles as 40°, 400°, 760°, 1120°, etc. (Fig. 3.03).

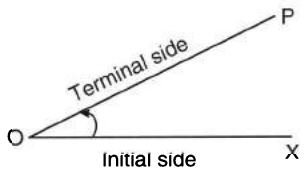


Fig. 3.01

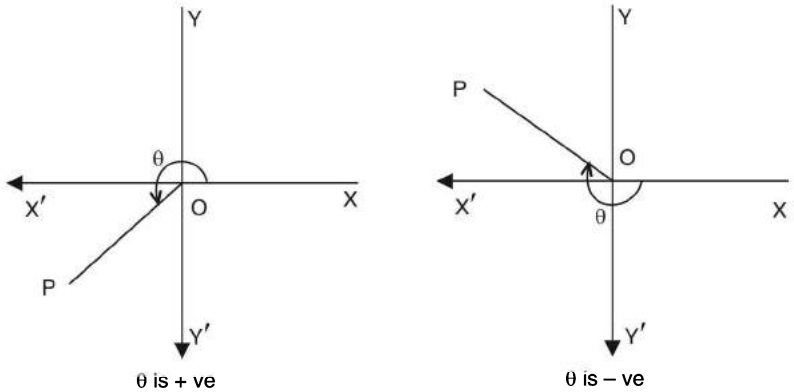


Fig. 3.02

3.04. Notation for angles

The following Greek letters are used in trigonometry to indicate in a general way the number of degrees in angles.

(1) θ (theta) (2) ϕ (phi) (3) α (alpha) (4) β (beta)
(5) γ (gamma) (6) δ (delta) (7) λ (lambda) (8) ω (omega), etc.

3.05. Magnitude of an angle

Let XOX′ and YOY′ be two mutually perpendicular lines in any plane. These lines divide the

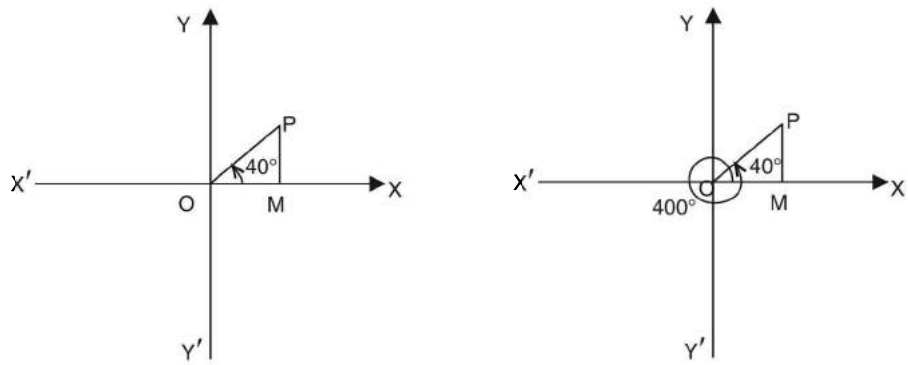


Fig. 3.03

plane into four regions, called *quadrants* numbered counter-clockwise. Thus the quadrants XOY, YOX′, X′OY′ and Y′OX are respectively called the **first quadrant,** the **second quadrant,** the **third quadrant,** and the **fourth quadrant.** The angle θ is said to be in the first, second, third or fourth

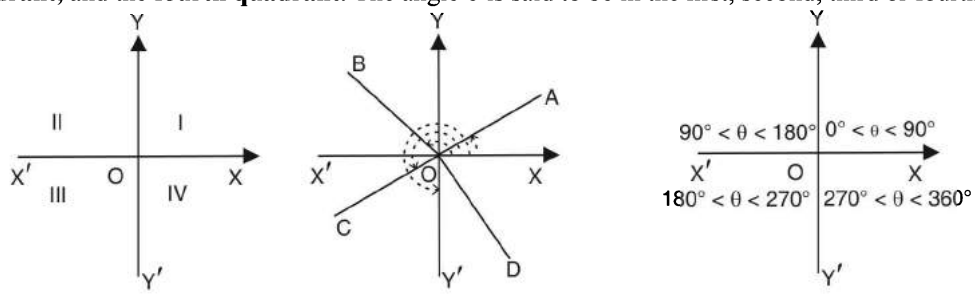


Fig. 3.04

quadrant, according as the terminal line is in the first, second, third or fourth quadrant.

As shown in the diagram, during the first revolution of the line.

1. θ will lie between 0° and 90°, if it is in the first quadrant.
2. θ will lie between 90° and 180°, if it is in the second quadrant.
3. θ will lie between 180° and 270°, if it is in the third quadrant.
4. θ will lie between 270° and 360°, if it is in the fourth quadrant.
5. θ will be more than 360°, if after making one complete revolution, the revolving line does not stop at the initial position OX but proceeds further on.

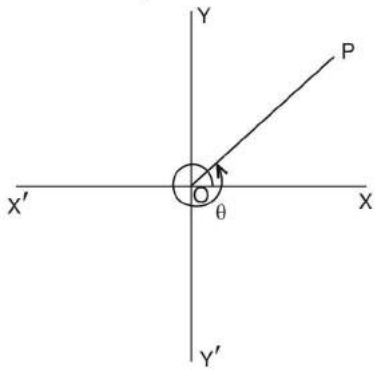


Fig. 3.05

Note. It is thus clear from the new definition of an angle, that the measure of ∠XOP will be $(360 \times n + \theta)°$ where n is the number of revolutions, which the revolving line has made about O.

Example. *Below are shown positions of the revolving line, when it has traced out the angles of 415°, – 130°, 957°, –1480°.*

How to do it ? First write as follows:

$415° = 360° + 55°$; $-130° = -90° - 40°$;

$957° = 2 \times 360° + 2 \times 90° + 57°$; $-1480° = -(4 \times 360° + 40°)$

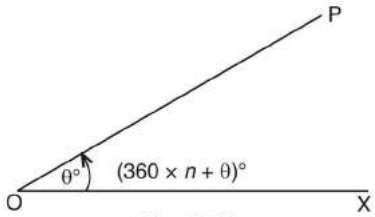


Fig. 3.06

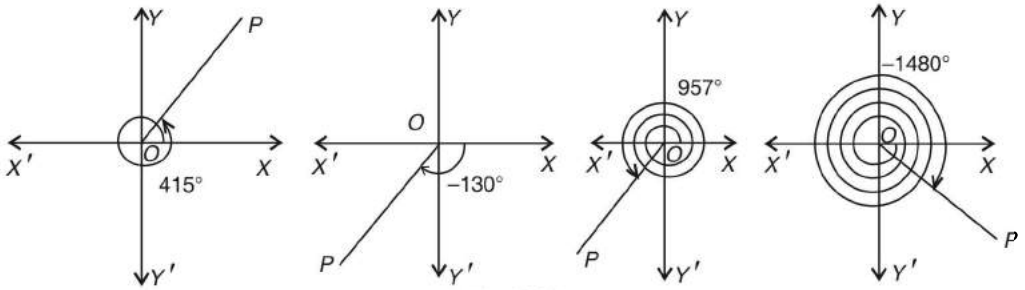


Fig. 3.07

Oral examples for class discussion

1. In what quadrants are the following angles?
   $-40°, 150°, -280°, 512°, -1872°$.
2. Draw diagrams for the following angles showing the exact position of the terminal line.
   (*i*) 125°, (*ii*) 810°, (*iii*) – 345°.
3. θ is a positive angle less than 360°. In what quadrant is $\frac{\theta}{2}$ if θ is in
   (*i*) the second quadrant, (*ii*) the third quadrant, (*iii*) the fourth quadrant?
4. Can two angles have the same initial and terminal sides and not be equal? Explain.

3.06. Measures of angles

There are two common systems of measuring angles, viz.

(1) *Sexagesimal system* (2) *Circular system*

(1) **Sexagesimal or English system.** In this system one complete revolution is divided into 360 equal parts, each called a **degree,** a degree is divided into sixty equal parts, each called a **minute** and each minute is again divided into sixty equal parts, each called a **second,** the symbols °, ′, ″ are used to express these divisions respectively. The 35° 54′ 43″ denotes the angle which contains 35 degrees 54 minutes and 43 seconds. As the subdivisions involve the number sixty, so the system is called sexagesimal. It is called English system because English people were the first to adopt it.

(2) **Circular system.** The unit of this system is a **radian.** You have learnt in plane geometry that the same circle or, in equal circles, the ratio of the lengths of any two arcs is equal to the ratio of the central angles which intercept these arcs. Thus, in Fig. 3.08,

$$\frac{\text{arc XY}}{\text{arc PQ}} = \frac{\angle XOY}{\angle POQ} = \frac{112°}{28°} = 4$$

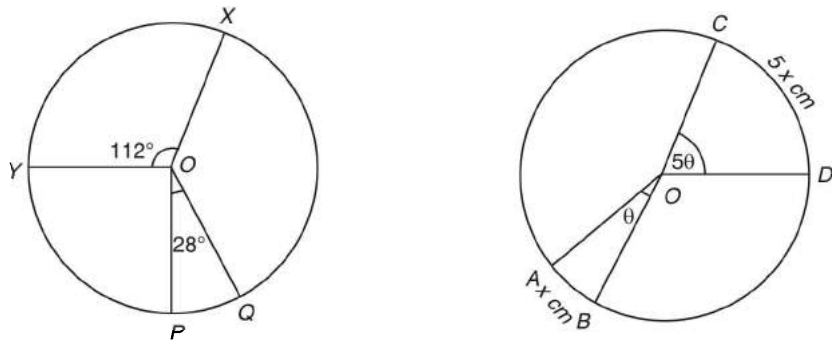


Fig. 3.08

Thus, arc XY is four times as long as arc PQ. Conversely, if arc CD is five times as long as arc AB, then $\angle$ COD will be five times as much as $\angle$ AOB.

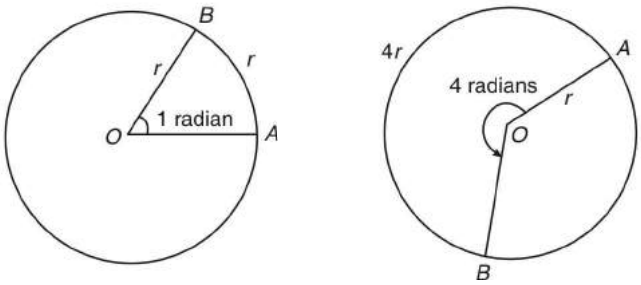


Fig. 3.09

Now, if there be an angle at the centre subtended by an arc AB whose length is equal to the radius of the circle, the **angle AOB** is called 1 **radian.**

Def. *A radian is the measure of the central angle subtended by an arc equal in length to the radius of the circle.*

An arc which is 4 times the radius of its own circle, will subtend an angle 4 radians at the centre and so on.

3.07. Circular measure – Definition

The circular measure of an angle is the number of radians it contains. Thus the circular measure of a radian is 1.

Notation. The SI symbol for radian is **rad.** When the unit in terms of which the angle is measured, is not mentioned, **radian is understood.**

3.08. Ratio of circumference of a circle to its diameter

The Circumference of a circle bears a constant ratio to its diameter. This constant ratio is denoted by the Greek letter π.

$$\frac{\text{Circumference of the circle}}{\text{Diameter of the circle}} = \pi \Rightarrow \frac{\text{Circumference}}{2r} = \pi$$

$\Rightarrow$ Circumference $= 2\pi r = \pi d$, where d is the diameter of the circle.

3.09. Relation between radians and degrees

$$\frac{\text{arc AB}}{\text{Circumference}} = \frac{1 \text{ radian}}{360°} \quad \therefore \quad \frac{r}{2\pi r} = \frac{1 \text{ radian}}{360°}$$

or $$\frac{1}{\pi} = \frac{1 \text{ radian}}{180°}$$

$\therefore$ $$1 \text{ radian} = \frac{180°}{\pi} = \text{a constant number}$$

or π **radians = 180 degrees**

For any angle, $$\frac{\text{no. of radians}}{\text{no. of degrees}} = \frac{\pi}{180}$$

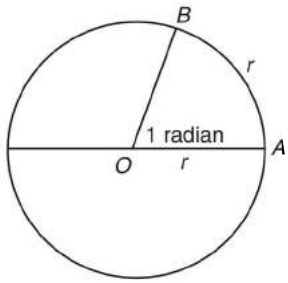


Fig. 3.10

Alternatively,

Arc equal in length to 1 radius subtends 1 rad at the centre.

Arc equal in length to 2π radius subtends 2π rad at the centre.

$\therefore$ Arc equal in length to circumference subtends 2π rad at the centre

$\therefore$ $360° = 2\pi$ rad or $\Rightarrow \pi$ rad $= 180°$

Procedure :

Hence, (1) to convert an angle in radians to its equivalent in degrees, multiply the number of radians by $\frac{180°}{\pi}$.

(2) to convert an angle in degrees to its equivalent in radians, multiply the number of degrees by $\frac{\pi}{180°}$.

Ex. 1. Express (*i*) 1 radian, (*ii*) $\frac{\pi}{3}$ radians, (*iii*) $\frac{\pi}{15}$ radians in degrees.

Sol. π rad $=180°$

$\therefore$ $1 \text{ rad} = \frac{180°}{\pi} = \frac{180°}{3.1416} = 57° 17' 45''$ (approximately) $\simeq 57.3°$;

$\frac{\pi}{3}$ rad $= \frac{180°}{\pi} \times \frac{\pi}{3} = 60°$; $\frac{\pi}{15}$ rad $= \frac{180°}{\pi} \times \frac{\pi}{15} = 12°$

Note. 1 rad $\simeq$ 57.3° and 1° $\simeq$ 0.0175 rad.

Ex. 2. Express (*i*) 45°, (*ii*) 30°, (*iii*) 9° in radians.

Sol. $180° = \pi$ radians

$\therefore$ $45° = \frac{\pi}{180°} \times 45° = \frac{\pi}{4}$ rad; $30° = \frac{\pi}{180°} \times 30° = \frac{\pi}{6}$ rad; $9° = \frac{\pi}{180°} \times 9° = \frac{\pi}{20}$ rad.

Caution. It should be carefully understood that π is always a number whose approximate value is 3.1416. When we say that $\pi = 180°$, we mean that π radians, (*i.e.* 3.1416 times one radian) are equivalent to 180°. It is wrong to understand that $\pi = 180°$.

A note on π. π is a transcendental number, *i.e.,* it is an irrational number which cannot be the root of any algebraic equation with integral coefficients. Another familiar transcendental number is *e*. The approximate value of π is $\frac{22}{7}$ or 3.1416, correct to four places of decimal only.

Note. The students may familiarise themselves with the following equivalents:

$$30° = \frac{\pi}{6}, 45° = \frac{\pi}{4}, 60° = \frac{\pi}{3}, 90° = \frac{\pi}{2}, 360° = 2\pi.$$

3.10. Length of an arc of a circle

Let XOP be a given angle. With O as centre and radius r draw a circle, cutting the sides of the angle XOP at A and B. Cut off an arc AC equal to the radius r. Join OC. Let the arc length ACB = s.

$\therefore$ $\angle AOC = 1$ rad (By def)

Now $\dfrac{\angle AOB}{\angle AOC} = \dfrac{\text{arc ACB}}{\text{arc AC}} = \dfrac{s}{r}$

(Since central angles are proportional to the arc lengths)

$\Rightarrow$ $\angle AOB = \dfrac{s}{r} \times \angle AOC = \dfrac{s}{r} \times 1 \text{ rad} = \dfrac{s}{r}$ rad

If θ be the circular measure of $\angle AOB$, then

$$\theta = \frac{s}{r} \Rightarrow \boxed{s = r\,\theta.}\ *$$

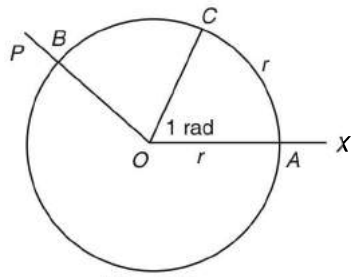


Fig. 3.11

3.11. Area of a sector of a circle

If the area of a circle is divided into 360 sectors, then obviously the angle of each sector will be 1°. Since the complete angle for the area of the circle is 2π rad, the area of the sector having anlge θ will be the fractions $\dfrac{\theta}{2\pi}$ of the area πr^2 of the full circle.

Let AOB be a sector of a circle, centre O, radius r. Let $\angle AOB = \theta$ radian and arc AB = s. Then

Area of the sector $AOB = \dfrac{\theta}{2\pi} \times \pi r^2 = \dfrac{1}{2} r^2\theta$

Area of the sector AOB $= \dfrac{1}{2} r^2\theta$ [Fig. 3.12]

Maths Alert! The above formulae work only if θ is in radians.

Corollary. Area of the sector AOB

$$\frac{1}{2}r^2\theta = \frac{1}{2}r^2 \times \frac{s}{r} = \frac{1}{2}rs \quad \left[\because \theta = \frac{s}{r} \text{ by Art 3.10}\right]$$

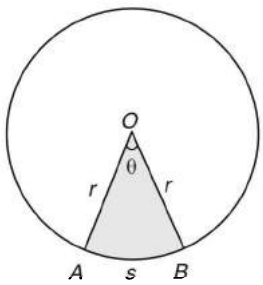


Fig. 3.12

i.e., **Area of the sector AOB** $= \dfrac{1}{2} rs.$

Ex. 3. Find the length of an arc of a circle of 3 cm radius if the angle subtended at the centre is 30°. ($\pi = 3.14$).

Sol. Length of the arc (s) $= r\theta = 3 \times \dfrac{\pi}{6} = \dfrac{1}{2}\ (3.14) = 1.57$ cm.

* The relation $s = \theta r$ shows that in the radian system of measure, the unit of length used in measuring the length of the arc is the length of the radius of the circle. The circumference of the circle contains 2π of these units ($C = 2\pi r$). That is why this is called circular system.

Ex. 4. What is the area of the sector shown in Fig. 3.13?

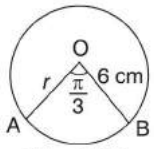


Fig. 3.13

Sol. Sector area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}\times 6^2 \times \frac{\pi}{3} = 6 \times 3.1416 \text{ cm}^2$

$= 18.85 \text{ cm}^2$ (to 2dp.)

Ex. 5. Taking the sun's distance as 1.4950×10^8 km and the angle subtended by the sun at a point *O* on earth as half a degree, find approximately the diameter of the sun.

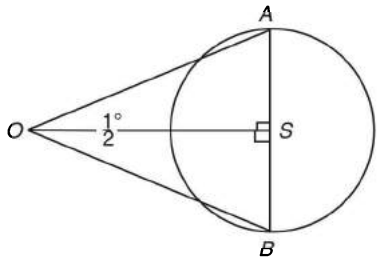


Fig. 3.14

Sol. $\theta = \left(\frac{1}{2}\right)^\circ = \frac{\pi}{180}\times\left(\frac{1}{2}\right)^\circ \text{ rad} = \frac{\pi}{360}\text{rad}$

As the distance between the earth and the sun is quite large, so we consider diameter AB as arc AB.

If d km be the diameter of the sun, we have from the formula $\frac{s}{r} = \theta$, $\frac{\text{AB}}{\text{OA}} = \frac{\pi}{360}$

$\Rightarrow \quad \frac{d}{1.4950\times 10^8} = \frac{\pi}{360}$ [d = arc AB, OA $\simeq$ OS = 1.495×10^8]

$\Rightarrow \quad d = 1.4950 \times 10^8 \times \frac{\pi}{360} = 1.4950 \times 10^8 \times \frac{3.1416}{360} = 1304637$

$= 1.304\,637 \times 10^6$ km approx. in standard or scientific notation.

Ex. 6. An arc *AB* of a circle subtends an angle *x* radians at the centre *O* of the circle. Given that the area of the sector *AOB* is equal to the square of the length of the arc *AB*, find the value of *x*. **[*SC*]**

Sol. From, Fig. 3.12, taking $\theta = x$, we get, area of the sector $AOB = \frac{1}{2}r^2x$.

By the question, $\frac{1}{2}r^2x = s^2 \Rightarrow \frac{1}{2}r^2x = r^2x^2$ [$\because s = rx$ by Art. 3.10]

$\Rightarrow \quad x = \frac{1}{2}$ rad.

Ex. 7. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.

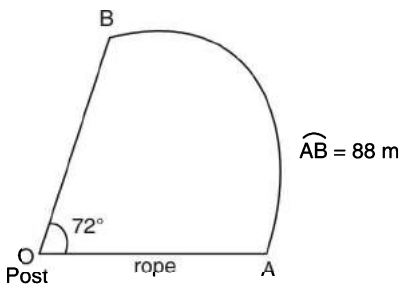


Fig. 3.15

Sol. Let point O denote the position of the point and OA the length of the rope in tight position, such that ∠AOB = 72° and arc AB = 88 m.

Let OA = r metre, then,

$\theta = 72° = \left(72\times\frac{2\pi}{180}\right)\text{rad} = \frac{2\pi}{5}\text{rad}$ and $s = 88$ m

$\therefore \quad$ Length of the rope, $(r) = \frac{s}{\theta} = 88\times\frac{5}{2\pi} = 88\times\frac{5}{2}\times\frac{7}{22} = 70$ m.

EXERCISE 3

$\left(\text{Take } \pi = \frac{22}{7}\right)$

1. Express the following angles in degrees : $\frac{\pi}{6}, \frac{14}{15}\pi, \frac{11}{18}\pi, \frac{7}{90}\pi$.
2. Express the following angles in radians (*i*) 1′ (*ii*) 20° (*iii*) 135°.
3. Express in radians and also in degrees the angle of a regular polygon of (*i*) 40 sides, (*ii*) *n* sides.
4. The perimeter of a certain sector of a circle is equal to the length of the arc of the semi-circle having the same radius, express the angle of the sector in degrees, minutes and seconds.
5. The length of a pendulum is 8 m while the pendulum swings through 1.5 rad, find the length of the arc through which the tip of the pendulum passes.
6. The minute hand of a clock is 15 cm long. How far does the tip of the hand move during 40 minutes ? (Take $\pi = 3.14$)
7. A central angle of a circle of radius 50 cm intercepts an arc of 10 cm. Express the central angle θ in radians and in degrees.
8. The moon's distance from the earth is 360000 km and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the moon.
9. A railway train is travelling on a curve of 750 m radius at the rate of 30 km/h, through what angle has it turned in 10 seconds ?
10. A horse is tethered to a stake by a rope 810 cm long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it will have gone when the rope has traced out an angle of 70°?
11. The area of a sector is 5.024 cm^2 and its angle is 36°. Find the radius, ($\pi = 3.14$).
12. Find the area of sector of a circle, radius 5 m bounded by an arc of length 8 m.
13. The diagram shows a windscreen wiper cleaning a car windscreen.
    (*i*) What is the length of the arc swept out?
    (*ii*) What area of the windscreen is not cleaned?

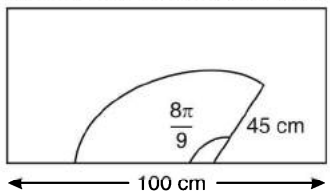


Fig. 3.16

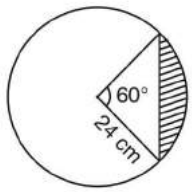


Fig. 3.17

14. Find the area of the shaded segment (Fig. 3.17).
15. What is the ratio of the areas of the major sector in diagram A to the minor sector in a diagram B?

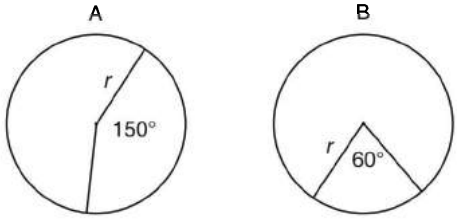


Fig. 3.18

ANSWERS

1. 30°, 168°, 110°,14°. **2.** $\frac{\pi}{10800}, \frac{\pi}{9}, \frac{3\pi}{4}$.

3. $\frac{19}{20}\pi, 171°; \left(\frac{n-2}{n}\right)\pi$ rad, $\left(\frac{n-2}{n}\right)180°$. **4.** 65° 27′ 16″ nearly.

5. 12 m **6.** 62.8 cm **7.** $\frac{1}{5}$ rad, 11°27′ 17″

8. 3247 $\frac{13}{21}$ km **9.** $\frac{1}{9}$ rad

10. 990 m **11.** 4.0 cm **12.** 20 m^2

13. (*i*) 126 cm (*ii*) 2670 cm^2 **14.** 52.2 cm^2 **15.** 7 : 2

HINTS TO SELECTED QUESTIONS

4. (*i*) Each angle of a regular polygon of 40 sides = $\pi - \frac{2\pi}{40} = \frac{19}{20}\pi$ rad

(*ii*) Each angle of a regular polygon of n sides = $\left(\pi - \frac{2\pi}{n}\right)$ rad

5. $2r + s = \pi r$ ... (1), $\frac{s}{r} = \theta$ ... (2)

CHAPTER SUMMARY

1. Degree measure

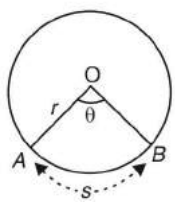
1 right-angle= 90 degrees (written as 90°)

1 degree = 60 minutes (written as 60′)

1 minute = 60 seconds (written as 60″)

2. Radian measure

Angle AOB = $\frac{\text{length of arc } AB}{\text{radius of the circle}}$



$\Rightarrow \quad \theta = \frac{s}{r} \Rightarrow s = r\,\theta.$

3. Relation between degrees and radians

π radians = 180 degrees

$\Rightarrow$ 1 radian = $\frac{180°}{\pi} = 57°16'$ (approx)

1 degree = $\frac{\pi}{180}$ rad

Ex. $30° = \frac{\pi}{180} \times 30$ rad $= \frac{\pi}{6}$ rad, $\quad 540° = \frac{\pi}{180} \times 540 = 3\pi$ rad

4. Area of sector AOB = $\frac{1}{2}r^2\theta = \frac{1}{2}rs$

CHAPTER TEST

1. Find the radian measure of (*i*) 25° (*ii*) 240°.

2. Find the degree measure of (*i*) $\frac{5\pi}{3}$ (*ii*) –4.

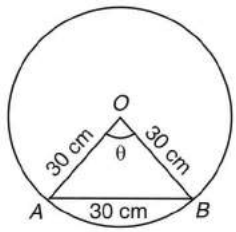
3. If an angle measures D degrees or C radians, show that $\frac{D}{90} = \frac{2C}{\pi}$. **[*WBHS*]**

4. One angle of a triangle in 54° and another angle is $\frac{\pi}{4}$ radians. Find the third angle in centesimal unit.

5. Express in circular measure and also in degrees the angle of a regular octagon.

6. If in two circles, arcs of the same length subtend angles of 60° and 75° at the centre, find the ratio of their radii. **[*WBHS, HB*]**

7. In a circle of diameter 60 cm the length of a chord is 30 cm. Find the length of the minor and major arcs of the chord.



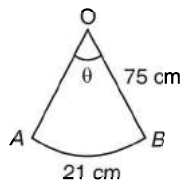
[Hint. ΔOAB being equilateral $\theta = 60° = \frac{\pi}{3}$ rad

Now, minor arc $AB = r\theta$

Also, major arc = $2\pi r$ – minor arc]

8. Find the angle in radian through which a pendulum swings and its length is 75 cm and the tip describes an arc of length 21 cm.

9. Find the area of the sector of a circle of radius 14 cm having central angle $\frac{\pi}{4}$. $\left(\text{Take } \pi = \frac{22}{7}\right)$.



10. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.

ANSWERS

1. (*i*) $\frac{5\pi}{36}$ (*ii*) $\frac{4\pi}{3}$ **2.** (*i*) 300° (*ii*) –229°5′ 27″. **4.** 81°

5. 135°; $\frac{3\pi}{4}$ rad **6.** 5 : 4

7. Minor arc 10π cm = 31.42 cm approx; Major arc = 50π cm

8. $\frac{7}{25}$ rad **9.** 77 cm^2 **10.** 70 m

CHAPTER 4

TRIGONOMETRICAL FUNCTIONS

4.01. A right-angled triangle

Let XOY be any given angle α. Take any point P on OY and draw PM perpendicular to OX. A right-angled ΔOMP is formed. If α is taken as the *angle of reference,* MP the side opposite to α is called *the perpendicular,* OP the side opposite to the right angle is called the *hypotenuse* and OM the third side is called the *base.*

Note. For angle MPO, the sides MP, OP, OM will represent the base, the hypotenuse, and the perpendicular respectively.

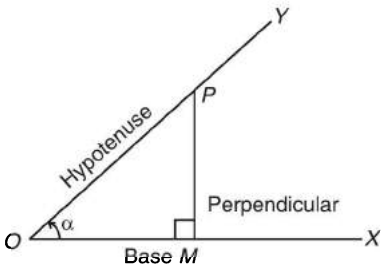


Fig. 4.01

4.02. Trigonometrical functions – Definition

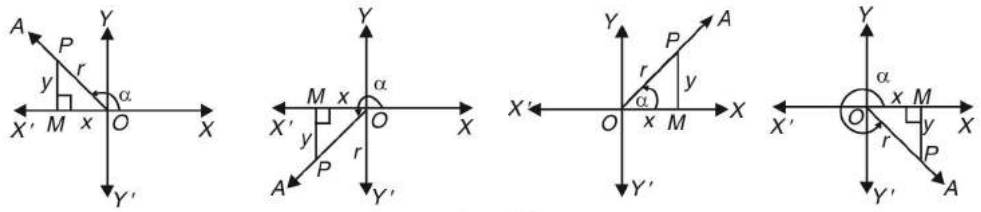


Fig. 4.02

Let a straight line OA, starting from the position OX and rotating round O, trace out an angle XOA. Let α be the measure of the angle XOA. This angle can be of any magnitude. From any point P in the final position of the revolving line OA, draw PM perpendicular to OX or XO produced, if necessary. Then the three sides OM, MP and OP can be arranged two at a time in six (${}^3P_2$) different ways and hence six ratios can be formed with them. These six ratios are called the *trigonometrical functions* and are given the following names.

(1) The ratio of the perpendicular to the hypotenuse is called the **sine** of the angle α and is written as **sin** α.

$$\sin\alpha = \frac{MP}{OP} = \frac{y}{r} \qquad \left[\frac{\textbf{Perp.}}{\textbf{Hyp.}}\right]$$

(2) The ratio of the base to the hypotenuse is called the **cosine** of the angle α and is written as **cos** α .

$$\cos\alpha = \frac{OM}{OP} = \frac{x}{r} \qquad \left[\frac{\textbf{Base}}{\textbf{Hyp.}}\right]$$

(3) The ratio of the perpendicular to the base is called the **tangent** of the angle α and is written as **tan** α.

$$\tan\alpha = \frac{MP}{OM} = \frac{y}{x} \qquad \left[\frac{\textbf{Perp.}}{\textbf{Base}}\right]$$

The following three ratios are **reciprocals** of these ratios.

(4) The ratio of the hypotenuse to the perpendicular is called the **cosecant** of the angle α and is written as **cosec** α.

$$\text{cosec } \alpha = \frac{\text{OP}}{\text{MP}} = \frac{r}{y} \qquad \left[\frac{\textbf{Hyp.}}{\textbf{Perp.}}\right]$$

(5) The ratio of the hypotenuse to the base is called the **secant** of the angle α and is written as **sec** α.

$$\sec \alpha = \frac{\text{PO}}{\text{OM}} = \frac{r}{x} \qquad \left[\frac{\textbf{Hyp.}}{\textbf{Base}}\right]$$

(6) The ratio of the base to the perpendicular is called the **cotangent** of the angle α and is written as **cot** α.

$$\cot \alpha = \frac{\text{OM}}{\text{MP}} = \frac{x}{y} \qquad \left[\frac{\textbf{Base}}{\textbf{Perp.}}\right]$$

Note 1. The student should not commit the mistake of regarding sin α as sin × α ; sin α means sine of the angle α. It is absolutely wrong to perform such operations as

sin (A + B) = sin A + sin B or sin 2x + sin x = sin (2x + x).

Note 2. It is easily seen that in the rt. ∠d Δ ABC,

(1) sin A = cos B, (2) cos A = sin B,

(3) tan A = cot B, (4) tan B = cot A.

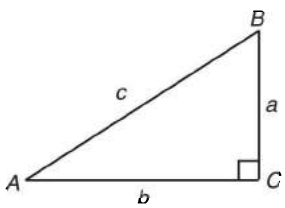


Fig. 4.03

Note 3. The values of t-functions are always the same for the same angle. For, if we take any other point P′ on the terminal line, then from the similar triangles OMP and OM′P′,

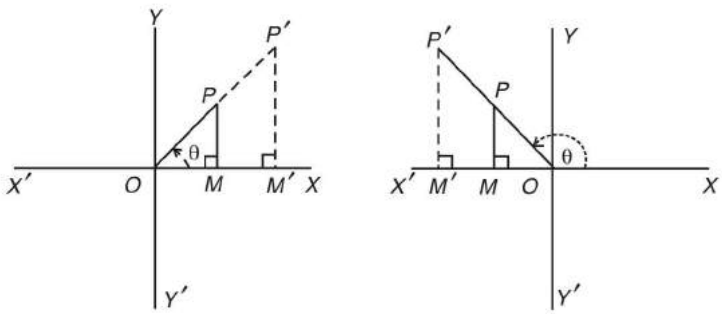


Fig. 4.04

$$\frac{\text{MP}}{\text{OP}} = \frac{\text{M′P′}}{\text{OP′}} \qquad (\because \text{ MP and M′P′ have the same sign})$$

Thus, the value of the ratio $\frac{\text{MP}}{\text{OP}}$, *i.e.,* sin θ is independent of the position of the point P and depends only on the magnitude of the angle XOP ; similarly, for other t-ratios.

Note 4. Power Notation for *t*-functions. $(\sin \theta)^2$ is written as $\sin^2\theta$ and is read as sine square θ.

$(\sin \theta)^3$ is written as $\sin^3 \theta$ and is read as sine cube θ.

Similarly, for powers of other t-functions : $(\cos \theta)^5 = \cos^5 \theta$, $(\tan \theta)^4 = \tan^4 \theta$, etc.

Caution. $(\sin \theta)^{-1}$ is not written as $\sin^{-1} \theta$. Thus $\sin^{-1} \theta \neq (\sin \theta)^{-1}$.

$\sin^{-1} \theta$ has a different meaning. It is read as 'sine inverse θ' and it means an angle whose sine is θ. Similar meanings are attached to $\cos^{-1} \theta$ and $\tan^{-1} \theta$, etc. You will study inverse functions in detail in Class XII.

4.03. Formulae expressing relations between the t-functions

*I. Reciprocal relations

(*i*) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ **(*ii*) $\sec\theta = \dfrac{1}{\cos\theta}$** **(*iii*) $\cot\theta = \dfrac{1}{\tan\theta}$**

In words: (*i*) *cosec* θ *is reciprocal of sin* θ ; (*ii*) *sec* θ *is reciprocal of cos* θ ; *and* (*iii*) *cot* θ *is reciprocal of tan* θ.

These relations are obvious from the definitions of *t*-ratios.

Cor. $\sin\theta \times \operatorname{cosec}\theta = 1$, $\cos\theta \times \sec\theta = 1$, $\tan\theta \times \cot\theta = 1$.

II. Quotient relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

From the figure of Art. 4.02, if α is replaced by θ, we have $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta \quad \text{and} \quad \frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot\theta.$$

III. Square relations. (Pythagorean identities)

(*i*) $\sin^2\theta + \cos^2\theta = 1$ **(*ii*) $\sec^2\theta = 1 + \tan^2\theta$** **(*iii*) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.**

From the figures of Art. 4.02, we have in rt. ∠*d* Δ OMP, $x^2 + y^2 = r^2$ ...(1)

(*a*) Dividing both sides of (1) by r^2 .

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \quad i.e., \ \sin^2\theta + \cos^2\theta = 1. \quad [\alpha \text{ is replaced by } \theta.]$$

(*b*) Dividing both sides of (1) by x^2

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \quad i.e., \ 1 + \tan^2\theta = \sec^2\theta.$$

(*c*) Dividing both sides of (1) by y^2,

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2, \quad i.e., \ \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

4.04. Defining trigonometric functions with the help of unit circle

Let $X'OX$ and $Y'OY$ be the coordinate axes. Let a circle be drawn with centre O and **unit radius** cutting the coordinate axes at A, B, A′ and B′ as shown in the figure. Then, the equation of the circle is $x^2 + y^2 = 1$. Let a moving point start from A and move along the circumference of the circle. Let it cover an arc length θ (measured in anticlockwise direction if θ > 0 and in clockwise direction if θ < 0) and subtend an angle | θ | at O. Then, the circular measure of $\angle AOP = \dfrac{arc\ AP}{radius} = \dfrac{\theta}{1} = \theta.$

---

* Such relations that are satisfied for *all* values of the *variable which are permissible* are also called *identities.* For example, $(x - 2)(x + 2) = 0$ is called an equation because it is satisfied by two values $x = 2, -2$, whereas $(x - 2)(x + 2) = x^2 - 4$ is called an identity.

Let the coordinates of P be (x, y) and let PN be perp. on the X-axis. Then,

$ON = x$ and $PN = y$.

The circular functions of θ are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\operatorname{cosec}\theta = \frac{1}{y}$, $\sec\theta = \frac{1}{x}$ and $\cot\theta = \frac{x}{y}$.

Thus, the circular functions of a real number θ and the trigonometric ratios of $\angle AOP$ are same.

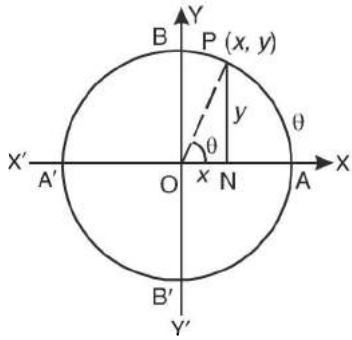


Fig. 4.05

4.05. The formula $\sin^2\theta + \cos^2\theta = 1, \forall\, \theta \in R$

By definition, $\cos\theta = x$ and $\sin\theta = y$.

Since the equation of the unit circle is $x^2 + y^2 = 1$ and the point (x, y), *i.e.*, $(\cos\theta, \sin\theta)$ lies on it, therefore, we have

$$\cos^2\theta + \sin^2\theta = 1.$$

4.06. Using the fundamental relations

The relations obtained in Art. 4.03 help us in many situations in the study of trigonometry. With their help we can express one t-ratio in terms of another, *viz.*

$$\sin^2\theta = 1 - \cos^2\theta, \tan\theta = \frac{\sin\theta}{\cos\theta}, \text{etc.}$$

Maths Alert! Very often you will come across expressions like $\sqrt{[f(x)]^2}$. Remember that $\sqrt{x^2} = |x|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

EXERCISE 4 (*a*)

Prove that

1. $1 - \cos^2\theta = \sin^2\theta$.

***2.** $\sqrt{1-\sin^2\theta} = \cos\theta$.

3. $\sec\alpha\sqrt{1-\sin^2\alpha} = 1$

4. $\frac{\sec^2\theta - 1}{\tan^2\theta} = 1$.

5. $\sec^2\theta - 1 - \tan^2\theta = 0$.

6. $\frac{\sin^2\theta + \cos^2\theta}{\sec^2\theta - \tan^2\theta} = 1$.

7. $1 - \cos^2\theta - \sin^2\theta = 0$.

8. $\sin^4\theta + \sin^2\theta\cos^2\theta = \sin^2\theta$.

9. $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta = 1$.

10. $\cos^2\theta(\operatorname{cosec}^2\theta - \cot^2\theta) = \cos^2\theta$.

11. $\frac{\sin\theta\cos\sec\theta\tan\theta\cot\theta}{\sin^2\theta + \cos^2\theta} = 1$.

12. $\frac{\sin^2 30° + \cos^2 30°}{\sec^2 57° - \tan^2 57°} = 1$.

13. $\cos A \tan A = \sin A$.

14. $\sin^4 A \operatorname{cosec}^2 A + \cos^4 A \sec^2 A = 1$.

15. $\sin^2 A \cot^2 A + \cos^2 A \tan^2 A = 1$.

---

* **Note.** We have $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ if $\cos\theta$ is +ve, *i.e.*, the results of Q. 2 and Q. 3 do not hold good if θ lies in 2nd or 3rd quadrant. Similarly, $\sec\alpha\sqrt{1-\sin^2\alpha} = \sec\alpha\sqrt{\cos^2\alpha} = \sec\alpha|\cos\alpha| = \sec\alpha.\cos\alpha$, if $\cos\alpha$ is +ve, *i.e.*, $\alpha \in$ I or IV quad.

4.07. Proving simple identities

You have proved some identities in the last Exercise. Here we take up more. Each identity has two sides—the **left side** and the **right side.**

Procedure:

1. *Take one side, generally the more complicated side as it is usually easier to reduce this side than to expand or enlarge the simpler one.*

Sometimes both sides, if both are complicated, may have to be taken and reduced, independently of each other, to the same thing.

We can prove an identity showing that the given equation is equivalent to a true statement or to another equation known to be an identity.

2. *Study the identity carefully, keeping in mind what is given and what you want to arrive at. Use the fundamental relations obtained in Art. 4.03 to make the necessary substitutions. Expressing all t-functions involved in the identity in terms of the sine and the cosine is the most commonly used method.*

3. *If an identity involves expressions involving only* $\tan^2$, $\sec^2$, $\cot^2$, cosec^2, *it may be more helpful to use the formulae* $\sec^2\theta = 1 + \tan^2\theta$ *and* $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

Ex. 1. Prove that $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A$.

Sol. Left side $= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A) = 1 \times (\cos^2 A - \sin^2 A)$

$= \cos^2 A - \sin^2 A =$ Right side.

Ex. 2. Prove that $(1 + \tan A)^2 + (1 - \tan A)^2 = 2\sec^2 A$.

Sol. Left side $= 1 + \tan^2 A + 2\tan A + 1 + \tan^2 A - 2\tan A$

$= 2 + 2\tan^2 A = 2(1 + \tan^2 A)$

$= 2\sec^2 A =$ Right side.

Ex. 3. Prove that $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$

Sol. Right side $= \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \operatorname{cosec}^2\theta(\operatorname{cosec}^2\theta - 1) = (1 + \cot^2\theta) \times \cot^2\theta$

$= \cot^2\theta + \cot^4\theta =$ Left side.

Ex. 4. Show that $\dfrac{\sec A + \tan A}{\operatorname{cosec} A + \cot A} = \dfrac{\operatorname{cosec} A - \cot A}{\sec A - \tan A}$.

Sol. We have

$\sec^2 A - \tan^2 A = \operatorname{cosec}^2 A - \cot^2 A$ (Each side = 1)

i.e. $(\sec A + \tan A)(\sec A - \tan A) = (\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)$

$\therefore \quad \dfrac{\sec A + \tan A}{\operatorname{cosec} A + \cot A} = \dfrac{\operatorname{cosec} A - \cot A}{\sec A - \tan A}$.

Ex. 5. Prove that $\dfrac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$.

Sol. Right side $= \left(\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}\right)^2$

$= \dfrac{(1+\sin\theta)^2}{\cos^2\theta} = \dfrac{(1+\sin\theta)^2}{1-\sin^2\theta} = \dfrac{(1+\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \dfrac{1+\sin\theta}{1-\sin\theta} =$ Left side.

EXERCISE 4 (*b*)

Prove that

1. $\sec\theta \cot\theta = \operatorname{cosec}\theta$

2. $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$.

3. $\dfrac{\cos\theta}{\sec\theta - \tan\theta} = 1 + \sin\theta$.

4. $\dfrac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$.

5. $\dfrac{3-4\sin^2\theta}{\cos^2\theta} = 3 - \tan^2\theta$.

6. $(\tan A + \cot A)\sin A \times \cos A = 1$

7. $\cos A = \dfrac{\cot A}{\operatorname{cosec} A} = \dfrac{\cot A}{\sqrt{(1+\cot^2 A)}}$ (A ∉ III or IV quad.)

8. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

9. $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A$.

10. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$.

11. $\sin^2\phi\,(1 + \cot^2\phi) = 1$.

12. $(\tan A + \cot A)^2 = \operatorname{cosec}^2 A + \sec^2 A$.

13. $\sec^4 A - \sec^2 A = \tan^2 A + \tan^4 A$.

14. $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$.
(NDA April 2007)

15. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$

16. $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha} = 2\operatorname{cosec}\alpha$.

17. $1 + \dfrac{1}{\cos A} = \dfrac{\tan^2 A}{\sec A - 1}$

18. $\dfrac{1+\cos A}{1-\cos A} = (\operatorname{cosec} A + \cot A)^2$

19. If $\tan\theta + \sin\theta = \alpha$ and $\tan\theta - \sin\theta = \beta$, show that $\alpha^2 - \beta^2 = 4\sqrt{\alpha\beta}$. ($\theta \notin$ II or III quad.)

4.08. Convention of signs of distances

In trigonometry, we measure distances in three directions :

(*i*) Horizontal direction (OX or OX′)

(*ii*) Vertical direction (OY or OY′)

(*iii*) Direction of the revolving line (OP).

The convention of sign is

(*a*) **distances measured along OX, OY, OP are positive ;**

(*b*) **distances measured along OX′, OY′ are negative.**

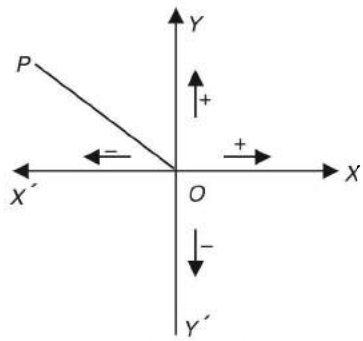


Fig. 4.06

4.09. Domain and range of circular functions

(*i*) In the unit circle, it is clear that x and y lie between –1 and 1, *i.e.*, $-1 \le x \le 1$ and $-1 \le y \le 1$.

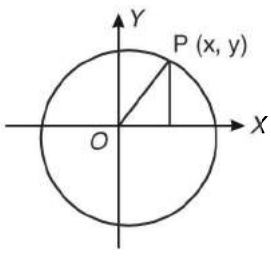


Fig. 4.07

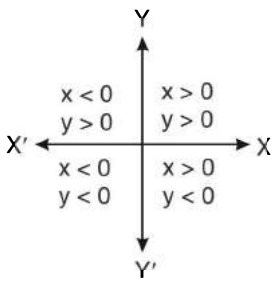


Fig. 4.08

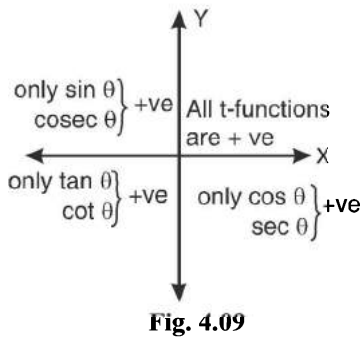


Fig. 4.09

(*ii*) Also, in the first quad. $x > 0, y > 0$;
in the 2nd quad. $x < 0, y > 0$;
in the 3rd quad. $x < 0, y < 0$;
in the 4th quad. $x > 0, y < 0$.

(*iii*) Hence, in the first quad., all the *t*-functions are +ve, in the 2nd quad., only sin θ and cosec θ are +ve, the rest are –ve. In the 3rd quad., only tan θ and cot θ are + ve and in the 4th quad., only cos θ and sec θ are + ve.

(*iv*) Further, from the definition of the *t*-functions, it is clear that while $\sin\theta = y$ and $\cos\theta = x$ are defined for all real values of θ, $\tan\theta = \frac{y}{x}$ and $\sec\theta = \frac{1}{x}$ are not defined when $x = 0$, *i.e.*, when OP coincides with the *y*-axis, *i.e.*, when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or $-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots$, *i.e.*, $\boldsymbol{\theta = (2n + 1)\frac{\pi}{2}}$, where *n* is any integer.

(*v*) Also, $\cot\theta = \frac{x}{y}$ and $\operatorname{cosec}\theta = \frac{1}{y}$, are not defined when $y = 0$, *i.e.*, when OP coincides with the X-axis, *i.e.*, when $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ or $-\pi, -2\pi, -3\pi$ ..... i.e., $\theta = \boldsymbol{n\pi}$, where *n* is any integer.

(*vi*) Again, as $-1 \le x \le 1$ and $\cos\theta = x$, we have $-1 \le \cos\theta \le 1$. Similarly, $-1 \le y \le 1$ and $\sin\theta = y$. Therefore, $-1 \le \sin\theta \le 1$.

(*vii*) Clearly, the other four trigonometric ratios may have all real values lying between $-\infty$ and ∞. **To summarise** the above discussion, we have

Circular function	Domain (θ in circular measure)	Range
sin	all real numbers	(–1, 1)
cos	all real numbers	(–1,1)
tan	all real numbers other than $(2n + 1)\frac{\pi}{2}$, where *n* is any integer.	all real values
cosec	all real numbers other than $n\pi$, *n* being and integer.	all real values except –1 and 1.
sec	all real numbers other than $(2n + 1)\frac{\pi}{2}$, *n* being an integer	all real values except –1 and 1.
cot	all real numbers other than $n\pi$, where *n* is any integer	all real values.

Ex. 6. If $\cos\theta = \frac{-5}{12}$, in which quadrant does θ lie?

Sol. As cos θ is negative, θ can be either in second or third quadrant.

Ex. 7. If $\tan\theta = \frac{4}{3}$, in which quadrant does θ lie?

Sol. As tan θ is positive, θ can be either in first or third quadrant.

Aid to memory. The sign of a particular *t*-function in any quadrant can be remembered by the word **"all-sin-tan-cos",** or "Add Sugar To Coffee". Whatever is written in a particular quadrant along with its reciprocal is positive and the rest are negative.

Note. From Art. 4.09 (*vi*), it is clear that **sin θ and cos θ cannot be numerically greater than 1.**

Also, sin θ and cos θ can have negative values also so the values of sin θ and cos θ will always be between –1 and +1 and so **the minimum and the maximum values of sin θ and cos θ will be –1 and +1 respectively.**

Again, as cosec θ and sec θ are respectively reciprocals of sin θ and cos θ, their values cannot be less than 1.

It follows from Art. 4.09 (*vii*) that tan θ and cot θ can have any value.

Ex. 8. The following equations are impossible:

$\sin\theta = -3$, $\cos\theta = \frac{5}{4}$, $\operatorname{cosec}\theta = \frac{3}{4}$, $\sec\theta = \frac{-4}{5}$. Why ?

Oral examples for class discussion

1. In what quadrant does an angle lie if

(*a*) its sine is positive and its cosine is negative?

(*b*) its tangent is positive and its sine is negative?

(*c*) its cosine is positive and its tangent is negative?

(*d*) its sine is negative and its cosine is positive?

2. Which is greater and why?

(*a*) sin 55° or cos 55°. (*b*) sin 30° or cos 30°. (*c*) sin 40° or cos 50°.

3. (*a*) Is it possible to construct an angle whose sine = $\frac{4}{3}$?

(*b*) Is it possible to construct an angle whose cosine = 5.7?

(*c*) Is it possible to construct an angle whose secant = $\frac{3}{7}$?

(*d*) Is it possible to construct an angle whose cosecant = $\frac{1}{2}$?

4.10. Determination of values of T-ratios from value of one T-ratio

If the value of one *t*-ratio is known, we can find the values of the remaining *t*-ratios.

Procedure. 1. *Draw a right-angled triangle containing the given angle and write down the value of the given t-ratio in terms of the sides of this triangle.*

2. *Use Pythagoras theorem to find out the third side of this triangle, taking the double sign* (±) *with the square root.*

3. *All the sides of the triangle being known the values of all the t-ratios can be written down for the given angle.*

Ex. 9. The cosecant of a certain angle is $\frac{13}{12}$. Evaluate other *t*-ratios of this angle.

Sol. $\operatorname{cosec}\theta = \frac{13}{12}$.

$\therefore$ In rt. $\angle d$ ΔOMP, $OP = 13$, $MP = 12$

$$OM = \pm\sqrt{(OP^2 - MP^2)} = \pm\sqrt{(13^2 - 12^2)} = \pm 5$$

As cosec θ is positive, so θ will be either in the first quadrant or in the second quadrant.

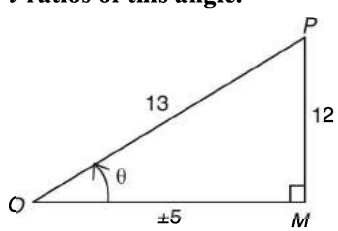


Fig. 4.10

$\therefore \quad \sin\theta = \frac{12}{13}, \quad \cos\theta = \pm\frac{5}{13}, \quad \sec\theta = \pm\frac{13}{5}, \quad \tan\theta = \pm\frac{12}{5}, \quad \cot\theta = \pm\frac{5}{12}.$

Ex. 10. If $\cos\alpha = \frac{5}{13}$ and α lies in the fourth quadrant, find the value of $\frac{2-3\cot\alpha}{4-9\sqrt{\sec^2\alpha-1}}$.

Sol. As α lies in the fourth quadrant, so sec α is +ve and cot α is –ve. By the method of the previous example,

$\cot\alpha = -\frac{5}{12}$, $\sec\alpha = \frac{13}{5}$,

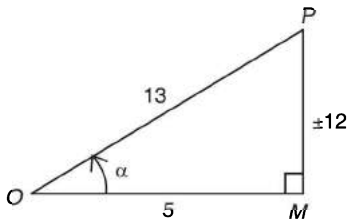


Fig. 4.11

$$\frac{2-3\cot\alpha}{4-9\sqrt{\sec^2\alpha-1}} = \frac{2-3\times\left(-\frac{5}{12}\right)}{4-9\sqrt{\left(\frac{169}{25}-1\right)}}$$

$$= \frac{2+\frac{15}{12}}{4-9\times\sqrt{\frac{144}{25}}} = \frac{\frac{39}{12}}{4-9\times\frac{12}{5}} = \frac{39}{12}\times\frac{-5}{88} = \frac{-65}{352}.$$

4.11. T-ratios of standard angles

To find the t-ratios of 0°, 30°, 45°, 60°, 90°

1. Angle of 45° $\left(\text{or } \frac{\pi}{4}\right)$

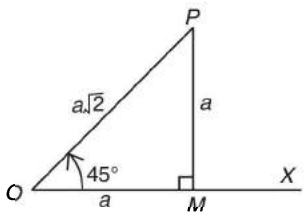


Fig. 4.12

Let the revolving line, starting from OX, trace out $\angle XOP = 45°$. Draw PM perpendicular to OX. Then,

$\angle OPM = 90° - \angle MOP = 90° - 45° = 45° = \angle MOP$

$\therefore \quad OM = MP = a$ (say)

$OP = \sqrt{OM^2 + MP^2}$ (OP is always +ve.)

$= \sqrt{a^2 + a^2} = a\sqrt{2}$

Now, $\quad \textbf{sin 45°} = \frac{MP}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \text{cosec } 45° = \sqrt{2}$

$\textbf{cos 45°} = \frac{OM}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \sec 45° = \sqrt{2}$

$\textbf{tan 45°} = \frac{PM}{OM} = \frac{a}{a} = 1 \Rightarrow \cot 45° = 1.$

2. Angles of 30° $\left(\text{or}\frac{\pi}{6}\right)$ and 60° $\left(\text{or}\frac{\pi}{3}\right)$

Let OPQ be an equilateral triangle. Draw PM perpendicular to OQ. Then, M is the middle point of OQ. Let the measure of each side of the triangle OPQ be $2a$. Then,

$OM = a$ and $MP = \sqrt{OP^2 - OM^2}$

$\therefore \quad MP = \sqrt{(4a^2 - a^2)} = \sqrt{(3a^2)} = a\sqrt{3}$

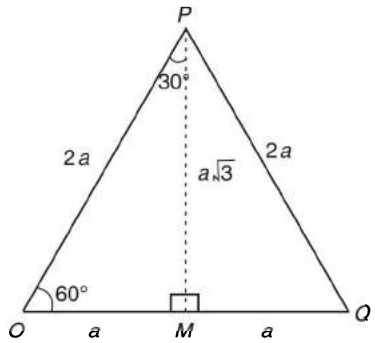


Fig. 4.13

$$\textbf{sin } 30° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2} \quad \therefore \text{ cosec } 30° = 2$$

$$\textbf{cos } 30° = \frac{PM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} \quad \therefore \text{ sec } 30° = \frac{2}{\sqrt{3}}$$

$$\textbf{tan } 30° = \frac{OM}{PM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \therefore \text{ cot } 30° = \sqrt{3}$$

$$\text{Now, } \textbf{sin } 60° = \frac{PM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} \quad \therefore \text{ cosec } 60° = \frac{2}{\sqrt{3}}$$

$$\textbf{cos } 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2} \quad \therefore \text{ sec } 60° = 2$$

$$\textbf{tan } 60° = \frac{MP}{OM} = \frac{a\sqrt{3}}{a} = \sqrt{3} \quad \therefore \text{ cot } 60° = \frac{1}{\sqrt{3}}$$

3. T-ratios of 0° and 90°

(*a*) Let MOP be any angle and PM be perpendicular to OM. Let ∠ MOP continually diminish. As this angle diminishes, MP decreases and OP comes nearer to OM. In the limit when ∠ MOP vanishes, OP coincides with OM. PM vanishes and OM becomes equal to OP, hence

$$\textbf{sin } 0° = \frac{0}{OP} = 0 \text{ , and } \textbf{cos } 0° = \frac{OP}{OP} = 1.$$

$$\therefore \quad \textbf{tan } 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0, \ \textbf{cot } 0° = \frac{1}{0} = \infty.$$

$$\textbf{cosec } 0° = \frac{1}{0} = \infty \text{ and } \textbf{sec } 0° = \frac{1}{1} = 1$$

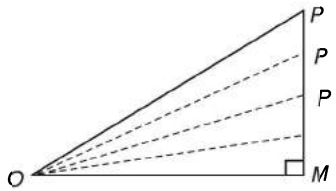


Fig. 4.14

(*b*) As angle XOP continuously increases, OM decreases and MP comes nearer and nearer to OP. In the limit when ∠XOP = 90°, OP coincides with OY, OM vanishes and MP becomes equal to OP, then

sin 90° = 1; **cosec 90° = 1**
cos 90° = 0; **sec 90° = ∞**
tan 90° = ∞ ; **cot 90° = 0.**

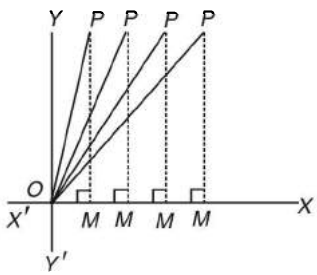


Fig. 4.15

Note. The *t*-ratios of angles 0° and 90° are valid only in the limiting sense. Thus, sin 0° = $\lim_{\theta \to 0°} \sin\theta$ or tan 90° = $\lim_{\theta \to 90°} \tan\theta$, etc.

Table of the results obtained above:

Table

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Aid to memory:

	0°	30°	45°	60°	90°
(*i*) sines are	$\frac{\sqrt{0}}{2}$,	$\frac{\sqrt{1}}{2}$,	$\frac{\sqrt{2}}{2}$,	$\frac{\sqrt{3}}{2}$,	$\frac{\sqrt{4}}{2}$,
(*ii*) cosines are	$\frac{\sqrt{4}}{2}$,	$\frac{\sqrt{3}}{2}$,	$\frac{\sqrt{2}}{2}$,	$\frac{\sqrt{1}}{2}$,	$\frac{\sqrt{0}}{2}$,

(the fractions for cosines written in the reverse order)

(*iii*) $\tan = \frac{\sin}{\cos}$, (*iv*) $\cot = \frac{\cos}{\sin}$ (*v*) $\sec = \frac{1}{\cos}$, (*vi*) $\operatorname{cosec} = \frac{1}{\sin}$.

Ex. 11. Verify:

(*i*) $\sin^2\theta + \cos^2\theta = 1$ if $\theta = 30°$; (*ii*) $\sec^2\theta = 1 + \tan^2\theta$ if $\theta = \frac{\pi}{4}$;

(*iii*) $\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$ if $\alpha = \frac{\pi}{3}$.

Sol. (*i*) $\sin^2\theta + \cos^2\theta = \sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1.$

(*ii*) Left side $= \sec^2\theta = \sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$

Right side $= 1 + \tan^2\theta = 1 + \tan^2\frac{\pi}{4} = 1 + 1^2 = 2$

$\therefore$ Left side = Right side

(*iii*) Left side $= \operatorname{cosec}^2\alpha = \operatorname{cosec}^2\frac{\pi}{3} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

Right side $= 1 + \cot^2\alpha = 1 + \cot^2\frac{\pi}{3} = 1 + \frac{1}{3} = \frac{4}{3}$

$\therefore$ Left side = Right side.

EXERCISE 4 (*c*)

1. If $\cot\theta = \frac{4}{3}$, find the values of other *t*-ratios of θ.

2. If $\cos A = 0.6$, find the value of $5\sin A - 3\tan A$.

3. If $\sin A = \frac{3}{5}$, prove that $\tan A + \frac{1}{\cos A} = 2$ or -2.

4. If $\tan\theta = \frac{1}{\sqrt{7}}$, find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$.

5. If $\sin\theta = \frac{21}{29}$, prove that $\sec\theta + \tan\theta = 2\frac{1}{2}$ if θ lies between 0 and $\frac{\pi}{2}$. What will be the value of the expression when θ lies (*i*) between $\frac{\pi}{2}$ and π and (*ii*) between π and $\frac{3\pi}{2}$?

[**Hint.** Part (*ii*) is impossible. Why ?]

6. If θ lies in the second quadrant and $\tan\theta = \frac{-5}{12}$, find the value of $\frac{2\cos\theta}{1 - \sin\theta}$.

7. If $\sin\theta \sec\theta = -1$ and θ lies in the second quadrant, find $\sin\theta$ and $\sec\theta$.

8. If A is in the fourth quadrant and $\cos A = \frac{5}{13}$, find the value of $\frac{13\sin A + 5\sec A}{5\tan A + 6\operatorname{cosec} A}$.

Verify that

9. $\sin 60° = \frac{2\tan 30°}{1+\tan^2 30°}$.

10. $\cos 60° = \frac{1-\tan^2 30°}{1+\tan^2 30°}$.

Prove that

11. $\sec 30° \tan 60° + \sin 45° \operatorname{cosec} 45° + \cos 30° \cot 60° = \frac{7}{2}$.

12. $\frac{4}{3}\cot^2 30° + 3\sin^2 60° - 2\operatorname{cosec}^2 60° - \frac{3}{4}\tan^2 30° = 3\frac{1}{3}$.

ANSWERS

1. $\sin\theta = \pm\frac{3}{5}$, $\cos\theta = \pm\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\sec\theta = \pm\frac{5}{4}$, $\operatorname{cosec}\theta = \pm\frac{5}{3}$. **2.** 0.

4. $\frac{3}{4}$. **5.** (*i*) $\frac{-5}{2}$ **6.** –3. **7.** $\sin\theta = \frac{1}{\sqrt{2}}$, $\sec\theta = -\sqrt{2}$. **8.** $-\frac{2}{37}$

T-FUNCTIONS OF ANGLES OF ANY MAGNITUDE

4.12. T-ratios of (–θ)

Let the revolving line OP starting from the position OX rotate in the anticlockwise (+ ve)

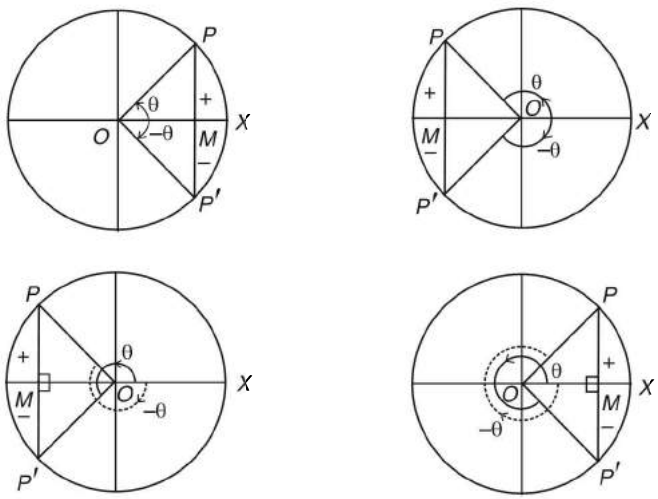


Fig. 4.16

direction and describe the angle XOP = θ. Draw PM, the perpendicular from P to the *x*-axis. Let another revolving OP′ (= OP) starting from the position OX rotate in the clockwise (– ve) direction and describe the angle XOP′ = – θ. Draw the perpendicular from P′ to the *x*-axis. This will also meet the *x*-axis at M. It is clear that in every quadrant

$$\angle POM = \angle P'OM \text{ and } \Delta OPM \equiv \Delta OP'M$$

$$\therefore \quad MP' = MP \text{ (numerically)} = -MP \text{ (algebraically)}$$

$$\therefore \quad \mathbf{\sin(-\theta)} = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\mathbf{\sin\theta}$$

$$\therefore \quad \textbf{cos}\ (-\theta) = \frac{OM}{OP'} = -\frac{OM}{OP} = -\textbf{cos}\ \theta \quad \text{and} \quad \textbf{tan}\ (-\theta) = \frac{MP'}{OM} = -\frac{MP}{OM} = -\tan\theta$$

Taking the reciprocals of these *t*-ratios, it is easily seen that cosec $(-\theta) = -$ cosec θ, sec $(-\theta) =$ sec θ, cot $(-\theta) = -$ cot θ.

Note I. The student will find it easier to draw the diagrams if he takes $\theta = 30°$ in the first, $\theta = 120°$ in the second, $\theta = 210°$ in the third and $\theta = 300°$ in the fourth quadrant. In the negative direction the angles would be $-30°$, $-120°$, $-210°$ and $-300°$.

Note II. In all articles of this chapter we will follow the following course:

1. First we will construct the triangles OMP and OM′P′.

2. Proving these triangles congruent, we will express values of OM′, M′P′ and OP′ (sides of ΔOM′P′) in terms of the sides OM, MP and OP (sides of Δ OMP).

3. After writing down the values of *t* ratios from Δ OM′P′, we will express them in terms of the sides of ΔOMP, using the relations obtained in (2).

4.13. T-ratios of (90° − θ)

Let the revolving line OP starting from the position OX and rotating in the positive direction describe the angle XOP = θ. Let another revolving line OP′ = OP starting from the same position OX coincide at first with OY so as to describe an angle of 90° in the positive direction. Let it now rotate in the negative direction through an angle equal to θ, then OP′ in its final position describes the angle XOP′ = 90° − θ. From P and P′ draw PM and P′M′ perpendiculars on XOX′. Then

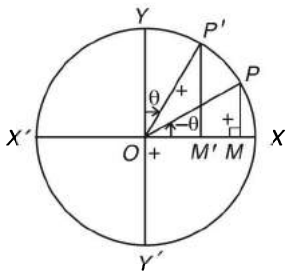


Fig. 4.17

$$\angle POM = \theta = \angle YOP' = \angle OP'M'; \ \angle OMP = \angle OM'P'$$

and $OP = OP'$

$\therefore \quad \Delta OPM \cong \Delta OP'M'$

$MP = OM'$ (algebraically); $OM = M'P'$ (algebraically)

$$\textbf{sin}\ (90° - \theta) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \textbf{cos}\ \theta$$

$$\textbf{cos}\ (90° - \theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \textbf{sin}\ \theta$$

$$\textbf{tan}\ (90° - \theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \textbf{cot}\ \theta$$

Similarly, cosec $(90° - \theta) =$ sec θ, sec $(90° - \theta) =$ cosec θ, cot $(90° - \theta) =$ tan θ.

Note. θ may lie in any one of the four quadrants and so there are four figures. The student is advised to draw the figures himself and extend the proof to other positions of the revolving line.

Oral Examples

Prove that

1. $\tan\theta \tan(90° - \theta) = 1$. **2.** $\sin\theta \cos(90° - \theta) = \sin^2\theta$. **3.** $\dfrac{\sin 18°}{\cos 72°} = 1$. **4.** $\dfrac{\sec 35°}{\text{cosec}\ 55°} = 1$

5. $\dfrac{\tan 75°}{\cot 15°} = 1$ **6.** $\sin^2 47° + \sin^2 43° = 1$ **7.** $\sec^2 33° - \cot^2 57° = 1$.

8. If AB, AC are the equal sides of an isosceles triangle, prove that $\tan\dfrac{A}{2} = \cot B$.

4.14. T-ratios of (90° + θ)

Let the revolving line OP starting from the position OX rotate in the positive direction to form ∠ XOP = θ. Let another revolving line OP′ = OP starting from the same position OX rotate in the positive direction, first through an angle θ, so as to coincide with OP, and then through an additional angle 90° to form ∠ XOP′ = 90° + θ. Then, for all positions of OP, ∠ POP′ = 90° ∴ ∠POM = complementary of ∠ P′OM′ = ∠ OP′M′

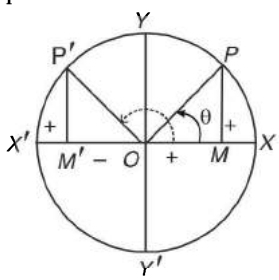


Fig. 4.18

∴ OP = OP′ and ∠ OMP = ∠ OM′P′ and (each =90°)

∴ Δ OMP ≡ OM′P′

∴ M′P′ = OM (numerically) = OM (algebraically)

OM′ = PM (numerically) = – MP (algebraically)

∴ **sin (90° + θ)** = $\frac{M'P'}{OP'} = \frac{OM}{OP}$ = **cos** θ

cos (90° + θ) = $\frac{OM'}{OP'} = \frac{-MP}{OP}$ = **– sin** θ

tan (90° + θ) = $\frac{M'P'}{OM'} = \frac{OM}{-MP}$ = **– cot** θ

Similarly, cosec (90° + θ) = sec θ, sec (90° + θ) = – cosec θ, cot (90° + θ) = – tan θ

Note. The student should draw figures for other positions of OP.

Ex. 12. sin 115° = sin (90° + 25°) = cos 25° **= 0.9063.**

cos 134° = cos (90° + 44°) = – sin 44° **= – 0.6947.**

4.15. T-ratios of (180° – θ)

Let the revolving line OP, starting from the position OX rotate in the positive direction and trace out ∠ XOP = θ. Let another revolving line OP′ = OP starting from the position OX rotate in the positive direction through an angle 180° to coincide with OX′ and then rotate in the negative direction through an angle θ. Then in its final position OP′ describes an angle XOP′ = 180° – θ.

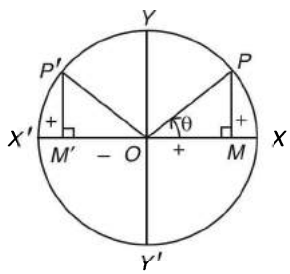


Fig. 4.19

∴ ∠ POM = ∠ P′OM′, OP = OP′

and ∠ PMO = ∠ P′M′O (each = 90°)

∴ Δ OPM ≅ Δ OP′M′ ∴ M′P′ = MP (algebraically)

OM′ = OM (numerically) = – OM (algebraically)

∴ **sin (180° – θ) =** $\frac{M'P'}{OP'} = \frac{MP}{OP}$ **=** **sin** θ ; sin 180° = sin (180° – 0°) = sin 0° = 0 ;

cos(180° – θ) = $\frac{OM'}{OP'} = -\frac{OM}{OP}$ **= – cos** θ ; cos 180° = cos (180° – 0°) = – cos 0° = – 1 ;

tan (180° – θ) = $\frac{M'P'}{OM'} = \frac{MP}{-OM}$ **= – tan** θ ; tan 180° = tan (180° – 0°) = – tan 0° = 0 ;

Similarly, cosec (180° – θ) = cosec θ, sec (180° – θ) = – sec θ, cot (180° – θ) = – cot θ.

Note. The student should draw figures for other positions of OP.

Ex. 13. sin 120° = sin (180° – 60°) = sin 60° = $\frac{\sqrt{3}}{2}$

cos 150° = cos (180° – 30°) = – cos 30° = $\frac{-\sqrt{3}}{2}$

tan 172° = tan (180° – 8°) = – tan 8° = **– 0.1405**

4.16. T-ratios of (180° + θ)

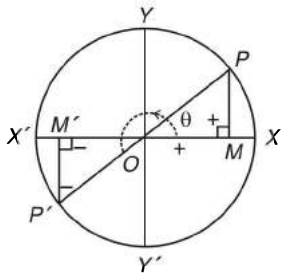


Fig. 4.20

Let the revolving line OP, starting from the position OX, rotate in the positive direction to describe an angle XOP = θ. Let another revolving line OP′ = OP starting from thesame position OX rotate through an angle 180° to position OX′ and then proceed further through an angle θ to make ∠ XOP′ = 180° + θ in its final position OP′. As the difference of the two angles is 180°, OP and OP′ are in the same straight line.

$\therefore \quad \angle POM = \angle P'OM'$

Δ s POM and P′OM′ are congruent.

$\therefore$ M′P′ = MP (numerically) = – MP (algebraically)

OM′ = OM (numerically) = – OM (algebraically)

$$\therefore \quad \textbf{sin (180°+ θ)} = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\textbf{ sin } \theta$$

$$\therefore \quad \textbf{cos (180}^{\textbf{0}}\textbf{ + θ)} = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\textbf{ cos } \theta$$

$$\textbf{tan (180° + θ)} = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \textbf{tan } \theta$$

Similarly, cosec (180° + θ) = – cosec θ, sec (180° + θ) = – sec θ, cot (180° + θ) = cot θ.

Note. The student should draw figures for other positions of OP.

Ex. 14. sin 225° = sin (180° + 45°) = – sin 45° = $-\frac{1}{\sqrt{2}}$

cos 246° = cos (180° + 66°) = – cos 66° = **– 0.4067**

tan 211° = tan (180° + 31°) = tan 31° = **0.6009.**

4.17. T-ratios of (n × 360° ± θ)

For an angle ($n \times 360° + \theta$), the revolving line occupies the same position which it would have occupied in revolving through an angle θ. Hence the values of *t*-ratios for the angles ($n \times 360° + \theta$) when n =1, 2, 3.... will be the same as those for θ. Similarly, for the angle $n \times 360° - \theta$, the final

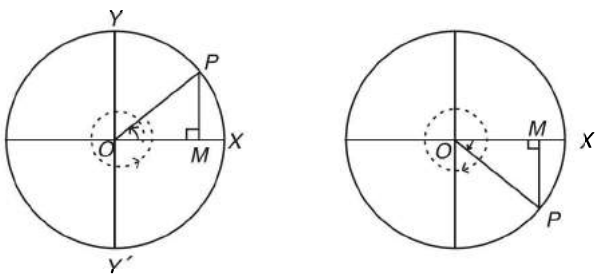


Fig. 4.21

position of the revolving line will be the same as for the angle (– θ). Hence, the values of *t*-ratios for the angle $n \times 360° - \theta$, will be the same as those for (– θ).

For example,

sin 748° = sin (2 × 360° + 28°) = sin 28° = 0.4695

cos 700° = cos (2 × 360° – 20°) = cos (–20°) = cos 20° = 0.9397

tan 675° = tan (2 × 360° – 45°) = tan (–45°) = – tan 45° = – 1

Aid to Memory

1. Any function of an angle expressed as 180° or 360° plus or minus an acute angle, *i.e.,* **[even number** $\times \frac{\pi}{2} \pm$ acute angle θ] equals numerically the **same function** of θ, with the same, or opposite sign according to the quadrant in which the revolving line terminates. The proper sign can be ascertained by

sin	all
tan	cos

2. Any function of an angle expressed as 90° or 270° plus or minus an acute angle, *i.e.,* $\left[\text{odd number} \times \frac{\pi}{2} \pm \text{acute angle } \theta\right]$ equals numerically the **co-function** of θ with the '+' or '–' sign depending upon the quadrant in which the revolving line terminates. The co-function of sin θ is cosine θ, of cosine θ is sine θ, of tan θ is cot θ, of cot θ is tan θ, etc.

Table

	$-\theta$	$90° - \theta$	$90° + \theta$	$180 - \theta$	$180° + \theta$	$2n\pi - \theta$	$2n\pi + \theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$\tan\theta$

4.18. Periods of trigonometrical functions

A function $f(x)$ is said to be periodic if there exists a real number $\lambda > 0$ such that $f(x + \lambda) = f(x)$

The smallest positive value of λ for which $f(x + \lambda) = f(x)$ is called the period of the function $f(x)$. As shown below, all *t*-ratios are periodic-functions.

1. $\sin(\theta + 2\pi) = \sin\theta$, $\cos(\theta + 2\pi) = \cos\theta$, $\text{cosec}(\theta + 2\pi) = \text{cosec}\,\theta$, $\sec(\theta + 2\pi) = \sec\theta$. Thus, the values of $\sin\theta$, $\cos\theta$, $\text{cosec}\,\theta$, $\sec\theta$ do not change when θ is changed to $\theta + 2\pi$. Hence 2π is the least positive constant.

$\therefore$ They are periodic functions and their period is 2π.

2. $\tan(\theta + \pi) = \tan\theta$, $\cot(\theta + \pi) = \cot\theta$

$\tan\theta$ and $\cot\theta$ are also periodic functions and their period is π.

4.19. Use of formulae

The main use of these formulas lies in reducing functions of any larger angle to functions of smaller angles.

Ex. 15. Find the values of each of the following:

$\tan 225°$, $\sin 315°$, $\tan(-1742°)$, $\cos(-1760°)$.

Sol. **$\tan 225°$** $= \tan(180° + 45°) = \tan 45° = 1$

$\sin 315°$ $= \sin(360° - 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

$\tan(-1742°)$ $= -\tan 1742° = -\tan(5 \times 360° - 58°) = -\tan(-58°) = \tan 58° = 1.6003$

$\cos(-1760°)$ $= \cos 1760° = \cos(4 \times 360° + 320°)$

$= \cos 320° = \cos(360° - 40°) = \cos(-40°) = \cos 40° = 0.7660.$

Procedure. The following steps of procedure are suggested for the guidance of the students. They are not to be treated as a hard and fast rule. The aim is to reduce the larger angle to a smaller angle and this can be achieved by the use of the formulae of this chapter in any manner.

1. *If the angle is of the form (– θ), express it in terms of* θ.
2. *If the angle is more than 360°, divide by 360 to separate from the given angle the multiples of 360°. Now use the formula for* $n \times 360° + \theta$.
3. *If the angle is now more than 180° either use the formula for* $180° + \theta$ *or for* $360° - \theta$, *whichever may leave a smaller angle.*
4. *If the angle is less than 180°, either use the formula for* $180° - \theta$ *or for* $90° + \theta$ *whichever may leave a smaller angle.*
5. *If need be, use the formula for* $90° - \theta$.

Ex. 16. Prove that tan 225° cot 405° + tan 765° cot 675° = 0.

Sol. Left side $= \tan(180° + 45°)\cot(360° + 45°) + \tan(2 \times 360° + 45°)\cot(2 \times 360° - 45°)$

$= \tan 45° \cot 45° + \tan 45° \cot(-45°)$

$= \tan 45° \cot 45° - \tan 45° \cot 45° = 0 =$ Right side

Ex. 17. If $\tan A = -\sqrt{3}$, find all the possible values of A between 0° and 360°.

Sol. As tan A is negative, so A lies either in the second or in the fourth quadrant.

The acute angle A for this value of $\tan A = 60°$.

$\therefore$ The required values of A between 0° and 360° are

$180° - 60°$ and $360° - 60°$

$120°$ and $300°$

Note. If for a given value of a *t*-ratio, the corresponding acute angle is α, it will be $180° - \alpha$, $180° + \alpha$ and $360° - \alpha$ if the angle lies respectively in second, third or fourth quadrant.

sin	all
tan	cos

$180° - \alpha$	α
$180° + \alpha$	$360° - \alpha$

Ex. 18. If $\theta = -1000°$, determine the sign of sin θ – cos θ.

Method. *Reduce to an angle less than 45° and use the fact that if* $\theta < 45°$, *cos* $\theta >$ *sin* θ.

Sol. $\sin\theta - \cos\theta = \sin(-1000°) - \cos(-1000°) = -\sin 1000° - \cos 1000°$

$= -\sin(3 \times 360° - 80°) - \cos(3 \times 360° - 80°) = \sin 80° - \cos 80°$

$= \sin(90° - 10°) - \cos(90° - 10°) = \cos 10° - \sin 10°$

As $\cos 10° > \sin 10°$, so the sign of the given expression is positive.

Ex. 19. Find the values of x and y, lying between 0 and 360, which satisfy the equations,

(i) **$\sin 2x° = 0.6428$.** ***(ii)*** **$\cos\left(\frac{1}{2}y° + 71°\right) = -0.3420$.** **[*SC*]**

Sol. (*i*) $\sin 2x° = 0.6428 = \sin 40°$ [From tables]

As $\sin 2x°$ is positive, the angle $2x°$ will either be in the first quadrant or in the second quadrant.

$\therefore$ The possible values of $2x°$ are

40° and $180° - 40°$, *i.e.* 140°. Here $360° + 40°$ and $360° + 140°$, *i.e.* 400° and 500° will also be possible as the angle is $2x°$ and x is to be found.

$x = 20°$ or $70°$ or $200°$ or $250°$.

Note. As x lies between 0° and 360°, so $2x$ lies between 0° and 720°, $3x$ lies between 0° and 1080°, etc.

(*ii*) $\cos\left(\frac{1}{2}y° + 71°\right) = -0.3420$

Now $0.3420 = \cos 70°$.

As $\left(\cos \frac{1}{2} y° + 71°\right)$ is given to be negative, so the angle $\frac{1}{2}y° + 71°$ will either lie in the second or third quadrant

$\therefore \quad \frac{1}{2}y° + 71° = 180° - 70°$ or $180° + 70° \quad \therefore \quad \frac{1}{2} y = 39°$ or $179° \quad \therefore y = 78°$ or $358°$.

4.20. Monotonicity

Function $f(x)$ defined in an interval $[a, b]$ is said to be increasing in this interval if for any numbers $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$, the inequality $f(x_1) < f(x_2)$ holds true.

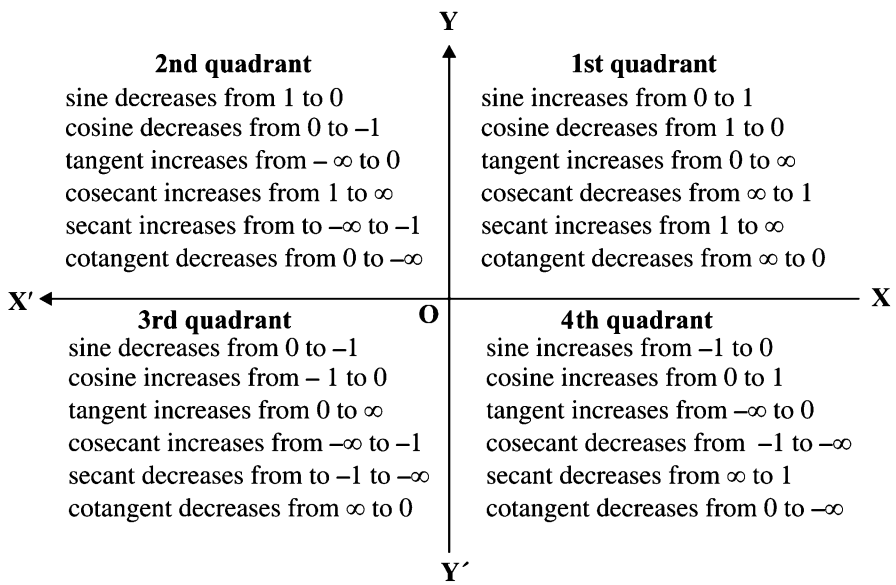
Similarly, for any two numbers $x_1 < x_2$, if $f(x_1) > f(x_2)$, then $f(x)$ is said to be decreasing in the interval. The property of increasing or decreasing of a function is also called monotonicity of the function.

We can investigate the behaviour of these functions in the first, second, third and fourth quadrants.

On the interval, $\left[0, \frac{\pi}{2}\right]$ the function $\sin x$ increases from 0 to 1, while $\cos x$ decreases from 1 to 0. On the interval $\left[\frac{\pi}{2}, \pi\right]$, the function $\sin x$ decreases from 1 to 0 while $\cos x$ decreases from 0 to –1. On the interval $\left[\pi, \frac{3\pi}{2}\right]$, the function $\sin x$ decreases from 0 to –1, while $\cos x$ increases from –1 to 0. On the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, the function $\sin x$ increases from –1 to 0 while $\cos x$ increases from 0 to 1.

The function $\tan x$ increases on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and the function $\cot x$ decreases on the interval $(0, \pi)$.

The foregoing results are given in the following table.



EXERCISE 4 (*d*)

1. Express the following as functions of angles less than 45° :

(*i*) sin 194°, (*ii*) sin 348°, (*iii*) cos 189°, (*iv*) sin (– 1785°),

(*v*) tan (3598°), (*vi*) cot (– 1952°), (*vii*) cosec (– 7498°).

2. Without using tables, give the value of each of the following :

(*i*) sin 120°, (*ii*) cot 330°, (*iii*) sec 210°, (*iv*) cos 315°,

(*v*) cosec 675°, (*vi*) cos 855°, (*vii*) sin 4530°, (*viii*) sec $\frac{15\pi}{4}$.

3. With the help of tables, find the values, correct to four places of decimals, of each of the following :

(*i*) cos 116° (*ii*) sin 267° (*iii*) sin (– 263°) (*iv*) cos 280° 10′

(*v*) tan (2015° 24′).

4. Find the value of sin 750° cos 300° + cos 1470° sin (– 1020°).

5. Evaluate $\frac{\cos 3\theta - 2\cos 4\theta}{\sin 3\theta + 2\sin 4\theta}$, when $\theta = 150°$.

6. Simplify : $\frac{\cos(-\theta)}{\sin(90° + \theta)}$.

7. $\frac{\tan(-\theta)}{\sin(540° + \theta)}$.

8. $\frac{\sin(90° - \theta)\sec(180° - \theta)\sin(-\theta)}{\sin(180° + \theta)\cot(360° - \theta)\operatorname{cosec}(90° + \theta)}$.

9. $\frac{\sin 150° - 5\cos 300° + 7\tan 225°}{\tan 135° + 3\sin 210°}$.

10. If $\sin(7\phi + 9°) = \cos 2\phi$, find a value of ϕ.

11. Find the values of θ lying between 0° and 360° when

(*i*) $\sin\theta = \frac{1}{2}$. (*ii*) $\tan\theta = -1$, (*iii*) $\sec\theta = -2$, (*iv*) $\sin\theta = \sin 21°$,

(*v*) $\tan\theta = -2.0145$, (*vi*) $\sin\theta = \cos 317°$, (*vii*) $\cos\theta = \sin 285°$.

12. If $0° < \theta < 90°$ and $\cos\theta = \frac{4}{5}$, find the values of

(*i*) cos (90° + θ) (*ii*) cosec (180° + θ) (*iii*) tan (360° – θ) (*iv*) sin (270° – θ).

13. Find six angles for which $\sin\theta = -\frac{\sqrt{3}}{2}$.

14. Find all the angles between 0° and 720° whose tangent is $-\frac{1}{\sqrt{3}}$.

15. Find the values of θ between 0° and 360° which satisfy the equations

(*i*) $\sin^2\theta = \frac{3}{4}$, (*ii*) $\cos 3\theta = \frac{1}{2}$. **[*SC*]**

16. If $\tan\theta = 0.4$, when θ lies between 0° and 360°, write down the possible values of θ and sin θ. **[*SC*]**

17. If $\cos x° = \sin 200°$, find the possible values of x between – 180° and 360°. **[*SC*]**

If A, B, C are the angles of a triangle, prove that

18. $\cos C = -\cos (A + B)$.

19. $\tan \frac{B+C}{2} = \cot \frac{A}{2}$.

20. $\frac{\tan (B + C) + \tan (C + A) + \tan (A + B)}{\tan (\pi - A) + \tan (2\pi - B) + \tan (3\pi - C)} = 1$.

21. If A, B, C, D are the angles of a quadrilateral, prove that $\cos \frac{1}{2}(A + B) + \cos \frac{1}{2}(C + D) = 0$.

ANSWERS

1. (*i*) $-\sin 14°$ (*ii*) $-\sin 12°$ (*iii*) $-\cos 9°$ (*iv*) $\sin 15°$ (*v*) $-\tan 2°$
(*vi*) $\cot 28°$ (*vii*) $\sec 28°$

2. (*i*) $\frac{\sqrt{3}}{2}$ (*ii*) $-\sqrt{3}$ (*iii*) $-\frac{2}{\sqrt{3}}$ (*iv*) $\frac{1}{\sqrt{2}}$ (*v*) $-\sqrt{2}$

(*vi*) $-\frac{1}{\sqrt{2}}$ (*vii*) $-\frac{1}{2}$ (*viii*) $\sqrt{2}$

3. (*i*) – 0.4384 (*ii*) – 0.9986 (*iii*) 0.9925 (*iv*) 0.1766 (*v*) 0.7107

4. 1 **5.** $-\frac{1}{2}(1+\sqrt{3})$ **6.** 1 **7.** $\sec \theta$ **8.** $\sin \theta$

9. – 2 **10.** 9° **11.** (*i*) 30°, 150° (*ii*) 135°, 315° (*iii*) 120°, 240°

(*iv*) 21°, 159° (*v*) 116° 24′, 296° 24′ (*vi*) 47°, 133° (*vii*) 165°, 195°

12. (*i*) $-\frac{3}{5}$ (*ii*) $\frac{-5}{3}$ (*iii*) $-\frac{3}{4}$ (*iv*) $-\frac{4}{5}$

13. 240°, 300°, 600°, 660°, 960°, 1020°. **14.** 150°, 330°, 510°, 690°.

15. (*i*) 60°, 120°, 240°, 300°. (*ii*) 20°, 100°, 140°, 220°, 260°, 340°.

16. 21° 48′, 201° 48′, 0.3714, – 0.3714 **17.** ± 110°, 250°.

4.21. Graphs of trigonometric functions

In Algebra, we draw graphs to express relationships between a variable x and its function $y = f(x)$. In trigonometry also, the relation between the values of variable angle x and the corresponding values of a trigonometric function y can be visualised by means of graphs. By assigning different values to x and evaluating the corresponding values of t-function, we obtain a set of points which, when plotted on a graph paper and joined, form a curve which is the graph of the given function. We come across such graphs in Physics while studying radiowaves, light waves, sound waves, alternating currents, simple harmonic motion, etc.

4.22. Useful notes

1. Take care in the proper choice of scale for the abscissa as well as for the ordinate, so that the graph may not be too long or too short. The size of the graph paper supplied is to be kept in view while choosing the scale.

2. In tabulating the values, the value of the function of acute angles can be obtained from the tables ; other values can be obtained with the help of the formulas , *viz.*

$$\sin (-x) = -\sin x, \cos (\pi - x) = -\cos x, \text{ etc.}$$

3. The angles involved may be regarded as measured either in degrees or in radians. If a trigonometric function is combined with an algebraic function, *e.g.*, $x = \sin 2x$, the angle will have to be expressed in radians.

Procedure.

1. *Draw two mutually perpendicular lines X ' OX and Y 'OY intersecting at the point O in the horizontal and vertical directions respectively.**
2. *Represent the angles along X 'OX and its given functions along Y 'OY.*
3. *Take* $\theta = 0°, 30°, 60°, 90°, \ldots$ *upto* $360°$ *(at intervals of* $30°$*), and find the corresponding values of the t-function correct to* 2 *places of decimals.*

   e.g., $\theta = 30°$, $\sin 30° = \frac{1}{2} = 0.50$; $\theta = 60°$, $\sin\theta = \sin 60° = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866 = 0.87$, etc.

4. *Scale.* The following scale is in most of the cases found convenient.

Let each small division along OX represent $10°$*, i.e., 3 small divisions* $30°$*. Let each small division along OY represent* 0.1.

5. *Construct a* **table** *of values and* **plot** *the points accordingly.*
6. *Draw a* **smooth curve** *through the points. This is the required graph.*

Note 1. Angles may be taken at intervals of 15°, *viz.,* 0°, 15°, 30°, 45° ..... etc. if the range in which the graph is to be drawn is small, say between 0° and 90°.

Note 2. The same rule applies when the angle increases from one given value to another, *e.g.,* from –180° to 180°.

4.23. The graph of sin θ

To draw the graph of sin θ **as** θ ***increases from 0° to 360°.***

1. Draw two straight lines X′OX and Y′OY at right angles to each other. Represent the values of θ along X′OX and those of sin θ along Y′OY.
2. Let each small division along X′OX = 10° and each small division along Y′OY = 0.1.
3. Taking θ = 0°, 30°, 60° ..... upto 360° and finding the corresponding values of sin θ, the following table is obtained.

Table

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

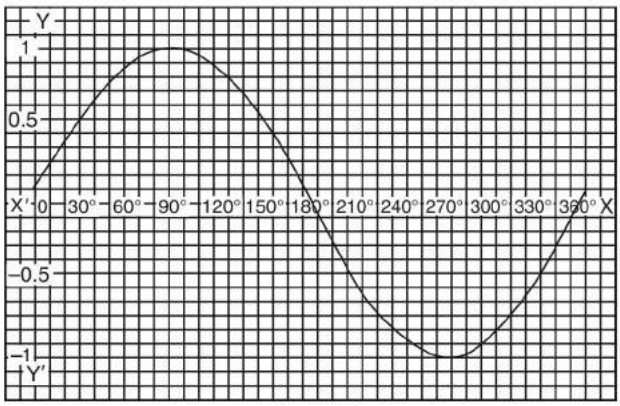


Fig. 4.22

Plotting these points and joining them by a free hand curve, we get the following graph.

The graph reveals the following properties of sin θ :

1. sin θ is a **continuous function** (as the graph has no break at any point).
2. The numerical value of sin θ cannot be greater than 1. The greatest value of sin θ is 1 and the least value is – 1 (∵ the graph does not go above the line $y = 1$ and below the line $y = -1$).

* Graph paper ruled in 2 mm squares should be used in conformity with the SI units.

4.24. The graph of cos θ

To draw the graph of cos θ as θ increases from 0° to 360°

Table

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1

Plotting these points and joining them by a free hand curve, we get the following graph. The graph reveals the following properties of cos θ :

1. The function cos θ is continuous. (∵ The graph has no break at any point.)
2. cos θ cannot be numerically greater than 1. The maximum value of cos θ is 1 and the minimum value is – 1. (∵ The graph does not go above the line $y = 1$ and below the line $y = -1$).

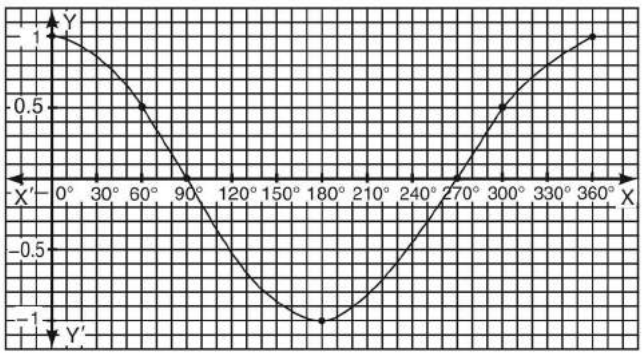


Fig. 4.23

4.25. The graph of tan θ

To draw the graph of tan θ as θ increases from 0° to 360°

Table of values:

θ	0°	30°	60°	90° – 0°	90° + 0°	120°	150°	180°
tan θ	0	0.58	1.73	$\to +\infty$	$\to -\infty$	–1.73	–0.58	0
θ	210°		240°	270°– 0°	270°+0°	300°	330°	360°
tan θ	0.58		1.73	$\to +\infty$	$\to -\infty$	–1.73	–0.58	0

Note. When $\theta = 90° - 0°$, $\tan\theta = \tan(90° - 0°) = \cot 0° = \infty$

$\theta = 90° + 0°$, $\tan\theta = \tan(90° + 0°) = -\cot 0° = -\infty$

$\theta = 270° - 0°$, $\tan\theta = \tan(270° - 0°) = \tan(180° + 90° - 0°) = \tan(90° - 0°) = \cot 0° = \infty$

$\theta = 270° + 0°$, $\tan\theta = \tan(270° + 0°) = \tan(180° + 90° + 0°) = \tan(90° + 0°) = -\cot 0° = -\infty$.

Graph:

The graph exhibits the following properties of tan θ.

1. The function tan θ is discontinuous at the points 90°, 270°,.....

(∵ The graph has a break at each of these points).

As θ passes through 90°, 270°,..... tan θ suddenly changes from $+\infty$ to $-\infty$.

2. tan θ can have any numerical value.

Note. Fig. 4.24 shows the graph of $y = \tan\theta$ between $\theta = 0°$ to $\theta = 990°$. The student will see that the curve consists of an unlimited number of similar branches.

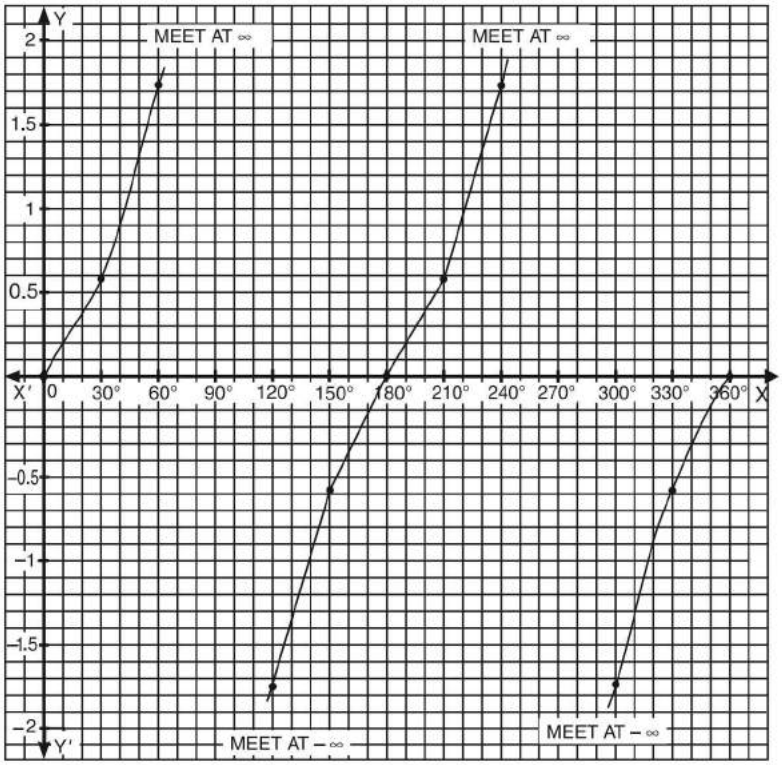
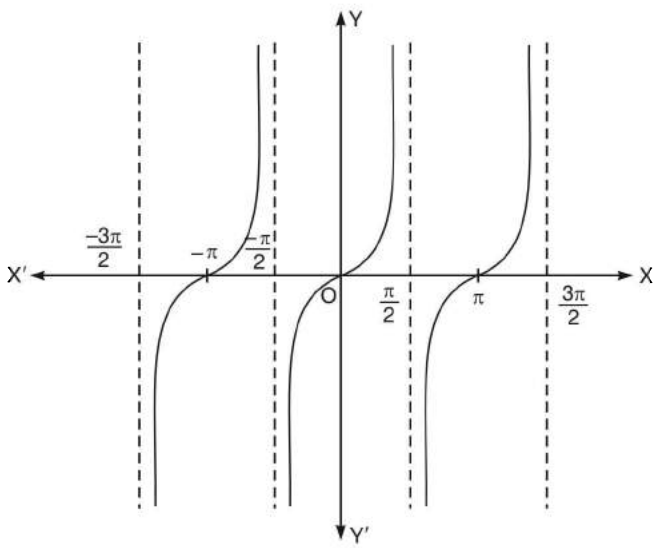


Fig. 4.24

Graph:



Graph of $y = \tan \theta$

Fig. 4.25

4.26. The graph of cot θ

To draw the graph of cot θ as θ increases from 0° to 360°

Table of values:—

θ	0°	30°	60°	90°	120°	150°	180° – 0°	
cot θ	+ ∞	1.73	0.58	0	–0.58	–1.73	– ∞	
θ	180°+ 0°		210°	240°	270°	300°	330°	360°
cot θ	+ ∞		1.73	0.58	0	–0.58	–1.73	– ∞

The graph shown reveals the following properties of cot θ:

1. cot θ can have any numerical value, positive or negative.
2. cot θ is a discontinuous function at 0° and 180°.
3. It suddenly changes from – ∞ to + ∞ at these values.

Graph:

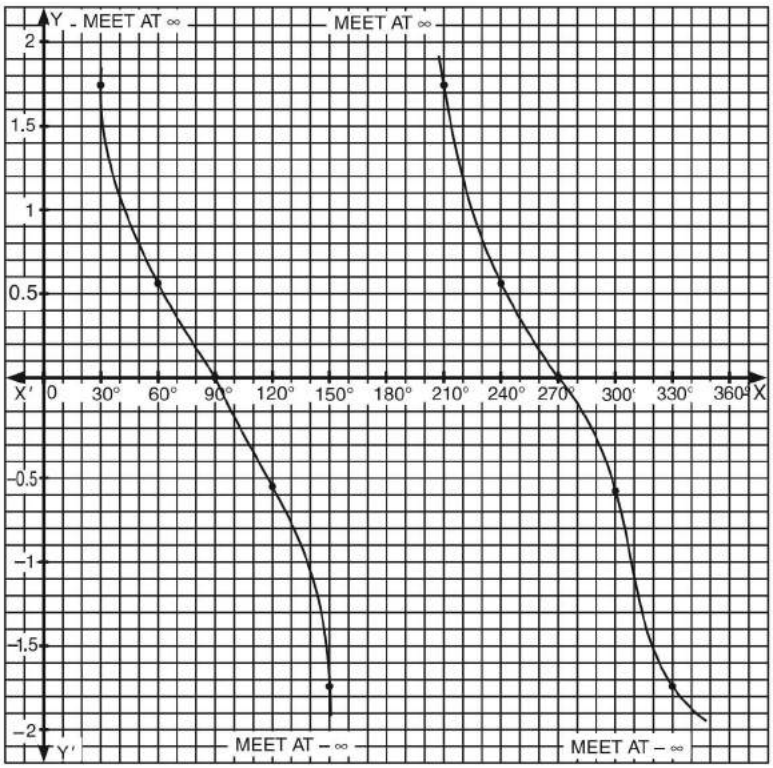
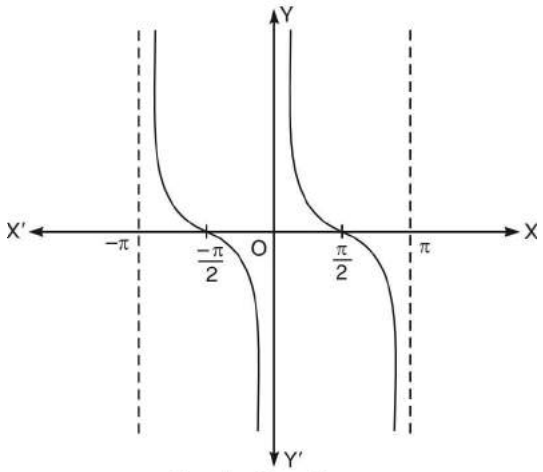


Fig. 4.26

Note. Below is shown the graph of cot θ between θ = 0° to θ = 360° so that the student may have a vivid picture of the graph.



Graph of cot θ

Fig. 4.27

4.27. The graph of sec θ

To draw the graph of sec θ as θ increases from 0° to 360°

Table of values:

θ	0°	30°	60°	90° – 0°	90° + 0°	120°	150°	180°
sec θ	1	1.15	2	+ ∞	– ∞	–2	–1.15	–1
θ	210°	240°	270°– 0°	270°+ 0°	300°	330°	360°	
sec θ	–1.15	–2	– ∞	+ ∞	2	1.15	1	

Graph:

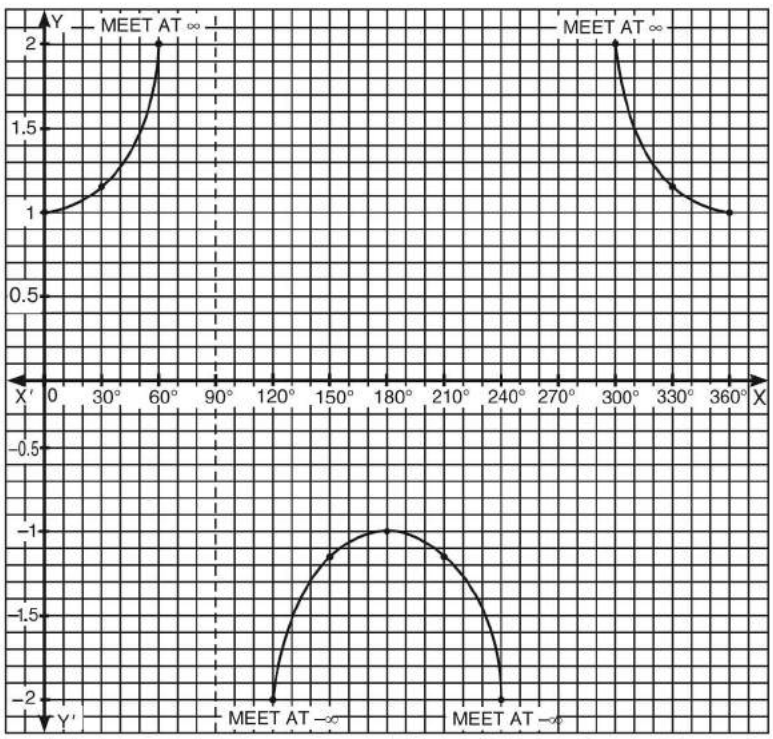


Fig. 4.28

The graph exhibits the following properties of sec θ :

1. sec θ cannot be numerically less than 1.
2. The function is discontinuous at θ = 90°, 270°,...
3. At these values, sec θ changes from + ∞ to – ∞ and then from – ∞ to + ∞. Below is shown the graph of sec θ between θ = 0° to θ = 990°.

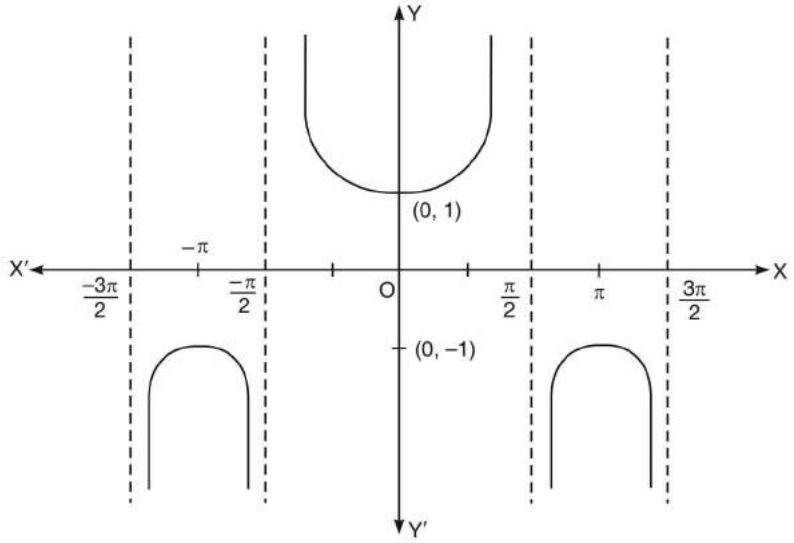


Fig. 4.29

4.28. The graph of cosec θ

To draw the graph of cosec θ as θ increases from 0° to 360°

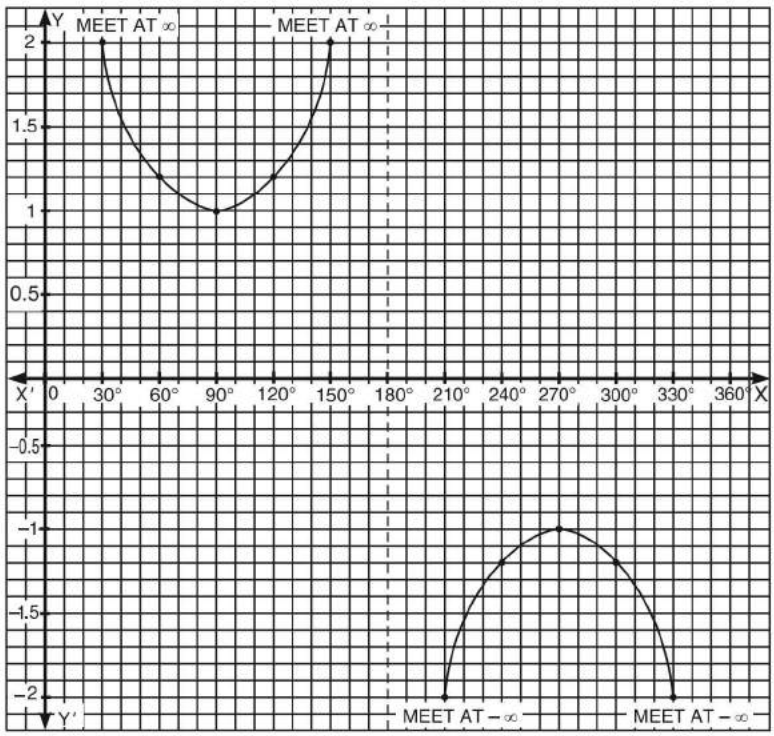


Fig. 4.30

Table of values:

θ	0°	30°	60°	90°	120°	150°	180°–0°	180°+0°	210°	240°	270°	300°	330°	360°
cosec θ	+ ∞	2	1.2	1	1.2	2	+ ∞	− ∞	–2	–1.2	–1	–1.2	–2	− ∞

The graph exhibits the following properties of cosec θ :

1. cosec θ cannot be numerically less than 1.
2. cosec θ is discontinuous at θ = 0°, 180°, ...

Fig. 4.31 shows the graph of cosec θ between θ = 0° to θ = 360°.

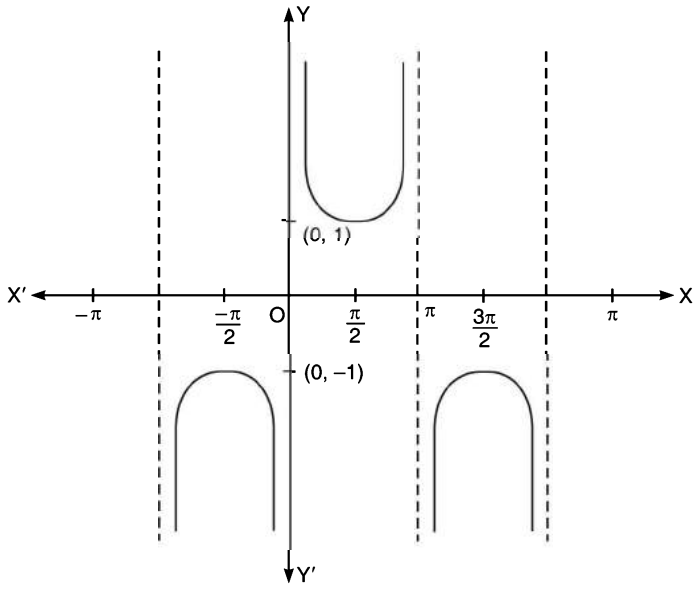
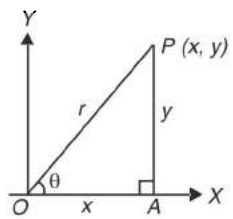


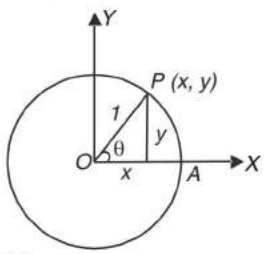
Fig. 4.31

Note. You know that the period of sin θ, cos θ, sec θ, is 2π or 360° and that the period of tan θ and cot θ is π or 180°. Graphically also you can find the period. The graph of sin θ from 0° to 360° is repeated for values of θ from 360° to 720° or from 720° to 1080°,etc. Similarly the graph of tan θ from 0° to 180° is repeated for values of θ from 180° to 360° or from 360° to 540°, etc. This means that the period of sin θ is 360°, whereas the period of tan θ is 180°. In the same way we can get the periods of cos θ, sec θ, cosec θ, and cot θ from their graphs.

CHAPTER SUMMARY



1. ΔOAP is a right triangle and let $\angle AOP = \theta$ degree. Then, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\text{cosec}\,\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$.



2. The figure shows a unit-circle. $\angle AOP = \frac{\text{arc } AP}{\text{radius}} = \frac{\theta}{1} = \theta$.

Let the coordinates of P be (x, y). Then, the circular functions of θ are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\text{cosec}\,\theta = \frac{1}{y}$, $\sec\theta = \frac{1}{x}$ and $\cot\theta = \frac{x}{y}$.

Note. The circular functions of a real number θ and the t-ratios of $\angle AOP$ are same.

3. Fundamental relations

(*a*) **Square relations**

(*i*) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$, $\sin^2\theta = 1 - \cos^2\theta$.

(*ii*) $1 + \tan^2\theta = \sec^2\theta \Rightarrow \tan^2\theta = \sec^2\theta - 1$, $\sec^2\theta - \tan^2\theta = 1$.

(*iii*) $1 + \cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow \cot^2\theta = \operatorname{cosec}^2\theta - 1$, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.

(*b*) **Quotient relations**

(*i*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, (*ii*) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

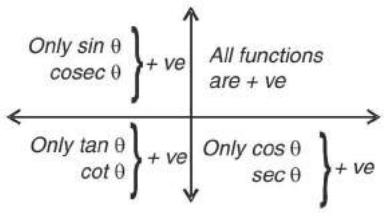
(*c*) **Reciprocal relations**

(*i*) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, (*ii*) $\sec\theta = \dfrac{1}{\cos\theta}$ (*iii*) $\tan\theta = \dfrac{1}{\cot\theta}$

4. Signs of trigonometric functions

The sign of a particular *t*-function in any quadrant can be remembered by the word **"all - sin - tan- cos"**,

"Add Sugar To Coffee". Whatever is written in a particular quadrant along with its reciprocal is positive and the rest are negative.



5. Domain and range of trigonometric functions

- $\sin\theta = y$ and $\cos\theta = x$ are defined for all real values of θ.
- $\tan\theta = \dfrac{y}{x}$ and $\sec\theta = \dfrac{1}{x}$ are not defined when $x = 0$, *i.e.*, when *OP* coincides with the *y*-axis, *i.e.*, when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots\ldots$ or $\dfrac{-\pi}{2}, \dfrac{-3\pi}{2}, \dfrac{-5\pi}{2}, \ldots$ *i.e.*, when $\theta = (2n + 1)\dfrac{\pi}{2}$.
- $\cot\theta = \dfrac{x}{y}$ and $\operatorname{cosec}\theta = \dfrac{1}{y}$ are not defined when $y = 0$, *i.e.*, when *OP* coincides with the *x*-axis, *i.e.*, when $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ or $-\pi, -2\pi, -3\pi, \ldots$, *i.e.*, when $\theta = n\pi$, *n* being any integer.
- $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$, *i.e.*, the value of $\sin\theta$ and $\cos\theta$ lies between –1 and +1, the minimum being –1 and the maximum +1.

  The other four *t*-functions may have all real values between $-\infty$ and $+\infty$.

6. Values of *t*-functions of standard angles.

t-function \ θ	0°	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	90° or $\frac{\pi}{2}$	180° or π
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*not defined	0

* Loosely we may write $\tan\theta = \infty$.

7. Trigonometric functions of angles of any magnitude

The following formulas are useful in reducing functions of any larger angle to functions of smaller angles.

	$-\theta$	$90^\circ - \theta$	$90^\circ + \theta$	$180^\circ - \theta$	$180^\circ + \theta$	$2n\pi - \theta$	$2n\pi + \theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$\tan\theta$

8. Periodic functions

- sine, cos, cosec and sec are t-functions with period 2π.
  Thus, $\sin(2\pi + \theta) = \sin\theta$, $\cos(2\pi + \theta) = \cos\theta$.
- $\tan\theta$ and $\cot\theta$ are periodic functions with period π.
  Thus, $\tan(\pi + \theta) = \tan\theta$, and $\cot(\pi + \theta) = \cot\theta$.

CHAPTER TEST

1. If $\cot A = \frac{3}{4}$, find the value of $3\cos A + 5\sin A$, where A lies in the first quadrant.

2. If $\cos 120^\circ = \frac{-1}{2}$ find the values of $\sin 120^\circ$ and $\tan 120^\circ$. ***[Tripura]***

3. Prove that $\sec(-1680^\circ).\sin 330^\circ = -1$. ***[WBHS]***

4. If A, B, C, D are the angles of a cyclic quadrilateral, show that
$\cos A + \cos B + \cos C + \cos D = 0$ ***[IIT, WBJEE, HP]***

[Hint. Since opposite angles of a cyclic quadrilateral are supplementary, therefore, $A + C = 180^\circ$, $B + D = 180^\circ \Rightarrow C = 180^\circ - A$ and $D = 180^\circ - B$.]

5. If $\tan 25^\circ = a$, prove that $\dfrac{\tan 155^\circ - \tan 115^\circ}{1 + \tan 155^\circ \cdot \tan 115^\circ} = \dfrac{1 - a^2}{2a}$. ***[AP]***

6. If A, B, C be the angles of a triangle, prove that $\dfrac{\sin(B+C) + \sin(C+A) + \sin(A+B)}{\sin(\pi + A) + \sin(3\pi + B) + \sin(5\pi + C)} = -1$.

7. Prove that $\dfrac{\cos\theta}{1 - \sin\theta} + \dfrac{1 - \sin\theta}{\cos\theta} = 2\sec\theta$.

8. If $\sec\theta = \sqrt{2}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, find the value of $\dfrac{1 + \tan\theta + \operatorname{cosec}\theta}{1 + \cot\theta - \operatorname{cosec}\theta}$.

ANSWERS

1. $5\frac{4}{5}$ **2.** $\frac{\sqrt{3}}{2}; -\sqrt{3}$ **8.** -1

CHAPTER 5

COMPOUND AND MULTIPLE ANGLES

5.01. *The addition formulae

(*a*) sin (A + B)

Let the revolving line, starting from the position OX, describe the first angle XOY = A, and then proceed further to trace out ∠ YOZ = B in the position OZ. OX, OY, OZ represent respectively the initial, middle and final positions of the revolving line and ∠ XOZ = A + B.

From any point P on OZ, the final position, draw two perpendiculars PL and PN on OX and OY respectively. From N, the foot of the perpendicular PN, draw perpendiculars NM and NK on OX and PL, respectively.

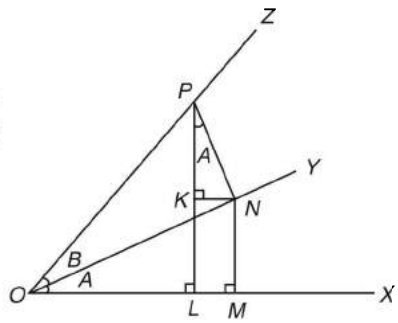


Fig. 5.01

∠ KPN = (90° – ∠ PNK) = ∠ KNO = ∠ NOM = ∠ A

$$\sin(A+B) = \frac{LP}{OP} = \frac{LK+KP}{OP} = \frac{MN+KP}{OP} = \frac{MN}{OP} + \frac{KP}{OP}$$

$$= \frac{MN}{ON} \times \frac{ON}{OP} + \frac{KP}{PN} \times \frac{PN}{OP}$$

$$= \sin A \cos B + \cos KPN \sin B$$

$$= \sin A \cos B + \cos A \sin B$$

$\therefore$ **sin (A + B) = sin A cos B + cos A sin B** ...(1)

(*b*) cos (A + B)

From the same figure,

$$\cos(A+B) = \frac{OL}{OP} = \frac{OM - LM}{OP} = \frac{OM - KN}{OP}$$

$$= \frac{OM}{OP} - \frac{KN}{OP}$$

$$= \frac{OM}{ON} \times \frac{ON}{OP} - \frac{KN}{PN} \times \frac{PN}{OP}$$

$$= \cos A \cos B - \sin KPN \sin B$$

$$= \cos A \cos B - \sin A \sin B$$

$\therefore$ ** **cos (A + B) = cos A cos B – sin A sin B** ...(2)

(*c*) tan (A + B)

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

---

* Here, the Addition formulae are proved for the case in which A and B are acute angles, but the same proof will be found to apply to angles of any size, due attention being paid to the signs of the quantities involved.

** This formula can also be obtained from (1) by changing B to (90° + B).

Dividing numerator and denominator by cos A cos B, we have

$$\mathbf{\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}} \qquad ...(3)$$

(*d*) cot (A + B)

$$\cot (A+B) = \frac{\cos (A+B)}{\sin (A+B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

Dividing numerator and denominator by sin A sin B, we have

$$\mathbf{\cot (A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}} \qquad ...(4)$$

(*e*) Replacing B by (–B) in (1), (2), (3) and (4) and recalling that sin (–B) = – sin B, cos (–B) = cos B, tan (– B) = – tan B, we obtain

$$\mathbf{\sin (A-B) = \sin A \cos B - \cos A \sin B} \qquad ...(5)$$

$$\mathbf{\cos (A-B) = \cos A \cos B + \sin A \sin B} \qquad ...(6)$$

$$\mathbf{\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}} \qquad ...(7)$$

$$\mathbf{\cot (A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}} \qquad ...(8)$$

Cor. 1. $\sin(A+B)\sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) = \sin^2 A \cos^2 B - \sin^2 B \cos^2 A$

$= \sin^2 A (1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A) = \sin^2 A - \sin^2 B.$

$\therefore$ $\mathbf{\sin (A+B)\sin (A-B) = \sin^2 A - \sin^2 B}$ ...(9)

i.e. sin (sum) × sin (diff.) = sin² (1st angle) – sin² (2nd angle)

Cor. 2. $\cos (A+B)\cos (A-B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B = \cos^2 A - \sin^2 B.$

$\therefore$ $\mathbf{\cos (A+B)\cos (A-B) = \cos^2 A - \sin^2 B}$ ...(10)

i.e. cos (sum) × cos (diff.) = cos² (1st angle) – sin² (2nd angle)

Cor. 3. $\tan (A+B)\tan (A-B)$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$$

$$\therefore \quad \mathbf{\tan (A+B)\tan (A-B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A . \tan^2 B}} \qquad ...(10A)$$

Cor. 4. $\tan (A+B+C) = \tan (\overline{A+B} + C)$

$$= \frac{\tan (A+B) + \tan C}{1 - \tan (A+B)\tan C} = \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B}\tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\therefore \quad \mathbf{\tan (A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}} \qquad ...(11)$$

Alternative proof of the formulae for cos (A ± B) and sin (A ± B).

We will prove the formula for cos (A + B) and derive other formulas from this.

To prove that cos (A + B) = cos A cos B – sin A sin B

Proof. Let O be the centre of a unit circle.

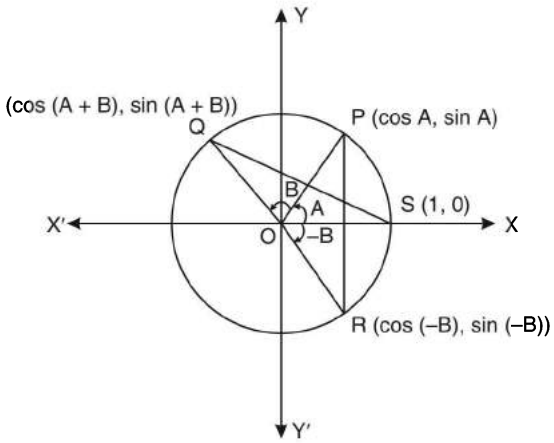


Fig. 5.02

Let ∠SOP = A radian, ∠POQ = B radian, ∠SOR (in clockwise direction) = – B radian

Now ∠SOQ = A + B and ∠ROP = A – B

Since radius of a circle is unity,

$\therefore$ arc SP = A, arc PQ = B, arc SR = |– B| = B [In formula $\theta = \frac{l}{r}$, θ is always taken as positive.]

Also as radius of the circle is 1.

$\therefore$ $P \equiv (\cos A, \sin A)$,

$Q \equiv \cos (A + B), \sin (A + B)$,

$R \equiv (\cos (-B), \sin (-B)$ or $R = (\cos B, -\sin B)$

$\because$ $\Delta SOQ \cong \Delta POR$

$\therefore$ $SQ = PR$

$\Rightarrow$ $SQ^2 = PR^2$

$\Rightarrow$ $[1 - \cos (A + B)]^2 + [0 - \sin (A + B)]^2 = [\cos A - \cos (-B)^2 + [\sin A - \sin (-B)]^2$

$\Rightarrow$ $1 + \cos^2 (A + B) - 2 \cos (A + B) + \sin^2 (A + B) = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$

$\Rightarrow$ $1 + \cos^2 (A + B) + \sin^2 (A + B) - 2 \cos (A + B)$

$= \cos^2 A + \cos^2 B - 2 \cos A \cos B + \sin^2 A + \sin^2 B + 2 \sin A \sin B$

$\Rightarrow$ $2 - 2 \cos (A + B) = (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2 (\cos A \cos B - \sin A \sin B)$

$\Rightarrow$ $2 - 2 \cos (A + B) = 2 - 2 (\cos A \cos B - \sin A \sin B)$

$\Rightarrow$ $\cos (A + B) = \cos A \cos B - \sin A \sin B$ ... (1)

2. Putting – B in place of B in (1), we get

$$\cos (A - B) = \cos A \cos (-B) - \sin A \sin (-B)$$

$$= \cos A \cos B + \sin A \sin B \qquad ... (2)$$

3. $\sin(A+B) = \cos\left[\frac{\pi}{2}-(A+B)\right]$

$= \cos\left[\left(\frac{\pi}{2}-A\right)-B\right]$

$= \cos\left(\frac{\pi}{2}-A\right)\cos B + \sin\left(\frac{\pi}{2}-A\right)\sin B$

$= \sin A\cos B + \cos A\sin B$

4. $\sin(A-B) = \cos(90^\circ - \overline{A-B})$ $\quad\because\ \sin\theta = \cos(90^\circ-\theta)$

$= \cos(\overline{90^\circ - A} + B)$

$= \cos(90^\circ - A)\cos B - \sin(90^\circ - A)\sin B$

[Using $\cos(A+B) = \cos A\cos B - \sin A\sin B$]

$\Rightarrow \quad \sin(A-B) = \sin A\cos B - \cos A\sin B$

Ex. 1. Compute sin 75°, cos 75° and tan 15°, from the functions of 30° and 45°.

Sol. $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{2.449+1.414}{4} = \frac{3.863}{4} \simeq 0.966$$

$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ\cos 30^\circ - \sin 45^\circ\sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{2.449-1.414}{4} \simeq 0.259$$

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$= \frac{1-\frac{1}{\sqrt{3}}}{1+1\times\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} \qquad \text{(By rationalising the denom.)}$$

$= 2 - 1.732 \simeq 0.268.$

Note. It should be committed to memory, that

$\cos 15^\circ = \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15^\circ$, in surd form.

Ex. 2. If α and β are acute angles and $\sin\alpha = \frac{3}{5}$, $\sin\beta = \frac{8}{17}$, find the values of $\sin(\alpha\pm\beta)$, $\cos(\alpha\pm\beta)$ and $\tan(\alpha\pm\beta)$.

Sol. As α is acute, so $\cos\alpha = \sqrt{\left(1-\frac{9}{25}\right)} = \frac{4}{5}$, $\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

As β is acute, so $\cos\beta = \sqrt{\left(1-\frac{64}{289}\right)} = \frac{15}{17}$, $\tan\beta = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

$\therefore \quad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \frac{3}{5}\times\frac{15}{17} \pm \frac{4}{5}\times\frac{8}{17} = \frac{77}{85}, \frac{13}{85}$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta = \frac{4}{5}\times\frac{15}{17} \mp \frac{3}{5}\times\frac{8}{17} = \frac{36}{85}, \frac{84}{85}.$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} = \frac{\frac{3}{4} \pm \frac{8}{15}}{1 \mp \frac{3}{4}\times\frac{8}{15}} = \frac{77}{36}, \frac{13}{84}.$$

Ex. 3. Prove that $\sin(\alpha + 30°) = \cos\alpha + \sin(\alpha - 30°)$ [*SC*]

Sol. Left side $= \sin(\alpha + 30°) = \sin\alpha\cos 30° + \cos\alpha\sin 30°$

$= (\sin\alpha\cos 30° - \cos\alpha\sin 30°) + 2\cos\alpha\sin 30°$

$= \sin(\alpha - 30°) + 2 \times \cos\alpha \times \frac{1}{2} = \cos\alpha + \sin(\alpha - 30°) =$ Right side.

Ex. 4. Prove that

$\sin(x+y)\sin(x-y) + \sin(y+z)\sin(y-z) + \sin(z+x)\sin(z-x) = 0.$

[*NDA April 2007*]

Sol. Left side $= \sin^2 x - \sin^2 y + \sin^2 y - \sin^2 z + \sin^2 z - \sin^2 x$ [By Formula 9]

$= 0 =$ Right side.

Ex. 5. Find the value of $\cos^2 45° - \sin^2 15°$.

Sol. $\cos^2 45° - \sin^2 15° = \cos(45° + 15°)\cos(45° - 15°)$ [By Formula 10]

$$= \cos 60°\cos 30° = \frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}.$$

Ex. 6. Prove that $\tan 11A \tan 7A \tan 4A = \tan 11A - \tan 7A - \tan 4A$.

Sol. $\tan 11A = \tan(7A + 4A) = \frac{\tan 7A + \tan 4A}{1 - \tan 7A \tan 4A}$

$\therefore \quad \tan 11A(1 - \tan 7A \tan 4A) = \tan 7A + \tan 4A$

$\Rightarrow \quad \tan 11A \tan 7A \tan 4A = \tan 11A - \tan 7A - \tan 4A.$

Ex. 7. Prove that $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$.

Sol. $\tan 56° = \tan(45° + 11°) = \frac{\tan 45° + \tan 11°}{1 - \tan 45°\tan 11°} = \frac{1 + \tan 11°}{1 - \tan 11°}$

$$= \frac{1 + \frac{\sin 11°}{\cos 11°}}{1 - \frac{\sin 11°}{\cos 11°}} = \frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$$

Ex. 8. If $\sin(\alpha + \beta) = \frac{4}{5}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $0 < \alpha < \frac{\pi}{4}$, $0 < \beta < \frac{\pi}{4}$; find $\tan 2\alpha$.

[*ISC 1990*]

Sol. $(\alpha + \beta) < \frac{\pi}{4} + \frac{\pi}{4}$, *i.e.* $(\alpha + \beta) < \frac{\pi}{2}$

$\therefore \quad (\alpha + \beta)$ is in 1st quadrant.

$(\alpha - \beta)$ can be positive or negative. Therefore, $\tan(\alpha + \beta)$ is always positive, whereas $\tan(\alpha - \beta)$ can be positive, or negative.

$$\tan 2\alpha = \tan\left[\overline{\alpha+\beta}+\overline{\alpha-\beta}\right]$$

$$= \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta).\tan(\alpha-\beta)} = \frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}} = \frac{63}{16}, \text{ if } \tan(\alpha-\beta) > 0$$

Also $\tan 2\alpha = \dfrac{\frac{4}{3}-\frac{5}{12}}{1+\frac{4}{3}.\frac{5}{12}} = \dfrac{33}{56}$, if $\tan(\alpha-\beta) < 0$.

Hence, $\tan 2\alpha = \dfrac{63}{16}$ or $\dfrac{33}{56}$.

EXERCISE 5 (*a*)

1. Compute :

(*i*) sin 15° from the functions of 60° and 45°, (*ii*) cos 345° from the functions of 300° and 45°, (*iii*) tan 105° from the functions of 45° and 60°, (*iv*) sin 135° from the functions of 180° and 45°, (*v*) cos 195° from the functions of 150° and 45°, (*vi*) cosec $(13\pi/12)$ **[*NDA April 2007*]**

2. Simplify by reducing to a single term :

(*i*) $\sin 3\alpha \cos 2\alpha + \cos 3\alpha \sin 2\alpha$. (*ii*) $\cos 5\theta \cos 2\theta - \sin 5\theta \sin 2\theta$.

(*iii*) $\sin 22° \cos 38° + \cos 22° \sin 38°$. (*iv*) $\sin 80° \cos 20° - \cos 80° \sin 20°$.

(*v*) $\sin(x-y)\cos x - \cos(x-y)\sin x$. (*vi*) $\cos(\theta+\alpha)\cos(\theta-\alpha) - \sin(\theta+\alpha)\sin(\theta-\alpha)$.

(*vii*) $\dfrac{\tan 69° + \tan 66°}{1-\tan 69° \tan 66°}$ **[*SC*]** (*viii*) $\dfrac{\tan\alpha - \tan(\alpha-\beta)}{1+\tan\alpha\tan(\alpha-\beta)}$.

Prove that:

3. $(\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2 = 1$.

4. $\sin(60° + \theta) - \sin(60° - \theta) = \sin\theta$. **5.** $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$.

6. $\sin(240° + \theta) + \cos(330° + \theta) = 0$. **7.** $\sin(A - 45°) = \dfrac{1}{\sqrt{2}}(\sin A - \cos A)$.

8. $\cos\left(\dfrac{\pi}{3}+x\right) = \dfrac{\cos x - \sqrt{3}\sin x}{2}$.

9. (*i*) $\tan(45° + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$. (*ii*) $\tan(45° - \theta) = \dfrac{1-\tan\theta}{1+\tan\theta}$.

10. (*i*) $\dfrac{\sin(\theta+\phi)}{\sin\theta\cos\phi} = \cot\theta\tan\phi + 1$. (*ii*) $\dfrac{\sin(\theta-\phi)}{\sin\theta\sin\phi} = \cot\phi - \cot\theta$.

11. $\dfrac{\sin(A-B)}{\sin A\sin B} + \dfrac{\sin(B-C)}{\sin B\sin C} + \dfrac{\sin(C-A)}{\sin C\sin A} = 0$. **12.** $\sin 105° + \cos 105° = \cos 45°$.

13. Find the value of $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, and $\tan(\alpha+\beta)$, given

(*i*) $\sin\alpha = \dfrac{3}{5}$, $\cos\beta = \dfrac{5}{13}$, α and β in Quadrant I.

(*ii*) $\cos\alpha = \dfrac{-12}{13}$, $\cot\beta = \dfrac{24}{7}$, α in Quadrant II, β in Quadrant III.

14. Find the values of $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$ and $\tan(\alpha - \beta)$, given

(*i*) $\sin\alpha = \frac{8}{17}$, $\tan\beta = \frac{5}{12}$, α and β in Quadrant I.

(*ii*) $\cos\alpha = \frac{-12}{13}$, $\cot\beta = \frac{24}{7}$, α in Quadrant II, β in Quadrant I.

15. If A and B are acute angles , find (A + B) given

(*i*) $\sin A = \frac{1}{\sqrt{5}}$, $\sin B = \frac{1}{\sqrt{10}}$. (*ii*) $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$.

16. Given that $\tan\alpha = \frac{m}{m+1}$, $\tan\beta = \frac{1}{2m+1}$ then what is the value of $\alpha + \beta$? **[*NDA April 2007*]**

17. In the ΔABC, the foot of the perpendicular from A to BC is D. Given that $\tan B = \frac{4}{3}$, $\cos C = \frac{15}{17}$ and that AB = 20 cm, calculate without using tables

(*i*) the lengths of the sides AC and BC; (*ii*) the value of sin A. **[*SC*]**

18. Given that $\tan(A + B) = 1$ and that $\tan(A - B) = \frac{1}{7}$, find without using tables the values of tan A and tan B. **[*SC*]**

19. If $\tan(A + B) = x$ and $\tan B = \frac{1}{2}$ prove that $\tan A = \frac{2x-1}{x+2}$ and obtain an expression for $\tan(A - B)$ in terms of x. If $\tan(A - B) = \frac{1}{3}$ and A is acute, find A without using tables. **[*SC*]**

20. If $\sin(\alpha + \beta) = \frac{4}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$, $\alpha + \beta$, $\alpha - \beta$ being acute angles prove that $\tan 2\alpha = \frac{63}{16}$.

21. Prove that $1 + \tan 2\theta \tan\theta = \sec 2\theta$. **22.** $4\sin(\frac{\pi}{3} - \theta)\sin(\frac{\pi}{3} + \theta) = 3 - 4\sin^2\theta$.

23. $\frac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$. **24.** $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$.

25. $\tan 75° - \tan 30° - \tan 75° \tan 30° = 1$.

26. $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta + \phi) = \cos(2\theta + 2\phi)$.

27. If $\sin(\theta + \alpha) = \cos(\theta + \alpha)$, prove that $\tan\theta = \frac{1 - \tan\alpha}{1 + \tan\alpha}$.

28. Given that A = B + C, prove that

$\tan A - \tan B - \tan C = \tan A \tan B \tan C$. **[*ISC*]**

29. If A + B = 45°, show that $\tan A + \tan B + \tan A \tan B = 1$.

Hence, or otherwise, express 22°30′ in surd form. **[*SC*]**

30. (*i*) Use the expansion of $\tan(A - B)$ to find tan 15° without the use of tables, leaving your answer in *surd* form with an integral denominator.

(*ii*) Prove that $\frac{2\tan A}{1 + \tan^2 A} = \sin 2A$. **[*ISC*]**

31. If A + B = 225°, prove that $\tan A + \tan B = 1 - \tan A \tan B$ **[*ISC*]**

ANSWERS

1. (*i*) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (*ii*) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (*iii*) $-(2+\sqrt{3})$ (*iv*) $\frac{1}{\sqrt{2}}$ (*v*) $-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ (*vi*) $-\sqrt{6}-\sqrt{2}$

2. (*i*) $\sin 5\alpha$ (*ii*) $\cos 7\theta$ (*iii*) $\frac{\sqrt{3}}{2}$ (*iv*) $\frac{\sqrt{3}}{2}$

(*v*) $-\sin y$ (*vi*) $\cos 2\theta$ (*vii*) -1 (*viii*) $\tan\beta$

13. (*i*) $\frac{63}{65}, \frac{-16}{65}, \frac{-63}{16}$ (*ii*) $\frac{-36}{325}, \frac{323}{325}, \frac{-36}{323}$

14. (*i*) $\frac{21}{221}, \frac{220}{221}, \frac{21}{220}$ (*ii*) $\frac{204}{325}, \frac{-253}{325}, \frac{-204}{253}$

15. (*i*) 45° or 135° (*ii*) 45° **16.** $\frac{\pi}{4}$ **17.** (*i*) 34 cm and 42 cm (*ii*) $\frac{84}{85}$

18. $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ or $\tan A = -2$, $\tan B = -3$.

19. $\frac{3x-4}{4x+3}$, 45° **29.** $\sqrt{2}-1$. **30.** (*i*) $2-\sqrt{3}$.

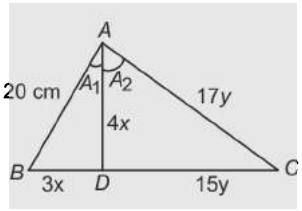
HINTS TO SELECTED QUESTIONS

17. (*ii*) To find $\sin A$, use $\sin A = \sin(A_1 + A_2)$
where $\angle BAD = A_1$ and $\angle CAD = A_2$

$(4x)^2 + (3x)^2 = 20^2 \Rightarrow x = 4 \Rightarrow AD = 16\,\text{cm}$

$(16)^2 + (15y)^2 = (17y)^2 \Rightarrow y = 2$

$\sin A = \sin(A_1 + A_2) = \sin A_1 \cos A_2 + \cos A_1 \sin A_2$.



18. $\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$, $\frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{1}{7}$. Let $\tan A = x, \tan B = y$.

Then, $x + y = 1 - xy$, $7(x - y) = 1 + xy$. Solve.

20. $\tan 2\alpha = \tan\left(\overline{\alpha+\beta} + \overline{\alpha-\beta}\right)$. **23.** $\tan 62° = \tan(45° + 17°)$.

28. Type solved Ex. 6. $A = B + C \Rightarrow \tan A = \tan(B + C) = \frac{\tan B + \tan C}{1 - \tan B \tan C}$. Now cross multiply.

PRODUCT FORMULAE

5.02. Converting products into sums or differences

By formula (1), $\sin(A + B) = \sin A \cos B + \cos A \sin B$

By formula (5), $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\therefore$ By addition,

$$\mathbf{2 \sin A \cos B = \sin(A + B) + \sin(A - B)} \qquad ...(12)$$

and by subtraction,

$$\mathbf{2 \cos A \sin B = \sin(A + B) - \sin(A - B)} \qquad ...(13)$$

By formula (2), $\cos(A + B) = \cos A \cos B - \sin A \sin B$

By formula (6), $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\therefore$ By addition,

$$\mathbf{2 \cos A \cos B = \cos (A + B) + \cos (A - B)} \quad ...(14)$$

and by subtracting formula (2) from formula (6),

$$\mathbf{2 \sin A \sin B = \cos (A - B) - \cos (A + B)} \quad ...(15)$$

Aid to Memory:

2 sin × cos = sin (sum) + sin (difference)
2 cos × sin = sin (sum) – sin (difference)
2 cos × cos = cos (sum) + cos (difference)
2 sin × sin = cos (difference) – cos (sum)

Sum stands for (1st angle + 2nd angle) and difference stands for (1st angle – 2nd angle).

Caution. (1) Each product should begin with 2. Thus $\sin\theta \sin 5\theta$ cannot be transformed unless we write it in the form $\frac{1}{2}(2 \sin\theta \sin 5\theta)$.

(2) In formula (15), cos (A – B) precedes cos (A + B).

Ex. 9. Find the value of sin 75° sin 15°.

Sol. $\sin 75° \sin 15° = \frac{1}{2}(2 \sin 75° \sin 15°)$.

$= \frac{1}{2}[\cos (75° - 15°) - \cos (75° + 15°)] = \frac{1}{2}(\cos 60° - \cos 90°) = \frac{1}{2}(\frac{1}{2} - 0) = \frac{1}{4}$.

Ex. 10. Prove that $\sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$.

Sol. Left side = $\sin 20° \sin 40° \sin 80° = \frac{1}{2}(2 \sin 20° \sin 40°) \sin 80°$

$= \frac{1}{2}(\cos 20° - \cos 60°) \sin 80°$

[$\because$ 2 sin × sin = cos (diff.) – cos (sum); cos (– 20°) = cos 20°]

$= \frac{1}{2}(\cos 20° - \frac{1}{2}) \sin 80° = \frac{1}{4}(2 \cos 20° \sin 80° - \sin 80°)$

$= \frac{1}{4}[\sin 100° - \sin (-60°) - \sin 80°]$

$= \frac{1}{4}(\sin 80° + \sin 60° - \sin 80°)$

$= \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$ = Right side.

$\because$ 2 cos × sin = sin (sum) – sin (difference)

sin 100° = sin of (its supplement)
= sin 80°

5.03. Converting sums or differences into products

By formula (12), $\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$

Let $A + B = C$ and $A - B = D$. Then

by addition, $A = \frac{C + D}{2}$

by subtraction, $B = \frac{C - D}{2}$

Therefore, substituting, we get

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \quad ...(16)$$

Making the same substitution in formulas (13), (14), and (15), we get

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \quad ...(17)$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \quad ...(18)$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \quad ...(19)$$

[By formula (15),

$$\cos (A + B) - \cos (A - B) = -2 \sin A \sin B = 2 \sin A \sin(-B)]$$

Notes. (1) These formulas convert algebraic sums of two sines or cosines into products. Hence, forth they will be called 'C, D' formulas.

(2) The first factor of every product is $\frac{\text{1st angle + 2nd angle}}{2}$ and the second factor is $\frac{\text{1st angle – 2nd angle}}{2}$ except in formula (19), in which the second factor is $\frac{\text{2nd angle – 1st angle}}{2}$. [Remember carefully]

(3) These formulas can be used to convert expressions of the type sin ± sin or cos ± cos, *i.e.*, the sum or difference of two sines or two cosines and not expressions of the type sin ± cos . If this be the case, we will first have to render them in the proper form by making use of the formulas for '90° ± θ.' Thus,

$$\sin \theta + \cos \theta = \sin \theta + \sin \left(\frac{\pi}{2} - \theta \right).$$

(4) These formulas cannot be used for expressions of the type sin ± *a* sin, or cos ± *a* cos, where *a* is any number.

Ex. 11. Convert the following into products:

(*i*) sin 53° + sin 21°, (*ii*) cos 83° – cos 17°, (*iii*) sin 50° – cos 80°.

Sol. (*i*) $\sin 53° + \sin 21° = 2 \sin \frac{53°+21°}{2} \cos \frac{53°-21°}{2} = 2 \sin \frac{74°}{2} \cos \frac{32°}{2}$

$$= 2 \sin 37° \cos 16°.$$

Note. After some practice, the student should be able to write the values of $\frac{C + D}{2}$ and $\frac{C-D}{2}$ directly.

(*ii*) $\cos 83° - \cos 17° = 2 \sin \frac{83°+17°}{2} \sin \frac{17° - 83°}{2}$ **[Note]**

$$= 2 \sin \frac{100°}{2} \sin \left(-\frac{66°}{2} \right) = -2 \sin 50° \sin 33°.$$

(*iii*) $\sin 50° - \cos 80° = \sin 50° - \sin 10°$ **[Change cos into sine]**

[$\cos 80° = \sin$ (its complementary ∠) $= \sin 10°$]

$$= 2 \cos \frac{50°+10°}{2} \sin \frac{50°-10°}{2} = 2 \cos 30° \sin 20°.$$

Ex. 12. Prove that $\dfrac{\sin 7A - \sin 5A}{\cos 5A + \cos 7A} = \tan A$.

Sol. Left side $= \dfrac{\sin 7A - \sin 5A}{\cos 5A + \cos 7A} = \dfrac{2\cos\dfrac{7A+5A}{2}\sin\dfrac{7A-5A}{2}}{2\cos\dfrac{5A+7A}{2}\cos\dfrac{5A-7A}{2}}$

$= \dfrac{\cos 6A \sin A}{\cos 6A \cos A} = \dfrac{\sin A}{\cos A} = \tan A =$ Right side

Ex. 13. Prove that

$$\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \cos 2A - \sin 2A \tan 3A.$$

Sol. Left side $= \dfrac{(\cos 3A + \cos 7A) + 2\cos 5A}{(\cos A + \cos 5A) + 2\cos 3A} = \dfrac{2\cos 5A \cos 2A + 2\cos 5A}{2\cos 3A \cos 2A + 2\cos 3A}$

$= \dfrac{2\cos 5A(\cos 2A + 1)}{2\cos 3A(\cos 2A + 1)} = \dfrac{\cos 5A}{\cos 3A} = \dfrac{\cos(2A + 3A)}{\cos 3A}$

$= \dfrac{\cos 2A \cos 3A - \sin 2A \sin 3A}{\cos 3A} = \dfrac{\cos 2A \cos 3A}{\cos 3A} - \dfrac{\sin 2A \sin 3A}{\cos 3A}$

$= \cos 2A - \sin 2A \tan 3A =$ Right side

Ex. 14. Prove that $\dfrac{\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta}{\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta} = \cot\dfrac{5}{2}\theta$.

[It may be noted here that $\theta + 4\theta = 2\theta + 3\theta$].

Sol. Left side $= \dfrac{(\cos\theta + \cos 4\theta) + (\cos 2\theta + \cos 3\theta)}{(\sin\theta + \sin 4\theta) + (\sin 2\theta + \sin 3\theta)}$

$$= \frac{2\cos\frac{5\theta}{2}\cos\left(-\frac{3\theta}{2}\right) + 2\cos\frac{5\theta}{2}\cos\left(-\frac{\theta}{2}\right)}{2\sin\frac{5\theta}{2}\cos\left(-\frac{3\theta}{2}\right) + 2\sin\frac{5\theta}{2}\cos\left(-\frac{\theta}{2}\right)}$$

$$= \frac{2\cos\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right)}{2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right)} = \frac{\cos\frac{5\theta}{2}}{\sin\frac{5\theta}{2}} = \cot\frac{5}{2}\theta$$

= Right side

EXERCISE 5 (*b*)

1. Convert the following products into sum or difference. If angles are given in degrees, evaluate from tables.

(*i*) $2\sin 48° \cos 12°$ (*ii*) $2\sin 54° \sin 66°$ (*iii*) $2\cos 5\theta \cos 3\theta$

(*iv*) $2\cos 72° \sin 56°$ (*v*) $\cos(\alpha + \beta)\cos(\alpha - \beta)$ (*vi*) $\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

2. Convert the following sums or differences into products:

(*i*) $\sin 12A + \sin 4A$ (*ii*) $\sin 37° + \sin 21°$ (*iii*) $\sin 12A - \sin 4A$

(*iv*) $\cos 79° + \cos 11°$ (*v*) $\cos 12\alpha + \cos 8\alpha$ (*vi*) $\cos 25° - \cos 37°$

(*vii*) $\sin 61° - \cos 39°$ (*viii*) $\sin 4x + \cos 2x$

Prove that

3. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$

4. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \dfrac{1}{\sqrt{3}}$.

5. $\dfrac{\sin 7x + \sin 3x}{\cos 7x + \cos 3x} = \tan 5x$.

6. $\dfrac{\cos 2B - \cos 2A}{\sin 2A + \sin 2B} = \tan (A - B)$.

7. $\dfrac{\sin (4A - 2B) + \sin (4B - 2A)}{\cos (4A - 2B) + \cos (4B - 2A)} = \tan (A+B)$

8. $\dfrac{\cos\alpha + 2\cos 3\alpha + \cos 5\alpha}{\cos 3\alpha + 2\cos 5\alpha + \cos 7\alpha} = \cos 3\alpha \sec 5\alpha$.

9. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$.

10. $\cos 20° + \cos 100° + \cos 140° = 0$

11. $\sin 10° + \sin 20° + \sin 40° + \sin 50° = \sin 70° + \sin 80°$.

12. (*i*) $\cos 15° - \sin 15° = \dfrac{1}{\sqrt{2}}$.

(*ii*) $\sin 36° + \cos 36° = \sqrt{2}\cos 9°$.

13. $\cos 20° \cos 40° \cos 80° = \dfrac{1}{8}$.

14. $\sin 10° \sin 50° \sin 70° = \dfrac{1}{8}$.

15. $4 \cos 12° \cos 48° \cos 72° = \cos 36°$

16. $\tan 20° \tan 40° \tan 80° = \tan 60°$

17. $2 \cos \dfrac{\pi}{13} \cos \dfrac{9\pi}{13} + \cos \dfrac{3\pi}{13} + \cos \dfrac{5\pi}{13} = 0$.

18. $\tan (A + 30°) + \cot (A - 30°) = \dfrac{1}{\sin 2A - \sin 60°}$.

19. $\cos A + \cos (120° - A) + \cos (120° + A) = 0$

20. $\cos (A + B) + \sin (A - B) = 2 \sin (45° + A) \cos (45° + B)$.

21. If $\cos A + \cos B = \dfrac{1}{3}$ and $\sin A + \sin B = \dfrac{1}{4}$ prove that $\tan \dfrac{1}{2}(A + B) = \dfrac{3}{4}$.

22. What is the value of $\dfrac{\cos 10° + \sin 20°}{\cos 20° - \sin 10°}$? **[*NDA April 2007*]**

ANSWERS

1. (*i*) 1.4538 (*ii*) 1.4781 (*iii*) $\cos 8\theta + \cos 2\theta$

(*iv*) 0.5124 (*v*) $\dfrac{1}{2}[\cos 2\alpha + \cos 2\beta]$ (*vi*) $\dfrac{1}{2}[\sin A + \sin B]$

2. (*i*) $2 \sin 8A \cos 4A$ (*ii*) $2 \sin 29° \cos 8°$ (*iii*) $2 \cos 8A \sin 4A$

(*iv*) $2 \cos 45° \cos 34°$ (*v*) $2 \cos 10\alpha \cos 2\alpha$ (*vi*) $2 \sin 31° \sin 6°$

(*vii*) $2 \cos 56° \sin 5°$ (*viii*) $2 \sin (45° + x) \cos (45° - 3x)$

22. $\sqrt{3}$

5.04. Double-angle formulae

In the formulas for sin (A + B) and cos (A + B), putting B = A, we obtain

$$\textbf{sin 2A = 2 sin A cos A} \quad ...(20)$$

$$\textbf{cos 2A = cos}^2\textbf{A – sin}^2\textbf{A} \quad ...(21)$$

In formula (21), substituting $1 - \sin^2 A$ for $\cos^2 A$, we get

$$\textbf{cos 2A = 1 – 2 sin}^2\textbf{A} \quad ...(22)$$

or putting $1 - \cos^2 A$ for $\sin^2 A$, we get

$$\textbf{cos 2A = 2 cos}^2\textbf{ A – 1} \quad ...(23)$$

In the formula for tan (A + B), Putting B = A, we obtain

$$\textbf{tan 2A} = \frac{\textbf{2 tan A}}{\textbf{1} - \textbf{tan}^2\textbf{A}} \quad ...(24)$$

Note. These formulas should be fully understood and remembered not in terms of 2A but in a more general form *i.e.,* **a double angle.** In all these formulas, the *t*-ratio of a double angle is expressed in terms of those of half the angle. Thus,

sin (a double angle) = 2 sin (half the angle) cos (half the angle).

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\ ;\quad \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$$

Ex. 15. (*i*) $\textbf{sin } 10x = \textbf{2 sin } 5x \textbf{ cos } 5x$

(*ii*) $\textbf{cos } 6x = \textbf{cos}^2\ 3x - \textbf{sin}^2\ 3x$

(*iii*) $\textbf{tan } 8x = \dfrac{\textbf{2 tan } 4x}{\textbf{1–tan}^2\ 4x}$

Cor. sin 2A and cos 2A in terms of tan A.

$$\sin 2A = 2\sin A\cos A = \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A}$$

Dividing the numerator and denominator by $\cos^2 A$,

$$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}.$$

Dividing the numerator and denominator by $\cos^2 A$, we have

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\therefore \quad \textbf{sin 2A} = \frac{\textbf{2 tanA}}{\textbf{1} + \textbf{tan}^2\textbf{A}} \quad ...(25)$$

$$\textbf{cos 2A} = \frac{\textbf{1} - \textbf{tan}^2\textbf{A}}{\textbf{1} + \textbf{tan}^2\textbf{A}} \quad ...(26)$$

Dividing (25) by (26), we get $\textbf{tan 2A} = \dfrac{\textbf{2 tanA}}{\textbf{1} - \textbf{tan}^2\textbf{A}}$

By formulas (22) and (23), we have by transposition

$$\mathbf{2\sin^2 A = 1 - \cos 2A} \qquad \therefore \quad \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\mathbf{2\cos^2 A = 1 + \cos 2A} \qquad \therefore \quad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\therefore \qquad \mathbf{\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}}$$

We use these formulas when we have to change squares of sines and cosines into their first degree terms. It should be noted that angle on the right side is double of the angle on the left side. Thus,

$$\cos^2 30° = \frac{1+\cos 60°}{2}, \quad \tan^2 60° = \frac{1-\cos 120°}{1+\cos 120°}.$$

Ex. 16. Given that $\sin A = \frac{3}{5}$ and that A is an acute angle, find without using tables, the values of sin 2A, cos 2A and tan 2A. Hence find the value of sin 4A. [*SC*]

Sol. $\sin A = \frac{3}{5}, \quad \therefore \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}, \quad \therefore \tan A = \frac{\sin A}{\cos A} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

$$\therefore \quad \sin 2A = 2\sin A\cos A = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}, \quad \cos 2A = 1 - 2\sin^2 A = 1 - 2\times\frac{9}{25} = \frac{7}{25}.$$

$$\therefore \quad \tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{3}{4}}{1-\frac{9}{16}} = \frac{24}{7} = 3\frac{3}{7}.$$

$$\left(\text{or dividing, } \frac{\sin 2A}{\cos 2A} = \frac{24}{25}\times\frac{25}{7} = \frac{24}{7}\right)$$

$$\sin 4A = 2\sin 2A\cos 2A = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625}.$$

Ex. 17. Prove that [*SC*]

(*i*) $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan x$, $x \in$ I or III quad.

(*ii*) $\tan A - \cot A = -2\cot 2A$.

Sol. (*i*) $\frac{1-\cos 2x}{1+\cos 2x} = \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x.$

[Apply that formula of cos 2*x* which cancels 1.]

$$\therefore \quad \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\tan^2 x} = |\tan x| = \tan x, \text{ since } \tan x \text{ is +ve in I and III quad.}$$

Maths alert! $\sqrt{x^2} = |x|$. We cannot write $\sqrt{a^2} = a$ unless we already know that $a \geq 0$.

(*ii*) $\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\cos A \sin A}$

$$= \frac{-2(\cos^2 A - \sin^2 A)}{2\cos A \sin A}$$

$$= -2\frac{\cos 2A}{\sin 2A} = -2\cot 2A.$$

1. Take minus common from the numerator to render it in the proper form.
2. Multiply numerator and denominator by 2 to have 2 cos A sin A in the denominator.

Ex. 18. Prove that

$$\frac{\cos A}{1-\sin A} = \tan\left(45° + \frac{A}{2}\right)$$

Sol. Left side $= \dfrac{\cos A}{1-\sin A} = \dfrac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}$

$$= \frac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)^2} = \frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}$$

$$= \frac{1 + \tan\frac{A}{2}}{1 - \tan\frac{A}{2}} \qquad \left[\text{Dividing throughout by } \cos\frac{A}{2}\right]$$

$$= \frac{\tan 45° + \tan\frac{A}{2}}{1 - \tan 45°.\tan\frac{A}{2}} = \tan\left(45° + \frac{A}{2}\right) = \text{Right side}$$

Ex. 19. Prove that **[*LU*]**

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 2\sec\theta.$$

Sol. Left side $= \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) + \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} + \dfrac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$

$$= \frac{\left(1 + \tan\frac{\theta}{2}\right)^2 + \left(1 - \tan\frac{\theta}{2}\right)^2}{1 - \tan^2\frac{\theta}{2}} = \frac{2\left(1 + \tan^2\frac{\theta}{2}\right)}{1 - \tan^2\frac{\theta}{2}} = \frac{2}{\cos\theta} = 2\sec\theta = \text{Right side}$$

[By formula (26)]

Ex. 20. Prove that

$$\sin^2\left(\frac{\pi}{8} + \frac{\theta}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{\theta}{2}\right) = \sin\frac{\pi}{4}\sin\theta.$$

Sol. Left side $= \dfrac{1}{2}\left\{1 - \cos\left(\dfrac{\pi}{4} + \theta\right)\right\} - \dfrac{1}{2}\left\{1 - \cos\left(\dfrac{\pi}{4} - \theta\right)\right\}$ $\left[\text{Using, } \sin^2\theta = \dfrac{1-\cos 2\theta}{2}\right]$

$$= \frac{1}{2}\left\{\cos\left(\frac{\pi}{4} - \theta\right) - \cos\left(\frac{\pi}{4} + \theta\right)\right\}$$

$$= \frac{1}{2} \times 2\sin\frac{\pi}{4}\sin\theta = \sin\frac{\pi}{4}\sin\theta = \text{Right side}$$

Method II. Using formula (9), we get, Left side $= \sin\dfrac{\pi}{4} \times \sin\theta$ immediately.

Ex. 21. Prove that: $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$, **where** $\theta \in \left[\frac{-\pi}{8}, \frac{\pi}{8}\right]$

Sol. L.H.S. $= \sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = \sqrt{2+\sqrt{2+\sqrt{2\,(2\cos^2 4\theta)}}}$

$= \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$

$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} = \sqrt{2+2\cos 2\theta}$

$= \sqrt{2(1+\cos 2\theta)} = \sqrt{2\cdot(2\cos^2\theta)} = \sqrt{4\cos^2\theta} = 2\,|\cos\theta|$

$= 2\cos\theta$ as $\theta \in \left[\frac{-\pi}{8}, \frac{\pi}{8}\right] \Rightarrow \cos\theta > 0 \Rightarrow |\cos\theta| = \cos\theta.$

Ex. 22. If $7A = \pi$, show that $\cos A \cos 2A \cos 3A = \frac{1}{8}$.

Sol. We have $\cos 3A = \cos(\pi - 4A) = -\cos 4A$ [$\because$ $\pi = 7A$, given]

L.H.S. $= \cos A \cos 2A \cos 3A = -\cos A \cos 2A \cos 4A$

$$= -\frac{2\sin A \cos A \cos 2A \cos 4A}{2\sin A}$$

[Multiplying num. and den. by $2\sin A$ **(Note this step)]**

$$= -\frac{\sin 2A \cos 2A \cos 4A}{2\sin A} = -\frac{\sin 4A \cos 4A}{4\sin A}$$

$$= -\frac{\sin 8A}{8\sin A} = -\frac{\sin(\pi + A)}{8\sin A} = -\frac{-\sin A}{8\sin A} = \frac{1}{8} = \text{R.H.S.} \quad (\because 7A = \pi \ \therefore \ 8A = \pi + A)$$

Ex. 23. Prove that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{2^4}$.

Sol. Let $\alpha = \frac{\pi}{9}$.

Let $C = \cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha$ and

$S = \sin\alpha\sin 2\alpha\sin 3\alpha\sin 4\alpha$. Then,

$C.S = (\sin\alpha\cos\alpha)(\sin 2\alpha\cos 2\alpha)(\sin 3\alpha\cos 3\alpha)(\sin 4\alpha\cos 4\alpha)$

$$= \frac{1}{2}\sin 2\alpha \cdot \frac{1}{2}\sin 4\alpha \cdot \frac{1}{2}\sin 6\alpha \cdot \frac{1}{2}\sin 8\alpha$$

But $\alpha = \frac{\pi}{9}$. *i.e.,* $9\alpha = \pi$

$\therefore$ $\sin 8\alpha = \sin(\pi - \alpha) = \sin\alpha$ and $\sin 6\alpha = \sin(\pi - 3\alpha) = \sin 3\alpha$

$\therefore$ $C.S = \frac{1}{2^4}\sin 2\alpha\sin 4\alpha\sin 3\alpha\sin\alpha = \frac{1}{2^4}.S \Rightarrow C = \frac{1}{2^4}$ $(\because S \neq 0)$

Ex. 24. Prove that

(*i*) $\cos^2 A + \cos^2(A + 120°) + \cos^2(A - 120°) = \frac{3}{2}$ **[*Roorkee*]**

(*ii*) $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$

[**Remark.** If squares or powers of squares of sine and cosine appear in an identity, change them into sines and cosines of multiples of angles by using the formulas $2\sin^2\theta = 1 - \cos 2\theta$ and $2\cos^2\theta = 1 + \cos 2\theta$]

Sol. (*i*) L.H.S. $= \frac{1}{2}[2\cos^2 A + 2\cos^2(A + 120°) + 2\cos^2(A - 120°)]$

$= \frac{1}{2}[1 + \cos 2A + \{1 + \cos 2(A + 120°)\} + \{1 + \cos 2(A - 120°)\}]$

$= \frac{1}{2}[1 + \cos 2A + 1 + \cos(2A + 240°) + 1 + \cos(2A - 240°)]$

$= \frac{1}{2}[3 + \cos 2A + \cos(2A + 240°) + \cos(2A - 240°)]$

$= \frac{1}{2}[3 + \cos 2A + 2\cos 2A \cos 240°]$

$[\because \cos(A + B) + \cos(A - B) = 2\cos A \cos B]$

$= \frac{1}{2}[3 + \cos 2A + 2\cos 2A(-\frac{1}{2})] = \frac{1}{2}[3 + \cos 2A - \cos 2A] = \frac{3}{2} = \text{R.H.S.}$

(*ii*) L.H.S. $= \left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2 + \left(\cos^2\frac{5\pi}{8}\right)^2 + \left(\cos^2\frac{7\pi}{8}\right)^2$

$= \left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1+\cos\frac{3\pi}{4}}{2}\right)^2 + \left(\frac{1+\cos\frac{5\pi}{4}}{2}\right)^2 + \left(\frac{1+\cos\frac{7\pi}{4}}{2}\right)^2$

$= \frac{1}{4}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right]$

$= 2\times\frac{1}{4}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right]$

$= \frac{1}{2}\times 2\times\left(1+\frac{1}{2}\right) = \frac{3}{2} = \text{R.H.S.}$

Ex. 25. Prove that $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta = \frac{\sin(2^n\theta)}{2^n(\sin\theta)}$. **(Remember this result)**

Sol. [**Note.** Here, each angle is double of the preceding angle.]

L.H.S. $= \frac{1}{2\sin\theta}.(2\sin\theta\cos\theta)\cos 2\theta \,.\, \cos 4\theta \ldots \cos 2^{n-1}\theta$

$= \frac{1}{2^2\sin\theta}.(2\sin 2\theta\cos 2\theta)\,.\,[\cos 4\theta \ldots \cos 2^{n-1}\theta]$

$$= \frac{1}{2^3 \sin\theta}.(2\sin 4\theta \cos 4\theta)\,.\,[\cos 8\theta \cos 16\theta \ldots \cos 2^{n-1}\theta]$$

$$= \frac{1}{2^4 \sin\theta}.\,(2\sin 8\theta \cos 8\theta)\,.\,[\cos 16\theta \ldots \cos 2^{n-1}\theta]$$

$$= \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$= \frac{1}{2^n \sin\theta}.[2\sin 2^{n-1}\theta \cos 2^{n-1}\theta] = \frac{\sin(2^n\theta)}{2^n \sin\theta}$$

Remark. You can use the above result to prove the following identities:

(*i*) $\cos A \cos 2A \cos 4A \cos 8A = \dfrac{\sin 2^4 A}{2^4 \sin A}$

(*ii*) $\cos 20° \cos 40° \cos 60° \cos 80° = \dfrac{1}{16}$

[**Hint.** Putting $\cos 60° = \dfrac{1}{2}$,

$$\text{L.H.S.} = \frac{1}{2}\left[\frac{\sin(2^3.20°)}{2^3 \sin 20°}\right] = \frac{\sin 160°}{16\sin 20°} = \frac{\sin(180° - 20°)}{16\sin 20°} = \frac{\sin 20°}{16\sin 20°} = \frac{1}{16}.]$$

(*iii*) $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = \dfrac{1}{16}$ **[*IIT*]**

(*iv*) $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}\sin\dfrac{7\pi}{14} = \dfrac{1}{8}$ **[*MNR 1992*]**

$$\left[\sin\frac{7\pi}{14} = \sin\frac{\pi}{2} = 1,\ \sin\frac{\pi}{14} = \sin\left(\frac{\pi}{2} - \frac{6\pi}{14}\right) = \cos\frac{6\pi}{14} = \cos\left(\pi - \frac{8\pi}{14}\right) = -\cos\frac{8\pi}{14}\right]$$

5.05. Triple-angle formulae

To prove that

$$\sin 3A = 3\sin A - 4\sin^3 A \qquad \ldots(27)$$

$$\cos 3A = 4\cos^3 A - 3\cos A \qquad \ldots(28)$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \qquad \ldots(29)$$

Sol. $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A = 3\sin A - 4\sin^3 A$

$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$

$= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A) = 4\cos^3 A - 3\cos A$

$\tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}\tan A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

5.06. One-third angle formulae

Replacing A by $\frac{A}{3}$ in the formulas for 3A, we obtain the following relations:

(*i*) $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

(*ii*) $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

(*iii*) $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$

I. Illustrations. (*i*) $\cos 81° = 4\cos^3 27° - 3\cos 27°$ (*ii*) $\sin 6A = 3\sin 2A - 4\sin^3 2A$

(*iii*) $\tan 45° = \dfrac{3\tan 15° - \tan^3 15°}{1 - 3\tan^2 15°}$

(*iv*) $8\cos^3 20° - 6\cos 20° = 2(4\cos^3 20° - 3\cos 20°)$

$= 2\cos(3 \times 20°) = 2\cos 60° = 2 \times \frac{1}{2} = 1.$ [*SC*]

Ex. 26. Prove that $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos 5\theta$.

Sol. R.H.S. $= \cos 5\theta = \cos(3\theta + 2\theta) = \cos 3\theta \cdot \cos 2\theta - \sin 3\theta \cdot \sin 2\theta$

$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)(2\sin\theta\cos\theta)$

$= (8\cos^5\theta - 4\cos^3\theta - 6\cos^3\theta + 3\cos\theta) - (6\sin^2\theta - 8\sin^4\theta)\cos\theta$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - [6(1 - \cos^2\theta) - 8(1 - \cos^2\theta)^2]\cos\theta$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - (-2 + 10\cos^2\theta - 8\cos^4\theta)\cos\theta$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta + 2\cos\theta - 10\cos^3\theta + 8\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \text{L.H.S.}$

Ex. 27. Prove that (*i*) $\sin 3A = 4\sin A \sin(60° - A)\sin(60° + A)$. **Hence deduce that**

(*ii*) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9} = \frac{3}{16}$.

Sol. (*i*) Right side $= 4\sin A\{\sin^2 60° - \sin^2 A\}$

$= 4\sin A\left(\frac{3}{4} - \sin^2 A\right)$

$= 4\sin A\left(\frac{3 - 4\sin^2 A}{4}\right)$

$= 3\sin A - 4\sin^3 A = \sin 3A = \text{Left side}$

(*ii*) We have proved above that

$$\sin A \sin\left(\frac{\pi}{3} + A\right)\sin\left(\frac{\pi}{3} - A\right) = \frac{1}{4}\sin 3A$$

Putting $A = \frac{\pi}{9}$ in the above relation, we get

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{\pi}{3} + \frac{\pi}{9}\right)\sin\left(\frac{\pi}{3} - \frac{\pi}{9}\right) = \frac{1}{4}\sin\left(\frac{\pi}{3}\right)$$

$\Rightarrow \sin\left(\frac{\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right) = \left(\frac{1}{4}\right)\sin\frac{\pi}{3}$

$\Rightarrow \sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{3\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = \left(\frac{1}{4}\right)\sin^2\frac{\pi}{3}$

$\Rightarrow \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9} = \frac{3}{16}$.

Ex. 28. Prove that $\cos\theta + \cos(120° + \theta) + \cos(\theta - 120°) = 0$.

Hence deduce that $\cos^3\theta + \cos^3(120° + \theta) + \cos^3(\theta - 120°) = \frac{3}{4}\cos 3\theta$.

Sol. L.H.S. $= \cos\theta + \cos(120° + \theta) + \cos(\theta - 120°)$

$= \cos\theta + \cos 120°\cos\theta - \sin 120°\sin\theta + \cos\theta\cos 120° + \sin\theta\sin 120°$

$= \cos\theta + 2\cos 120°\cos\theta = \cos\theta + 2\cos(180° - 60°)\cos\theta = \cos\theta - 2\cos 60°\cos\theta$

$= \cos\theta - 2\cdot\left(\frac{1}{2}\right)\cos\theta = \cos\theta - \cos\theta = 0 = \text{R.H.S.}$

Now let $a = \cos\theta$, $b = \cos(120° + \theta)$, $c = \cos(\theta - 120°)$

$\because\ a + b + c = 0\ \therefore\ a^3 + b^3 + c^3 = 3abc$. [From algebra]

$\Rightarrow \cos^3\theta + \cos^3(120° + \theta) + \cos^3(\theta - 120°) = 3\cos\theta\cdot\cos(120° + \theta)\cos(\theta - 120°)$

$= 3\cos\theta(\cos^2\theta - \sin^2 120°)$ [$\because \cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$]

$= 3\cos\theta\left(\cos^2\theta - \frac{3}{4}\right)$ [$\because \sin 120° = \sin(180 - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$]

$= \frac{3}{4}(4\cos^3\theta - 3\cos\theta) = \frac{3}{4}\cos 3\theta$.

5.07. Trigonometrical ratios of sub-multiple angles

Half angle formulae

By formula (22),

$\cos 2A = 1 - 2\sin^2 A\ \therefore\ \sin^2 A = \frac{1 - \cos 2A}{2}$

$\therefore\quad \mathbf{\sin A = \pm\sqrt{\frac{1 - \cos 2A}{2}}}$ .....(30)

Similarly, $\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$

By formula (23), $\cos 2A = 2\cos^2 A - 1\ \therefore\ \cos^2 A = \frac{1 + \cos 2A}{2}$

$\therefore\quad \mathbf{\cos A = \pm\sqrt{\frac{1 + \cos 2A}{2}}}$ ....(31)

Similarly, $\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$

By the quotient relation $\mathbf{\tan A = \pm\sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}}$ ...(31)

Similarly, $\tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$

The sign on the right hand sides of the formulas (30), (31) and (32) will be determined by the quadrant in which the angle $\frac{A}{2}$ lies.

Aid to Memory!

$$\cos(\text{any angle}) = \pm\sqrt{\frac{1+\cos(\text{double angle})}{2}}$$

$$\sin(\text{any angle}) = \pm\sqrt{\frac{1-\cos(\text{double angle})}{2}}$$

Ex. 29. Find $\cos\frac{7\pi}{8}$.

Sol. $\cos\frac{7\pi}{8} = \cos\left(\pi-\frac{\pi}{8}\right) = -\cos\frac{\pi}{8}\ \left(\text{double of } \frac{\pi}{8}=\frac{\pi}{4}\right)$

$$= -\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{1}{2}\sqrt{2+\sqrt{2}}$$

Ex. 30. Using the value of cos 315°, find the values of sin $157\frac{1}{2}°$ and cos $157\frac{1}{2}°$.

Sol. As $157\frac{1}{2}°$ lies in the second quadrant, so its sine is positive and cosine negative.

$\because\quad \frac{1}{2}A = 157\frac{1}{2}° \quad \therefore \quad A = 315°.$

$$\sin 157\frac{1}{2}° = \sqrt{\left(\frac{1-\cos 315°}{2}\right)}$$

$$= \sqrt{\left\{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\right\}} = \sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)} = \frac{1}{2}\sqrt{(2-\sqrt{2})}.$$

$$\because\quad \cos 157\frac{1}{2}° = -\sqrt{\left(\frac{1+\cos 315°}{2}\right)} = -\sqrt{\left\{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\right\}}$$

$$= -\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)} = -\sqrt{\left(\frac{\sqrt{2}(\sqrt{2}+1)}{4}\right)} = -\frac{1}{2}\sqrt{2+\sqrt{2}}$$

5.08. T-ratios of special angles

Ex. 31. Find the values of

(*i*) sin $22\frac{1}{2}°$ **(*ii*) cos $22\frac{1}{2}°$** **(*iii*) tan $22\frac{1}{2}°$**

(*iv*) sin $7\frac{1}{2}°$ **(*v*) cos $7\frac{1}{2}°$** **(*vi*) tan $11\frac{1}{4}°$**

Sol. (*i*) $\sin^2 A = \frac{1-\cos 2A}{2}$, Taking $A = 22\frac{1}{2}°$, we have,

$$\sin^2 22\frac{1}{2}° = \frac{1-\cos 45°}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{(\sqrt{2}-1)\sqrt{2}}{2\sqrt{2}\times\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

$\therefore\quad \sin 22\frac{1}{2}° = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{1}{2}\sqrt{(2-\sqrt{2})}$, since $\sin 22\frac{1}{2}°$ is +ve.

(*ii*) Similarly, find out the value of $\cos 22\frac{1}{2}°$ by using $\cos^2 A = \frac{1+\cos 2A}{2}$. You will obtain

$$\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{(2+\sqrt{2})}.$$

(*iii*) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

Let $\theta = 45°$, then $\frac{\theta}{2} = 22\frac{1}{2}°$ $\quad\therefore\quad 1 = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

or $\quad 2\tan\frac{\theta}{2} = 1 - \tan^2\frac{\theta}{2}$ or $\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 1 = 0$

$$\therefore\quad \tan\frac{\theta}{2} = \frac{-2\pm\sqrt{4-4\times1(-1)}}{2\times1} = \frac{-2\pm\sqrt{8}}{2} = \frac{-2\pm2\sqrt{2}}{2} = -1\pm\sqrt{2}$$

As $22\frac{1}{2}°$ is an angle in the first quadrant, so $\tan 22\frac{1}{2}°$ is positive.

$\therefore\quad \tan\frac{\theta}{2}$, *i.e.*, $\tan 22\frac{1}{2}° = \sqrt{2} - 1$.

You can also find out $\tan 22\frac{1}{2}°$ by putting

$$\tan 22\frac{1}{2}° = \frac{\sin 22\frac{1}{2}°}{\cos 22\frac{1}{2}°}$$

(*iv*) $\sin^2 7\frac{1}{2}° = \dfrac{1-\cos 15°}{2} = \frac{1}{2} - \dfrac{\sqrt{3}+1}{4\sqrt{2}} = \dfrac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}$ [See solved Ex. 1 in Art. 5.01]

$$= \frac{4-\sqrt{6}-\sqrt{2}}{8}$$

$\therefore\quad \sin 7\frac{1}{2}° = \dfrac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$, since $\sin 7\frac{1}{2}°$ is + ve

(*v*) Similarly, find out $\cos 7\frac{1}{2}°$ by putting $\cos^2 7\frac{1}{2}° = \dfrac{1+\cos 15°}{2} = \frac{1}{2} + \dfrac{\sqrt{3}+1}{4\sqrt{2}}$. You will obtain

$$\cos 7\frac{1}{2}° = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}},$$ [See Solved Ex 1, in Art. 5.01]

(*vi*) $\tan A = \dfrac{2\tan A/2}{1-\tan^2\frac{A}{2}} \Rightarrow \tan 22\frac{1}{2}° = \dfrac{2\tan 11\frac{1}{4}°}{1-\tan^2 11\frac{1}{4}°}$

$\Rightarrow \sqrt{2}-1 = \dfrac{2x}{1-x^2}$, putting $\tan 11\frac{1}{4}° = x$

$\Rightarrow (\sqrt{2}-1)x^2+2x-(\sqrt{2}-1)=0,$

$\Rightarrow \quad x=\dfrac{-2\pm\sqrt{(4+4(\sqrt{2}-1)^2}}{2(\sqrt{2}-1)} \qquad \left[\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$

$\Rightarrow \quad x=\dfrac{-1\pm\sqrt{(4-2\sqrt{2})}}{(\sqrt{2}-1)}$

$\Rightarrow \quad \tan 11\dfrac{1^\circ}{4}=\dfrac{-1+\sqrt{(4-2\sqrt{2})}}{(\sqrt{2}-1)}\times\dfrac{\sqrt{2}+1}{\sqrt{2}+1}$, since $\tan 11\dfrac{1^\circ}{4}$ is +ve.

$\Rightarrow -(\sqrt{2}+1)+(\sqrt{2}+1)\sqrt{4-2\sqrt{2}}=-(\sqrt{2}+1)+\sqrt{(\sqrt{2}+1)^2}.\sqrt{4-2\sqrt{2}}$

$\Rightarrow -(\sqrt{2}+1)+\sqrt{3+2\sqrt{2}}\,.\sqrt{4-2\sqrt{2}}=-(\sqrt{2}+1)+\sqrt{(3+2\sqrt{2})(4-2\sqrt{2})}$

$\Rightarrow -(\sqrt{2}+1)+\sqrt{12-6\sqrt{2}+8\sqrt{2}-8}=-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$

$\Rightarrow \quad \tan 11\dfrac{1^\circ}{4}=-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}.$

Ex. 32. Prove that (*i*) $\tan 7\dfrac{1^\circ}{2}=\sqrt{2}-\sqrt{3}-\sqrt{4}+\sqrt{6}=(\sqrt{3}-\sqrt{2})\,(\sqrt{2}-1)$. [*EAMCET 95*]

(*ii*) $\cot 7\dfrac{1^\circ}{2}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=(\sqrt{3}+\sqrt{2})\,(\sqrt{2}+1)$.

Sol. (*i*) $\tan 7\dfrac{1^\circ}{2}=\dfrac{\sin 7\frac{1^\circ}{2}}{\cos 7\frac{1^\circ}{2}}=\dfrac{2\sin^2 7\frac{1^\circ}{2}}{2\sin 7\frac{1^\circ}{2}\cos 7\frac{1^\circ}{2}}=\dfrac{1-\cos 15^\circ}{\sin 15^\circ}=\dfrac{1-\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$ [See Solved Ex. 1]

$=\dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}=\dfrac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{3-1}$

$=\dfrac{2\sqrt{6}+2\sqrt{2}-3-\sqrt{3}-\sqrt{3}-1}{2}=\dfrac{2\sqrt{6}-2\sqrt{3}+2\sqrt{2}-4}{2}$

$=\sqrt{6}-\sqrt{3}+\sqrt{2}-2=\sqrt{2}-\sqrt{3}-\sqrt{4}+\sqrt{6}$

$=\sqrt{3}(\sqrt{2}-1)-\sqrt{2}(\sqrt{2}-1)=(\sqrt{3}-\sqrt{2})\,(\sqrt{2}-1)\,.$

(*ii*) $\cot 7\dfrac{1^\circ}{2}=\dfrac{\cos 7\frac{1^\circ}{2}}{\sin 7\frac{1^\circ}{2}}=\dfrac{2\cos^2 7\frac{1^\circ}{2}}{2\sin 7\frac{1^\circ}{2}\cos 7\frac{1^\circ}{2}}=\dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

Now proceed as in (*i*) above.

5.09. Trigonometric ratios of 18°, 36°, 54° and 72°

(i) T-ratios of 18°

Let $A=18^\circ \Rightarrow 5A=90^\circ \Rightarrow 2A=90^\circ-3A$

$\Rightarrow \quad \sin 2A=\sin(90^\circ-3A)=\cos 3A$

$\Rightarrow \quad 2\sin A\cos A=4\cos^3 A-3\cos A$

$\Rightarrow \quad 2\sin A=4\cos^2 A-3$ [As $A=18^\circ$, so $\cos A\neq 0$]

$\Rightarrow \quad 2 \sin A = 4 (1 - \sin^2 A) - 3 \Rightarrow 4 \sin^2 A + 2 \sin A - 1 = 0$

$\Rightarrow \quad \sin A = \dfrac{-2 \pm 2\sqrt{5}}{8} = \dfrac{\sqrt{5}-1}{4}$ or $\dfrac{-\sqrt{5}-1}{4}$

But sin A, *i.e.,* sin 18° cannot be negative in value, therefore rejecting $\dfrac{-\sqrt{5}-1}{4}$, we have

$$\mathbf{\sin 18^\circ = \frac{\sqrt{5}-1}{4}.}$$

Now, $\cos 18^\circ = +\sqrt{1 - \sin^2 18^\circ}$

(Only + sign is taken before the square root as cos 18° is positive.)

$\therefore \quad \cos 18^\circ = \sqrt{1 - \left(\dfrac{\sqrt{5}-1}{4}\right)^2} \qquad \therefore \ \mathbf{\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}}$

$$\tan 18^\circ = \frac{\sin 18^\circ}{\cos 18^\circ} = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$$

$$\cot 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}; \sec 18^\circ = \frac{4}{\sqrt{10+2\sqrt{5}}}; \operatorname{cosec} 18^\circ = \frac{4}{\sqrt{5}-1}$$

(ii) T-ratios of 36°

Let $A = 36^\circ \quad \Rightarrow \quad 5A = 180^\circ \ \therefore \ 2A = 180^\circ - 3A$

$\therefore \quad \sin 2A = \sin (180^\circ - 3A) = \sin 3A$

$\Rightarrow \quad 2 \sin A \cos A = 3 \sin A - 4 \sin^3 A$

$\Rightarrow \quad 2 \cos A = 3 - 4 \sin^2 A \qquad (\because \sin A \neq 0)$

$\Rightarrow \ 2 \cos A = 3 - 4 (1 - \cos^2 A) \ \Rightarrow \ 4 \cos^2 A - 2 \cos A - 1 = 0$

$\Rightarrow \ \cos A = \dfrac{2 \pm \sqrt{4+16}}{8} = \dfrac{\sqrt{5}+1}{4}$ or $\dfrac{-\sqrt{5}+1}{4}$

But cos A, *i.e.* cos 36° cannot be negative, therefore rejecting $\dfrac{-\sqrt{5}+1}{4}$, we have

$$\mathbf{\cos 36^\circ = \frac{\sqrt{5}+1}{4}}$$

Now $\sin 36^\circ = +\sqrt{1 - \cos^2 36^\circ} = +\sqrt{1 - \left(\dfrac{\sqrt{5}+1}{4}\right)^2} \Rightarrow \mathbf{\sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}}$

$$\tan 36^\circ = + \frac{\sin 36^\circ}{\cos 36^\circ} = + \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}; \cot 36^\circ = \frac{\sqrt{5}+1}{10-2\sqrt{5}}$$

$$\sec 36^\circ = \frac{4}{\sqrt{5}+1}; \operatorname{cosec} 36^\circ = \frac{4}{\sqrt{10 - 2\sqrt{5}}}$$

Method II. $\cos 2A = 1 - 2 \sin^2 A$

Putting $A = 18^\circ$, we get

$$\cos 36^\circ = 1 - 2 \sin^2 18^\circ = 1 - 2 \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \frac{6-2\sqrt{5}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4} \quad \therefore \ \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

Now sin 36°, tan 36°, etc. can be found as above.

(iii) T-ratios of 54° and 72°

54° and 72° are complements of 36° and 18° respectively.

$\therefore$ $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$; $\cos 54° = \sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

Similarly, $\sin 72° = \cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$; $\cos 72° = \sin 18° = \dfrac{\sqrt{5}-1}{4}$.

EXERCISE 5 (*c*)

1. Evaluate:

(*i*) $2 \sin 15° \cos 15°$ (*ii*) $1 - 2\sin^2 22.5°$ (*iii*) $2\cos^2 157.5° - 1$

(*iv*) $\cos^2 \dfrac{\pi}{12} - \sin^2 \dfrac{\pi}{12}$ (*v*) $\dfrac{1}{2} - \sin^2 \dfrac{7\pi}{12}$ (*vi*) $\cos \dfrac{\pi}{8} \sin \dfrac{\pi}{8}$

(*vii*) $\dfrac{2\tan 22\frac{1°}{2}}{1 - \tan^2 22\frac{1°}{2}}$ (*viii*) $8\cos^3 \dfrac{\pi}{9} - 6\cos \dfrac{\pi}{9}$

2. Find the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given :

(*i*) $\sin \theta = \dfrac{3}{5}$, θ in Quadrant I. (*ii*) $\sin \theta = \dfrac{3}{5}$, θ in Quadrant II .

(*iii*) $\sin \theta = -\dfrac{1}{2}$, θ in Quadrant IV. (*iv*) $\tan \theta = -\dfrac{1}{5}$, θ in Quadrant II.

3. ABC is an acute-angled triangle inscribed in a circle of radius 5 cm and centre O. The sine of angle A is equal to $\dfrac{3}{5}$. Calculate without using tables:

(*i*) the length of BC (*ii*) sin OBC (*iii*) sin BOC (*iv*) cos BOC. **[*SC*]**

4. Derive functions of 120° from functions of 60° and check by using relations between functions of supplementary angles.

5. If $\sin \theta = a$ and $\sin 2\theta = b$, find an expression for $\cos \theta$ in terms of a and b. Hence find a relation between a and b not involving θ. **[*SC*]**

6. (*i*) Given that $\tan A = \dfrac{1}{5}$. find the values of $\tan 2A$, $\tan 4A$ and $\tan(45° - 4A)$. **[*SC*]**

(*ii*) If A is an obtuse angle whose sine is $\dfrac{5}{13}$ and B is an acute angle whose tangent is $\dfrac{3}{4}$, without using tables find the values of

(*a*) $\sin 2B$, (*b*) $\tan(A - B)$. **[*SC*]**

7. Express

(*i*) $\cos 6\alpha$ in terms of $\cos 3\alpha$; (*ii*) $\sin 10\theta$ in terms of functions of 5θ ;

(*iii*) $\tan 8\alpha$ in terms of $\tan 4\alpha$; (*iv*) $\cos 2\theta$ in terms of $\cos 4\theta$;

(*v*) $\tan 4\phi$ in terms of $\cos 8\phi$; (*vi*) $\sin \dfrac{5\theta}{2}$ in terms of $\cos 5\theta$;

(*vii*) $\cos 20\,\theta$ in terms of $\sin 5\theta$.

[**Hint.** $20\,\theta$ is double of $10\,\theta$ and $10\,\theta$ is double of 5θ. Hence apply double angle formulas].

8. Using the half angle formulas, find the exact value of (*i*) $\sin 15°$ (*ii*) $\sin 292\frac{1}{2}°$.

9. In the triangle ABC, in which C is the right angle, prove that :

$$\sin 2A = \frac{2ab}{c^2},\ \cos 2A = \frac{b^2 - a^2}{c^2},\ \sin\frac{A}{2} = \sqrt{\frac{c-b}{2c}},\ \cos\frac{A}{2} = \sqrt{\frac{c+b}{2c}}.$$

10. If $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{4}{5}$, find the value of $\cos\frac{\alpha-\beta}{2}$, assuming α and β to be acute angles.

11. Given that $\cos\frac{A}{2} = \frac{12}{13}$, calculate without the use of tables, the values of sin A, cos A and tan A. **[*ISC*]**

12. Given that $\tan x = \frac{12}{5}$, $\cos y = \frac{-3}{5}$, and the angles x and y are in the same quadrants, calculate without the use of tables the value of (*i*) $\sin(x + y)$, (*ii*) $\cos\frac{y}{2}$. **[*ISC*]**

13. Given that $\sin^2\beta = \sin\alpha\cos\alpha$, show that $\cos 2\beta = 2\cos^2\left(\frac{\pi}{4} + \alpha\right)$. **[*ISC 1991*]**

14. Derive formulas for the following in terms of functions of 2θ and then of θ.
(*i*) $\sin 4\theta$ (*ii*) $\cos 4\theta$, (*iii*) $\tan 4\theta$

15. If $\sin\alpha = \frac{3}{5}$, find the value of (*i*) $\sin 3\alpha$, (*ii*) $\cos 3\alpha$, (*iii*) $\tan 3\alpha$.

16. If $2\cos\theta = x + \frac{1}{x}$, prove that $2\cos 3\theta = x^3 + \frac{1}{x^3}$.

17. Calculate without using tables

$\tan 20°\tan 40°\tan 80°$. **[*ISC 1992*]**

[**Hint.** Given exp. = $\tan 20°.\tan(60° - 20°).\tan(60° + 20°)$

$$= \tan 20°\left[\frac{\tan^2 60° - \tan^2 20°}{1 - \tan^2 60°.\tan^2 20°}\right] = \tan 20°\left[\frac{3 - \tan^2 20°}{1 - 3\tan^2 20°}\right]$$

$$= \frac{3\tan 20° - \tan^3 20°}{1 - 3\tan^2 20°} = \tan 3(20°) = \tan 60° = \sqrt{3}.$$

Note. Here, we have used Art. 5.01, Cor. 3., viz. $\tan(A + B)\tan(A - B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A.\tan^2 B}$]

ANSWERS

1. (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{\sqrt{2}}$ (*iii*) $\frac{1}{\sqrt{2}}$ (*iv*) $\frac{\sqrt{3}}{2}$ (*v*) $\frac{-\sqrt{3}}{4}$

(*vi*) $\frac{1}{2\sqrt{2}}$ (*vii*) 1 (*viii*) 1

2. (*i*) $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$ (*ii*) $-\frac{24}{25}, \frac{7}{25}, \frac{-24}{7}$. (*iii*) $\frac{-\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$; (*iv*) $\frac{-5}{13}, \frac{12}{13}, \frac{-5}{12}$.

3. $\angle BOC = 2\angle BAC = 2A \Rightarrow \angle BOD = A = \angle DOC$

$\frac{BD}{OB} = \sin BOD = \sin A = \frac{3}{5}$ (Given)

(*i*) $\Rightarrow BD = OB\sin A = 5\left(\frac{3}{5}\right) = 3$

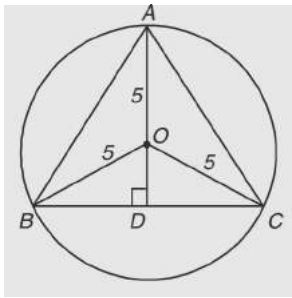
(*ii*) $\angle OBC = \dfrac{180° - 2A}{2} = 90° - A$

$\Rightarrow \sin OBC = \sin(90° - A) = \cos A$

$= \sqrt{1 - \sin^2 A}$

(*iii*) $\sin BOC = \sin 2A = 2 \sin A \cos A$

(*iv*) $\cos BOC = \cos 2A = 1 - 2\sin^2 A$



4. $\sin 120° = \dfrac{\sqrt{3}}{2}$, $\cos 120° = -\dfrac{1}{2}$, $\tan 120° = -\sqrt{3}$.

$\operatorname{cosec} 120° = \dfrac{2}{3}\sqrt{3}$, $\sec 120° = -2$, $\cot 120° = -\dfrac{1}{3}\sqrt{3}$.

5. $\cos\theta = \dfrac{b}{2a}$, $4a^4 + b^2 = 4a^2$. **6.** (*i*) $\dfrac{5}{12}, \dfrac{120}{119}, \dfrac{-1}{239}$; (*ii*) (*a*) $\dfrac{24}{25}$ (*b*) $-\dfrac{56}{33}$.

7. (*i*) $2\cos^2 3\alpha - 1$ (*ii*) $2 \sin 5\theta \cos 5\theta$. (*iii*) $\dfrac{2\tan 4\alpha}{1 - \tan^2 4\alpha}$ (*iv*) $\pm\sqrt{\dfrac{1 + \cos 4\theta}{2}}$

(*v*) $\pm\sqrt{\dfrac{1 - \cos 8\phi}{1 + \cos 8\phi}}$ (*vi*) $\pm\sqrt{\dfrac{1 - \cos 5\theta}{2}}$ (*vii*) $1 - 8\sin^2 5\theta + 8\sin^4 5\theta$

8. (*i*) $\dfrac{1}{2}\sqrt{2 - \sqrt{3}}$ (*ii*) $-\dfrac{1}{2}\sqrt{2 + \sqrt{2}}$ **10.** $\dfrac{7\sqrt{2}}{10}$

11. $\sin A = \pm\dfrac{120}{169}, \cos A = \dfrac{119}{169}, \tan A = \pm\dfrac{120}{119}$ **12.** (*i*) $\dfrac{56}{65}$ (*ii*) $\dfrac{-1}{\sqrt{5}}$

14. (*i*) $2\sin 2\theta \cos 2\theta$, $4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

(*ii*) $2\cos^2 2\theta - 1$, $8\cos^4\theta - 8\cos^2\theta + 1$ (*iii*) $\dfrac{2\tan 2\theta}{1 - \tan^2 2\theta}, \dfrac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$

15. (*i*) $\dfrac{117}{125}$ (*ii*) $\dfrac{-44}{125}$ if α is in Quadrant I; $\dfrac{44}{125}$ if α is in Quadrant II.

(*iii*) $\dfrac{-117}{44}$ if α is in Quadrant I; $\dfrac{117}{44}$ if α is in Quadrant II.

EXERCISE 5 (*d*)

Prove that

1. (*i*) $(\sin\phi - \cos\phi)^2 = 1 - \sin 2\phi$. (*ii*) $\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$.

2. $\dfrac{\sin\theta \times \tan\dfrac{\theta}{2}}{2} = \sin^2\dfrac{\theta}{2}$ **3.** $\dfrac{1}{\sec 2\theta} = \cos^4\theta - \sin^4\theta$. **4.** $\dfrac{2}{\cot\alpha \tan 2\alpha} = 1 - \tan^2\alpha$.

5. $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$ **6.** $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$ **7.** $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

8. $\dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = \dfrac{2 + \sin 2A}{2}$. **9.** $\dfrac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$. ***[SC]***

10. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$. **11.** $\operatorname{cosec} A - 2\cot 2A \cos A = 2\sin A$.

12. $\dfrac{1 + \sin 2x + \cos 2x}{\cos x + \sin x} = 2\cos x$ ***[SC]*** **13.** $\dfrac{1 - \cos 2x + \sin x}{\sin 2x + \cos x} = \tan x$. ***[SC]***

14. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta.$

15. $\cot A=\dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right).$

16. $\dfrac{\cos A+\sin A}{\cos A-\sin A}-\dfrac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A.$

17. $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A.$

18. $\dfrac{\sin 2A}{1-\cos 2A}.\dfrac{1-\cos A}{\cos A}=\tan\dfrac{A}{2}.$

19. $2\cos A=\sqrt{[2+\sqrt{2(1+\cos 4A)}]}$, $A\in I$ or IV Quad.

20. $\tan 2A=(\sec 2A+1)\sqrt{(\sec^2 A-1)}$, A in I or IV Quad.

21. $\dfrac{\sec 8A-1}{\sec 4A-1}=\dfrac{\tan 8A}{\tan 2A}.$

22. $\tan 2A-\sec A\sin A=\tan A\sec 2A.$

23. (*i*) $(\cos A+\cos B)^2+(\sin A+\sin B)^2=4\cos^2\left(\dfrac{A-B}{2}\right).$

(*ii*) $(\cos A-\cos B)^2+(\sin A-\sin B)^2=4\sin^2\left(\dfrac{A-B}{2}\right).$

24. (*i*) $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8}=2.$

(*ii*) $\sin^4\dfrac{\pi}{8}+\sin^4\dfrac{3\pi}{8}+\sin^4\dfrac{5\pi}{8}+\sin^4\dfrac{7\pi}{8}=\dfrac{3}{2}.$

25. $\cos\alpha\cos(60°-\alpha)\cos(60°+\alpha)=\dfrac{1}{4}\cos 3\alpha.$

26. $\dfrac{\cos^3\alpha-\cos 3\alpha}{\cos\alpha}+\dfrac{\sin^3\alpha+\sin 3\alpha}{\sin\alpha}=3.$

27. $\cos^2 2\theta-\sin^2\theta=\cos\theta\,.\cos 3\theta.$

28. $1+\dfrac{\cos 2\theta+\cos 6\theta}{\cos 4\theta}=\dfrac{\sin 3\theta}{\sin\theta}.$ **[*CWB*]**

29. $\cos^3\left(x-\dfrac{2\pi}{3}\right)+\cos^3 x+\cos^3\left(x+\dfrac{2\pi}{3}\right)=\dfrac{3}{4}\cos 3x.$ **[*OC*]**

30. $\dfrac{1}{\sin 10°}-\dfrac{\sqrt{3}}{\cos 10°}=4.$

31. $\tan 70°-\tan 20°-2\tan 40°=4\tan 10°.$

32. $\tan\theta+2\tan 2\theta+4\tan 4\theta+8\cot 8\theta=\cot\theta.$

33. (*i*) $4(\cos^3 20°+\cos^3 40°)=3(\cos 20°+\cos 40°).$

(*ii*) $4(\cos^3 10°+\sin^3 20°)=3(\cos 10°+\sin 20°).$

34. $\cos^3 x\sin^2 x=\dfrac{1}{16}(2\cos x-\cos 3x-\cos 5x).$ **[*Roorkee*]**

35. $\cot A+\cot(60°+A)+\cot(120°+A)=3\cot 3A.$

36. (*i*) $\cos 36° - \sin 18° = \frac{1}{2}$. (*ii*) $\cos^2 36° + \sin^2 18° = \frac{3}{4}$.

(*iii*) $3\cos 72° - 4\sin^3 18° = \cos 36°$. (*iv*) $\cos^2 48° - \sin^2 12° = \frac{\sqrt{5}+1}{8}$.

(*v*) $\sin \frac{1}{10}\pi \cos\frac{1}{5}\pi = \frac{1}{4}$. (*vi*) $\sec 72° - \sec 36° = 2$.

(*vii*) $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$. (*viii*) $\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{16}$.

37. (*i*) $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$. **[*IIT*]** (*ii*) $4\cos 6° \cos 42° \cos 66° \cos 78° = \frac{1}{4}$.

(*iii*) $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{4\pi}{5}\sin\frac{3\pi}{5} = \frac{5}{16}$. (*iv*) $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = \frac{1}{16}$.

(*v*) $\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\frac{3\pi}{5}+\cos^2\frac{9\pi}{10} = 2$ (*vi*) $\tan 6° \tan 42° \tan 66° \tan 78° = 1$

38. $\cos\alpha \cos 2\alpha \cos 4\alpha \cos 8\alpha = \frac{1}{16}$, if $\alpha = 24°$

39. $\cos 12° \cos 24° \cos 36° \cos 48° \cos 72° \cos 84° = \frac{1}{2^6}$

40. If $\theta = \frac{\pi}{2^n+1}$ prove that $2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = 1$

41. If $\tan\frac{x-y}{2}$, $\tan z$, $\tan\frac{x+y}{2}$ are in G.P., then show that $\cos x = \cos y \,.\, \cos 2z$

42. If $\sin\theta$ is G.M. of $\sin\phi$ and $\cos\phi$, then prove that $\cos 2\theta = 2\cos^2\left(\frac{\pi}{4}+\phi\right)$.

43. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$. **[*IIT*]**

44. If $\sin\alpha = \lambda\sin(\theta - \alpha)$ then prove that $\tan\left(\alpha - \frac{\theta}{2}\right) = \frac{\lambda-1}{\lambda+1}\tan\frac{\theta}{2}$.

45. If $\tan\frac{\alpha}{2} = \sqrt{\left\{\left(\frac{1-e}{1+e}\right)\right\}}\tan\frac{\beta}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1 - e\cos\alpha}$. **[*Dhanbad*]**

HINTS TO SELECTED QUESTIONS

24. (*i*) L.H.S $= \frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)+\frac{1}{2}\left(1+\cos\frac{3\pi}{4}\right)+\frac{1}{2}\left(1+\cos\frac{5\pi}{4}\right)+\frac{1}{2}\left(1+\cos\frac{7\pi}{4}\right)$

$= 2+\frac{1}{2}\left[\cos\frac{\pi}{4}+\cos\left(\pi-\frac{\pi}{4}\right)+\cos\left(\pi+\frac{\pi}{4}\right)+\cos\left(2\pi-\frac{\pi}{4}\right)\right]$

$= 2+\frac{1}{2}\left[\cos\frac{\pi}{4}-\cos\frac{\pi}{4}-\cos\frac{\pi}{4}+\cos\frac{\pi}{4}\right] = 2+\frac{1}{2}\times 0 = 2$

(*ii*) $\sin^4\frac{\pi}{8} = \left(\frac{1-\cos\frac{\pi}{4}}{2}\right)^2$, etc.

30. L.H.S. $= \frac{4\left(\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°\right)}{2\sin 10°\cos 10°}$. Put $\sin 30°$ for $\frac{1}{2}$ and $\cos 30°$ for $\frac{\sqrt{3}}{2}$.

31. L.H.S. $= (\cot 20° - \tan 20°) - 2\tan 40° = 2\cot 40° - 2\tan 40°$
$= 2(\cot 40° - \tan 40°) = 2(2\cot 80°) = 4\tan 10°$.

32. It is easy to prove that $\cot\theta - \tan\theta = 2\cot 2\theta$. Then,

L.H.S. $= \cot\theta - (\cot\theta - \tan\theta) + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$
$= \cot\theta - 2\cot 2\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$
$= \cot\theta - 2(\cot 2\theta - \tan 2\theta) + 4\tan 4\theta + 8\cot 8\theta$ etc.

33. (*i*) Consider $\cos 60° = \cos 3(20°)$

$\Rightarrow \frac{1}{2} = 4\cos^3 20° - 3\cos 20°$ ....(1)

$\cos 120° = \cos 3(40°) \Rightarrow -\frac{1}{2} = 4\cos^3 40° - 3\cos 40°$. ....(2)

Substitute in L.H.S. the values of $4(\cos^3 20° + \cos^3 40°)$ from (1) and (2).

34. R.H.S. $= \frac{1}{16}(2\cos x - 2\cos 4x\cos x)$

$= \frac{1}{16}\,2\cos x(1 - \cos 4x)$

$= \frac{1}{8}\cos x(2\sin^2 2x)$

$= \frac{1}{4}\cos x(2\sin x\cos x)^2$

35. L.H.S. $= \cot A + \cot(60° + A) + \cot(180° - \overline{60° - A})$

$= \cot A + \cot(60° + A) - \cot(60° - A)$

$= \frac{1}{\tan A} + \frac{1 - \sqrt{3}\tan A}{\sqrt{3} + \tan A} - \frac{1 + \sqrt{3}\tan A}{\sqrt{3} - \tan A}$

$= \frac{1}{\tan A} - \frac{8\tan A}{3 - \tan^2 A} = 3\left(\frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}\right) = \frac{3}{\tan 3A} = 3\cot 3A.$

38. $\alpha = 24° \Rightarrow 15\alpha = 360°$.

39. Let $\alpha = 12°$ so that $15\alpha = 180°$.

Let $C = \cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 6\alpha\cos 7\alpha$

$S = \sin\alpha\sin 2\alpha\sin 3\alpha\sin 4\alpha\sin 6\alpha\sin 7\alpha$

S.C. $= \frac{1}{2^6}\sin 2\alpha\sin 4\alpha\sin 6\alpha\sin 8\alpha\sin 12\alpha\sin 14\alpha$

$= \frac{1}{2^6}\sin 2\alpha\sin 4\alpha\sin 6\alpha\sin(180° - 7\alpha)\sin(180° - 3\alpha)\sin(180° - \alpha)$

Now complete.

40. L.H.S. $= \frac{1}{\sin\theta}(2^n \sin\theta \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta)$

$= \frac{1}{\sin\theta}(2^{n-1} \sin 2\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta)$

$= \frac{1}{\sin\theta}(2^{n-2} \sin 2^2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta)$

$= \frac{2\sin 2^{n-1}\theta \cos 2^{n-1}\theta}{\sin\theta} = \frac{\sin 2^n\theta}{\sin\theta} = \sin\frac{2^n\pi}{2^n+1} \div \sin\frac{\pi}{2^n+1}\left[\because \theta = \frac{\pi}{2^n+1}, \text{given}\right]$

$= \sin\left(\pi - \frac{\pi}{2^n+1}\right) \div \sin\frac{\pi}{2^n+1} = 1.$

41. $b^2 = ac \Rightarrow \frac{\sin^2 z}{\cos^2 z} = \frac{\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{\cos\frac{x+y}{2}\cos\frac{x-y}{2}}$

$\Rightarrow \frac{\sin^2 z}{\cos^2 z} = \frac{\cos y - \cos x}{\cos y + \cos x} \Rightarrow \cos x \,.1 = \cos y\,(\cos^2 z - \sin^2 z) = \cos y \cos 2z.$

42. $\sin^2\theta = \sin\phi\cos\phi \Rightarrow 2\sin^2\theta = \sin 2\phi \Rightarrow 1 - \cos 2\theta = -\cos\left(\frac{\pi}{2} + 2\phi\right)$

$\Rightarrow \cos 2\theta = 1 + \cos\left(\frac{\pi}{2} + 2\phi\right) \Rightarrow \cos 2\theta = 2\cos^2\left(\frac{\pi}{4} + \phi\right)$

43. $\frac{m}{n} = \frac{\tan(\theta+120°)}{\tan(\theta-30°)} = \frac{3-\tan^2\theta}{1-3\tan^2\theta} \therefore \tan^2\theta = \frac{m-3n}{3m-n}$ etc.

44. $\frac{\sin\alpha}{\sin(\theta-\alpha)} = \lambda$. Apply componendo – dividendo.

45. $\frac{1}{\tan^2\frac{\beta}{2}} = \frac{1-e}{1+e}.\frac{1}{\tan^2\frac{\alpha}{2}} = \frac{(1-e)}{(1+e).\tan^2\frac{\alpha}{2}}$. Apply componendo – dividendo.

CHAPTER SUMMARY

1. Sum formulae

(1) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(2) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(3) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$ (4) $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$

(5) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\ \tan B\ \tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

2. Difference formulae

(1) $\sin(A-B) = \sin A\cos B - \cos A\sin B$ (2) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(3) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$ (4) $\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$

3. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

4. $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$ $\qquad$ $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$ $\qquad$ $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

5. Product formulae

$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

6. Double-angle formulae

$\sin 2A = 2 \sin A \cos A$ $\qquad$ $\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2 \sin^2 A$ $\qquad$ $\cos 2A = 2 \cos^2 A - 1$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ $\qquad$ $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$

$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$2 \sin^2 A = 1 - \cos 2A$ $\qquad$ $2 \cos^2 A = 1 + \cos 2A$

$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

7. Half-angle formulae

$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$ $\qquad$ $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

$\cos A = 2 \cos^2 \frac{A}{2} - 1$ $\qquad$ $\cos A = 1 - 2 \sin^2 \frac{A}{2}$

$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$ $\qquad$ $\sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$

$\cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$ $\qquad$ $\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$

$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$ $\qquad$ $\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$

Aid to memory! $\cos (\text{any angle}) = \pm \sqrt{\frac{1 + \cos (\text{double angle})}{2}}$

$\sin (\text{any angle}) = \pm \sqrt{\frac{1 - \cos (\text{double angle})}{2}}$

$\tan (\text{any angle}) = \pm \sqrt{\frac{1 - \cos (\text{double angle})}{1 + \cos (\text{double angle})}}$

8. Triple-angle formulae

$\sin 3A = 3\sin A - 4\sin^3 A$ $\qquad$ $\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

9. One-third angle formulae

$$\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$$

$$\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$$

$$\tan A = \frac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$$

10. $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ $\qquad$ $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

$\cos 18° = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$ $\qquad$ $\sin 36° = \dfrac{1}{4}\sqrt{10-2\sqrt{5}}$

CHAPTER TEST

1. Show that $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$. Hence deduce that $\tan 75° - \cot 75° = 4\sin 60°$. **[*AP, HB, CBSE*]**

2. Prove that $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$

3. If $A + B + C = \pi$, and $\cos A = \cos B \cos C$, show that $2\cot B \cot C = 1$. **[*WBHS*]**

4. Show that $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = 3$, given that $\tan\alpha = 2\tan\beta$.

5. Show that $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \tan 55°$.

6. If $\sin 2A = \dfrac{4}{5}$, find the value of $\tan A$, $(0° \le A \le \dfrac{\pi}{4})$ **[*Tripura, WB, AP*]**

7. Express (*i*) $\cot A$ in terms of $\cos 2A$, (*ii*) $\cos 4\theta$ in terms of $\cos\theta$.

8. A positive acute angle is divided into two parts whose tangents are $\dfrac{1}{2}$ and $\dfrac{1}{3}$. Show that the angle is $\dfrac{\pi}{4}$.

9. Show that $\cos 10° + \cos 110° + \cos 130° = 0$ **[*WBHS*]**

10. Show that $\dfrac{\sin 5A + 2\sin 8A + \sin 11A}{\sin 8A + 2\sin 11A + \sin 14A} = \dfrac{\sin 8A}{\sin 11A}$.

11. Show that $\dfrac{1}{2\sin 10°} - 2\sin 70° = 1$. **[*NMOC, WBJEE*]**

12. Show that $\sin 19° + \sin 41° + \sin 83° = \sin 23° + \sin 37° + \sin 79°$. **[*Raj. PET, WBJEE*]**

13. If $\sin A = \dfrac{1}{\sqrt{3}}$ and $\sin B = \dfrac{1}{\sqrt{5}}$ find the value of $\tan\dfrac{1}{2}(A+B).\cot\dfrac{1}{2}(A-B)$.

14. If $\sin\theta = n\sin(\theta + 2\alpha)$, show that $(n-1)\tan(\theta+\alpha) + (n+1)\tan\alpha = 0$.

15. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of the equation $8x^2 - 26x + 15 = 0$, then find the value of $\cos(\alpha+\beta)$.

16. Prove that $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{if } n \text{ is even.} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$

17. Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following cases:

(*i*) $\sin x = \frac{1}{4}$, x in II quadrant.

(*ii*) $\tan x = \frac{-4}{3}$, x in II quadrant.

18. Prove that $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

19. Prove that $\sin^2 72° - \sin^2 60° = \frac{\sqrt{5}-1}{8}$.

20. If $\tan x$, $\tan y$, $\tan z$ are in G.P., show that $\cos 2y = \frac{\cos(x+z)}{\cos(x-z)}$. ***[EAMCET, TN]***

21. If $\tan\frac{\alpha}{2} : \tan\frac{\beta}{2} = 1 : \sqrt{3}$ show that $\cos\beta = \frac{2\cos\alpha - 1}{2-\cos\alpha}$. ***[WBHS]***

ANSWERS

6. $\frac{1}{2}$ **7.** (*i*) $\sqrt{\frac{1+\cos 2A}{1-\cos 2A}}$ (*ii*) $8\cos^4\theta - 8\cos^2\theta + 1$

13. $4+\sqrt{15}$ **15.** $-\frac{627}{725}$

17. (*i*) $\sin\frac{x}{2} = \frac{1}{4}\sqrt{8+2\sqrt{15}}$, $\cos\frac{x}{2} = \frac{1}{4}\sqrt{8-2\sqrt{15}}$, $\tan\frac{x}{2} = 4+\sqrt{15}$

(*ii*) $\sin\frac{x}{2} = \frac{2\sqrt{5}}{5}$, $\cos\frac{x}{2} = \frac{1}{5}\sqrt{5}$, $\tan\frac{x}{2} = 2$

HINTS TO SELECTED QUESTIONS

7. $\cot A = \sqrt{\frac{2\cos^2 A}{2\sin^2 A}}$ **11.** $\frac{1-4\sin 70°\sin 10°}{2\sin 10°} = \frac{1-2(\cos 60°-\cos 80°)}{2\sin 10°}$

12. LHS = 2 sin 30° cos 11° + cos (90° – 83°) = cos 11° + cos 7°. Similarly, RHS

14. $\frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{1}{n} \Rightarrow \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \frac{1+n}{1-n}$

20. $\tan^2 y = \frac{\sin x}{\cos x}.\frac{\sin z}{\cos z}$. Apply componendo and dividendo

21. $\tan^2\frac{\beta}{2} = 3\tan^2\frac{\alpha}{2} \Rightarrow \frac{1-\cos\beta}{1+\cos\beta} = \frac{3(1-\cos\alpha)}{1+\cos\alpha} \Rightarrow \frac{1+\cos\beta}{1-\cos\beta} = \frac{1+\cos\alpha}{3-3\cos\alpha}$

Apply componendo and dividendo.

CHAPTER 6

TRIGONOMETRIC EQUATIONS

6.01. Trigonometric equation

Definition. A trigonometric equation is an equation involving the trigonometric function or functions of unknown angles, *e.g.,* $\sin x = 0$, $\cos^2 x - \sin x = \frac{1}{4}$, $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}$, etc., are all trigonometric equations.

A solution of a trigonometric equation is a value of the unknown angle that satisfies the equation. A trigonometric equation may have an unlimited number of solutions, *e.g.,* if $\sin x = 0$, $x = 0, \pi, 2\pi, 3\pi, \ldots$ The solution lying between 0 and 360° is called the **principal solution.** Since the trigonometric functions are periodic, a solution generalised by means of periodicity is known as the **general solution.** Every equation will have principal solution as well as general solution. We will always find out the general solution if not otherwise stated. The principal solution can always be obtained from the general solution.

Ex. 1. Solve $\sin x = -1$ for principal solution as well as general solution :

(*i*) $x \in$ { angle with measures given in radians }

(*ii*) $x \in$ {angle with measures given in degrees }.

Sol. In the interval, $0 \le x < 2\pi$, $x = \frac{3\pi}{2}$, which is the principal solution.

Therefore, the general solutions are

(*i*) $\left\{x \mid x = \left(\frac{3\pi}{2} + 2k\pi\right), k \in I\right\}$ (*ii*) $\{x \mid x = (270° + k \,.\, 360°), k \in I\}$.

6.02. The equations sin θ = 0, cos θ = 0 and tan θ = 0

(*i*) sin θ = 0

Since $\sin\theta = \sin 0 = \sin\pi = \sin(-\pi) = \sin 2\pi = \sin(-2\pi) = \ldots\ldots = 0$

therefore, the equation $\sin\theta = 0$ is satisfied by the following values of θ

$\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi\ldots\ldots\ldots$

In short, we can write the above solution as

$$\theta = n\pi, n \in I \qquad \textbf{[Remember]}$$

which is the general solution.

(*ii*) cos θ = 0

This is satisfied by the following values of π:

$$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$$

because $\cos\theta = \cos(\pm\frac{\pi}{2}) = \cos(\pm\frac{3\pi}{2}) = \ldots = 0$

Hence, the general solution is

$$\theta = (2n+1)\frac{\pi}{2}, n \in I.$$

***(iii)* $\tan\theta = 0$**

This is satisfied by $\theta = 0, \pm\pi, \pm 2\pi$.....

$\therefore$ $\theta = n\pi, n \in I.$

6.03. The equation $\sin\theta = \sin\alpha$

$\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

or $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

Hence, either $\cos\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\cos\frac{\theta+\alpha}{2} = 0$	$\sin\frac{\theta-\alpha}{2} = 0$
$\Rightarrow \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}, m \in I$	$\Rightarrow \frac{\theta-\alpha}{2} = m\pi, m \in I,$
$\Rightarrow \theta + \alpha = (2m+1)\pi$	$\Rightarrow \theta - \alpha = 2m\pi,$
$\Rightarrow \theta = (2m+1)\pi - \alpha$ ...(1)	
$\Rightarrow \theta = 2m\pi + \alpha$ ...(2)	

Hence, either $\theta = -\alpha$ + any odd multiple of π

or $\theta = \alpha$ + any even multiple of π

Thus, the general value of θ such that $\sin\theta = \sin\alpha$ is given by

$$\theta = n\pi + (-1)^n\alpha, \text{ where } n \in I \quad \ldots(3)$$

Note. If n is odd, *i.e.,* $n = 2m + 1$, we get solutions given by (1), and if n is even, *i.e.,* $n = 2m$, we get solutions given by (2). The solution (3) includes all possible solutions given by (1) as well as (2).

6.04. The equation $\cos\theta = \cos\alpha$

$\cos\theta - \cos\alpha = 0$

$\Rightarrow 2\sin\frac{\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2} = 0$

$\Rightarrow -\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow -\sin\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

Hence, either $\sin\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\sin\frac{\theta+\alpha}{2} = 0$	$\sin\frac{\theta-\alpha}{2} = 0$
$\Rightarrow \frac{\theta+\alpha}{2} = n\pi$	$\Rightarrow \frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \theta + \alpha = 2n\pi$ $\Rightarrow \theta - \alpha = 2n\pi$

$\Rightarrow \theta = 2n\pi - \alpha$ $\Rightarrow \theta = 2n\pi + \alpha$

Combining these two solutions, we get the general solution as

$$\theta = 2n\pi \pm \alpha, n \in I$$

6.05. The equation tan θ = tan α

$$\tan\theta = \tan\alpha \text{ or } \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$ or $\sin(\theta - \alpha) = 0$,

$\Rightarrow \theta - \alpha = n\pi, n \in I$

Hence, the general solution for θ satisfying $\tan\theta = \tan\alpha$ is

$$\theta = n\pi + \alpha, n \in I$$

Note 1. In all the above results, α is numerically the least angle which should be expressed in radians, as far as possible.

Note 2. $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ follows $\sin\theta = \sin\alpha$;

$\sec\theta = \sec\alpha$ follows $\cos\theta = \cos\alpha$;

$\cot\theta = \cot\alpha$ follows $\tan\theta = \tan\alpha$.

6.06. The equations sin² θ = sin² α, cos² θ = cos² α, tan² θ = tan² α

(*i*) $\sin^2\theta = \sin^2\alpha$

$$\Rightarrow \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2} \Rightarrow \cos 2\theta = \cos 2\alpha$$

$$\therefore 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$$

(*ii*) $\cos^2\theta = \cos^2\alpha$

$$\Rightarrow \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2} \Rightarrow \cos 2\theta = \cos 2\alpha$$

$$\therefore 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$$

(*iii*) $\tan^2\theta = \tan^2\alpha \Rightarrow \frac{\tan^2\theta}{1} = \frac{\tan^2\alpha}{1}$

$$\Rightarrow \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$ (By componendo and dividendo)

$\Rightarrow \cos 2\theta = \cos 2\alpha$

$$\therefore 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$$

Method II. $\tan^2\theta = \tan^2\alpha \Rightarrow \tan\theta = \pm\tan\alpha = \tan(\pm\alpha) \Rightarrow \theta = n\pi \pm \alpha, n \in I$

Note. The following points should be kept in mind about the general values obtained so far:

(1) α is numerically the least angle and should as far as possible be expressed in radians.

(2) If $\sin\theta = -k$, then $\theta = -\sin^{-1}k$. Thus, if $\sin\theta = -\frac{1}{\sqrt{2}}$, then $\theta = -45° = -\frac{\pi}{4}$. The same applies to cosec θ, tan θ and cot θ.

(3) If $\cos\theta = -k$, then $\theta = \pi - \cos^{-1}k$. Thus, if $\cos\theta = -\frac{1}{\sqrt{2}}$, then $\theta = 180° - 45° = 135° = \frac{3\pi}{4}$. The same applies to sec θ.

6.07. Procedure for solving trigonometric equations

No particular method can be laid down for solving a trigonometric equation. The aim is to extract the value of the unknown angle from the equation involving it and this is done by throwing the given equation in a suitable form with the help of the various formulae that we learnt in previous chapters. Below we give some suggestions, which will take care of most of the cases.

Type 1. Equations in which only one function of a single angle is involved

Procedure. *Solve algebraically for the values of the function.*

Ex. 2. Solve for x, $\sin x = \frac{-\sqrt{3}}{2}$, $(0 < x < 2\pi)$.

Sol. $\sin x = -\frac{\sqrt{3}}{2} = -\sin 60° = \sin(180° + 60°) = \sin(360° - 60°) \Rightarrow x = 240°, 300°$.

Ex. 3. Find all values of x which satisfy $5 \sin x = 4 \cos x$, $(0° < x < 360°)$

Sol. $5 \sin x = 4 \cos x$

$\therefore \quad \tan x = \frac{4}{5} = 0.8 = \tan 38° 39' = \tan(180° + 38° 39')$

$\therefore \quad x = 38° 39'$ and $218° 39'$ (to the nearest minute).

Ex. 4. Calculate the values of x between 0° and 360° which satisfy $\sin(270° - x) = \cos 292°$. ***[SC]***

Sol. $\cos 292° = \cos(360° - 68°) = \sin 22°$

$\therefore \quad \sin(270° - x) = \sin 22° \quad$ or $\quad \sin(270° - x) = \sin(180° - 22°)$

$\therefore \quad 270° - x = 22° \qquad 270° - x = 180° - 22°$

$\therefore \quad x = 248° \qquad x = 112°$

Therefore, $\theta = 112°, 248°$.

Ex. 5. Solve $4 \cos^2 \theta = 3$, $(\theta° \leq \theta < 360°)$.

Sol. $\cos^2 \theta = \frac{3}{4}$

Hence, $\cos \theta = \frac{\sqrt{3}}{2}$ | $\cos \theta = -\frac{\sqrt{3}}{2}$

$\theta = 30°, 330°$ | $\theta = 150°, 210°$

Therefore, (arranging the values in order of size) $\theta = 30°, 150°, 210°, 330°$.

Ex. 6. Find the general values of θ if

(*i*) $2 \sin \theta - 1 = 0$ **(*ii*) $\sin \theta = -\frac{\sqrt{3}}{2}$** **(*iii*) $\cos \theta = \frac{1}{\sqrt{2}}$** **(*iv*) $\cos \theta = -\frac{1}{2}$**

(*v*) $4 \sin^2 \theta = 1$ **(*vi*) $\sqrt{3}$ cosec $\theta = 2$** **(*vii*) $\tan 2x - \sqrt{3} = 0$** **(*viii*) $2 \cot^2 \theta = \text{cosec}^2 \theta$**

Sol. (*i*) $2 \sin \theta - 1 = 0 \Rightarrow \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6}$. Hence, $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

(*ii*) $\sin \theta = -\frac{\sqrt{3}}{2} = \sin(-\frac{\pi}{3})$. Hence, $\theta = n\pi + (-1)^n(-\frac{\pi}{3}) = n\pi - (-1)^n \frac{\pi}{3} \in I$

(*iii*) $\cos \theta = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$. Hence, $\theta = 2n\pi \pm \frac{\pi}{4}, n \in I$.

(*iv*) $\cos \theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3}$. Hence, $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in I$.

(*v*) $4\sin^2\theta = 1 \quad \Rightarrow \quad 2(1-\cos 2\theta) = 1 \quad \Rightarrow \quad \cos 2\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

Hence, $2\theta = 2n\pi \pm \frac{\pi}{3}$ or $\theta = \frac{2n\pi}{2} \pm \frac{\pi}{6}$, *i.e.* $n\pi \pm \frac{\pi}{6}$

(*vi*) $\sqrt{3}\ \text{cosec}\,\theta = 2$ or $\text{cosec}\,\theta = \frac{2}{\sqrt{3}}$ or $\sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$

Hence, $\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in I$

(*vii*) $\tan 2x - \sqrt{3} = 0$ or $\tan 2x = \sqrt{3} = \tan\frac{\pi}{3}$

$\therefore \quad 2x = n\pi + \frac{\pi}{3}$ or $x = \frac{n\pi}{2} + \frac{\pi}{6}, n \in I$

(*viii*) $2\cot^2\theta = \text{cosec}^2\theta$ or $2\cot^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = 1$

$\Rightarrow \quad \cot\theta = \pm 1 = \cot(\pm\frac{\pi}{4})$. Hence, $\theta = n\pi \pm \frac{\pi}{4}, n \in I$

Ex. 7. Solve $2\sin^2\phi = \sin\phi$, $(0° \le \phi < 360°)$.

Sol. $2\sin^2\phi - \sin\phi = 0$; $\sin\phi(2\sin\phi - 1) = 0$

Hence, $\sin\phi = 0$ | $2\sin\phi - 1 = 0$ or $2\sin\phi = 1$

$\therefore \quad \phi = 0°, 180°$ | $\therefore \quad \sin\phi = \frac{1}{2} \ \therefore \ \phi = 30°, 150°$

$\therefore \quad \phi = 0°, 30°, 150°, 180°.$

Ex. 8. Solve $\sin^2 x + \sin x - 2 = 0$, $(0° < \theta < 360°)$.

Sol. Factorising $(\sin x + 2)(\sin x - 1) = 0$.

Hence,

$\sin x + 2 = 0$ | $\sin x - 1 = 0$

$\sin x = -2$ | $\sin x = 1$

This is inadmissible, as $\sin x$ cannot be numerically greater than 1. $\therefore \quad x = 90°$

$\therefore \quad x = 90°$

Note. If the expression of the given equation cannot be factorised, use formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Ex. 9. Solve for the two smallest positive values of θ, to the nearest minute, $\cot^2\theta - 3\cot\theta - 2 = 0$, $(0° < \theta < 360°)$.

Sol. $\cot\theta = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2} = \frac{3 \pm 4.1231}{2}$

$= \frac{7.1231}{2}, \frac{-1.1231}{2}$

$= 3.562, -0.5616$

If $\cot\theta = 3.562, \quad \Rightarrow \quad \theta = 15°41' \text{ or } 195°41'$

If $\cot\theta = -0.5616 \ \Rightarrow \ \theta = 119°19' \text{ or } 299°19'$

Hence, $\theta = 15°41' \text{ or } 119°19'$.

Type 2. Equations expressible in terms of one *t*-ratio of the unknown angle

Procedure. *The following formulas help in expressing all the t-functions in terms of a single t-function.*

(*a*) *Square relations* (*b*) *Double angle formulas*

Ex. 10. Solve $\cos^2\theta - \sin\theta - \frac{1}{4} = 0$ $(0° < \theta < 360°)$.

Sol. $\cos^2\theta - \sin\theta - \frac{1}{4} = 0$ | Using the square relation $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad 1 - \sin^2\theta - \sin\theta - \frac{1}{4} = 0$

$\Rightarrow \quad 4\sin^2\theta + 4\sin\theta - 3 = 0$

Factorising, $(2\sin\theta + 3)(2\sin\theta - 1) = 0$

Hence, $2\sin\theta + 3 = 0$ or $2\sin\theta - 1 = 0$

$\therefore \quad \sin\theta = \frac{-3}{2}$ | $\sin\theta = \frac{1}{2}$

This value is inadmissible. | $\theta = 30°, 150°$.

$\therefore \quad \theta = 30°, 150°$

Ex. 11. Find the values of θ between 0° and 360°, which satisfy the equation

$3\cos 2\theta - \sin\theta = 2$. **[*SC*]**

Sol. $3\cos 2\theta - \sin\theta = 2$ | Using the double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \quad 3(1 - 2\sin^2\theta) - \sin\theta = 2$

$\Rightarrow \quad 6\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad (3\sin\theta - 1)(2\sin\theta + 1) = 0$

Hence,

$3\sin\theta - 1 = 0$ | $2\sin\theta + 1 = \theta$

$\Rightarrow \quad \sin\theta = \frac{1}{3} = 0.3333$ | or $\sin\theta = -\frac{1}{2}$

$\Rightarrow \quad \theta = 19° 28', 160° 32'$ | $\Rightarrow \quad \theta = 210°, 330°$

Hence, $\theta = 19° 28', 160° 32', 210°, 330°$.

Ex. 12. Solve $2\sin\theta\cos\theta = \cos\theta$. $(0° < \theta < 360°)$.

Sol. The given equation may be written as $\cos\theta(2\sin\theta - 1) = 0$

Hence,

$\cos\theta = 0$ | $2\sin\theta - 1 = 0$

$\theta = 90°, 270°$ | $\sin\theta = \frac{1}{2}$

$\therefore \quad \theta = 30°, 150°$

Hence, $\theta = 30°, 90°, 150°, 270°$.

Ex. 13. Solve : $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$ for the general solution.

Sol. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$

$\Rightarrow \quad (1 - \cos^2\theta) - 2\cos\theta + \frac{1}{4} = 0 \Rightarrow 4\cos^2\theta + 8\cos\theta - 5 = 0$

$\Rightarrow$ Factorising, $(2\cos\theta + 5)(2\cos\theta - 1) = 0$

Hence, either $2\cos\theta + 5 = 0$, or $2\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{-5}{2}$, or $\cos\theta = \frac{1}{2}$; Rejecting the first value, since numerical value of cos cannot be >1,

Now, $\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$. Hence, $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in I$.

Ex. 14. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$, $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

(*a*) zero (*b*) one (*c*) two (*d*) four ***[IIT JEE 2007]***

Sol. $2\sin^2\theta - \cos 2\theta = 0 \Rightarrow 2\sin^2\theta - (1 - 2\sin^2\theta) = 0 \Rightarrow 4\sin^2\theta = 1 \Rightarrow \sin\theta = \pm\frac{1}{2}$

In the interval $[0, 2\pi]$, the solutions are $\theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$ ... (A)

Now, $2\cos^2\theta - 3\sin\theta = 0 \Rightarrow 2(1 - \sin^2\theta) - 3\sin\theta = 0 \Rightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow 2\sin^2\theta + 4\sin\theta - \sin\theta - 2 = 0 \Rightarrow (\sin\theta + 2)(2\sin\theta - 1) = 0$

We know that $-1 \le \sin\theta \le 1 \Rightarrow 1 \le \sin\theta + 2 \le 3$ $\quad \therefore \sin\theta + 2 \neq 0$

$\therefore 2\sin\theta - 1 = 0 \Rightarrow \sin\theta = 1/2$. Therefore, the solutions for this equation in $[0, 2\pi]$ are

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{... (B)}$$

From (A) and (B), we find that $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ satisfy both the equations.

Hence, there are two solutions for the given pair of equations in the interval $[0, 2\pi]$. **Ans.** (*c*)

Ex. 15. Solve: $\tan^2\alpha + \sec\alpha - 1 = 0$, $0 \le \alpha < 2\pi$.

Sol. $\tan^2\alpha + \sec\alpha - 1 = 0 \Rightarrow \sec^2\alpha - 1 + \sec\alpha - 1 = 0$

$\Rightarrow \sec^2\alpha + \sec\alpha - 2 = 0 \Rightarrow (\sec\alpha + 2)(\sec\alpha - 1) = 0$

$\Rightarrow$	$\sec\alpha + 2 = 0$	or $\sec\alpha - 1 = 0$
$\Rightarrow$	$\sec\alpha = -2$	$\Rightarrow \sec\alpha = 1$
	$\cos\alpha = \frac{-1}{2}$	$\Rightarrow \cos\alpha = 1$
$\Rightarrow$	$\alpha = \frac{2\pi}{3}, \frac{4\pi}{3}$	$\Rightarrow \alpha = 0$

The required solutions are $0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

Type 3. Equations involving multiple angles

Case I. When the equation involves only one multiple angle

Ex. 16. Solve $\sin 3x = -\frac{1}{\sqrt{2}}$, $0 < x < 2\pi$.

Sol. Since we require x such that $0 < x < 2\pi$, $3x$ must be such that $0 < 3x < 6\pi$.

Now $\sin 3x = \frac{-1}{\sqrt{2}} = -\sin\frac{\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = \sin\left(2\pi - \frac{\pi}{4}\right) = \sin\left(2\pi + \frac{5\pi}{4}\right)$

$$= \sin\left(2\pi + \frac{7\pi}{4}\right) = \sin\left(4\pi + \frac{5\pi}{4}\right) = \sin\left(4\pi + \frac{7\pi}{4}\right)$$

$\therefore \quad 3x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4} \quad \therefore x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$.

Case II. Equations involving two multiple and sub-multiple angles

These equations can be of the following types :

(*i*) $\sin n\theta = \sin m\theta$ (*ii*) $\sin n\theta = \cos m\theta$ (*iii*) $\cos n\theta = \cos m\theta$

(*iv*) $\tan n\theta = \tan m\theta$ (*v*) $\tan n\theta = \cot m\theta$ (*vi*) $\cot n\theta = \cot m\theta$

Method I.

1. *Express the given equation in terms of one t-function, i.e., to the form*

$\sin(\quad) = \sin(\quad)$ or $\cos(\quad) = \cos(\quad)$ or $\tan(\quad) = \tan(\quad)$.

For example, $\sin n\theta = \cos m\theta \Rightarrow \sin n\theta = \sin\left(\frac{\pi}{2} - m\theta\right)$

2. *Apply the formula for general value for sin* θ *= sin* α, $\cos\theta = \cos\alpha$, $\tan\theta = \tan\alpha$ *as the case may be.*

3. *Collect the terms involving the unknown angle on one side and find the value.*

Note. If the equation is of the form $\sin\theta = \sin\alpha$, then we should discuss the cases when n is even and when n is odd separately. (Art. 6.03)

Ex. 17. Solve : $\tan 3\theta = \tan\theta$.

Sol. $\tan 3\theta = \tan\theta \quad \therefore \quad 3\theta = n\pi + \theta \Rightarrow 2\theta = n\pi$

$$\theta = \frac{1}{2}n\pi, n \in I$$

But for odd multiples of $\frac{\pi}{2}$, both $\tan\theta$ and $\tan 3\theta$ are not defined; hence θ = even multiples of $\frac{\pi}{2}$. $\quad \therefore \theta = m\pi, m \in I.$

Ex. 18. Solve : $\sin 9\theta = \sin\theta$.

Sol. $\sin 9\theta = \sin\theta$

$\therefore \quad 9\theta = 2n\pi + \theta$ or $9\theta = (2n+1)\pi - \theta, n \in I$ (Art. 6.03)

$\therefore \quad \theta = \frac{2n\pi}{8}$ or $(2n+1)\frac{\pi}{10}$

$\Rightarrow \quad \theta = \frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{10}, n \in I$

Method II. $\sin 9\theta = \sin\theta$

$\Rightarrow \quad \sin 9\theta - \sin\theta = 0 \quad \Rightarrow \quad 2\cos 5\theta \sin 4\theta = 0$

$\therefore \quad$ either $\sin 4\theta = 0$ or $\cos 5\theta = 0$

$\therefore \quad 4\theta = n\pi$, or $5\theta = (2n+1)\frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{4}$ or $\theta = (2n+1)\frac{\pi}{10}$ where $n \in I$.

Ex. 19. Solve : $\cos p\theta = \sin q\theta$.

Sol. $\cos p\theta = \sin q\theta$

$$\Rightarrow \quad \cos p\theta = \cos\left(\frac{\pi}{2} - q\theta\right); p\theta = 2n\pi \pm \left(\frac{\pi}{2} - q\theta\right), n \in I$$

Taking +ve sign, $p\theta = 2n\pi + \frac{\pi}{2} - q\theta \Rightarrow (p+q)\theta = 2n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{\left(2n + \frac{1}{2}\right)\pi}{p+q}, n \in I$

Taking –ve sign, $p\theta = 2n\pi - \frac{1}{2}\pi + q\theta \Rightarrow (p-q)\theta = \left(2n - \frac{1}{2}\right)\pi \Rightarrow \theta = \frac{\left(2n - \frac{1}{2}\right)\pi}{p-q}, n \in I$

Ex. 20. Solve : $\sin 2\theta = \cos 3\theta$.

Sol. $\sin 2\theta = \cos 3\theta \Rightarrow \cos 3\theta = \sin 2\theta$

$\Rightarrow \cos 3\theta = \cos\left(\frac{\pi}{2} - 2\theta\right) \Rightarrow 3\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right), n \in \mathrm{I}$

Taking +ve sign, $3\theta = 2n\pi + \left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow 5\theta = 2n\pi + \frac{\pi}{2}, \theta = \left(2n + \frac{1}{2}\right)\frac{\pi}{5}$

Taking –ve sign, $3\theta = 2n\pi - \left(\frac{\pi}{2} - 2\theta\right) \Rightarrow \theta = \left(2n - \frac{1}{2}\right)\pi$

$\therefore \theta = \left(2n + \frac{1}{2}\right)\frac{\pi}{5}$ or $\left(2n - \frac{1}{2}\right)\pi, n \in I$

Ex. 21. What is the general value of θ which satisfies both the equations $\sin\theta = -\frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$?

Sol. $\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right) \quad \therefore \theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

Therefore, considering only angles between 0 and 2π the only values of θ satisfying $\sin\theta = -\frac{1}{2}$ are $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

Now, $\cos\theta = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right)$

Therefore, considering only angles between 0 and 2π, the only values of θ satisfying $\cos\theta = -\frac{\sqrt{3}}{2}$ are $\frac{5\pi}{6}$ or $\frac{7\pi}{6}$.

Thus, $\frac{7\pi}{6}$ is the only angle which satisfies both the equations. Hence adding $2n\pi$, the general expression for θ is $\theta = 2n\pi + \frac{7\pi}{6}, n \in I$.

1. The function $\sin\theta$ being of period 2π, various possible solutions are obtained on adding 2π to each solution.
2. We have to take those solutions for which $8\theta \le 16\pi$ and $10\theta \le 20\pi$.

Ex. 22. Solve $\sin 9\theta = \sin\theta$, for $0° < \theta < 360°$.

Sol. $\sin 9\theta = \sin\theta$

or $\sin 9\theta = \sin(\pi - \theta)$.

First solution: $9\theta = \theta, 2\pi + \theta, 4\pi + \theta, 6\pi + \theta, 8\pi + \theta, 10\pi + \theta, \ldots$

$8\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi, 14\pi, 16\pi$

$\theta = 0, \frac{2\pi}{8}, \frac{4\pi}{8}, \frac{6\pi}{8}, \frac{8\pi}{8}, \frac{10\pi}{8}, \frac{12\pi}{8}, \frac{14\pi}{8}, \frac{16\pi}{8}$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

Second solution: $9\theta = \pi - \theta, 3\pi - \theta, 5\pi - \theta, 7\pi - \theta, 9\pi - \theta, 11\pi - \theta, \ldots$

$\therefore 10\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, 11\pi, 13\pi, 15\pi, 17\pi, 19\pi$

$\therefore \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10} \ldots\ldots, \frac{19\pi}{10}$.

Method II. $\sin 9\theta = \sin\theta$

or $\sin 9\theta - \sin\theta = 0$

or $2\cos 5\theta \sin 4\theta = 0$ [Applying 'C, D' formula]

$\therefore$ either $\sin 4\theta = 0$ or $\cos 5\theta = 0$

$\sin 4\theta = 0 = \sin 0 = \sin\pi = \sin 2\pi = \sin 3\pi = \sin 4\pi = \sin 5\pi = \sin 6\pi = \sin 7\pi = \sin 8\pi$

$\therefore \quad 4\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$, or $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

Now $\cos 5\theta = 0 = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \cos\frac{7\pi}{2} = \cos\frac{9\pi}{2}$

$= \cos\frac{11\pi}{2} = \cos\frac{13\pi}{2} = \ldots = \cos\frac{19\pi}{2}$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \ldots\ldots, \frac{19\pi}{2}$ or $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, \ldots\ldots\frac{19\pi}{10}$.

Ex. 23. Solve $\sin 3x = \cos 2x$. $(0 < x < 2\pi)$.

Method. The given equation should first be reduced to the form sin ( ) = sin ( ) or cos ( ) = cos ( ) or tan ( ) = tan ( ).

Sol. $\cos 2x = \sin\left(\frac{1}{2}\pi - 2x\right)$

and also $\cos 2x = \sin\left(\frac{1}{2}\pi + 2x\right)$,

(*a*) When $\sin 3x = \sin\left(\frac{1}{2}\pi - 2x\right)$, giving $3x = \frac{\pi}{2} - 2x, \frac{5\pi}{2} - 2x, \frac{9\pi}{2} - 2x\ldots\ldots$; and

(*b*) From $\sin 3x = \sin\left(\frac{1}{2}\pi + 2x\right)$, giving $3x = \frac{1}{2}\pi + 2x, \frac{5\pi}{2} + 2x, \frac{9\pi}{2} + 2x, \ldots$

From (*a*) $5x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

or $x = \frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$ (Since $5x < 10\pi$)

From (*b*) $x = \frac{\pi}{2}$

Hence, the required solutions are $x = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$.

Ex. 24. Solve $\tan 3x = \cot 5x$, $(0 < x < 2\pi)$.

Sol. The given equation may be written as

$\tan 3x = \tan\left(\frac{\pi}{2} - 5x\right) \quad \therefore \; 3x = \frac{\pi}{2} - 5x, \frac{3\pi}{2} - 5x, \frac{5\pi}{2} - 5x, \frac{7\pi}{2} - 5x, \ldots\ldots, \frac{31\pi}{2} - 5x,$

or $8x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots., \frac{31\pi}{2}$.

$\therefore \quad x = \frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \ldots\ldots, \frac{31\pi}{16}$.

Case III. Equations involving more than two multiple angles

Method. *Apply 'C, D' formulas to combine the two.*

Ex. 25. (*i*) Solve $\sin 3\theta + \sin\theta = \sin 2\theta$, $0 \le \theta \le 2\pi$. (*ii*) Give the general solution.

Sol. (*i*) $(\sin 3\theta + \sin\theta) - \sin 2\theta = 0$ or $2 \sin 2\theta \cos\theta - \sin 2\theta = 0$ or $\sin 2\theta (2\cos\theta - 1) = 0$

Hence,

$\sin 2\theta = 0$	$2\cos\theta - 1 = 0$
$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$	$\cos\theta = \frac{1}{2}$
$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$	$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore \quad \theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{5\pi}{3}, 2\pi$

(*ii*) For general solution, we have

$\sin 2\theta = 0$	$2\cos\theta - 1 = 0$
$\therefore \quad 2\theta = n\pi$	$\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$
$\therefore \quad \theta = \frac{n\pi}{2}, n \in I$	$\therefore \quad \theta = 2m\pi \pm \frac{\pi}{3}, m \in I$

Hence, $\theta = \frac{n\pi}{2}$ or $2m\pi \pm \frac{\pi}{3}$, where $n, m \in I$.

Ex. 26. Solve : $\cos 3x + \cos 2x = \sin\frac{3}{2}x + \sin\frac{1}{2}x$, $0 < x \le \pi$. **[*IIT*]**

Sol. $\cos 3x + \cos 2x = \sin\frac{3}{2}x + \sin\frac{1}{2}x \Rightarrow 2\cos\frac{5}{2}x \cos\frac{1}{2}x = 2\sin x \cos\frac{1}{2}x$

$\Rightarrow \quad \cos\frac{1}{2}x(\cos\frac{5}{2}x - \sin x) = 0 \quad \therefore \quad \cos\frac{1}{2}x = 0$ or $\cos\frac{5}{2}x - \sin x = 0$

$\cos\frac{1}{2}x = 0 \Rightarrow \frac{1}{2}x = \frac{\pi}{2} \Rightarrow x = \pi$

$\cos\frac{5}{2}x - \sin x = 0 \Rightarrow \cos\frac{5}{2}x = \sin x = \cos\left(\frac{\pi}{2} - x\right)$ or $\cos\left(2\pi + \frac{\pi}{2} - x\right)$

$\Rightarrow \quad \frac{5}{2}x = \frac{\pi}{2} - x$ or $\frac{5\pi}{2} - x \Rightarrow \frac{7}{2}x = \frac{\pi}{2}$ or $\frac{5\pi}{2} \Rightarrow x = \frac{\pi}{7}$ or $\frac{5\pi}{7}$

Hence, $x = \frac{\pi}{7}$ or $\frac{5\pi}{7}$ or π.

Type 4. Equations involving compound angles

Procedure. *Expand by applying 'A ± B' formulas.*

Ex. 27. Find the values of θ between 0° and 360° that satisfy the equation

$\sin(\theta + 30°) = 2\cos(45° + \theta)$

Sol. The given equation can be written as

$\sin\theta \cos 30° + \cos\theta \sin 30° = 2(\cos 45° \cos\theta - \sin 45° \sin\theta)$

$$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = 2\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$$

i.e., $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = \sqrt{2}\cos\theta - \sqrt{2}\sin\theta$

$$\therefore \quad \left(\frac{\sqrt{3}}{2}+\sqrt{2}\right)\sin\theta = (\sqrt{2}-\frac{1}{2})\cos\theta\,;$$

$$\tan\theta = \frac{2\sqrt{2}-1}{\sqrt{3}+2\sqrt{2}} = 0.4009 = \tan 21°\,51' = \tan(180° + 21°51') = \tan 201°51'$$

$$\Rightarrow \qquad \theta = 21°\,51',\ 201°\,51'$$

Type 5. Linear equations of the form $a\cos\theta + b\sin\theta = c$ where $|c| \le \sqrt{a^2+b^2}$.

Ex. 28. Solve : $a\cos\theta + b\sin\theta = c$.

Sol. Let $a = r\cos\alpha$ and $b = r\sin\alpha$. Then

$$r = \sqrt{a^2+b^2}. \qquad \text{(Taking only +ve value)}$$

$$\therefore \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}, \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$$

$\therefore$ The given eqn. is reduced to the form $r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = c$

or $$\cos(\theta-\alpha) = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta \quad \text{(suppose)} \qquad (|c| \le \sqrt{a^2+b^2})$$

$$\therefore \quad (\theta-\alpha) = 2n\pi \pm \beta \Rightarrow \theta = 2n\pi + \alpha \pm \beta$$

Method 1. *Divide both sides of the given equation by* $\sqrt{a^2+b^2}$.

2. *Express left side in the form cos* $(\theta \pm \alpha)$ *or sin* $(\theta \pm \alpha)$. *Reduce right side to the same form (cos or sine).*

3. *Now use the formulas for general values of sin* $\theta = \sin\alpha$ *or cos* $\theta = \cos\alpha$ *as the case may be.*

Ex. 29. Solve : $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

Sol. Dividing both sides of the equation by $\sqrt{\{(\sqrt{3})^2 + (-1)^2\}}$, *i.e.,* 2, we get

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \cos\frac{\pi}{6}\sin\theta - \sin\frac{\pi}{6}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin(\theta - \frac{\pi}{6}) = \sin\frac{\pi}{4} \Rightarrow \theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4},\ n \in I$$

$$\therefore \quad \theta = n\pi + \frac{\pi}{6} + (-1)^n\frac{\pi}{4},\ n \in I.$$

Ex. 30. Solve : $\tan\theta + \sec\theta = \sqrt{3}$.

Sol. $\tan\theta + \sec\theta = \sqrt{3} \Rightarrow \sin\theta + 1 = \sqrt{3}\cos\theta \Rightarrow \sqrt{3}\cos\theta - \sin\theta = 1$

Dividing both sides by $\sqrt{\{(\sqrt{3})^2 + (-1)^2\}}$, *i.e.,* 2 we get

$$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$$

or $$\cos\left(\theta + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\therefore \qquad \theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3},\ n \in I \Rightarrow \theta = 2n\pi + \frac{\pi}{6},\ 2n\pi - \frac{1}{2}\pi$$

$$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{6},\ n \in I, \text{ as } 2n\pi - \frac{1}{2}\pi, \text{ is rejected, because this does not satisfy the equation.}$$

Ex. 31. Solve $\cos x - \sqrt{3}\ \sin x = 1$, $0° \le x \le 360°$.

Sol. Dividing both sides of the equation by $\sqrt{\{(1)^2 + (-\sqrt{3})^2\}}$ *i.e.* 2, we get

$$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}$$

$\Rightarrow \cos 60° \cos x - \sin 60° \sin x = \frac{1}{2}$

$\Rightarrow \cos (x + 60°) = \cos 60°$ or $\cos (360° - 60°)$ or $\cos (360° + 60°)$

$\therefore \quad x + 60° = 60°$ or $300°$ or $420° \quad \Rightarrow \quad x = 0°, 240°, 360°$

taking values such that $0° \le x \le 360°$.

Ex. 32. Express $2 \cos \theta + 3 \sin \theta$ in the form $r \cos (\theta - \alpha)$, where r is positive, stating the values of r and α. What is the maximum value of $2 \cos \theta + 3 \sin \theta$? ***[NW]***

Sol. Here, $2 = r \cos \alpha$, $3 = r \sin \alpha \ \therefore \ \tan \alpha = \frac{3}{2} = 1.5000 \Rightarrow \alpha = 56° 19'$

The given expression is

$2 \cos \theta + 3 \sin \theta$

$= \sqrt{13}\left(\frac{2}{\sqrt{13}}\cos\theta + \frac{3}{\sqrt{13}}\sin\theta\right)$

Multiply and divide by $r = \sqrt{2^2 + 3^2}$, *i.e.* $\sqrt{13}$.

$= \sqrt{13}\ (\cos 56° 19' \cos \theta + \sin 56° 19' \sin \theta) = \sqrt{13}\ \cos (\theta - 56°19')$

$\therefore \quad r = \sqrt{13}$ and $\alpha = 56° 19'$.

$\because$ The max. value of $\cos (\theta - 56° 19') = 1$

$\therefore$ The maximum value of the given expression is $\sqrt{13}$

Note. As the minimum value of cos of an angle is -1, the minimum value of the given expression is $-\sqrt{13}$.

Ex. 33. Find all the values of θ satisfying the equation $\sin \theta + \sin 5\theta = \sin 3\theta$, such that $0 \le \theta \le \pi$. ***[ISC]***

Sol. $\sin \theta + \sin 5\theta = \sin 3\theta \ \Rightarrow \ 2 \sin 3\theta \cos 2\theta - \sin 3\theta = 0$

$\Rightarrow \qquad \sin 3\theta\ (2 \cos 2\theta - 1) = 0.$

Either $\sin 3\theta = 0$ or $2 \cos 2\theta - 1 = 0$. Thus,

(*i*) $\sin 3\theta = 0 = \sin 0 = \sin \pi = \sin 2\pi = \sin 3\pi$

$\therefore \quad 3\theta = 0, \pi, 2\pi, 3\pi \quad \therefore \quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

(*ii*) $2 \cos 2\theta - 1 = 0$

$\Rightarrow \quad \cos 2\theta = \frac{1}{2} = \cos \frac{\pi}{3}$ or $\cos\left(2\pi - \frac{\pi}{3}\right)$

$\Rightarrow \quad 2\theta = \frac{\pi}{3}, \frac{5\pi}{3} \ \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, all possible values of θ are $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$.

EXERCISE 6

1. Solve the following equations for $0 \le x \le 2\pi$.

(*i*) $2 \sin x - 1 = 0$ (*ii*) $\sin x \cos x = 0$ (*iii*) $\tan \theta + \sqrt{3} = 0$ (*iv*) $\sin \theta \cos \theta = \frac{1}{2}$

(*v*) $2 \sin^2 \theta = 3 \cos \theta$ **[*IIT*]** (*vi*) $2 + 7 \tan^2 \theta = 3.25 \sec^2 \theta$

2. $\cos \theta + \sin \theta - \sin 2\theta = \frac{1}{2}, 0 < \theta < \frac{\pi}{2}$ **3.** $\sin 5\theta = \cos 2\theta, 0° < \theta < 180°$.

4. $\cot^2 \theta - (1 + \sqrt{3}) \cot \theta + \sqrt{3} = 0, 0 < \theta < \frac{\pi}{2}$

5. $\sin x + \cos (x + 30°) = 0, 0° < x < 360°$. **[*ISC*]**

6. $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0, 0° < \theta < 180°$ **[*IIT*]**

7. $\sin 7\theta + \sin 4\theta + \sin \theta = 0, \ 0 < \theta < \frac{\pi}{2}$ **[*IIT*]**

Solve, giving the general value.

8. $2 \cos^2 \theta - 5 \cos \theta + 2 = 0$ **9.** $2 \sin^2 x + \sqrt{3} \cos x + 1 = 0$ **10.** $2 + \sqrt{3} \sec x - 4 \cos x = 2\sqrt{3}$

11. $\tan^2 \theta - (1 + \sqrt{3}) \tan \theta + \sqrt{3} = 0$ **12.** $\tan \theta + 4 \cot 2\theta + 1 = 0$

13. $\tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}$ **14.** $\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta$

15. $2 \cos \theta + \cos 3\theta = 0$ **16.** $2 \sin 2x - \sin x = 0$ **17.** $\tan 2x + 2 \tan x = 0$

18. $\sin 7\theta + \sin 4\theta + \sin \theta = 0$ **19.** $\cos \theta + \cos 2\theta + \cos 3\theta = 0$ **20.** $\sin \theta + \cos \theta = \sqrt{2}$

21. $\sin \theta + \sqrt{3} \cos \theta = \sqrt{2}$ **22.** $\sqrt{2} \sec \theta + \tan \theta = 1$

23. $3 - 2 \cos \theta - 4 \sin \theta - \cos 2\theta + \sin 2\theta = 0$ **[*IIT*]**

24. If the equation $a \cos 2\theta + b \sin 2\theta = c$ has θ_1, θ_2 as its roots, prove that

(*i*) $\tan \theta_1 + \tan \theta_2 = \frac{2b}{c+a}$ (*ii*) $\tan \theta_1 . \tan \theta_2 = \frac{c-a}{c+a}$ **[*IIT*]**

25. If α, β are two different values of θ lying between 0 and 2π which satisfy the equation $6 \cos \theta + 8 \sin \theta = 9$, find the value of $\sin (\alpha + \beta)$. **[*IIT*]**

26. Find all the values of θ satisfying the equation $\cos 2\theta - \cos 8\theta + \cos 6\theta = 1$, such that $0 \le \theta \le \pi$. **[*ISC, 1989*]**

27. $\sec \theta - \operatorname{cosec} \theta = \frac{4}{3}$ **[*Roorkee 1994*]**

28. Find the smallest positive number p for which the equation $\cos (p \sin x) = \sin (p \cos x)$ has a solution when $x \in [0, 2\pi]$. **[*IIT 1995*]**

ANSWERS

1. (*i*) $\frac{\pi}{6}, \frac{5\pi}{6}$ (*ii*) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (*iii*) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (*iv*) $\frac{\pi}{4}, \frac{5\pi}{4}$ (*v*) $\frac{\pi}{3}, \frac{5\pi}{3}$

(*vi*) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **2.** $\frac{\pi}{6}, \frac{\pi}{3}$ **3.** $30°, 150°, 12\frac{6°}{7}, 64\frac{2°}{7}, 115\frac{5°}{7}, 167\frac{1°}{7}$

4. $\frac{\pi}{6}, \frac{\pi}{4}$ **5.** $120°, 300°$ **6.** $30°, 45°, 90°, 135°, 150°$ **7.** $\frac{\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}$

8. $2n\pi \pm \frac{\pi}{3}$ **9.** $2n\pi \pm \frac{5\pi}{6}$ **10.** $2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{5\pi}{6}$ **11.** $n\pi + \frac{\pi}{3}$, $n\pi + \frac{\pi}{4}$

12. $n\pi - \frac{\pi}{4}$, $n\pi + \alpha$, where $\alpha = \tan^{-1}2$ **13.** $\frac{n\pi}{3} + \frac{\pi}{9}$ **14.** $2n\pi \pm \frac{1}{3}\pi$

15. $(2n+1)\frac{\pi}{2}, n\pi \pm \frac{\pi}{3}$ **16.** $n\pi$, $2n\pi \pm \alpha$ when $\cos\alpha = \frac{1}{4}$ **17.** $n\pi$, $n\pi \pm \alpha$ when $\tan^2\alpha = 2$

18. $\frac{n\pi}{4}$ or $\frac{2}{3}\left(n \pm \frac{1}{3}\right)\pi$ **19.** $\left(n + \frac{1}{2}\right)\frac{\pi}{2}, 2n\pi \pm \frac{2\pi}{3}$ **20.** $\left(2n + \frac{1}{4}\right)\pi$

21. $\left(n - \frac{1}{3}\right)\pi + (-1)^n \frac{1}{4}\pi$ **22.** $2n\pi - \frac{\pi}{4}$ **23.** $n\pi + (-1)^n \frac{\pi}{2}$, $2n\pi$, $2n\pi + \frac{\pi}{2}$

25. $\frac{24}{25}$. **26.** $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

27. $\frac{1}{2}\left[n\pi + (-1)^n \sin^{-1}\frac{3}{4}\right]$ **28.** $\frac{\pi}{2\sqrt{2}}$

HINTS TO SELECTED QUESTIONS

23. $3 - 2\cos\theta - 4\sin\theta - (1 - 2\sin^2\theta) + 2\sin\theta\cos\theta = 0 \Rightarrow (\sin\theta - 1)(\sin\theta - 1 + \cos\theta) = 0$

24. $a.\frac{1-\tan^2\theta}{1+\tan^2\theta} + b.\frac{2\tan\theta}{1+\tan^2\theta} = c \Rightarrow (a+c)\tan^2\theta - 2b\tan\theta + (c-a) = 0$

25. $6\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) + 8\left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = 9 \Rightarrow 15\tan^2\frac{\theta}{2} - 16\tan\frac{\theta}{2} + 3 = 0$

$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = ?$, $\tan\frac{\alpha}{2}.\tan\frac{\beta}{2} = ?$, $\tan\frac{\alpha+\beta}{2} = ?$, $\sin(\alpha+\beta) = \frac{2\tan\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}}$

26. Given equation $\Rightarrow$ $2\cos4\theta \cdot \cos2\theta - (2\cos^2 4\theta - 1) - 1 = 0$
$\Rightarrow 2\cos4\theta(\cos2\theta - \cos4\theta) = 0 \Rightarrow \cos4\theta \cdot \sin3\theta \cdot \sin\theta = 0$
$\Rightarrow \cos4\theta = 0$ or $\sin3\theta = 0$ or $\sin\theta = 0$

$\cos4\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{8}$, $\sin3\theta = 0 \Rightarrow \theta = \frac{n\pi}{3}$,

$\sin\theta = 0 \Rightarrow \theta = n\pi$, where $n = 0, \pm1, \pm2, \ldots\ldots$
But since $0 \le \theta \le \pi$, therefore, putting $n = 0, 1, 2, 3, \ldots\ldots$

$\cos4\theta = 0 \Rightarrow \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$; $\sin3\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi$

$\therefore$ Reqd. values of θ in $0 \le \theta \le \pi$ are $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

27. $\frac{1}{\cos\theta} - \frac{1}{\sin\theta} = \frac{4}{3} \Rightarrow 3(\sin\theta - \cos\theta) = 2\sin2\theta$.

Squaring and simplifying : $4\sin^2 2\theta + 9\sin2\theta - 9 = 0$

28. $\cos(p\sin x) = \sin(p\cos x) \Rightarrow p\sin x = \frac{\pi}{4}$ and $p\cos x = \frac{\pi}{4}$ or $p\sin x = \frac{5\pi}{4}$ and $p\cos x = \frac{5\pi}{4}$.

Square and add.

CHAPTER SUMMARY

Trigonometric equation	General solution
1. $\sin\theta = 0$	$\theta = n\pi$
2. $\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}$
3. $\tan\theta = 0$	$\theta = n\pi$
4. $\cot\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}$
5. $\sec\theta = 0$	No solution
6. $\operatorname{cosec}\theta = 0$	No solution
7. $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha$
8. $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
9. $\tan\theta = \tan\alpha$	$\theta = n\pi \pm \alpha$
10. $\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
11. $\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
12. $\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$

CHAPTER TEST

1. Find the general value of θ which satisfies the equation $\sin^2\theta = \frac{3}{4}$.

2. Find the general solution of $\sin 2\theta = \cos\theta$.

3. Solve : $\sin 5x - \sin 3x - \sin x = 0$, $0^\circ < x < 360^\circ$. **[*WBHS*]**

4. Find the general solution of $\tan^2\theta = \frac{1}{3}$, and hence find those values of θ for which $-\pi \le \theta \le \pi$.

5. Solve : $\tan\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}-\theta\right) = 4$ **[*EAMCET, WBHS*]**

6. Solve the equation $\sqrt{3}\cos x + \sin x = 1$ for $-2\pi < x < 2\pi$.

ANSWERS AND HINTS

1. $\theta = n\pi \pm \frac{\pi}{3}$ **2.** $n\pi + (-1)^n\frac{\pi}{6}$

3. $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \pi, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$. [**Hint.** $x = \frac{1}{4}\left(2n\pi \pm \frac{\pi}{3}\right)$, Put $n = 0, 1, 2, 3, 4$]

4. $\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$ [**Hint.** $\theta = n\pi \pm \frac{\pi}{6}$, where n is 0 or any other integer. Put $n = 0, -1, 1$.]

5. $\theta = n\pi \pm \frac{\pi}{6}$, where n is 0 or any integer.

6. $x = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{6}$, where n is 0 or any integer.

CHAPTER 7

PROPERTIES OF TRIANGLE

7.01. The law of sines (sine formula)

Theorem 1. In any triangle, the sides are proportional to the sines of the opposite angles, *i.e.*,

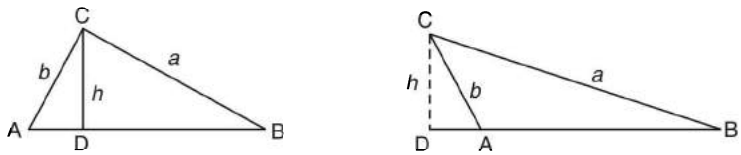
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof. Let ABC be any oblique triangle. Draw the altitude $CD = h$. Two cases are possible: all angles acute or one angle obtuse.

In either case;

$\sin B = \frac{h}{a}$; hence $h = a \sin B$, $\sin A = \frac{h}{b}$; hence $h = b \sin A$

$$\left[\text{In Fig. 7.01 } (ii),\ \sin A = \sin(\pi - A) = \sin \angle CAD = \frac{h}{b}\right]$$



(*i*) (*ii*)

Fig. 7.01

Equating the two values of h, we get

$$a \sin B = b \sin A \ \Rightarrow\ \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, by drawing the perpendicular from A to BC, at BC produced, we can prove that

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

Therefore, in any triangle $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Note 1. This law can be written in the form $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Note 2. This is equivalent to the following three equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \frac{a}{\sin A} = \frac{c}{\sin C}, \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note 3. These formulae help us to change sines of angle to sides and vice *versa*. Thus,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (suppose)} \Rightarrow \sin A = ak, \sin B = bk, \sin C = ck.$$

Ex. 1. In a ΔABC, $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$, find $\angle$B.

Sol. By the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\frac{2}{\sin A} = \frac{3}{\sin B} \Rightarrow 2 \sin B = 3 \sin A = 3 \times \frac{2}{3} = 2$$

$$\therefore \quad \sin B = \frac{2}{2} = 1 \Rightarrow B = 90°$$

Ex. 2. The angles of a triangle ABC are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, find $\angle$A. (*IIT*)

Sol. In ΔABC, $\angle A + \angle B + \angle C = 180°$ ... (1)

It is given that $\angle A$, $\angle B$, $\angle C$ are in A.P. $\therefore 2\angle B = \angle A + \angle C$ ... (2)

(1) and (2) $\Rightarrow 2\angle B + \angle B = 180° \Rightarrow \angle B = 60°$

By the sine formula, $\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{\sin C} = \frac{b}{c} \Rightarrow \frac{\sin 60°}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$

$$\Rightarrow \quad \sin C = \frac{\sqrt{2}}{\sqrt{3}} \sin 60° = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}} = \sin 45° \therefore \angle C = 45°$$

$\therefore$ From (1), $\angle A + 60° + 45° = 180° \Rightarrow \angle A = 75°$.

7.02. Relations between three sides and two angles

Theorem 2. In any triangle, to prove that
***a* = *b* cos *C* + *c* cos *B*, *b* = *c* cos *A* + *a* cos *C*, *c* = *a* cos *B* + *b* cos *A* (*Projection Formulae*)**

In Fig. 7.01 (*i*), $\frac{AD}{AC} = \cos A \therefore AD = AC \cos A = b \cos A$

$$\frac{DB}{BC} = \cos B \therefore DB = BC \cos B = a \cos B$$

Now, $AB = AD + DB = b \cos A + a \cos B \Rightarrow c = a \cos B + b \cos A$

Similarly, we can prove the other relations.

Ex. 3. In any Δ ABC, prove that, $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$.

Sol. R.H.S. $= \frac{(a\cos B + b\cos A) - a\cos B}{(c\cos A + a\cos C) - a\cos C} = \frac{b\cos A}{c\cos A} = \frac{b}{c} = \frac{\sin B}{\sin C}$.

7.03. The law of Cosines

Theorem 3. The square on any side of a triangle is equal to the sum of the squares on the other two sides minus twice the product of those two sides and the cosine of the included angle, *i.e.*,

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof. Consider any oblique triangle ABC with the altitude CD drawn from the vertex C to the opposite side. In Fig. 7.02 (*i*), $AD = b\cos A$, $CD = b\sin A$; $DB = AB - AD = c - b\cos A$.
In Fig. 7.02 (*ii*);

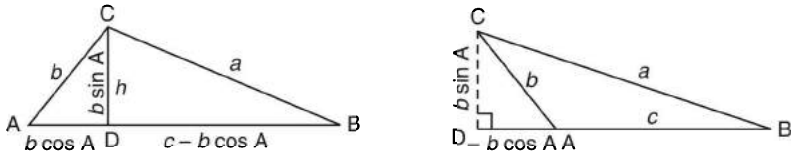


Fig. 7.02

$$\cos A = \cos(\pi - \angle CAD) = -\cos\angle CAD = -\frac{AD}{b} \Rightarrow AD = -b\cos A;\ CD = b\sin A;$$

$$DB = DA + AB = c - b\cos A$$

Thus, in both figures, $DB = c - b\cos A$, $CD = b\sin A$

$\therefore$ $a^2 = DB^2 + CD^2$ (Pythagoras' Theorem) $= (c - b\cos A)^2 + b^2\sin^2 A$

$\Rightarrow$ $a^2 = c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A = c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A)$

$\therefore$ $a^2 = b^2 + c^2 - 2bc\cos A$.

Other results can be proved in like manner.

By changing the subject, the above three formulae can be written in the form:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note. The student should remember only the first of these six formulae and since they are all symmetrical he can write them by taking any side of the triangles as 'a'.

Ex. 4. In any ΔABC, prove that $a(b\cos C - c\cos B) = b^2 - c^2$.

Sol. Using the cosine formula, we have

$$\text{L.H.S.} = a(b\cos C - c\cos B) = ab\cdot\frac{a^2 + b^2 - c^2}{2ab} - ac\cdot\frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{1}{2}(a^2 + b^2 - c^2 - c^2 - a^2 + b^2) = \frac{1}{2}(2b^2 - 2c^2)$$

$$= b^2 - c^2 = \text{R.H.S.}$$

Ex. 5. In any ΔABC, prove that, $\frac{b^2 - c^2}{a^2}\sin 2A + \frac{c^2 - a^2}{b^2}\sin 2B + \frac{a^2 - b^2}{c^2}\sin 2C = 0$.

Sol. L.H.S $= \frac{b^2 - c^2}{a^2}\times 2\sin A\cos A + \frac{c^2 - a^2}{b^2}\times 2\sin B\cos B + \frac{a^2 - b^2}{c^2}\times 2\sin C\cos C$

$$= \frac{b^2 - c^2}{a^2}\cdot 2ka\cdot\frac{b^2 + c^2 - a^2}{2bc} + \frac{c^2 - a^2}{b^2}2kb\cdot\frac{c^2 + a^2 - b^2}{2ca} + \frac{a^2 - b^2}{c^2}\cdot 2kc\cdot\frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{k}{abc}\left[(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(c^2 + a^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)\right]$$

$$= \frac{k}{abc}[b^4 - c^4 - a^2(b^2 - c^2) + c^4 - a^4 - b^2(c^2 - a^2) + a^4 - b^4 - c^2(a^2 - b^2)]$$

$$= \frac{k}{abc}\times 0 = 0.$$

7.04. Area of a Triangle

Theorem 4. The area of a triangle is given by $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$.

Proof. Let there be a ΔABC and AD its altitude while BC or BC produced at D. Then,

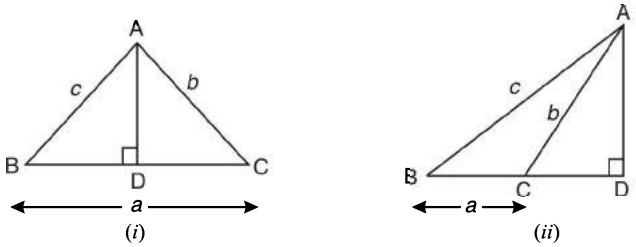


Fig. 7.03

$$\text{Area of } \Delta ABC = \frac{1}{2} \times BC \times AD \qquad \left[\frac{1}{2} \times \text{base} \times \text{height}\right]$$

In Fig. (i) $\quad \frac{AD}{AC} = \sin C \Rightarrow \frac{AD}{b} = \sin C \Rightarrow AD = b \sin C$

In Fig. (ii), $\quad \frac{AD}{AC} = \sin \angle ACD = \sin(\pi - C) = \sin C \Rightarrow AD = b \sin C$

Thus, in both cases $AD = b \sin C$

$\therefore$ $\quad \text{area of } \Delta ABC = \frac{1}{2} \times BC \times AD$

$$= \frac{1}{2} \times a \times b \sin C = \frac{1}{2}ab\sin C$$

Hence, $\quad \text{area of } \Delta ABC = \frac{1}{2}ab\sin C$

Similarly, it can be proved that

$$\Delta = \frac{1}{2}ac\sin B \text{ and } \Delta = \frac{1}{2}bc\sin A$$

Hence, $\quad \text{area of } \Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

Ex. 6. Find the area of a ΔABC in which $\angle A = 30°$, $b = 6$ cm, $c = 2$ cm.

Sol. $$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2} \times 6 \times 2 \times \sin 30° = \frac{1}{2} \times 6 \times 2 \times \frac{1}{2}\text{cm}^2 = 3 \text{ cm}^2$$

Ex. 7. In any ΔABC, prove that $\Delta = \frac{b^2 + c^2 - a^2}{4\cot A}$.

Sol. R.H.S. $= \frac{b^2 + c^2 - a^2}{4\cot A} = \frac{b^2 + c^2 - a^2}{4\cos A}\sin A$

$$= \frac{b^2 + c^2 - a^2}{4} \cdot \frac{2bc}{b^2 + c^2 - a^2}\sin A$$

$$= \frac{1}{2}bc\sin A = \Delta = \text{L.H.S.}$$

Ex. 8. In any ΔABC, prove that $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A = 4\Delta$.

Sol. L.H.S $= b^2 \sin 2C + c^2 \sin 2B$

$= b^2 \,.\, 2 \sin C \cos C + c^2 \,.\, 2 \sin B \cos B$

$= b^2 \,.\, 2 \sin C \cos C + c^2 \,.\, 2 \sin B \cos B$

$= 2b^2 \,.\, kc \cos C + 2c^2 \,.\, kb \cos B,$ (Using sine formule)

$= 2bck\,(b \cos C + c \cos B)$

$= 2bck.\ a = 2bc.\ (ka)$ (Using projection formula)

$= 2\text{bc. sin A}$ (Using sine formula)

$= 4 \cdot \left(\frac{1}{2} bc \sin A\right) = 4\Delta = \text{R.H.S.}$

MORE SOLVED EXAMPLES

Ex. 9. In any ΔABC, prove that $a \cos \frac{1}{2}(B - C) = (b + c) \sin \frac{1}{2} A$.

[**Note.** In such questions in which we can collect the sides at one place we should do so and work with the expression so obtained].

In ΔABC, $\angle A + \angle B + \angle C = 180° \Rightarrow \angle\frac{A}{2} + \angle\frac{B}{2} + \angle\frac{C}{2} = 90°$ ... (1)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (suppose)}$$

Now, $$\frac{b+c}{a} = \frac{k \sin B + k \sin C}{k \sin A}$$

$$= \frac{\sin B + \sin C}{\sin A} = \frac{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}}$$

$$= \frac{\sin\left(90° - \frac{A}{2}\right) \cos \frac{B-C}{2}}{\sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\cos \frac{A}{2} \cos \frac{B-C}{2}}{\sin \frac{A}{2} \cos \frac{A}{2}}$$

$\therefore$ $$\frac{b+c}{a} = \frac{\cos \frac{B-C}{2}}{\sin \frac{A}{2}} \Rightarrow (b+c) \sin \frac{A}{2} = a \cos \frac{B-C}{2}.$$

Ex. 10. In any ΔABC, prove that

(*i*) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$.

(*ii*) If $\angle B = 90°$, prove that $\tan \frac{1}{2} A = \sqrt{\left(\frac{b-c}{b+c}\right)}$.

(*iii*) If $\angle C = 90°$, prove that $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b}$.

Sol. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k(\text{suppose})$

$$\therefore \quad \frac{b-c}{b+c}=\frac{k\sin B-k\sin C}{k\sin B+k\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}=\cot\frac{B+C}{2}\tan\frac{B-C}{2}$$

$$=\cot\left(90°-\frac{A}{2}\right)\tan\frac{B-C}{2}=\tan\frac{A}{2}\tan\frac{B-C}{2}$$

$$\left[\text{Using } A+B+C=180°\Rightarrow\frac{B+C}{2}=90°-\frac{A}{2}.\right]$$

$$\therefore \quad \left(\frac{b-c}{b+c}\right)\cdot\frac{1}{\tan\dfrac{A}{2}}=\tan\frac{B-C}{2}\Rightarrow\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}=\tan\frac{B-C}{2}$$

Hence, $\quad \boldsymbol{\tan\dfrac{B-C}{2}=\left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2}}$ **[Remember]**

Similarly, $\boldsymbol{\tan\dfrac{C-A}{2}=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}}$, **and** $\boldsymbol{\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}}$.

Note. The above formula is called **Napiers Analogy** and is used in solving triangles.

(*ii*) $\angle B=90°$. Also, $\angle A+\angle B+\angle C=180°$

$\therefore \quad \angle C=180°-\angle A-\angle B=180°-90°-\angle A=90°-\angle A.$

Substituting in the formula,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{90°-(90°-A)}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}\Rightarrow\tan\frac{A}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\Rightarrow \quad \tan^2\frac{A}{2}=\frac{b-c}{b+c} \quad \therefore \ \tan\frac{A}{2}=\sqrt{\frac{b-c}{b+c}}$$

(*iii*) It is given that $\angle C=90°$.

$\therefore$ By the formula $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{90°}{2}\Rightarrow\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot 45°$$

$$\Rightarrow \quad \tan\frac{A-B}{2}=\frac{a-b}{a+b}.$$

Ex. 11. The sides of a triangle are three consecutive numbers and its largest angle is twice the smallest one. Find the sides of the triangle.

Sol. Let $BC=n$, $CA=n+1$ and $AB=n+2$ in a ΔABC. Then, clearly $C>B>A$. So, $C=2A$ (given).

$\therefore \ B=180°-(A+C)=180°-3A.$

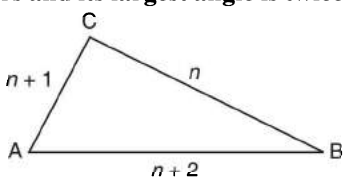


Fig. 7.04

By sine formula, $\quad \dfrac{\sin A}{n}=\dfrac{\sin B}{n+1}=\dfrac{\sin C}{n+2}$

Now, $\frac{\sin A}{n} \Rightarrow \frac{\sin C}{n+2} = \frac{\sin A}{n} = \frac{\sin 2A}{n+2}$ (Putting $C = 2A$)

$= \frac{2\sin A\cos A}{n+2} \Rightarrow \cos A = \frac{n+2}{2n}$ ... (1)

Also, $\frac{\sin A}{n} = \frac{\sin B}{n+1} \Rightarrow \frac{\sin A}{n} = \frac{\sin(180° - 3A)}{n+1} = \frac{\sin 3A}{n+1} = \frac{3\sin A - 4\sin^3 A}{n+1}$...

$\Rightarrow \frac{\sin A}{n} = \frac{\sin A(3 - 4\sin^2 A)}{n+1} \Rightarrow \frac{1}{n} = \frac{3 - 4\sin^2 A}{n+1} \Rightarrow 3 - 4\sin^2 A = \frac{n+1}{n}$

$\Rightarrow 4\sin^2 A = 3 - \frac{n+1}{n} = \frac{2n-1}{n} \Rightarrow \sin^2 A = \frac{2n-1}{4n}$ ... (2)

(1) and (2) $\Rightarrow \left(\frac{n+2}{2n}\right)^2 + \left(\frac{2n-1}{4n}\right) = 1$ which on solving gives, $n = 4$.

$\therefore$ The sides are 4, 5 and 6.

Ex. 12. In any ΔABC prove that, $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$.

Sol. By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say) $\therefore a = k\sin A, b = k\sin B, c = k\sin C$

First term on L.H.S $= a^3 \sin(B - C) = k^3 \sin^3 A \sin(B - C)$

$= k^3 \sin^2 A \,.\, \sin A \sin(B - C) = k^3 \sin^2 A \sin(180° - \overline{B + C}) \sin(B - C)$

$= k^3 \sin^2 A \sin(B + C) \sin(B - C) = k^3 \sin^2 A (\sin^2 B - \sin^2 C)$. Similarly, second term of L.H.S. $= k^3 \sin^2 B (\sin^2 C - \sin^2 A)$, third term on L.H.S $= k^3 \sin^2 C (\sin^2 A - \sin^2 B)$

$\therefore$ L.H.S. $= k^3[\sin^2 A (\sin^2 B - \sin^2 C) + \sin^2 B (\sin^2 C - \sin^2 A) + \sin^2 C (\sin^2 A - \sin^2 B)]$

$= k^3 \times 0 = 0 =$ R.H.S.

Ex. 13. If $a\cos A = b\cos B$, then either the triangle is isosceles or right angled.

(*Roorkee*)

Sol. $a\cos A = b\cos B \Rightarrow k\sin A\cos A = k\sin B\cos B \Rightarrow \sin 2A = \sin 2B$

$\Rightarrow \sin 2A - \sin 2B = 0 \Rightarrow 2\cos(A + B)\sin(A - B) = 0$

$\because$ Either $\cos(A + B) = 0$	or $\sin(A - B) = 0$
$\Rightarrow \cos(A + B) = \cos 90°$	$\Rightarrow \sin(A - B) = \sin 0$
$\Rightarrow A + B = 90°$	$\Rightarrow A - B = 0$
$\therefore \angle C = 90°$	$\Rightarrow$ A = B
i.e., the triangle is rt $\angle d$.	*i.e.,* the triangle is isosceles.

Ex. 14. If A, B, C are the angles of a triangle ABC and if $\cos A = \frac{\sin B}{2\sin C}$, show that the triangle is isosceles. **(*C.B.S.E., Roorkee*)**

Sol. By sine formula, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$(say) $\Rightarrow \sin A = ak, \sin B = bk, \sin C = ck$

Now $\cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{kb}{2kc} \Rightarrow b^2 + c^2 - a^2 = b^2$

$\Rightarrow c^2 = a^2 \Rightarrow c = a$ (in magnitude). Hence, the triangle is isosceles.

Ex. 15. If a^2, b^2, c^2 are in A.P., prove that cot A, cot B, cot C are in A.P.

Sol. cot A, cot B, cot C will be in A.P. if $2\cot B = \cot A + \cot C$.

$$\Rightarrow \quad 2\frac{\cos B}{\sin B} = \frac{\cos A}{\sin A}+\frac{\cos C}{\sin C} \Rightarrow \frac{2\cos B}{kb} = \frac{\cos A}{ka}+\frac{\cos C}{kc}$$

$$\Rightarrow \quad \frac{2\cos B}{b} = \frac{c\cos A + a\cos C}{ac} = \frac{b}{ac}$$

$$\Rightarrow \quad b^2 = 2ac\cos B = 2ac\frac{c^2+a^2-b^2}{2ca} \Rightarrow 2b^2 = c^2 + a^2 \Rightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

Ex. 16. In any ΔABC, prove that $\Delta = \frac{a^2-b^2}{2}\frac{\sin A\sin B}{\sin(A-B)}$

Sol. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, then,

$$a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

Now, R.H.S. $= \frac{a^2-b^2}{2}\cdot\frac{\sin A\sin B}{\sin(A-B)}$

$$= \frac{k^2\sin^2 A - k^2\sin^2 B}{2}\cdot\frac{\sin A\sin B}{\sin(A-B)}$$

$$= \frac{k^2}{2}\sin(A+B)\sin(A-B)\cdot\frac{\sin A\sin B}{\sin(A-B)}$$

$$= \frac{k^2}{2}\sin(A+B)\cdot\sin A\sin B$$

$$= \frac{1}{2}\cdot(k\sin A)(k\sin B)\cdot\sin(\pi - C)$$

$$= \frac{1}{2}ab\sin C = \Delta = \text{L.H.S.}$$

EXERCISE 7

1. In ΔABC,

(*i*) if $a = 2$, $b = 3$, $c = 4$, prove that $\cos A = \frac{7}{8}$.

(*ii*) if the sides are 7, $4\sqrt{3}$ and $\sqrt{13}$ cm, prove that the smallest angle is 30°.

(*iii*) if $a = 9$, $b = 8$, $c = 4$, prove that $6\cos C = 4 + 3\cos B$.

(*iv*) The sines of the angles of a triangle are in the ratio of 4 : 5 : 6; prove that the cosines of the angles are 12 : 9 : 2.

(*v*) If the two angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm, find the area of the triangle. **[*I.I.T*]**

(*vi*) If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A-B) = \frac{4}{5}$, find its area. **[*Roorkee 1997*]**

(*vii*) In a triangle ABC, $\angle C = 60°$ and $\angle A = 75°$. If D is a point on AC such that the area of the ΔBAD is $\sqrt{3}$ times the area of the ΔBCD, find the $\angle ABD$. **[*Rookee 1996*]**

In any ΔABC, prove that

2. $\frac{\sin A}{\sin(A+B)} = \frac{a}{c}$

3. $\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$

4. $ac\cos B - bc\cos A = a^2 - b^2$ **5.** $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{c^2}$

6. $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

7. $a\cos(A+B+C) - b\cos(B+A) - c\cos(A+C) = 0$

8. $a(\cos C - \cos B) = 2(b-c)\cos^2\dfrac{1}{2}A$ **9.** $a\sin\dfrac{1}{2}(B-C) = (b-c)\cos\dfrac{1}{2}A$

10. $a\sin\left(\dfrac{A}{2}+B\right) = (b+c)\sin\dfrac{A}{2}$ **11.** $c^2 = (a-b)^2\cos^2\dfrac{1}{2}C + (a+b)^2\sin^2\dfrac{1}{2}C$

12. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$ **13.** $\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2B}{b^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$

14. $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \dfrac{a^2+b^2}{a^2+c^2}$

15. $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$

16. $a^3\sin(B-C)\operatorname{cosec}^2 A + b^3\sin(C-A)\operatorname{cosec}^2 B + c^3\sin(A-B)\operatorname{cosec}^2 C = 0$

17. $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$

18. (*i*) In a ΔABC, if $\dfrac{2\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2\cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ca}$, prove that $\angle A = 90°$ **[*I.I.T. 1993*]**

(*ii*) In a ΔABC, AD is the altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \dfrac{abc}{(b^2-c^2)}$, find $\angle B$. **[*I.I.T. 1994*]**

19. In ΔABC, $\dfrac{a^2+b^2}{a^2-b^2} = \dfrac{\sin(A+B)}{\sin(A-B)}$, prove that the triangle is isosceles or right triangle.

20. If $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in A.P.

[**Hint.** $\dfrac{a}{c} = \dfrac{a\cos B - b\cos A}{b\cos C - c\cos B} \Rightarrow b(a\cos C + c\cos A) = 2ca\cos B$]

21. If $\sin 2A + \sin 2B = \sin 2C$, prove that $A = 90°$ or $B = 90°$. **[*U.P. Board*]**

22. If $C = 60°$, prove that $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$ **[*I.I.T. 1991*]**

23. (*i*) In a ΔABC the angles A, B, C are in A.P. show that $2\cos\dfrac{A-C}{2} = \dfrac{a+c}{\sqrt{(a^2-ac+c^2)}}$ **[*I.I.T*]**

(*ii*) If the angles A, B, C of ΔABC are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, show that $A = 75°$.

(*iii*) If $A = 45°$ and $B = 75°$, show that $a + c\sqrt{2} = 2b$. **[*I.I.T*]**

24. The angles A, B, C of a triangle are in the ratio 3 : 5 : 4, prove that $a + c\sqrt{2} = 2b$.

25. In a ΔABC, if $\cos A = \dfrac{17}{22}$, $\cos C = \dfrac{1}{14}$, prove that the ratio of the sides is 7 : 9 : 11.

26. The angle of a triangle are in the ratio 1 : 2 : 7, prove that the ratio of the greatest side to the least side is $(\sqrt{5}+1) : (\sqrt{5}-1)$.

27. If the sides of ΔABC are in the ratio 4 : 5 : 6, prove that one angle is twice that of the other.

28. Two sides and included angles of a triangle are respectively $3+\sqrt{3}$, $3-\sqrt{3}$ and 60°. Show that the remaining elements of the triangle are 105°, 15°, $3\sqrt{2}$.

29. In a ΔABC, if $B = 3C$, prove that

(*i*) $\cos C = \sqrt{\left(\frac{b+c}{4c}\right)}$ (*ii*) $\sin\frac{A}{2} = \frac{b-c}{2c}$.

ANSWERS

1. (*v*) $\frac{1}{2}(\sqrt{3}+1)$ sq. units (*vi*) 9 sq. units (*vii*) 30°

18. $\angle B = 113°$

HINTS TO SELECTED QUESTIONS

1. (*iv*) Since the sines of the angles of the Δ are in the ratio of 4 : 5 : 6 their opposite sides are also in the same ratio. Let the sides be $a = 4k$, $b = 5k$, $c = 6k$. Now find cos *A*, cos *B*, and cos *C*.

(*v*) Let $\angle B = 30°$, $\angle C = 45°$, then $A = 105°$ and $a = BC = (\sqrt{3}+1)$ cm. Use $\frac{a}{\sin A} = \frac{b}{\sin B}$ and obtain $b = \sqrt{2}$: Get $\sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ by using $\sin 105° = \sin(60° + 45°)$.

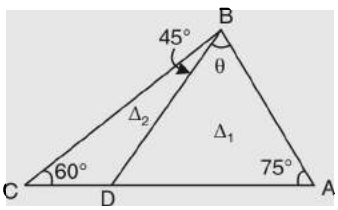
Use $\Delta = \frac{1}{2}ab\sin C$ to find area.

(*vi*) $\Delta = \frac{1}{2}ab\sin C = 9\sin C$

$\cos(A-B) = \frac{1-t^2}{1+t^2} = \frac{4}{5} \Rightarrow t = \frac{1}{3}.\left(\text{where } t = \tan\frac{A-B}{2}\right) \therefore \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{1}{2}$ (Napier' rule).

This gives C = 90°.

(*vii*) $\frac{1}{2}BD\cdot BA\sin\theta = \sqrt{3}\cdot\frac{1}{2}BD\cdot BC\sin(45°-\theta)$



$\Rightarrow \frac{BA}{BC}\sin\theta = \sqrt{3}\sin(45°-\theta) \Rightarrow \frac{c}{a}\sin\theta = \sqrt{3}\sin(45°-\theta)$

$\Rightarrow \frac{\sin C}{\sin A}\sin\theta = \sqrt{3}\sin(45°-\theta) \Rightarrow \frac{\sin 60°}{\sin 75°}\sin\theta = \sqrt{3}\sin(45°-\theta)$

$\Rightarrow \frac{\sqrt{6}}{\sqrt{3}+1}\sin\theta = \sqrt{3}\sin(45°-\theta) \Rightarrow \frac{\sqrt{2}}{\sqrt{3}+1}\sin\theta = \sin(45°-\theta)$

17. $a^3\cos(B-C) = a^2k\sin(B+C)\cos(B-C) = \frac{a^2k}{2}(\sin 2B + \sin 2C)$

$= a^2k(\sin B\cos B + \sin C\cos C) = a^2(b\cos B + c\cos C)$

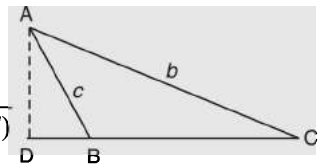
Therefore, left side $= a^2(b\cos B + c\cos C) + b^2(c\cos C + a\cos A) + c^2(a\cos A + b\cos B)$

$= ab(a\cos B + b\cos A) + bc(b\cos C + c\cos B) + ca(a\cos C + c\cos A)$

$= abc + bca + cab = 3abc.$

18. $AD = \dfrac{abc}{(b^2-c^2)} = b\sin 23° \Rightarrow \dfrac{a}{b^2-c^2} = \dfrac{\sin 23°}{c}$ also $\dfrac{\sin A}{a} = \dfrac{\sin C}{c} = \dfrac{\sin 23°}{c}$

$\therefore \dfrac{\sin A}{a} = \dfrac{a}{b^2-c^2}$



$\Rightarrow \sin A = \dfrac{a^2}{b^2-c^2} = \dfrac{\sin^2 A}{\sin^2 B - \sin^2 C} = \dfrac{\sin^2 A}{\sin(B+C)\sin(B-C)}$

$= \dfrac{\sin A}{\sin(B-C)} \Rightarrow \sin(B-C) = 1.$

22. $\dfrac{b+c+a+c}{(a+c)(b+c)} = \dfrac{3}{a+b+c}$

$\Rightarrow (a+b+2c)(a+b+c) = 3(a+c)(b+c) \Rightarrow a^2+b^2-c^2 = ab \Rightarrow \dfrac{a^2+b^2-c^2}{2bc} = \dfrac{1}{2}$

23. (*i*) $2B = A + C = 180° - B \Rightarrow B = 60° \Rightarrow b^2 = a^2 + c^2 - ac$

R.H.S. $= \dfrac{a+c}{b} = \dfrac{\sin A + \sin C}{\sin B}$, etc.,

(*ii*) As in (*i*), $B = 60°$, $\dfrac{b}{c} = \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{\sin B}{\sin C} = \dfrac{\sqrt{3}}{2\sin C} \Rightarrow \sin C = \dfrac{1}{\sqrt{2}}$

(*iii*) $\angle C = 60°$

L.H.S. $= a + c\sqrt{2} = k(\sin A + \sqrt{2}\sin 60°) = k \cdot \dfrac{(1+\sqrt{3})}{2} = 2k \sin 75°$

R.H.S. $= 2b = 2.k \sin B = 2k \sin 75°$

24. $\dfrac{A}{3} = \dfrac{B}{5} = \dfrac{C}{4} = \dfrac{A+B+C}{3+5+4} = \dfrac{180°}{12} = 15°$

$\Rightarrow$ $A = 45°, B = 75°, C = 60°$ etc.,

25. Find sin *A*, sin *B*, sin *C*.

$\sin A = \dfrac{\sqrt{195}}{22}$, $\sin C = \dfrac{\sqrt{195}}{14}$, $\sin B = \sin(A + C) = \dfrac{9\sqrt{195}}{154}$

26. Angles are 18°, 36°, 126°.

Greatest side : least side = sin 126° : sin 18° = sin 54° : sin 18°

27. Find cos *A*, cos *C* and cos 2 *A* = 2 cos^2 *A* – 1. You will find that cos 2A = cos C $\Rightarrow$ 2*A* = *C*.

29. $\dfrac{b+c}{4c} = \dfrac{\sin B + \sin C}{4\sin C} = \dfrac{\sin 3C + \sin C}{4\sin C} = \dfrac{2\sin 2C\cos C}{4\sin C} = \cos^2 C$

$\dfrac{b-c}{2c} = \dfrac{\sin B - \sin C}{2\sin C} = \dfrac{\sin 3C - \sin C}{2\sin C} = \cos 2C$

$A + B + C = 180° \Rightarrow A + 4C = 180° \Rightarrow 4C = 180° - A \Rightarrow 2C = 90° - \dfrac{A}{2}$

$\therefore \dfrac{b-c}{2c} = \cos\left(90° - \dfrac{A}{2}\right) = \sin\dfrac{A}{2}$

CHAPTER 8

MATHEMATICAL INDUCTION

8.01. Method of mathematical induction

(*a*) **Introduction.** Reasoning or drawing conclusions can be classified in two categories, namely, *inductive* reasoning and *deductive* reasoning. When a person makes observations or experiments and on their basis reaches conclusions, he is said to reason *inductively.* For example, a person, on the basis of a very few observations, concludes that a passenger in the bus is thrown forwards whenever the bus driver applies brakes suddenly. Deductive reasoning, on the other hand, proceeds from assumptions, rather than experience. *It is usually by inductive reasoning that mathematical results are discovered. By deductive reasoning, they are proved.*

Take an example from geometry. By drawing a number of triangles and measuring the angles a person finds that more or less, the sum of the angles of a triangle is equal to 180°. "More or less" because we cannot measure the angles with absolute accuracy and our measurements are likely to vary by a degree or two.

On the basis of the above measurements, in spite of the fact that angles cannot be measured with absolute accuracy, an experimenter would conclude tentatively that *the sum of the angles of a triangle is equal to 180°.* Thus, he arrives at a **generalisation** (general statement). This is called the **Inductive method.**

The experimenter knows that he has not **proved** anything but, he can be reasonably sure that his conclusion is valid. Later he will develop a formal proof for this relationship, which is called the **Deductive method.** After he has *proved* the general statement, the result can be applied to any particular triangle.

Thus, *inductive reasoning starts with specific cases and ends with a general statement about all such cases, whereas deductive reasoning starts with a general statement and ends with a specific case.*

(*b*) **Mathematical Induction.** Consider the following :

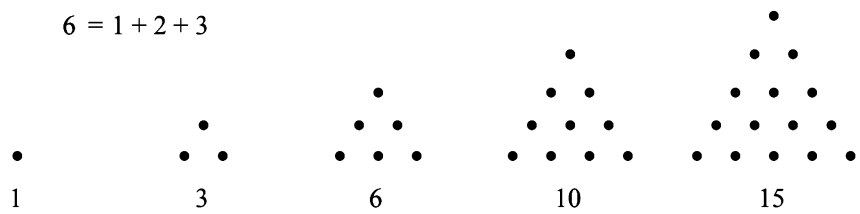
About 2500 years ago, Pythagoreans represented numbers by sets of points. If the set of points formed a triangle, the corresponding numbers were called **triangular numbers.**

Each triangular number is the sum of the number of dots in its rows.

$1 = 1$

$3 = 1 + 2$

$6 = 1 + 2 + 3$



$$10 = 1 + 2 + 3 + 4$$
$$15 = 1 + 2 + 3 + 4 + 5$$

and so on.

Each triangular number is equal to the sum of n consecutive natural numbers, beginning with 1. Instead of adding each natural number, we can use a formula to find the triangular number. If there are n consecutive natural numbers starting with 1, then their sum is

$$S_n = \frac{n(n+1)}{2}$$

We can verify that this formula works for several numbers.

$$S_1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \quad ; \quad S_2 = \frac{2(2+1)}{2} = \frac{6}{2} = 3$$

(1) It is impossible, however, to test all possible numbers. Perhaps the formula works for the first 85 natural numbers, but fails when n is 86 or may be it fails when n is $10^{100} + 7$. We need a method of proof that will show that the formula holds for *all* the natural numbers.

(2) No one has so far been able to invent a formula that would always produce a prime number. One effort produced the formula
$n^2 - n + 41$.

If we substitute the numbers 1, 2, 3,....., 40 in this formula for n, a prime number will be produced. This may lead us to think that this formula will always produce prime numbers for all $n \in$ N. However, we do not get a prime number for $n = 41$.

(3) Pierre de Fermat thought he had a formula, the expression $2^{2^n} + 1$.

Prime numbers are produced for n = 1, 2, 3 and 4. About a century later another mathematician Eular discovered that for $n = 5$ this formula gives the number 4 294 967 297 which is not a prime number.*

Thus if a proposition happens to be true for some particular values, it will be erroneous to conclude that it will be true for all values.

The examples given above show that showing a proposition to be true for some values does not serve any purpose.

We need a method which may enable us to move from a particular statement to a general statement. Method of **mathematical induction** is the answer to our problem and is one of the most useful and most powerful tools of mathematics. Here is a proposition so fundamental that requires no proof and is an essential characteristic of the positive integers.

8.02. Proof by mathematical induction

It is a proof which depends on a process that is much like climbing a ladder. First we must get on to the first step ; then we must show that we can always advance from one step to the next. Thus, if we can get on to the first step, we can certainly climb to the second ; if we are on the second step, we can climb to the third ; if we are on the third step, we can climb to the fourth; and so on, indefinitely, for all steps.

The following analogy may also be given to understand this method of proof.

Assume that you are in line at a cinema house, waiting to buy a ticket. An indefinite number of tickets will be sold, and an unspecified number of persons are in line ahead of you. You can prove that you will get a ticket if you can establish the following two statements:

(1) The first person in line gets a ticket.

(2) If any person in line gets a ticket, the person following him gets a ticket.

Do you see how (1) and (2) assure you of getting a ticket; by statement (1), the first person gets a ticket; by statement (2) since the first person gets a ticket, the second person gets a ticket. Again by

* It is divisible by 641 and the quotient is 6700417.

statement (2) since the second person gets a ticket, the third person gets a ticket, and so on, until the conclusion : You get a ticket.

The same reasoning is used to prove theorems.

The proof of a proposition $T(n)$ by mathematical induction consists of the following three steps:

Step 1. (*Basic step*). Actual verification of the proposition $[T(1)]$, $[T(2)]$, etc., for particular positive integral values of n, say $n = 1, 2$, etc.

Step 2. (*Induction step*). Assuming the proposition to be true for some positive integral value k of n, *i.e.,* $T(k)$ and then proving that it is true for the value $(k + 1)$ which is the next higher integer, *i.e.,* proving $T(k + 1)$ true whenever $T(k)$ holds.

Step 3. The reasoning that if the theorem is true for the value k of n, it is also true for the value $(k + 1)$, which is the next higher integer. But, by the first step if the theorem is true for the value 1, it is true for the value 1+1, *i.e.* 2 and so on.

8.03. Statement of the principle of mathematical induction

A statement T(n) is true for all n ∈ N, where N is the set of natural numbers, provided (i) T (1) is true and (ii) T(k) is true ⇒ T(k + 1) is true.

Note 1. The power of a proof by mathematical induction is great, particularly for formulas about number relations. Its most serious disadvantage is that the formula to be proved must be known or discovered before the method can be applied.

Note 2. Induction in Verse

You want to prove a theorem then
To prove it for every integer n
You prove it first for n equal one
And then the induction is begun
The proof goes on in the following way
You assume it next for n equal k
If you can show it then for k + 1
Then the induction is truly done.

(L. MOSER)

8.04. Illustrative solved examples

Type 1. Summation of Series

Ex. 1. Prove that for all +ve integral values of n, $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Sol. Let T(n) be the statement: $1 + 3 + 5 + ... + (2n - 1) = n^2$

Basic step. For $n = 1$, $1 = 1^2$. Thus T(l) is true.

Induction step. Assume that T(k) is true, that is

$$1 + 3 + 5 + ... + (2k - 1) = k^2$$

To obtain T($k + 1$), add the $(k + 1)$th term

i.e., $(2k - 1) + 2$ or $2k + 1$ to both sides. Then

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = k^2 + 2k + 1$$

i.e., $1 + 3 + 5 + ...$ to $(k + 1)$ terms $= (k + 1)^2$.

Thus the statement is true for $n = k + 1$ under the assumption that the statement is true for $n = k$. Therefore by the principle of induction, the statement is true for every +ve integer n.

Hence, $1 + 3 + 5 ....$ to n terms $= n^2$.

Ex. 2. Use the principle of mathematical induction to prove the following statement for all $n \in N$. $x + 4x + 7x + \ldots\ldots\ldots + (3n - 2)\,x = \frac{1}{2} n\,(3n - 1)\,x$.

Sol. Let T(n) be the statement :

$$x + 4x + 7x + \ldots\ldots + (3n - 2)\,x = \frac{1}{2}\,n\,(3n - 1)\,x.$$

Basic step. *For* $n = 1$,

$$x = \frac{1}{2} \times 1 \times (3 \times 1 - 1)\,x = \frac{1}{2} \times 2 \times x = x$$

Thus T(l) is true.

Induction step. Assume that T(k) is true, that is,

$$x + 4x + 7x + \ldots\ldots\ldots + (3k - 2)\,x = \frac{1}{2}\,k\,(3k - 1)\,x \qquad \ldots(i)$$

We shall show that $T(k + 1)$ is also true.

Now, $x + 4x + 7x + \ldots\ldots + (3k - 2)x + (3k + 1)\,x$

$$= \frac{1}{2}k(3k - 1)\,x + (3k + 1)\,x \qquad [\textit{Using } (i)]$$

$$= \frac{1}{2}\,[k\,(3k - 1)\,x + 2\,(3k + 1)x] = \frac{1}{2}\,(3k^2 + 5k + 2)\,x = \frac{1}{2}\,(k + 1)\,(3k + 2)\,x$$

$$= \frac{1}{2}\,(k + 1)\,[3\,(k + 1) - 1]\,x$$

$\Rightarrow$ T($k + 1$) is true

Since T (k) is true implies T($k + 1$) is true, therefore, by the principle of induction, T (n) holds for all positive integral values of n.

Ex. 3. Prove that for all +ve integral values of n.

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \qquad [\textit{ISC 1995}]$$

Sol. Let T(n) be the statement :

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Basic step. For $n = 1$, $\frac{1}{1.2} = \frac{1}{1+1}$

Thus T(l) is true.

Induction step. Assume that T(k) is true, that is,

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots + \frac{1}{k\,(k+1)} = \frac{k}{k+1}$$

To obtain T ($k + 1$), add the ($k + 1$)th term, that is $\frac{1}{(k+1)\,(k+2)}$ to both sides. Then

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{k\,(k+1)} + \frac{1}{(k+1)\,(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)\,(k+2)} = \frac{k\,(k+2)+1}{(k+1)\,(k+2)}$$

i.e., $$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots + \frac{1}{(k+1)\,(k+2)} = \frac{(k+1)^2}{(k+1)\,(k+2)} = \frac{k+1}{k+2}.$$

Thus, $T(k+1)$ is true with the assumption that $T(k)$ is true.

Hence, by the principle of induction $T(n)$ holds for all positive integral values of n.

Ex. 4. Prove by principle of induction that

$$\mathbf{1.4.7 + 2.5.8 + 3.6.9 + ... + n\,(n+3)\,(n+6) = \frac{n}{4}\,(n+1)\,(n+6)\,(n+7).}$$

Sol. Let $T(n)$ denote the given statement, then

$$T(n) = 1.4.7 + 2.5.8 + 3.6.9 + ... + n\,(n+3)\,(n+6) = \frac{n}{4}\,(n+1)\,(n+6)\,(n+7) \qquad ...(A)$$

Basic step. For $n = 1$, L.H.S. $= 1.4.7 = 28$, R.H.S. $= \frac{1}{4}(1+1)(1+6)(1+7) = 28$

$\therefore$ L.H.S. = R.H.S.

Thus, $T(1)$ is true.

Induction step. Assume that $T(k)$, is true, $(k \in N)$, *i.e.,*

$$1.4.7 + 2.5.8 + 3.6.9 + ... + k\,(k+3)\,(k+6) = \frac{k}{4}\,(k+1)\,(k+6)\,(k+7) \qquad ...(i)$$

We will now show that $T(k+1)$ is also true.

To obtain $T(k+1)$, add the $(k+1)$th term, *i.e.,* $(k+1)(k+4)(k+7)$ to both sides. Then

$1.4.7 + 2.5.8 + 3.6.9 + ... + k\,(k+3)\,(k+6) + (k+1)(k+4)(k+7)$

$$= \frac{k}{4}(k+1)(k+6)(k+7) + (k+1)(k+4)(k+7) \qquad \text{[Using } (i)\text{]}$$

$$= (k+1)(k+7)\left[\frac{k}{4}(k+6) + k + 4\right] = \frac{(k+1)(k+7)}{4}(k^2 + 6k + 4k + 16)$$

$$= \frac{(k+1)}{4}(k+7)(k^2 + 10k + 16) = \frac{1}{4}(k+1)(k+7)(k+2)(k+8)$$

$$= \frac{1}{4}(k+1)(k+2)(k+7)(k+8) = \text{R.H.S. of } (A) \text{ for } n = k+1...$$

Showing that $T(k+1)$ is true. Since $T(k) \Rightarrow T(k+1)$ is true, therefore, by the principle of mathematical induction $T(n)$ is true for all $n \in N$.

Ex. 5. Prove by the method of induction ($n \in N$)

$$\mathbf{\frac{1}{3.6} + \frac{1}{6.9} + \frac{1}{9.12} + ... + \frac{1}{3n\,(3n+3)} = \frac{n}{9\,(n+1)}} \qquad \textbf{[}\textit{ISC 1999}\textbf{]}$$

Sol. Let $T(n)$ be the statement

$$\frac{1}{3.6} + \frac{1}{6.9} + \frac{1}{9.12} + ... + \frac{1}{3n\,(3n+3)} = \frac{n}{9\,(n+1)}.$$

Basic step : For $n = 1$, $\quad \frac{1}{3.6} = \frac{1}{9(1+1)} \Rightarrow \frac{1}{18} = \frac{1}{18}$

Thus, T (1) is true.

Induction step : Assume that $T(k)$, is true, $(k \in N)$, *i.e.*,

$$\frac{1}{3.6}+\frac{1}{6.9}+\frac{1}{9.12}+...+\frac{1}{3k\,(3k+3)}=\frac{k}{9\,(k+1)}.$$

To obtain $T(k+1)$, add $(k+1)$th term, *i.e.*, $\dfrac{1}{3\,(k+1)\,[3\,(k+1)+3]}$

or $\dfrac{1}{3\,(k+1)\,(3k+6)}$, to both sides. Then

$$\frac{1}{3.6}+\frac{1}{6.9}+\frac{1}{9.12}+...+\frac{1}{3k\,(3k+3)}+\frac{1}{3\,(k+1)\,(3k+6)}$$

$$=\frac{k}{9\,(k+1)}+\frac{1}{3\,(k+1)\,(3k+6)}=\frac{k\,(k+2)+1}{9\,(k+1)\,(k+2)}=\frac{k^2+2k+1}{9\,(k+1)\,(k+2)}$$

$$=\frac{(k+1)^2}{9\,(k+1)\,(k+2)}=\frac{k+1}{9\,(k+2)}=\frac{k+1}{9\,[(k+1)+1]}.$$

Thus, $T(k+1)$ is true with the assumption that $T(k)$ is true. Hence, by the principle of mathematical induction, $T(n)$ holds for all positive integral values of n.

Ex. 6. Prove that for every natural number *n*,

$$\mathbf{1^3+2^3+3^3+...+n^3=\left[\frac{n(n+1)}{2}\right]^2}$$

Sol. Let $T(n)$ be the statement

$$1^3+2^3+3^3+...+n^3=\frac{n^2(n+1)^2}{4},\ n\in N.$$

Basic step. For $n=1$; $\quad 1^3=\dfrac{1^2(1+1)^2}{4}=1$

Thus $T(1)$ holds.

Induction step. Let $T(k)$ hold for a natural number k, that is

$$1^3+2^3+3^3+...+k^3=\frac{k^2(k+1)^2}{4}$$

Now $\quad (1^3+2^3+3^3+...+k^3)+(k+1)^3=\dfrac{k^2(k+1)^2}{4}+(k+1)^3$

$$=\frac{(k+1)^2}{4}[k^2+4\,(k+1)]=\frac{(k+1)^2\,(k+2)^2}{4}$$

Thus, $T(k+1)$ is true. Hence, by the principle of induction, $T(n)$ is true for every natural number n.

Maths Alert! A proof by mathematical induction requires the application of both the basic and the inductive steps. We give below two examples to show that if one of them does not hold, the proposition may not be true for all n.

(1) Let $T(n)$ be the statement

$$\forall\, n\in N,\ 4+6+8+...+2n=n(n+1)$$

Assume $\quad T(k)$ is true. Then,

$$4+6+8+...+2k=k\,(k+1)$$

$$4+6+8+...+2k+2k+2=k(k+1)+2k+2$$

$$=(k+1)\,(k+2)=(k+1)\,[(k+1)+1]$$

Thus, $T(k)$ is true $\Rightarrow T(k+1)$ is true. This shows that the induction step holds.

For $n = 1$, $4 = 1\,(1+1) = 2$ which is not true,

i.e., the basic step does not hold.

Thus, despite the apparent "proof", it turns out that

$4 + 6 + 8 + \ldots + 2n = n(n+1)$ is false for all natural numbers n.

(2) Let us try to establish by mathematical induction that for all natural numbers n, $n^2 + n + 11$ is a prime number.

(*i*) ***Basic step.*** When $n = 1$, $n^2 + n + 11 = 13$, is prime . Thus $T(1)$ holds.

(*ii*) ***Induction step.*** Assume $T(k)$ true, *i.e.,* $k^2 + k + 11$ is prime.

Now $(k+1)^2 + (k+1) + 11 = k^2 + 3k + 13$ which is not a prime if k is 13, since $(13)^2 + 3\,(13) + 13$ has the factor 13.

Hence, $T(k+1)$ does not hold.

In fact, $n^2 + n + 11$ is not a prime when $n = 11$.

Ex. 7. Using the principle of mathematical induction prove that

$$1\lfloor\underline{1} + 2\lfloor\underline{2} + 3\lfloor\underline{3} + \ldots + n\lfloor\underline{n} = \lfloor\underline{n+1} - 1 \text{ for all } n \in N.$$ **[*ISC 1994*]**

Sol. If $n = 1$, then $1\lfloor\underline{1} = 1 \times 1 = 1 = \lfloor\underline{1+1} - 1$.

Thus, the result is true for $n = 1$

We assume that the result is true for $n = m$, say. $(\because\ m \in N)$

$\therefore\quad 1\lfloor\underline{1} + 2\lfloor\underline{2} + 3\lfloor\underline{3} + \ldots + m\lfloor\underline{m} = \lfloor\underline{m+1} - 1$

Adding $(m+1)\ \lfloor\underline{m+1}$ to both sides, we get

$$1\lfloor\underline{1} + 2\lfloor\underline{2} + 3\lfloor\underline{3} + \ldots + m\lfloor\underline{m} + (m+1)\lfloor\underline{m+1} = \lfloor\underline{m+1} - 1 + (m+1)\lfloor\underline{m+1}$$

$$= \left[\lfloor\underline{m+1} + (m+1)\lfloor\underline{m+1}\right] - 1$$

$$= \lfloor\underline{m+1}\,.\,(1 + m + 1) - 1 = (m+2)\lfloor\underline{m+1} - 1 = \lfloor\underline{m+2} - 1 = \lfloor\underline{(m+1)+1} - 1$$

$\therefore$ The result is true for $n = m+1$ if it is true for $n = m$. But the result is true for $n = 1$, therefore, it is true for $n = 1 + 1$, *i.e.,* 2. If it is true for $n = 2$, it is true for $n = 2 + 1$, *i.e.,* 3. Continuing this reasoning we conclude that it is true for $n = 2, 3, 4, \ldots$ *i.e.,* for all positive integral values of n.

Note 1. From the above examples it will be seen that the only theorems to which method of induction can be applied are those which admit of successive cases corresponding to the order of the natural numbers $1, 2, 3, \ldots n$.

Note 2. See Art. 12.04 in Chapter 12 where the Binomial Theorem is proved by induction.

EXERCISE 8 (*a*)

Use the principle of mathematical induction to prove the following statements for all $n \in N$.

1. $1 + 2 + 3 + \ldots + n = \dfrac{1}{2} n\,(n+1)$.

2. $2 + 4 + 6 + \ldots + 2n = n(n+1)$.

3. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{1}{6}\, n\,(n+1)\,(2n+1)$.

4. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{1}{3}\, n\,(2n-1)\,(2n+1)$.

5. $2 + 2^2 + 2^3 + ... + 2^n = 2(2^n - 1)$.

6. $1.2 + 2.3 + 3.4 + ... + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$. **[*ISC BM 2004*]**

7. $1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n-1)2^{n+1} + 2$.

8. $3.2^2 + 3^2.2^3 + 3^3.2^4 + ... + 3^n.\ 2^{n+1} = \dfrac{12}{5}(6^n - 1)$.

9. $1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$.

10. (*i*) $a + (a+d) + (a+2d) + ... + [a + (n-1)d] = \dfrac{n}{2}[2a + (n-1)d]$.

(*ii*) $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$, $r \neq 1$.

11. $5 + 15 + 45 + ... + 5(3)^{n-1} = \dfrac{5}{2}(3^n - 1)$.

12. $\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + ... + \dfrac{1}{(2n+1)(2n+3)} = \dfrac{n}{3(2n+3)}$.

13. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + ... + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$.

14. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$. **[*ISC 1996*]**

15. $1 + 5 + 12 + 22 + 35 + ... + \dfrac{n}{2}(3n-1) = \dfrac{n^2(n+1)}{2}$.

16. Prove by the method of mathematical induction that

$\dfrac{1}{4.7} + \dfrac{1}{7.10} + \dfrac{1}{10.13} + ... + \dfrac{1}{(3n+1)(3n+4)} = \dfrac{n}{4(3n+4)}$, for all $n \in N$. **[*ISC 2003*]**

17. Let $S(k) = 1 + 3 + 5 + ... + (2k-1) = 3 + k^2$, then which of the following is true ?

(*a*) $S(k) = S(k+1)$ (*b*) $S(k) \neq S(k+1)$ (*c*) $S(1)$ is correct

(*d*) Principle of mathematical induction can be used to prove the formula. **[*AIEEE 2004*]**

[Hint. $S(k) = k^2 = 3 + k^2$ (given) $\because$ Sum of the first n odd natural numbers $= k^2$

$S(k+1) = 1 + 3 + 5 + ... + (2k+1) = (k+1)^2 = k^2 + 2k + 1 = k^2 + 3 + 2k + 1 = 3 + (k+1)^2$

$\Rightarrow$ $S(k) = S(k+1)$. Hence, (*a*) is correct.]

Type 2. Proving divisibility

Ex. 8. Prove that $x^n - y^n$ is divisible by $x - y$ when n is a +ve integer.

Sol. Let T(n) be the statement :

$x^n - y^n$ is divisible by $x - y$.

Basic step. For $n = 1$,

$x - y$ is divisible by $x - y$

Thus, T(l) is true.

Induction step. Assume that T(k) is true, that is, ($\because k \in N$)

$x^k - y^k$ is divisible by $x - y$.

Now $x^{k+1} - y^{k+1} = x^k . x - y^k . y$

$= x^k . x - x^k . y + x^k . y - y^k . y = x^k (x - y) + y (x^k - y^k)$ **[Note this step]**

Since $x^k(x - y)$ is divisible by $(x - y)$ and $y (x^k - y^k)$ is divisible by $x - y$. [By $T(k)$]

[As $x^k - y^k$ is divisible by $x - y$]

$\therefore$ $x^{k+1} - y^{k+1}$ is divisible by $x - y$.

Thus, T(k) is true implies T($k + 1$) is true.

Hence, by the principle of induction, T(n) holds for all positive integral values of n.

Ex. 9. Show that $n (n + 1) (n + 2)$ is a multiple of 6.

Sol. Let T(n) be the statement: $n (n + 1) (n + 2)$ is a multiple of 6. ($n \in N$).

Basic step. For $n = 1$; $1(1 + 1) (1 + 2) = 6$

Thus T(l) is true.

Induction step. Assume that T(k) holds for $n = k$; that is $k(k + 1) (k + 2)$ is a multiple of 6.

Let $k(k + 1) (k + 2) = 6p.$ $(p \in N)$

Now $T(k + l) = (k + l) (k + 2)(k + 3)$

$= k(k + 1) (k + 2) + [3(k + 1) (k + 2)] = 6p + 3k^2 + 9k + 6$

$= 6p + 3k^2 + 3k + 6k + 6 = 6p + 6(k + 1) + 3k (k + 1).$

Now, $k (k + 1)$ is always even for every $k \in N$. Let $k(k + 1) = 2m$.

Then, $T(k + 1) = 6p + 6 (k + 1) + 6m$ $(\because m \in N)$

$= 6[p + (k + 1) + m]$ = a multiple of 6.

Therefore, the truth of $T(k)$ implies the truth of $T(k + 1)$.

Hence, by the principle of induction, T(n) is true for every natural number n.

Ex. 10. Prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for any positive integer n. **[*ISC 2001*]**

Sol. Let T(n) be the statement : $3^{2n + 2} - 8n - 9$ is divisible by 64.

Basic step. For $n = 1$; $3^{2\times l+2} - 8 \times 1 - 9 = 81 - 8 - 9 = 64$

Thus, T(l) holds.

Induction step. Let T(k), $k \in N$, hold; that is,

$3^{2k+2} - 8k - 9$ is divisible by 64.

$T(k + l) = 3^{2(k+1) + 2} - 8(k + l) - 9 = 3^2 . 3^{2k + 2} - 8k - 17$

$= 9(3^{2k+2} - 8k - 9) + 64k + 64 = 9\ T(k) + 64 (k + 1)$

Hence, $T(k + 1)$ is also divisible by 64, *i.e.* $T(k + 1)$ is true.

Since the truth of T(k) implies the truth of $T(k + 1)$, therefore, $T(n)$ is true for every natural number n.

Ex. 11. Apply the method of induction to prove that no natural number is equal to its successor.

Sol. Let k be a natural number.

Its successor is $k + 1$. Let $k = k + 1$.

Then, $k + 1 = k + 1 + 1 = k + 2$

This means that (*By induction*)

$1 = 2$ which is not true. Hence, $k \neq k + l$.

Ex. 12. Use the principle of mathematical induction to prove, $\forall n \in N$ that $10^{2n-1} + 1$ is divisible by 11.

Sol. Let the given statement be $T(n)$:

i.e., $10^{2n-1} + 1 = M(11)$.

Basic step. When $n = 1$, $10^{2n-1} + 1 = 10 + 1 = 11 = M(11)$

$\Rightarrow$ $T(1)$ is true.

Induction step. Assume that $T(k)$ is true

i.e., Let $10^{2k-1} + 1 = M(11)$ ...(*i*)

We shall show that T($k + 1$) is true

i.e., $10^{2(k+1)-1} + 1 = 10^{2k+1} + 1 = M(11)$.

Now $10^{2k+1} + 1 = 10^{(2k-1)+2} + 1 = 10^{2k-1} \cdot 10^2 + 1$

$= 100 \cdot 10^{2k-1} + 1 = 100\,[M(11) - 1] + 1$ [Using (*i*)]

$= 100M(11) - 99 = 100\,M(11) - M(11)$ ($\because 99 = 11 \times 9$)

$= M(11)$ $\Rightarrow$ $T(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction, $T(n)$ is true for all $n \in N$.

Ex. 13. Prove by the method of induction ($n \in N$); that $3^{2n+2} - 9$ is divisible by 72. **[*ISC 2000*]**

Sol. Let $T(n)$ be the statement

$3^{2n+2} - 9 = M(72)$

Basic step. When $n = 1$, $3^{2n+2} - 9 = 3^4 - 9 = 72 = M(72)$.

Induction step. Assume that $T(k)$, $k \in N$, is true, *i.e.,*

$3^{2k+2} - 9 = M(72)$ ...(*i*)

We shall prove that $T(k + 1)$ is true, *i.e.,*

$3^{2(k+1)+2} - 9 = 3^{2k+4} - 9 = M(72)$.

Now $3^{2k+4} - 9 = 3^{2k+2} \cdot 3^2 - 9 = 9.3^{2k+2} - 9 = 9\,[M(72) + 9] - 9$[$\because$ From (*i*), $3^{2k+2} = M(72) + 9$]

$\Rightarrow 3^{2k+4} - 9 = 9 \cdot M(72) + 81 - 9 = 9.M(72) + 72 = 9.M(72) + M(72)$

$\Rightarrow 3^{2k+4} - 9 = M(72) \Rightarrow T(k + 1)$ is true.

By the principle of mathematical induction $T(n)$ is true for all positive integral values of n.

Ex. 14. By the principle of mathematical induction, prove that for all $n \in N$, $11^{n+2} + 12^{2n+1}$ is divisible by 133. [*Roorkee*]

Sol. Let the given statement be $T(n)$:

i.e., $11^{n+2} + 12^{2n+1} = M(133)$. :

Basic step. When $n = 1$,

$11^{n+2} + 12^{2n+1} = 11^3 + 12^3 = 1331 + 1728 = 3059 = 133 \times 23 = \text{M}(133)$.

$\Rightarrow$ T(1) is true.

Induction step. Assume that $T(k)$ is true ($\because k \in N$)

i.e., Let $11^{k+2} + 12^{2k+1} = M(133)$ ...(*i*)

We shall show that T($k + 1$) is true

i.e., $11^{k+3} + 12^{2k+3} = M(133)$.

Now, $11^{k+3} + 12^{2k+3} = 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 12^2 = 11 \cdot 11^{k+2} + (11 + 133) \cdot 12^{2k+1}$

$= 11\,(11^{k+2} + 12^{2k+1}) + 133 \,.\, 12^{2k+1} = 11M(133) + M(133)$ [Using (i)]

$= M(133) \quad \Rightarrow \quad T(k+1)$ is true.

Since the truth of T(k) $\Rightarrow$ The truth of T($k+1$), therefore, T(n) is true for all $n \in N$.

Ex. 15. Using the principle of mathematical induction, prove that $10^n + 3.4^{n+2} + 5$ is divisible by 9, for all $n \in N$. **[*ISC 1993*]**

Sol. Method I. Let the given statement be T(n), *i.e.*, T(n) : $10^n + 3 \,.\, 4^{n+2} + 5 = M(9)$.

Basic step. When $n = 1$, $10^n + 3 \,.\, 4^{n+2} + 5 = 10^1 + 3 \,.\, 4^{1+2} + 5 = 10 + 192 + 5 = 207 = M(9)$

$\Rightarrow$ T(1) is true.

Induction step. Assume that T(k), $k \in N$, is true, *i.e.*, let $10^k + 3 \,.\, 4^{k+2} + 5 = M(9)$ ...(i)

We shall show that T($k+1$) is true.

i.e., $10^{k+1} + 3.4^{k+3} + 5 = M(9)$

Now $(10^{k+1} + 3 \,.\, 4^{k+3} + 5) - (10^k + 3 \,.\, 4^{k+2} + 5) = (10^{k+1} - 10^k) + (3 \,.\, 4^{k+3} - 3 \,.\, 4^{k+2})$

$= 10^k\,(10 - 1) + 3 \,.\, 4^{k+2}\,(4 - 1) = 10^k \times 9 + 9 \times 4^{k+2} = 9\,(10^k + 4^{k+2})$

$\therefore \quad 10^{k+1} + 3.\, 4^{k+3} + 5 = 9 \,.\, (10^k + 4^{k+2}) + (10^k + 3 \,.\, 4^{k+2} + 5)$

which is divisible by 9, by virtue of (i). $\Rightarrow$ T($k+1$) is true.

Since the truth of T(k) $\Rightarrow$ the truth of T($k+1$), therefore, T(n) is true for all $n \in N$.

Method II.

$T\,(k+1) = 10^{k+1} + 3 \,.\, 4^{k+3} + 5 = 10^k \,.\, 10 + 3.4^{k+2} \,.\, 4 + 5 = 10^k \,.\, 10 + 12.4^{k+2} + 5$

$= (9 + 1) \,.\, 10^k + (9 + 3) \,.\, 4^{k+2} + 5 = 9\,(10^k + 4^{k+2}) + (10^k + 3.4^{k+2} + 5)$

$= 9\,(10^k + 4^{k+2}) + M(9)$[By (i)]

= a multiple of 9.

Type 3. Proving inequalities

Ex. 16. Using the principle of mathematical induction, prove that $1 + 2 + 3 + \ldots\ldots + n < (2n + 1)^2$.

Sol. Let $T(n)$ be the statement $1 + 2 + 3 + \ldots\ldots + n < (2n + 1)^2$.

Basic step. For $n = 1$; $1 < (2 \times 1 + 1)^2$, *i.e.*, $1 < 9$ which is true.

Thus $T(1)$ holds.

Induction step. Assume that $T(k)$ is true; that is $1 + 2 + 3 + \ldots.3 + k < (2k + 1)^2$ ... (i)

We shall show that $T(k+1)$ is also true.

$1 + 2 + 3 + \ldots. + k + (k + 1) < (2k + 1)^2 + (k + 1)$ [Using (k)]

$\Rightarrow \quad 1 + 2 + 3 + \ldots. + k + (k + 1) < 4k^2 + 5k + 2$

$\Rightarrow \quad 1 + 2 + 3 + \ldots. + k \; + (\mathrm{k} + 1) < (4k^2 + 12k + 9) - (7k + 7)$

$\Rightarrow \quad 1 + 2 + 3 + \ldots\ldots + k + (k + 1) < (2k + 3)^2$ $\quad \because \quad k \geq 1$

$\Rightarrow \quad T(k+1)$ is true. $\quad \therefore \quad 7k + 7 \geq 0$

Since the truth of $(k) \Rightarrow$ the truth of $T(k+1)$, therefore, $T(n)$ is true for all $n \in N$.

Ex. 17. Prove that: $1^2 + 2^2 + 3^2 + \ldots n^2 > \dfrac{n^3}{3}$ *for all* $n \in N$.

Sol. Let $T(n)$ be the statement given by

$T(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 > \dfrac{n^3}{3}$

Step I

$T(1)$: $1^2 > \frac{1^3}{3}$

$\because$ $1^2 = 1 > \frac{1}{3} = \frac{1^3}{3}$

$\therefore$ T(1) is true.

Step II

Let $T(n)$ be true for $n = k$. Then,

$$1^2 + 2^2 + 3^2 + ... + k^2 > \frac{k^3}{3}$$

we shall now prove that $P(k + 1)$ is true i.e,

$$1^2 + 2^2 + 3^2 + ... + 3k^2 + (k + 1)^2 > \frac{(k+1)^3}{3}$$

Now, $T(n)$ is true.

$\Rightarrow$ $1^2 + 2^2 + 3^2 + ... + k^2 > \frac{k^3}{3}$

$\Rightarrow$ $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 > \frac{k^3}{3} + (k + 1)^2$

$\Rightarrow$ $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 > \frac{1}{3}(k^3 + 3k^2 + 6k + 3)$

$\Rightarrow$ $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 > \frac{1}{3}\{(k^3 + 3k^2 + 3k + 1) + (3k + 2)\}$

$\Rightarrow$ $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 > \frac{1}{3}\{(k + 1)^3 + (3k + 2)\} > \frac{(k+1)^3}{3}$

$\Rightarrow$ $T(k + 1)$ is true.

Thus, $T(k)$ is true.

Hence, by the principle of mathematical induction, $T(n)$ is true for all $n \in N$.

Ex. 18. Prove that $x > -1$, $(1 + x)^n \geq 1 + nx$ of all natural number $n \geq 2$. **[*C.B.S.E.*]**

Sol. Let $T(n)$ be the statement, $(1 + x)^n \geq 1 + nx$, $n \geq 2$.

Basic step: For $n = 2$, $(1 + x)^2$ *i.e.*, $1 + x^2 + 2x \geq 1 + 2x$ which is true.

Thus $T(2)$ is true. ($\because$ x^2 is a positive quantity)

Induction step. Assume that $T(k)$ is true; that is

$(1 + x)^k \geq 1 + kx$, $x > -1$ ... (i)

We shall show that $T(k + 1)$ is also true, i.e., $(1 + x)^{k+1} \geq 1 + (k + 1)x$ [using (i)]

Now, $(1 + x)^{k+1} = (1 + x)^k (1 + x) \geq (1 + kx)(1 + x)$ $\because$ $k \in N$

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + kx + x + kx^2$ $\therefore$ $k > 0$

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x + kx^2$ Also $x^2 > 0$

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x$ $\therefore$ $kx^2 \geq 0$

$\Rightarrow$ $T(k + 1)$ is true.

Since the truth $T(k) \Rightarrow$ the truth of $T(k + 1)$, therefore, $T(n)$ is true for all values of $n \geq 2$.

Ex. 19. Prove by mathematical induction that $\dfrac{2n!}{2^{2n}(n!)^2} \le \dfrac{1}{(3n+1)^{\frac{1}{2}}}$. **[*I.I.T.*]**

Sol. We have to prove that $\dfrac{2n!}{2^{2n}(n!)^2} \le \dfrac{1}{\sqrt{3n+1}}$

Putting $n = 1$, $T(1)$ is $\dfrac{2!}{2^2(1!)^2} \le \dfrac{1}{\sqrt{4}} \Rightarrow \dfrac{1}{2} \le \dfrac{1}{2}$

$\therefore$ The given statement is true for $n = 1$; now, suppose it is true for $n = k$.

i.e., $T(k) \Rightarrow \dfrac{2k!}{2^{2k}(k!)^2} \le \dfrac{1}{\sqrt{3k+1}}$ is true. ... (*i*)

$$T(k+1) = \frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2} \le \frac{1}{\sqrt{3k+4}} \quad \text{... (ii)}$$

Now
$$\text{L.H.S.} = \frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2} = \frac{(2k+2)(2k+1)(2k!)}{2^2 \cdot 2^{2k} \cdot (k+1)^2 \cdot (k!)^2}$$

$$= \frac{(2k+2)(2k+1)}{2^2 \cdot (k+1)^2} \cdot \frac{(2k!)}{2^{2k} \cdot (k!)^2} = \frac{2k+1}{2k+2} \cdot \frac{(2k!)}{2^{2k} \cdot (k!)^2} \quad \text{[By (i)]}$$

$$\le \frac{2k+1}{2k+2} \cdot \frac{1}{\sqrt{3k+1}}$$

From (*ii*) and (*iii*), we have to prove that

$$T(k+1) \text{ is true if } \frac{(2k+1)}{(2k+2)} \cdot \frac{1}{\sqrt{3k+1}} \le \frac{1}{\sqrt{3k+4}}$$

$$\text{Squaring, } \frac{(2k+1)^2}{(2k+2)^2 \cdot (3k+1)} \le \frac{1}{3k+4}$$

or $(2k+1)^2 \,.\, (3k+4) \le (2k+2)^2\,(3k+1)$

$\Rightarrow$ $(2k+1)^2\,(3k+4) - (2k+2)^2\,(3k+1) \le 0$

$\Rightarrow$ $[(4k^2+4k+1)\,(3k+4)] - 4\,(k^2+2k+1)\,(3k+1) \le 0$

$\Rightarrow$ $(12k^3+28k^2+19k+4) - 4\,(3k^3+7k^2+5k+1) \le 0$

$\Rightarrow$ $-k \le 0 \Rightarrow k \ge 0$

Which is true as k is a positive integer. Hence, $T(k+1)$ is true.

This proves that if $\dfrac{2n!}{2^{2n}(n!)^2} \le \dfrac{1}{\sqrt{3n+1}}$ is true for any value $n = k$ then it is true for the next value $n = k + 1$. Again since it is true for $n = 1$, hence it is true $n \forall \in N$.

EXERCISE 8 (*b*)

Using the principle of mathematical induction prove that :

1. $n\,(n+1)\,(n+5)$ is a multiple of 6. [See Solved Ex. 9] **[*ISC 1998 Type*]**

2. $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

[Hint. $T(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = M\,(9)$, is true.

$T(k)$: $k^3 + (k+1)^3 + (k+2)^3 = M(9)$. ... (1) Assume true

$T(k+1) : (k+1)^3 + (k+2)^3 + (k+3)^3$

$= (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27 = k^3 + (k+1)^3 + (k+2)^3 + 9k^2 + 27k + 27$

$= M(9) + 9(k^2 + 3k + 3)$ [using (1)]

$= M(9) + M(9) = M(9)$

$\Rightarrow$ $T(k+1)$ is true.]

3. $2^{3n} - 1$ is divisible by 7.

4. $3^{2n} - 1$ is divisible by 8.

5. $4^n + 15n - 1$ is divisible by 9. **[*Roorkee, 1994*]**

6. $3^{4n+2} + 5^{2n+1}$ is a multiple of 14.

7. $7^{2n} + (2^{3n-3})\, 3^{n-1}$ is divisible by 25. **[*IIT*]**

8. $x^n - y^n$ is divisible by $x - y$ when n is an *even* positive integer.

[Hint. Type solved Ex. 8]

9. Prove by the method of mathematical induction that $3^{2n-1} + 3^n + 4$ is divisible by 2 for all $n \in N$. **[*ISC 2002*]**

[**Sol.** We have to prove that $3^{2n-1} + 3^n + 4$ is divisible by 2, *i.e.*, $3^{2n-1} + 3^n$ is divisible by 2.

Basic step. $T(1) = 3 + 3 = 6 = M(2)$

$\Rightarrow$ $T(1)$ is true.

Induction step. Assume that $T(k)$, $k \in N$, is true, *i.e.*,

$$3^{2k-1} + 3^k = M(2) \quad \ldots (i)$$

We shall show that $T(k+1)$ is true, *i.e.*,

$3^{2(k+1)-1} + 3^{k+1} = M(2)$, *i.e.*, $3^{2k+1} + 3^{k+1} = M(2)$

Now $3^{2k+1} + 3^{k+1} = 3^{2k-1} \cdot 3^2 + (3^2 - 6)3^k = (3^{2k-1} \cdot 3^2) + 3^2 \cdot 3^k - 6 \cdot 3^k$

$= 3^2(3^{2k-1} + 3^k) - 6 \cdot 3^k = 9M(2) - M(2) = M(2)$ [using (i)]

$\Rightarrow$ $T(k+1)$ is true.

Since $T(k) \Rightarrow T(k+1)$, therefore

$T(n)$ is true for for all $n \in N$.]

Prove by the principle of mathematical induction.

10. $n < 2^n$ for all $n \in N$

11. $(2n + 7) < (n + 3)^2$ for all natural numbers.

12. $1^2 + 2^2 + 3^2 + \ldots + n^2 > \dfrac{n^3}{3}$ for all $n \in N$.

HISTORICAL NOTE

The principle of mathematical induction was known to the Pythagoreans but Francesco Maurolico was the first to make fairly explicit use of it (in his Arithmetic, 1575). Blaise Pascal was the next person to use the idea, as he did repeatedly in his work on the so-called Pascal triangle, which is called the "arithmetic triangle". The induction proofs of Maurolico are given in a rather sketchy style not easily followed. Pascal's style is more nearly along modern lines.

CHAPTER 9

COMPLEX NUMBERS

9.01. Imaginary numbers

Necessity is the mother of invention. It was found out that the real numbers were also not sufficient to meet the demands of civilization in general and of mathematics in particular. Just as the need for more freedom in computations led to the use of negative integers and rational numbers, so too did the desire to solve quadratic and cubic equations led to the gradual introduction of complex numbers. When man came across such equations as $x^2 + 1 = 0$, $x^2 + 4 = 0$, etc., he found himself unable to solve them, as there is no real number whose square is a negative real number. Thus the real numbers were also found inadequate. To obtain solutions of such equations it became necessary to extend the number system. About 400 years ago mathematicians proposed the introduction of a number i with the property that

$$\boldsymbol{i^2 + 1 = 0} \quad \text{or} \quad \boldsymbol{i^2 = -1.}$$

The solution of the equation $x^2 + 1 = 0$ is then $x = i$ or $-i$. Also since $i^2 = -1$, we may write $\sqrt{-1} = i$ and call i "a square root of -1". Using i we can now introduce numbers that are square roots of any negative number. The necessary definition is as follows:

For every positive real number a, $\sqrt{-a} = \sqrt{a}\, i$

For example,

$\sqrt{-2} = \sqrt{2}\, i$ $\quad\therefore\ (\sqrt{2}i)^2 = -2$

$\sqrt{-36} = \sqrt{36}\, i = 6i$ $\quad\therefore\ (6i)^2 = -36$

$\sqrt{-12} = \sqrt{12}\, i = 2\sqrt{3}\, i$ $\quad\therefore\ (2\sqrt{3}i)^2 = -12$

The preceding examples suggest the following fact:

For every non-zero real number b, bi is a number whose square is $-b^2$; that is, $(bi)^2 = -b^2$. For $b \neq 0$, bi is called a **pure imaginary number.** *The number i is called the* **imaginary unit**. *0.i is defined to be 0.*

9.02. Integral powers of i

$i = \sqrt{-1}$; $i^2 = -1$, $i^3 = i^2 \times i = -i$; $i^4 = (i^2)^2 = (-1)^2 = 1$; $i^5 = i^4.i = 1 \times i = i$; $i^6 = i^4 \times i^2 = 1 \times -1 = -1$; $i^7 = i^4 \times i^3 = -i$, $i^8 = (i^4)^2 = 1^2 = 1$; $i^9 = i^8.i = (i^4)^2.i = 1^2 \times i = i$, and so on.

This shows that the **values repeat in cycles of four, according to the pattern i, -1, $-i$, 1.** Since $i^{4p} = 1$, therefore if $p, q \in I$, then

$i^{4p+q} = i^{4p} \cdot i^q = 1.\ i^q = i^q$, where $q = 0, 1, 2,$ or 3. i^q will be simplified to $1, i, -1, -i$.

$$\boldsymbol{i^{4n} = 1, \text{ where } n \in W}$$

Note. If we have any number i^p, $p \in W$ and $p > 4$, then we first divide p by 4, to separate the greatest multiple of 4 less than p.

For example, $i^{439} = i^{4\times 109+3} = (i^4)^{109}.i^3 = 1.i^2.i = -i.$

Notice also that $i(-i) = -1(i^2) = -1 \times -1 = 1$. Thus, i and $-i$ are reciprocals, that is,

$$\frac{1}{i} = -i \quad \text{and} \quad \frac{1}{-i} = i$$

You can also argue that
$\frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

For example, $\frac{15}{18i} = \frac{15}{18}\cdot\frac{1}{i} = \frac{5}{6}(-i) = \frac{-5}{6}i$

$$\frac{35}{i^{19}} = \frac{35}{i^{16}.i^3} = \frac{35}{(i^4)^4.i^3} = \frac{35}{i^3} = \frac{35}{i^4}\cdot i = \frac{35.i}{1} = 35i.$$

Ex. 1. Simplify (*i*) i^{38} (*ii*) i^{15} (*iii*) i^{-6} (*iv*) $\frac{1}{i}$.

Sol. (*i*) $i^{38} = i^{4\times 9+2} = (i^4)^9 \cdot i^2 = 1 \cdot i^2 \cdot = 1.(-1) = -1.$

(*ii*) $i^{15} = i^{4\times 3+3} = (i^4)^3 \cdot i^3 = 1 \cdot i^3 = i^3 = -i.$

(*iii*) $i^{-6} = i^{-8}.i^2 = (i^4)^{-2} \cdot i^2 = (1)^{-2}.i^2 = 1 \cdot i^2 = -1.$

(*iv*) $\frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$

Note. The (*iv*) above shows that i and $-i$ are reciprocal of each other, *i.e.*, $\frac{1}{i} = -i$ or $\frac{1}{-i} = i$.

Also $(i)^{-1} = -i$.

Ex. 2. Show that *i* is neither 0, nor greater than 0, nor less than 0.

Sol. Consider the two elements i and 0.

If $i = 0$, then $i^2 = 0$, but $i^2 = -1$, therefore $i \neq 0$.

If $i > 0$, then $i \,.\, i > 0$. i, or $i^2 > 0$, or $-1 > 0$. This is not true, therefore $i \ngtr 0$.

If $i < 0$, *i.e.*, i is a negative number; then $i \cdot i > 0 \cdot i$, or $-1 > 0$, since the inequality is multiplied by a negative number, it changes from 'less than' to 'greater than'. Now $-1 > 0$ is not true, therefore $i \not< 0$.

Hence i is neither 0, nor greater than 0, nor less than 0.

Note. We quite often read $-i$ as negative i. This implies $-i < 0$ which is not true. *We should read $-i$ as the inverse of i.*

Ex. 3. Simplify the following:

(i) $(5i) \times 7$ ***(ii)*** $(3i)\,(4i)$ ***(iii)*** $\frac{21}{14i}$ ***(iv)*** $\frac{5}{i^3}$

(v) $\sqrt{-9} + \sqrt{-16}$ ***(vi)*** $\frac{21}{4}\sqrt{-48} - 5\sqrt{-27}$ ***(vii)*** $\sqrt{-18}\cdot\sqrt{-2}$ ***(viii)*** $\frac{20}{\sqrt{-5}}$

Sol. (*i*) $(5i) \times 7 = (5 \times 7)i = 35i$

(*ii*) $(3i)\,(4i) = (3 \times 4)\,(i \cdot i) = 12\, i^2 = 12\,(-1) = -12$

(*iii*) $\frac{21}{14i} = \frac{21}{14}\cdot\frac{1}{i} = \frac{3}{2}(-i) = \frac{-3}{2}i$ ($\because -i$ is the reciprocal of i.)

(*iv*) $\dfrac{5}{i^3} = \dfrac{5}{i^2 \cdot i} = \dfrac{5}{-i} = 5i$ ($\because$ i is the reciprocal of $-i$.)

(*v*) $\sqrt{-9} + \sqrt{-16} = i\sqrt{9} + i\sqrt{16} = 3i + 4i = 7i$

(*vi*) $\dfrac{21}{4}\sqrt{-48} - 5\sqrt{-27} = \dfrac{21}{4} \cdot i \cdot \sqrt{48} - 5 \cdot i \cdot \sqrt{27}$

$= \dfrac{21}{4} \cdot i \cdot 4\sqrt{3} - 5 \cdot i \cdot 3\sqrt{3} = (21 - 15) \cdot i \cdot \sqrt{3} = 6\sqrt{3}\, i$

(*vii*) $\sqrt{-18} \cdot \sqrt{-2} = i \cdot \sqrt{18} \cdot i\sqrt{2} = \sqrt{18 \times 2} \cdot i^2 = 6(-1) = -6$

(*viii*) $\dfrac{20}{\sqrt{-5}} = \dfrac{20}{i\sqrt{5}} = -\dfrac{20i}{\sqrt{5}} = -\dfrac{20\sqrt{5}\, i}{5} = -4\sqrt{5}\, i.$

Caution. In (*vii*) above the student should not commit the mistake of writing $\sqrt{-18} \cdot \sqrt{-2} = \sqrt{(-18)(-2)} = \sqrt{36} = 6$. This is an incorrect result obtained by applying the property $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ which holds good only if $\sqrt{a}, \sqrt{b}$ are real numbers. It does not hold good if a, b are negative numbers, *i.e.*, $\sqrt{a}, \sqrt{b}$ are imaginary numbers. In this case $\sqrt{a}.\sqrt{b} = -\sqrt{ab}$.

To avoid difficulty with this point, you should write all expressions of the form $\sqrt{-b}$, $(b > 0)$, in the form $\sqrt{b}i$ before writing expressions involving imaginary numbers.

Ex. 4. Evaluate : $\left[i^{18} + \left(\dfrac{1}{i}\right)^{25} \right]^3$.

Sol. $\left[i^{18} + \left(\dfrac{1}{i}\right)^{25} \right]^3 = \left[i^{16}.i^2 + \left(\dfrac{i}{i^2}\right)^{25} \right]^3$

$= \left[(i^2 - i^{25}) \right]^3 = \left[(-1 - i^{24}.i) \right]^3 = (-1 - i)^3$

$= (-1)^3 (1 + i)^3 = -\left[1 + 3i + 3i^2 + i^3 \right]$ $\left[(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \right]$

$= -(1 + 3i - 3 - i) = -(-2 + 2i) = \mathbf{2 - 2i}.$

EXERCISE 9 (*a*)

Express each of the following in the form *b* or *bi*, where *b* is a real number:

1. $3i \cdot 2$ **2.** $i(-i)$ **3.** $-i(-i)$ **4.** $5i(-8i)$

5. $\dfrac{20i}{4}$ **6.** $\sqrt{-25}$ **7.** $\sqrt{-8}$ **8.** $\sqrt{\dfrac{-1}{3}}$

9. $\dfrac{1}{2}\sqrt{\dfrac{-3}{4}}$ **10.** $\dfrac{6}{-i}$ **11.** $\sqrt{-144}$ **12.** $\dfrac{x}{i}$

Simplify:

13. i^{13} **14.** i^{28} **15.** i^{18} **16.** i^{23}

17. $\sqrt{-4} + \sqrt{-16} - \sqrt{-25}$ **18.** $\sqrt{-20} + \sqrt{-12}$ **19.** $-\sqrt{\dfrac{-7}{4}} - \sqrt{\dfrac{-1}{7}}$

20. $\frac{\sqrt{-2}}{\sqrt{-8}}$ **21.** $\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}$ **22.** $\frac{1}{i}-\frac{1}{i^2}+\frac{1}{i^3}-\frac{1}{i^4}$ **23.** $i+2i^2+3i^3+i^4$

24. $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$ **25.** $\sqrt{\frac{-x}{4}}+\sqrt{\frac{-x}{16}}-\sqrt{\frac{-x}{64}}$, where x is a positive real number.

26. $\sqrt{-5x^8}-\sqrt{-20x^8}+\sqrt{-45x^8}$, where x is a positive real number.

27. If $i=\sqrt{-1}$, prove the following:

$(x+1+i)\,(x+1-i)\,(x-1+i)\,(x-1-i)=x^4+4.$ **[*ISC 2004*]**

ANSWERS

1. $6i$ **2.** 1 **3.** -1 **4.** 40 **5.** $5i$

6. $5i$ **7.** $2\sqrt{2}\,i$ **8.** $\frac{1}{3}\sqrt{3}\,i$ **9.** $\frac{1}{4}\sqrt{3}\,i$ **10.** $6i$

11. $12i$ **12.** $-xi$ **13.** i **14.** 1 **15.** -1

16. $-i$ **17.** i **18.** $2(\sqrt{5}+\sqrt{3})i$ **19.** $\frac{-9\sqrt{7}i}{14}$ **20.** $\frac{1}{2}$

21. 0 **22.** 0 **23.** $-1-2i$ **24.** $2(1-i)$ **25.** $\frac{5}{8}\sqrt{x}\,i$

26. $2\sqrt{5}\,x^4\,i$

HINTS TO SELECTED QUESTIONS

25. Given exp. $=\frac{1}{2}\sqrt{x}\,i+\frac{1}{4}\sqrt{x}\,i-\frac{1}{8}\sqrt{x}\,i$

26. Given exp. $=\sqrt{5}x^4i-2\sqrt{5}x^4i+3\sqrt{5}x^4i$

27. Given exp. $=\{(x+1)^2-i^2\}\,\{(x-1)^2-i^2\}=(x^2+2+2x)\,(x^2+2-2x)$

9.03. Complex numbers

A complex number is the sum of a real number and a pure imaginary number. The standard form of a complex number is $\boldsymbol{a+bi}$, where a and b are real numbers, a is called the *real part* and b, the *imaginary part*, of $a+bi$. If $b\neq 0$, $a+bi$ is also called an imaginary number. We use the letter C to refer to the set of all complex numbers. Thus,

$$\boldsymbol{C=\{a+bi\},\ a\in R \text{ and } b\in R \text{ and } i^2=-1.}$$

The real and imaginary parts of a complex number C may be denoted by **Re(*C*)** and **Im(*C*)** respectively.

For example, if $(i)\ z=7+8i$, then $\mathrm{Re}(z)=7$, $\mathrm{Im}(z)=8$

$(ii)\ z=-\sqrt{3}-11i$, then $\mathrm{Re}(z)=-\sqrt{3}$, $\mathrm{Im}(z)=-11$.

Remarks:

1. By identifying the real number a with the complex number $a+0i$, we can say that every real number belongs to z. Similarly, for the complex number $0+bi=bi$, it follows that every pure imaginary number is also a complex number.

2. In the complex number $a + bi$, if $b = 0$, then the number $a + 0i = a$ is a purely real number and if $a = 0$, then the number $0 + bi$, is a purely imaginary number. If $a = 0$ and $b = 0$, then the complex number reduces to $0 + 0i$, which is called the **zero complex number**. It follows from the above that **0 is both a purely real number and purely imaginary number**.
3. $a + bi = 0$ iff $a = 0$ and $b = 0$, *i.e.*, **a complex number is zero if and only if its real part is zero and its imaginary part is zero**.

9.04. Properties of complex numbers

Property 1: A real number cannot be equal to imaginary number.

Proof : Let a, b ($a \neq 0$, $b \neq 0$) be numbers such that $a = ib$

Then $a = ib \Rightarrow a^2 = (ib)^2 = i^2b^2 = -b^2$

$\Rightarrow a^2 + b^2 = 0 \Rightarrow a = 0$ and $b = 0$

This contradicts the hypothesis that $a \neq 0$ and $b \neq 0$

Since this contradiction arises by assuming that $a = ib$, therefore, $a \neq ib$.

Property 2. $a + ib = 0$ if $a = 0$ and $b = 0$.

i.e. a complex number is zero if and only if its real part is zero and its imaginary part is zero.

Proof : $a + ib = 0 \Rightarrow a = -ib$

Squaring $a^2 = i^2b^2 \Rightarrow a^2 = -b^2$

$\Rightarrow$ $a^2 + b^2 = 0$ which is only possible when $a = b = 0$

Property 3. Two complex numbers are equal when their real parts are equal and their imaginary parts are equal, *i.e.*, $a + ib = x + iy$ iff $a = x$ and $b = y$.

For example, if $a + bi = 2 + 7i$, then $a = 2$ and $b = 7$.

Also, $7 + 5i \neq 5 + 7i$, because $7 \neq 5$.

Proof : $a + ib = x + iy \Rightarrow (a + ib) - (x + iy) = 0$

$\Rightarrow$ $(a - x) + i(b - y) = 0$, which is only possible when

$a - x = 0$ and $b - y = 0$ $\therefore$ $a = x$, $b = y$. [By Property 2]

Ex. 5. Find the values of x and y if $2x + 4iy = -i^3x - y + 3$.

So. Given : $2x + 4iy = -i^3x - y + 3 = -(i^2 . i)x - y + 3 = ix - y + 3$

By property of equality of complex numbers, we have

$$2x = -y + 3 \text{ and } 4y = x$$

Solving, we get $x = \frac{4}{3}, y = \frac{1}{3}$.

Ex. 6. Write the values of x and y if $(3 - 4i)(x + yi) = 1 + i(0)$. (***Type ISC 2008, CBSE***)

Sol. We have $(3 - 4i)(x + iy) = 1 + i(0)$

$\Rightarrow$ $3x + 3yi - 4ix - 4yi^2 = 1 + i(0)$ $\Rightarrow$ $3x + 3yi - 4ix - 4y(-1) = 1 + i(0)$

$\Rightarrow$ $(3x + 4y) + (3y - 4x)i = 1 + i(0)$

By equality of complex numbers, we have

$$\left.\begin{aligned} 3x + 4y = 1 \quad \ldots(1) \\ \text{and } 3y - 4x = 0 \quad \ldots(2) \end{aligned}\right\} \textit{Equating real and imaginary parts.}$$

Solving (1) and (2), we get $x = \frac{3}{25}, y = \frac{4}{25}$.

Remark. Order relations 'greater than' and 'less than' do not hold in the set of complex numbers, *i.e.*, the relations > and < cannot be used to compare pairs of complex numbers.

9.05. Definition of complex numbers in terms of ordered pairs

The complex number $z = x + yi$ is defined by the ordered pair (x, y) of the real numbers x and y.

We know that every complex number can be written in standard form as $a + bi$, where a and b are real numbers. The number $b + ai$ is different from $a + bi$ unless $a = b$ is true. Therefore, any ordered pair of real numbers (a, b) can be used to specify exactly one complex number. The first component of the ordered pair is the *real part* of the complex number. The *imaginary part* of a complex number is the real coefficient of i.

Thus, the ordered pair (3, 4) is associated with the complex number $3 + 4i$ and no other complex number. Any ordered pair of real numbers is a way of naming a complex number.

For example : $(7, 8) = 7 + 8i$; $(11, -4) = 11 + (-4)i = 11 - 4i$

$98 + \sqrt{3}i = (98, \sqrt{3})$, $i - 15 = -15 + 1.\ i = (-15, 1)$.

9.06. Geometrical representation of complex numbers — The argand diagram

Every complex number $x + yi$ can be represented geometrically as a unique point, (x, y) in the XOY plane with x-coordinate representing its real part and y-coordinate representing imaginary part.

The point $(x, 0)$ on the x-axis represents the complex number $x + 0i$, *i.e.*, the real number x and every real number is represented as a point on the axis. As such the x-axis is called the **real axis**.

Similarly, the point $(0, y)$ on the y-axis represents the complex number $0 + yi$, *i.e.*, the imaginary number yi. Therefore, the y-axis is called the imaginary axis. All imaginary numbers are represented as points on this axis.

Positive real numbers are represented by points on the positive x-axis while the negative real numbers by points on the negative x-axis and the real number 0 by the origin. Imaginary numbers with $\text{Im}\, z > 0$ are represented on the positive y-axis and imaginary numbers with $\text{Im}\, z < 0$ are represented on the negative y-axis.

The plane on which complex numbers are thus represented as points is called complex plane and also the **Argand plane**.

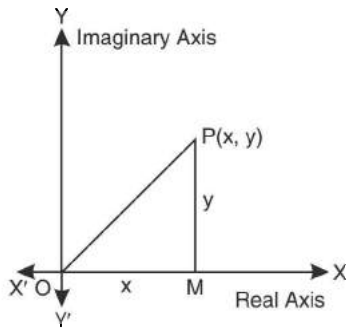


Fig. 9.01

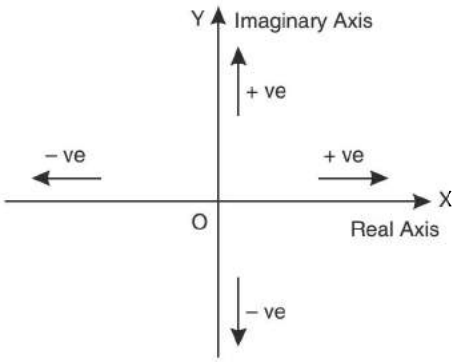


Fig. 9.02

Ex. 7. Represent the following complex numbers in the complex plane.

(i) $2 + 3i$ ***(ii)*** $3 - 5i$ ***(iii)*** $0 + 0i$ ***(iv)*** i

Sol. (*i*) Here, $z = 2 + 3i$. This represents the point (2, 3). The point is plotted as shown in Fig. 9.03 (*i*).

(*ii*) Here, $z = 3 - 5i$. This represents the point (3, – 5). This is plotted as shown in Fig. 9.03 (*ii*).

(*iii*) Here, $z = 0 + i.\ 0$. This represents the point O (0, 0). This is shown in Fig. 9.03 (*iii*).

(*iv*) Here, $z = i = 0 + i$. This represents the point P (0, 1). This is shown in Fig. 9.03 (*iv*).

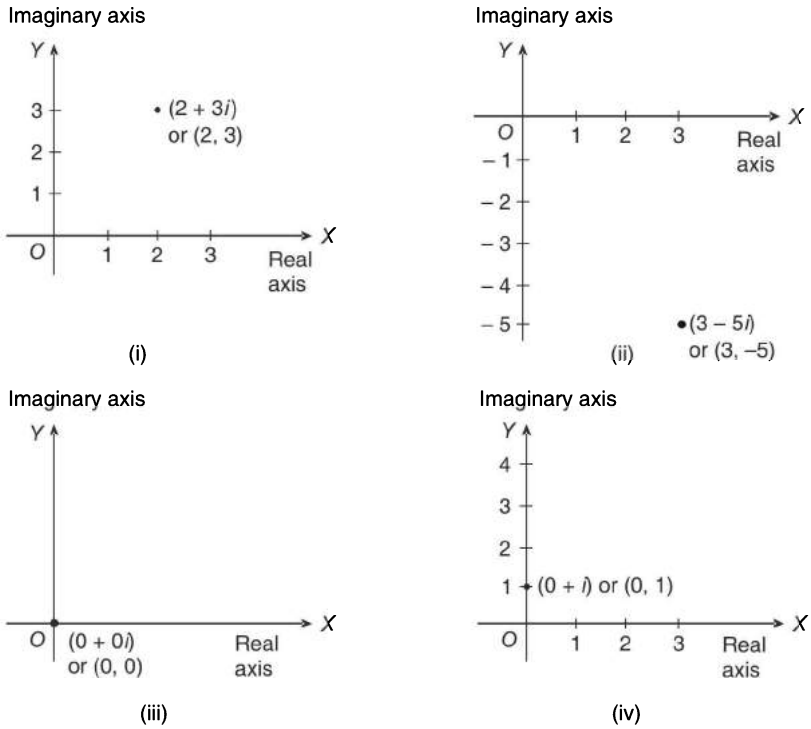


Fig. 9.03

Ex. 8. Can two different points in the complex plane represent the same complex number? Give reasons for your answer.

Sol. No.

Suppose (a, b) and (c, d) are two different points in the complex plane, *i.e.*, $a \neq c$ and $b \neq d$.

Now $(a, b) = (c, d) \Rightarrow a + ib = c + id$ which is possible iff $a = c$ and $b = d$.

Since this is not possible, therefore, two different points in the complex plane cannot represent the same complex number.

9.07. Operations with complex numbers

It is a remarkable fact that the number obtained on adding, subtracting, multiplying or dividing two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is another complex number.

1. **Addition:** The sum of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is defined by the relation.

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

Since x_1, x_2, y_1, y_2 are real numbers, so are $x_1 + x_2$ and $y_1 + y_2$. Therefore, the expression on the R.H.S is also a complex number of the form $p + qi$.

2. **Negative of a complex number.**

If $z = x + yi$, then $-z$ is defined as $-z = -x + (-y)i$.

3. **Subtracting :** The subtraction of z_1 and z_2 is defined by the relation.

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i$$

4. **Additive inverse.**

z_2 is the additive inverse of z_1 if $z_1 + z_2 = 0$ or if $(x_1 + y_1 i) + (x_2 + y_2 i) = 0$

$\Rightarrow$ If $(x_1 + x_2) + (y_1 + y_2)i = 0$

$\Rightarrow$ If $x_1 + x_2 = 0$ and $y_1 + y_2 = 0 \Rightarrow x_2 = -x_1, y_2 = -y_1$

$\therefore \quad z_2 = -x_1 - y_1 i = -(x_1 + y_1 i)$

Thus, if $x_1 + y_1 i$ is a complex number (x_1 and y_1 real), then its additive inverse is $-(x_1 + y_1 i) = -x_1 + (-y_1)i$.

5. (*i*) **Multiplication and division of a complex number by an integer**

If m is an integer and $z = x + yi$, a complex number, then

$$m(x + yi) = mx + myi \text{ and } \frac{1}{m}(x + yi) = \frac{x}{m} + \frac{y}{m}i.$$

e.g., $-7(3 + 5i) = -21 - 35i$, $\frac{1}{6}(12 + 21i) = \frac{12}{6} + \frac{21}{6}i = 2 + \frac{7}{2}i$.

(*ii*) **Product of a rational number and a complex number**

If $z = x + yi$ is a complex number and $\frac{p}{q}$, a rational number, then the product of $\frac{p}{q}$ and z is defined as multiplication of z by p and its division by q, *i.e.*,

$$\frac{p}{q}(x + yi) = \frac{p}{q}x + \frac{p}{q}yi$$

(*iii*) If a is a real number, then $a(x + yi) = ax + ayi$.

6. Multiplication of two complex numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are two complex numbers, then by the usual rules for multiplying out brackets, we have

$$\begin{aligned}(x_1 + y_1 i) \times (x_2 + y_2 i) &= x_1 x_2 + x_1(y_2 i) + (y_1 i)x_2 + (y_1 i)(y_2 i) \\ &= x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2 \\ &= x_1 x_2 + (x_1 y_2 + y_1 x_2)i + y_1 y_2(-1) \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i\end{aligned}$$

Since x_1, x_2, y_1, y_2 are real numbers, so are $x_1 x_2 - y_1 y_2$ and $x_1 y_2 + y_1 x_2$. The product is therefore, of the form $a + bi$ where a and b are real.

Cor. An important case is

$$\begin{aligned}(x_1 + y_1 i)\,(x_1 - y_1 i) &= (x_1 x_1 - y_1(-y_1)) + (x_1(-y_1) + y_1 x_1)i \\ &= (x_1^2 + y_1^2) + 0i = x_1^2 + y_1^2 + 0 = x_1^2 + y_1^2, \text{ a real number.}\end{aligned}$$

So with complex numbers the sum of two squares $a^2 + b^2$, can be factorised as $(a + bi)\,(a - bi)$.

7. Division of complex numbers

The quotient $\frac{z_2}{z_1}$ of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is obtained by multiplying numerator and denominator by $x_1 - y_1 i$ and using the result proved above, *i.e.*, $(x_1 + y_1 i)\,(x_1 - y_1 i) = x_1^2 + y_1^2$.

Thus, $\frac{z_2}{z_1} = \frac{x_2 + y_2 i}{x_1 + y_1 i} = \frac{x_2 + y_2 i}{x_1 + y_1 i} \times \frac{x_1 - y_1 i}{x_1 - y_1 i}$

$$= \frac{(x_2 + y_2 i)(x_1 - y_1 i)}{x_1^2 + y_1^2} = \frac{x_1 x_2 - x_2 y_1 i + y_2 x_1 i - (y_2 y_1)i^2}{x_1^2 + y_1^2}$$

$$= \frac{x_1x_2 + (x_1y_2 - x_2y_1)i - (y_2y_1)(-1)}{x_1^2 + y_1^2} = \frac{(x_1x_2 + y_1y_2) + (x_1y_2 - x_2y_1)i}{x_1^2 + y_1^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2} + \frac{x_1y_2 - x_2y_1}{x_1^2 + y_1^2}i$$

Note. Do not try to remember the formulae for $(x_1 + y_1i)(x_2 + y_2i)$ and $\frac{x_2 + y_2i}{x_1 + y_1i}$. Just understand the method and apply it when you need to do it.

8. Reciprocal of a complex number

If $z_1 = x_1 + y_1i$, then $\frac{1}{z_1} = \frac{1}{x_1 + y_1i} = \frac{1}{x_1 + y_1i} \times \frac{x_1 - y_1i}{x_1 - y_1i}$

$$\Rightarrow \qquad \frac{1}{z_1} = \frac{x_1 - y_1i}{x_1^2 + y_1^2} = \frac{x_1}{x_1^2 + y_1^2} + \frac{(-y_1)}{x_1^2 + y_1^2}i$$

The reciprocal of z_1 is called the multiplicative inverse of z_1 as $z_1 \times \frac{1}{z_1} = 1$.

Thus, if $x_1 + y_1i$ is a complex number (x_1 and y_1 are real), then its multiplicative inverse is $\frac{x_1}{x_1^2 + y_1^2} + \frac{(-y_1)}{x_1^2 + y_1^2}i = \frac{x_1 - y_1i}{x_1^2 + y_1^2}$.

9.08. Properties of operations with complex numbers

Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ be two complex numbers in the set of complex numbers C.

1. ***The set is closed under both addition and multiplication.***
2. ***Addition and multiplication are commutative in C,***
   i.e., $z_1 + z_2 = z_2 + z_1$ **and** $z_1z_2 = z_2z_1$.
3. ***Addition and multiplication are associative in C, i.e., for any three complex numbers*** z_1, z_2 ***and*** z_3, ***we have*** $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
4. ***There is an identity for both operations '+' and '×'.***
   (i) The complex number $0 + 0i$ is the additive identity in C.
   Thus, if $a + bi$ is any complex number, then $a + bi + (0 + 0i) = (a + 0) + (b + 0)i = a + bi$.
   (ii) The complex number $1 + 0i$ is the multiplicative identity in C.
   Thus, $(a + bi)(1 + 0i) = (a.1 - 0.b) + (a.0 + b.1)i = a + bi$.
5. ***Each element in C has an additive inverse.***
   Thus, the additive inverse of $z = a + bi$ is $-z = -(a + b)i = -a - bi$.
6. ***Every non-zero complex number has a multiplicative inverse.***
   The multiplicative inverse of a complex number z (except $0 + 0i$), is denoted by z^{-1}. Thus the **multiplicative inverse of** $z = \frac{1}{z} = z^{-1}$.
7. ***Multiplication is distributive over addition in C.***
   If $z_1 = a_1 + b_1i, z_2 = a_2 + b_2i, z_3 = a_3 + b_3i$ are three complex numbers, then $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

Note. It is impossible to define an ordering in the system of complex numbers. The phrases 'less than' or 'greater than' are meaningless in relation between two complex numbers. Inequalities can occur only in relations between the moduli (plural of modulus) of complex numbers.

9.09. Identities

1. ***To prove that*** $(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$.

Proof. $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2) = z_1^2 + z_1z_2 + z_2z_1 + z_2^2$

($\because$ Multiplication distributes over addition and multiplication is associative in C)

$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2$ ($\because$ Multiplication is commulative in C)

$= z_1^2 + 2z_1z_2 + z_2^2$.

Similarly, we can prove that following identities.

2. $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$ **3.** $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

4. $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$ **5.** $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$.

Ex. 9. Express the following in the form $a + bi$.

(*i*) $(8 + 7i)(8 - 7i)$ **(*ii*)** $(5 - 6i)^2$ **(*iii*)** $(2 + 3i)(3 + 7i)$

(*iv*) $\left(-2 - \frac{1}{3}i\right)^3$ **(*v*)** $(1 - i)^4$ **(*vi*)** $(\sqrt{3} + 5i)(\sqrt{3} - 5i)^2 + (-4 + 5i)^2$

Sol. (*i*) $(8 + 7i)(8 - 7i) = 8^2 - (7i)^2 = 64 - 49i^2$

$= 64 - 49 \times -1 = 64 + 49 = 113 = 113 + 0i$.

(*ii*) $(5 - 6i)^2 = 5^2 - 2(5)(6i) + (6i)^2$

$= 25 - 60i + 36i^2 = 25 - 60i - 36$ ($\because i^2 = -1$)

$= -11 - 60i$.

(*iii*) $(2 + 3i)(3 + 7i) = 6 + 14i + 9i + 21i^2$

$= 6 + 23i - 21 = -15 + 23i$.

(*iv*) $\left(-2 - \frac{1}{3}i\right)^3 = (-1)^3\left(2 + \frac{1}{3}i\right)^3$

$$= -\left(2^3 + 3 \times 2^2 \times \frac{1}{3}i + 3 \times 2 \times \left(\frac{1}{3}i\right)^2 + \left(\frac{1}{3}i\right)^3\right)$$

$$= -\left(8 + 4i + \frac{2}{3}i^2 + \frac{1}{27}i^3\right) = -\left(8 + 4i + \frac{2}{3}(-1) + \frac{1}{27}(-i)\right)$$

$$= -\left(8 - \frac{2}{3} + 4i - \frac{1}{27}i\right) = -\left(\frac{22}{3} + \frac{107}{27}i\right) = -\frac{22}{3} - \frac{107}{27}i.$$

(*v*) $(1 - i)^4 = [(1 - i)^2]^2 = (1 - 2i + i^2)^2 = (1 - 2i - 1)^2$

$= (-2i)^2 = 4i^2 = 4 \times -1 = -4$.

(*vi*) $(\sqrt{3} + 5i)(\sqrt{3} - 5i)^2 + (-4 + 5i)^2$

$= [(\sqrt{3} + 5i)(\sqrt{3} - 5i)](\sqrt{3} - 5i) + (-4 + 5i)^2$

$= (3 - 25i^2)(\sqrt{3} - 5i) + (16 - 40i + 25i^2)$

$= (3 - 25 \times -1)(\sqrt{3} - 5i) + (16 - 40i - 25)$

$= 28(\sqrt{3} - 5i) - 9 - 40i = 28\sqrt{3} - 140i - 9 - 40i$

$= (28\sqrt{3} - 9) - 180i$.

Ex. 10. If $(x + yi)^3 = u + vi$, prove that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.

Sol. $(x + yi)^3 = u + vi. \Rightarrow x^3 + 3x^2yi + 3xy^2i^2 + y^3i^3 = u + vi \Rightarrow x^3 + 3x^2yi - 3xy^2 - y^3i = u + vi$

$\Rightarrow \quad x^3 - 3xy^2 + (3x^2y - y^3)i = u + vi$

Equating real and imaginary parts, we have

$x^3 - 3xy^2 = u,\ 3x^2y - y^3 = v$

$\Rightarrow \quad \frac{u}{x} = x^2 - 3y^2,\ \frac{v}{y} = 3x^2 - y^2$ ....(1)

$\Rightarrow \quad \frac{u}{x} + \frac{v}{y} = x^2 - 3y^2 + 3x^2 - y^2 = 4x^2 - 4y^2 = 4(x^2 - y^2).$ [using (1)]

Ex. 11. If $x = -5 + \sqrt{-16}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Sol. $x = -5 + \sqrt{-16}$,

$\therefore \quad x + 5 = 4i \Rightarrow (x + 5)^2 = -16 \Rightarrow x^2 + 25 + 10x = -16 \Rightarrow x^2 + 10x + 41 = 0$ (1)

Now we may write $x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$= (x^2 \times 0) - (x \times 0) + (4 \times 0) - 160 = -160.$ [using (1)]

Ex. 12. Express the following in the form $a + bi$, where a and b are real numbers.

(i) $\frac{1+i}{1-i}$ **(ii)** $\left(\frac{1-i}{1+i}\right)^2$ **(iii)** $\left(\frac{1+i}{1-i}\right)^3$ **(iv)** $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

Sol. $(i)\ \frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2 - i^2} = \frac{1+2i+i^2}{1+1} = \frac{1+2i-1}{2} = i = \mathbf{0 + 1i}.$

$(ii)\ \left(\frac{1-i}{1+i}\right)^2 = \frac{1-2i+i^2}{1+2i+i^2} = \frac{1-2i-1}{1+2i-1} = \frac{-2i}{2i} = -1 = \mathbf{-1 + 0i}.$

$(iii)\ \left(\frac{1+i}{1-i}\right) = \left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right] = \frac{(1+i)^2}{1-i^2} = \frac{1+i^2+2i}{1-(-1)} = \frac{1-1+2i}{2} = i$ ....(1)

$\therefore \quad \left(\frac{1+i}{1-i}\right)^3 = i^3 = i.i^2 = -i = \mathbf{0 - 1i}.$ [Using (1)]

$(iv)\ \frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)} = \frac{9-5i^2}{2\sqrt{2}i} = \frac{9+5}{2\sqrt{2}i}$

$= \frac{14}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} \cdot \frac{i}{i} = \frac{14i}{2\sqrt{2}i^2} = -\frac{7\sqrt{2}}{2}i.$

Ex. 13. Find the multiplicative inverse of $\frac{3+4i}{4-5i}$.

Sol. Multiplicative inverse of $\frac{3+4i}{4-5i} = \frac{4-5i}{3+4i}$

$$= \frac{(4-5i)(3-4i)}{(3+4i)(3-4i)} = \frac{12-16i-15i+20i^2}{9-16i^2} \qquad \because (i^2=-1)$$

$$= \frac{12-31i-20}{9+16} = \frac{-8-31i}{25} \quad \frac{-8}{25} - \frac{31i}{25}.$$

Ex. 14. Find the smallest positive integer n, for which $\left(\frac{1+i}{1-i}\right)^n = 1$.

(MP PET 2002, UPSEAT 2001, IIT, MNR)

Sol $\left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow \left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^n = 1 \Rightarrow \left(\frac{1+2i+i^2}{1-i^2}\right)^n = 1$

$$\Rightarrow \left(\frac{1+2i-1}{1-(-1)}\right)^n = 1 \Rightarrow \left(\frac{2i}{2}\right)^n = 1 \Rightarrow i^n = 1$$

Now, we know that $i^1 = i$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.
Hence, the smallest positive integral value of $n = 4$.

Ex. 15. If $a + bi = \frac{c+i}{c-i}$, $a, b, c \in R$, show that $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

Sol. $a + bi = \frac{c+i}{c-i} = \frac{c+i}{c-i}\cdot\frac{c+i}{c+i} = \frac{c^2+2ci+i^2}{c^2-i^2} = \frac{c^2+2ci-1}{c^2+1}$

$$= \frac{c^2-1}{c^2+1} + \frac{2c}{c^2+1}i$$

Equating real and imaginary parts, we have

$$a = \frac{c^2-1}{c^2+1}, \quad b = \frac{2c}{c^2+1}$$

$$\therefore \quad a^2 + b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2 + \left(\frac{2c}{c^2+1}\right)^2 = \frac{(c^2-1)^2+4c^2}{(c^2+1)^2} = \frac{(c^2+1)^2}{(c^2+1)^2} = 1$$

$$\frac{b}{a} = \frac{2c}{c^2+1} \times \frac{c^2+1}{c^2-1} = \frac{2c}{c^2-1}.$$

Ex. 16. Solve the equation

$$\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i \;;\; x\,y \in R;\; i = \sqrt{-1}\,.$$ ***(IIT, ISC 1998)***

Sol. $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

$$\Rightarrow \quad \frac{(3-i)(1+i)x-2i(3-i)+(3+i)(2-3i)y+i(3+i)}{9-i^2} = i$$

$$\Rightarrow \quad \frac{(3+2i-i^2)x-6i+2i^2+(6-7i-3i^2)y+3i+i^2}{10} = i$$

$\Rightarrow \quad (4 + 2i)x - 3i - 3 + (9 - 7i)y = 10i$

$\Rightarrow \quad (4 + 2i)x + (9 - 7i)y = 13i + 3$

$\Rightarrow \quad (2x - 7y)i + 4x + 9y = 13i + 3$

$\Rightarrow \quad 2x - 7y = 13$ and $4x + 9y = 3$

[Equating the real parts and imaginary parts]

Solving them simultaneously, we get $x = 3, y = -1$.

9.10. Conjugate of a complex number

If $z = a + bi$, then the number $a - bi$ is called the complex conjugate or, simply conjugate of $a + bi$ and is denoted by $\overline{z}$.

For example, (*i*) $z = 7 + 2i$, then $\overline{z} = \overline{7 + 2i} = 7 - 2i$,

(*ii*) If $z = -1 - 5i$, then $\overline{z} = -1 + 5i$,

(*iii*) Conjugate of $i\sqrt{9}$ is $-i\sqrt{9}$.

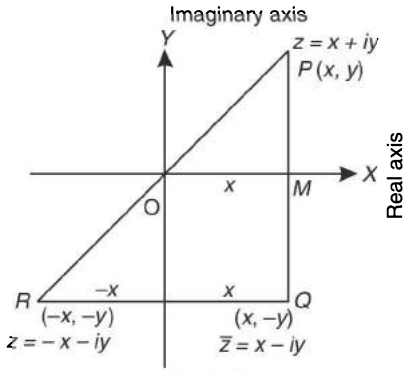


Fig. 9.04

The conjugate of a complex number is formed by changing the sign of the imaginary part.

Notation: The conjugate of a complex number $z = (a + bi) = \overline{z} = \overline{a + bi} = a - bi$.

Geometrically, if a point P represents number $z = x + yi = (x, y)$, the point Q, which represents the conjugate number $= \overline{z} = x - yi = (x, -y)$, is the image or reflection of P in the real axis.

If R denotes the complex number $-z = -x - yi = (-x, -y)$, then $-z$ is the reflection or image of $\overline{z}$ in the imaginary axis. (Fig. 9.04)

9.11. Some properties of conjugates

Property 1. ***The conjugate of the conjugate of a complex number is the complex number itself, i.e., if z is a complex number. Then***

$$\boxed{\overline{(\overline{z})} = z}$$

Proof. Let $z = x + yi$, then $\overline{z} = x - yi$

$\Rightarrow \quad (\overline{z}) = \overline{(x - yi)} = x + yi = z.$

Property 2. $z = \overline{z} \Leftrightarrow$ ***z is purely real, i.e. A complex number is equal to its conjugate, if it is a purely real number.***

Proof. Let $z = x + yi$, then $z = \overline{z} \Leftrightarrow x + yi = \overline{x + yi} \Leftrightarrow x + yi = x - yi$

$\Leftrightarrow \quad x - x + yi + yi = 0 \Leftrightarrow 2yi = 0 \Leftrightarrow y = 0$

$\Leftrightarrow \quad \text{Im}(z) = 0 \Leftrightarrow z$ is purely real.

Property 3. ***The conjugate of the sum of two complex numbers is the sum of their conjugates, i.e., if z_1 and z_2 are two complex numbers, then***

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Proof. Let $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i$. Then $\overline{z_1} = x_1 - y_1 i, \ \overline{z_2} = x_2 - y_2 i$

$\therefore \quad \overline{z_1} + \overline{z_2} = x_1 - y_1 i + x_2 - y_2 i = (x_1 + x_2) - (y_1 + y_2)i \qquad \ldots(1)$

Now $\overline{z_1 + z_2} = [\overline{(x_1 + y_1 i) + (x_2 + y_2 i)}] = \overline{(x_1 + x_2) + (y_1 + y_2)i}$

$= (x_1 + x_2) - (y_1 + y_2)i = (x_1 - y_1 i) + (x_2 - y_2 i) = \overline{z_1} + \overline{z_2}$

In general, $\overline{z_1 + z_2 + z_3 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \ldots + \overline{z_n}$.

Property 4. ***The conjugate of the difference of two complex numbers is equal to the difference of their conjugates, i.e., if z_1 and z_2 are two complex numbers, then***

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

Proof. $\overline{z_1 - z_2} = \overline{[(x_1 + y_1 i) - (x_2 + y_2 i)]} = \overline{(x_1 - x_2) + (y_1 - y_2)i}$

$= (x_1 - x_2) - (y_1 - y_2)i = (x_1 - y_1 i) - (x_2 - y_2 i) = \overline{z_1} - \overline{z_2}$

Property 5. ***The conjugate of the product of two complex numbers is the product of their conjugates, i.e., $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$ where z_1 and z_2 are two complex numbers.***

Proof. $\overline{z_1 z_2} = \overline{(x_1 + y_1 i)(x_2 + y_2 i)} = \overline{(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i}$

$= (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1)i$ …(1)

$\overline{z_1}.\overline{z_2} = (x_1 - y_1 i)(x_2 - y_2 i) = (x_1 x_2 - y_1 y_2) - (x_1 y_2 + y_1 x_2)i$ …(2)

From (1) and (2) it follows that $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$

In general, $\overline{z_1 . z_2 . z_3 \ldots z_n} = \overline{z}_1 . \overline{z}_2 . \overline{z}_3 \ldots \overline{z}_n$.

Property 6. ***The conjugate of the quotient of two complex numbers (non-zero denominator) is the quotient of their conjugates, i.e.,***

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}.$$

Proof. $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{x_1 + y_1 i}{x_2 + y_2 i}\right)} = \overline{\left\{\frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)}\right\}}$

$= \frac{\overline{(x_1 x_2 + y_1 y_2) + (-x_1 y_2 + x_2 y_1)i}}{x_2^2 + y_2^2} = \frac{(x_1 x_2 + y_1 y_2) - (-x_1 y_2 + x_2 y_1)i}{x_2^2 + y_2^2}$ …(1)

$\frac{\overline{z}_1}{\overline{z}_2} = \frac{x_1 - y_1 i}{x_2 - y_2 i} = \frac{(x_1 - y_1 i)(x_2 + y_2 i)}{(x_2 - y_2 i)(x_2 + y_2 i)} = \frac{(x_1 x_2 + y_1 y_2) - (-x_1 y_2 + y_1 x_2)i}{x_2^2 + y_2^2}$ …(2)

(1) and (2) $\Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$

In general, $\overline{\left(\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} \cdot \frac{z_5}{z_6} \cdots \frac{z_{n-1}}{z_n}\right)} = \frac{\overline{z}_1}{\overline{z}_2} \cdot \frac{\overline{z}_3}{\overline{z}_4} \cdot \frac{\overline{z}_5}{\overline{z}_6} \cdots \frac{\overline{z}_{n-1}}{\overline{z}_n}$.

Property 7. ***The square of the conjugate of a complex number is equal to the conjugate of the square of that complex number, i.e.,***

$$(\overline{z})^2 = \overline{(z^2)}.$$

Proof. Let $z = x + yi$. Then $\overline{z} = x - yi$.

$(\overline{z})^2 = (x - yi)^2 = (x^2 - y^2) - 2xyi$ …(1)

$z^2 = (x + yi)^2 = (x^2 - y^2) + 2xyi$

$\therefore \quad \left(\overline{z^2}\right) = (x^2 - y^2) - 2xyi$ …(2)

(1) and (2) $\Rightarrow (\overline{z})^2 = (\overline{z^2})$.

Ex. 17. Write the conjugate of

(*i*) $\sqrt{-16}-3$ **(*ii*)** i^7 **(*iii*)** $(3+4i)^2$

(*iv*) $\sqrt{-25}\,(7+\sqrt{-576})$ **(*v*)** $\dfrac{1-i}{1+i}$

Sol. (*i*) $\sqrt{-16}-3 = \sqrt{-1} \times 4 - 3 = 4i - 3.$

$\therefore$ Conjugate of $\sqrt{-16}-3$ is $-4i-3$.

(*ii*) $i^7 = i^4.i^2.i = 1 \times -1 \times i = -i$

$\therefore$ Conjugate of i^7 is i.

(*iii*) $(3+4i)^2 = 9 + 24i + 16i^2 = 9 + 24i - 16 = -7 + 24i$

$\therefore$ Conjugate of $(3+4i)^2$ is $-7 - 24i$.

(*iv*) $\sqrt{-25}(7+\sqrt{-576}) = 5i(7+24i) = 35i + 120i^2 = 35i - 120$

$\therefore$ Conjugate of $\sqrt{-25}(7+\sqrt{-576})$ is $-35i - 120$.

(*v*) $\dfrac{1-i}{1+i} = \dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{(1-i)^2}{1-i^2} = \dfrac{1-2i+i^2}{1-(-1)} = \dfrac{1-2i-1}{2} = \dfrac{-2i}{2} = -i$

$\therefore$ Conjugate of $\dfrac{1-i}{1+i}$ is i.

Ex. 18. Find the real numbers x and y if $(x - yi)(3 + 5i)$ is the conjugate of $-6 - 24i$. **(*NMOC*)**

Sol. $(x - yi)(3 + 5i)$ is the conjugate of $-6 - 24i$

$\therefore$ $(x - yi)(3 + 5i) = -6 + 24i \Rightarrow 3x + 5xi - 3yi - 5yi^2 = -6 + 24i$

$\Rightarrow$ $3x + (5x - 3y)i - 5y(-1) = -6 + 24i \Rightarrow (3x + 5y) + (5x - 3y)i = -6 + 24i$

$\Rightarrow$ $3x + 5y = -6,\ 5x - 3y = 24$ (on equating real and imaginary part)

Solving these equations, we obtain $\boldsymbol{x = 3, y = -3.}$

Ex. 19. Given $z_1 = 1 - i$, $z_2 = -2 + 4i$, calculate the values of a and b if $a + bi = \dfrac{z_1 z_2}{\overline{z_1}}$.

(*ISC 1994*)

Sol. $a + bi = \dfrac{z_1 z_2}{\overline{z_1}} = \dfrac{(1-i)(-2+4i)}{1+i} = \dfrac{-2+4i+2i-4i^2}{1+i} = \dfrac{-2+6i+4}{1+i}$

$$= \frac{2+6i}{1+i} = \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{2-2i+6i-6i^2}{1-i^2} = \frac{2+4i-6(-1)}{1-(-1)} = \frac{8+4i}{2} = 4 + 2i$$

$\Rightarrow$ $a = 4, b = 2.$

Ex. 20. If z be a non-zero complex number, show that $\overline{(z^{-1})} = (\overline{z})^{-1}$.

Sol. Let $z = a + bi$, then

$$z^{-1} = \frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2 - b^2 i^2} = \frac{a-bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i \quad (\text{Since } z \neq 0, \text{ therefore, } a^2 + b^2 > 0)$$

$$\therefore \quad \overline{(z^{-1})} = \overline{\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right)} = \frac{a}{a^2+b^2} + \frac{b}{a^2+b^2}i \qquad \ldots(1)$$

$$\text{Also, } (\overline{z})^{-1} = \frac{1}{\overline{z}} = \frac{1}{\overline{a+bi}} = \frac{1}{a-bi} = \frac{1}{a-bi} \times \frac{a+bi}{a+bi}$$

$$= \frac{a+bi}{a^2-b^2i^2} = \frac{a+bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{b}{a^2+b^2}i \qquad \ldots(2)$$

(1) and (2) $\Rightarrow \overline{(z^{-1})} = (\overline{z})^{-1}$.

Ex. 21. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{107} + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{107}$, then show that Im(z) = 0.

Sol. $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{107} + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{107}$

$$\Rightarrow \quad \overline{z} = \overline{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{107}} + \overline{\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{107}} = \overline{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{107}} + \overline{\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{107}} \quad (\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2})$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{107} + \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{107} = z$$

Now, $\overline{z} = z \quad \Rightarrow$ z is purely real. [(Property 2 $\Rightarrow$ Im $(z) = 0$.]

Ex. 22. If $\frac{(a+i)^2}{2a-i} = p + qi$, show that $p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}$.

Sol. $p + qi = \frac{(a+i)^2}{2a-i} \Rightarrow \overline{p+qi} = \overline{\left(\frac{(a+i)^2}{2a-i}\right)}$

$$\Rightarrow \; p - qi = \frac{\overline{(a+i)}^2}{\overline{2a-i}} = \frac{(a-i)^2}{2a+i} \quad \left(\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}\right)$$

$$\therefore \quad (p+qi)(p-qi) = \frac{(a+i)^2}{2a-i} \times \frac{(a-i)^2}{2a+i} = \frac{[(a+i)(a-i)]^2}{(2a-i)(2a+i)}$$

$$\Rightarrow \; p^2 - q^2i^2 = \frac{(a^2-i^2)^2}{4a^2-i^2} \Rightarrow p^2 - q^2(-1) = \frac{[a^2-(-1)]^2}{4a^2-(-1)}$$

$$\Rightarrow \qquad p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}.$$

EXERCISE 9 (*b*)

1. **In each of the following find $r + s$, $r - s$, rs, $\frac{r}{s}$ if r denotes the first complex number and s denotes the second complex number:**
   (*i*) $3 + 7i, i$ (*ii*) $-i, 5 + 2i$ (*iii*) $3i, 1 - i$ (*iv*) $-7, -1 - 3i$ (*v*) $7 + 3i, 3i - 7$
2. **Solve each of the following equations for real x and y:**
   (*i*) $(x + yi) + (3 - 2i) = 1 + 4i$ (*ii*) $(x + yi) - (7 + 4i) = 3 - 5i$
   (*iii*) $2x + yi = 1 + (2 + 3i)$ (*iv*) $x + 2yi = i - (-3 + 5i)$

3. Determine the conjugate and the reciprocal of each complex number given below:

(*i*) i (*ii*) i^3 (*iii*) $3 - i$ (*iv*) $\sqrt{-1} - 3$ (*v*) $\sqrt{-9} - 1$

4. Simplify:

(*i*) $(3 - 7i)^2$ (*ii*) $\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^2$ (*iii*) $(9 + 4i)\left(\frac{3}{2} - i\right)(9 - 4i)$ **(*CBSE, Pb*)**

5. Determine real values of x and y for which each statement is true.

(*i*) $\frac{x+y}{i} + x - y + 4 = 0$ (*ii*) $-(x + 3y)i + (2x - y + 1) = \frac{8}{i}$

(*iii*) $(x - yi) = \frac{2+i}{1+i}$ (*iv*) $(3 - 4i)(x + yi) = 1 + 0i$ **(*CBSE*)**

(*v*) $(x - yi)(2 + 3i) = \frac{x-2i}{1-i}$ **(*ISC*)** (*vi*) $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$ **(*CBSE*)**

6. Write the conjugate of $(6 + 5i)^2$.

7. Write the additive inverse of the following:

(*i*) $-2 + 3i$ **(*HB*)** (*ii*) $3 - 4i$

8. Find the multiplicative inverse of each of the following complex numbers when it exists.

(*i*) $2 + 2i$ **(*CBSE*)** (*ii*) $-7 + 0i$ (*iii*) $0 + 0i$

(*iv*) -16 (*v*) $\frac{i}{1+i}$ (*vi*) $(1 + i)^2$

(*vii*) $\frac{3+4i}{4-5i}$ **(*CBSE*)** (*viii*) $(6 + 5i)^2$ **(*CBSE*)** (*ix*) $\frac{(2+3i)(3+2i)i}{5+i}$

9. Simplify:

(*i*) $(1 + i)^{-1}$ (*ii*) $\sqrt{-\frac{49}{25}}\sqrt{-\frac{1}{9}}$ (*iii*) $\sqrt{-64} \cdot (3 + \sqrt{-361})$

(*iv*) $(3 - 7i)^2$ (*v*) $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2$ (*vi*) $\frac{(1-i)^3}{(1-i^3)}$

(*vii*) $\left(\frac{1+i}{1-i}\right)^{4n+1}$ (n, is a + ve integer) (*viii*) $\frac{\sqrt{(5+12i)} + \sqrt{(5-12i)}}{\sqrt{(5+12i)} - \sqrt{(5-12i)}}$

10. Prove that $\left[\left(\frac{3+2i}{2-5i}\right) + \left(\frac{3-2i}{2+5i}\right)\right]$ is rational.

11. Show that $\frac{1+2i}{3+4i} \times \frac{1-2i}{3-4i}$ is real.

12. Perform the indicated operation and give your answer in the form $x + yi$, where x and y are real numbers and $i = \sqrt{-1}$.

(*i*) $(3 + 4i)^{-1}$ (*ii*) $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ **(*CBSE*)** (*iii*) $\frac{5+2i}{-1+\sqrt{3}i}$ **(*CBSE*)**

(*iv*) $\frac{5-3i}{6+i}$ **(*CBSE*)** (*v*) $(\sqrt{5} - 7i)(\sqrt{5} - 7i)^2 + (-2 + 7i)^2$ **(*CBSE*)**

13. If $x + yi = \frac{u+vi}{u-vi}$, prove that $x^2 + y^2 = 1$.

14. Prove that: $[4+3\sqrt{-20}]^{\frac{1}{2}}+[4-3\sqrt{-20}]^{\frac{1}{2}}=6$.

15. Express the following in the form $a + bi$:

(*i*) $\sqrt{\dfrac{5(2+i)}{2-i}}$ (*ii*) $\dfrac{(3-i)^2}{2+i}$ (*iii*) $(1+i)^{-3}$ (*iv*) $\left(\dfrac{4i^3-i}{2i+1}\right)^2$

(*v*) $\dfrac{i-1}{i+1}$ (*vi*) $\dfrac{2+i}{(3-i)(1+2i)}$ (*vii*) $\dfrac{5}{2i-7i^2}$

16. Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$ is a positive integer.

17. If one of the values of x of the equation $2x^2 - 6x + k = 0$ be $\dfrac{1}{2}(a + 5i)$, find the values of a and k.

18. Define conjugate complex numbers and show that their sum and product are real numbers.

19. If $\bar{z} = -z \neq 0$, show that z is necessarily a purely imaginary number.

20. z and z' are complex numbers such that their product $zz' = 3 - 4i$.
Given that z' is $5 + 3i$, express z in the form $a + bi$ where a and b are rational numbers.

21. If $a + bi = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

22. Let $z_1 = 2 - i$, $z_2 = -2 + i$, find (*i*) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$, (*ii*) $\text{Im}\left(\dfrac{1}{z_1\bar{z}_2}\right)$

23. If $z_1 = 3 + 5i$ and $z_2 = 2 - 3i$, then verify that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

24. If $x = -2 - \sqrt{3}i$, where $i = \sqrt{-1}$, find that value of $2x^4 + 5x^3 + 7x^2 - x + 41$.

25. If $z = -3 + \sqrt{-2}$, then prove that $z^4 + 5z^3 + 8z^2 + 7z + 4$ is equal to -29.

ANSWERS

1. (*i*) $3 + 8i, 3 + 6i, 3i - 7, 7 - 3i$ (*ii*) $5 + i, -5 - 3i, 2 - 5i, \dfrac{-2}{29} - \dfrac{5}{29}i$

(*iii*) $1 + 2i, 4i - 1, 3 + 3i, \dfrac{3i}{2} - \dfrac{3}{2}$ (*iv*) $-8 - 3i, -6 + 3i, 7 + 21i, \dfrac{7}{10} - \dfrac{21}{10}i$

(*v*) $6i, 14, -58, \dfrac{-20}{29} - \dfrac{21}{29}i$

2. (*i*) $x = -2, y = 6$ (*ii*) $x = 10, y = -1$ (*iii*) $x = \dfrac{3}{2}, y = 3$ (*iv*) $x = 3, y = -2$

3. (*i*) $-i, -i$ (*ii*) i, i (*iii*) $3 + i, \dfrac{3}{10} + \dfrac{1}{10}i$ (*iv*) $-i - 3, \dfrac{-3}{10} - \dfrac{1}{10}i$

(*v*) $-3i - 1, -\dfrac{1}{10} - \dfrac{3}{10}i$ **4.** (*i*) $-40 - 42i$ (*ii*) $\dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$ (*iii*) $\dfrac{291}{2} - 97i$

5. (*i*) $x = -2, y = 2$ (*ii*) $x = \dfrac{5}{7}, y = \dfrac{17}{7}$ (*iii*) $x = \dfrac{3}{2}, y = \dfrac{1}{2}$ (*iv*) $x = \dfrac{3}{25}, y = \dfrac{4}{25}$

(*v*) $x = -\dfrac{2}{21}, y = \dfrac{8}{21}$ (*vi*) $x = 2, y = 3; x = -2, y = \dfrac{1}{3}$

6. $11 - 60i$ **7.** (*i*) $2 - 3i$ (*ii*) $-3 + 4i$

8. (*i*) $\frac{1}{4} - \frac{1}{4}i$ (*ii*) $-\frac{1}{7} + 0i$ (*iii*) Does not exist (*iv*) $-\frac{1}{16} + 0i$

(*v*) $1 - i$ (*vi*) $0 - \frac{1}{2}i$ (*vii*) $-\frac{8}{25} - \frac{31}{25}i$ (*viii*) $\frac{11}{3721} - \frac{60}{3721}i$

(*ix*) $\frac{-5-i}{13}$

9. (*i*) $\frac{1-i}{2}$ (*ii*) $\frac{-7}{15}$ (*iii*) $-152 + 24i$ (*iv*) $-40 - 42i$

(*v*) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (*vi*) -2 (*vii*) i (*viii*) $-\frac{3}{2}i$

12. (*i*) $\frac{3}{25} - \frac{4}{25}i$ (*ii*) $\frac{22}{17} + \frac{3}{17}i$ (*iii*) $\frac{-5+2\sqrt{3}}{4} + \frac{-5\sqrt{3}-2}{4}i$

(*iv*) $\frac{27}{37} - \frac{23}{37}i$ (*v*) $-(142\sqrt{5} + 45) + 210i$

15. (*i*) $2 + i$ (*ii*) $2 - 4i$ (*iii*) $\frac{-1}{4} - \frac{1}{4}i$ (*iv*) $3 + 4i$

(*v*) $0 + 1i$ (*vi*) $\frac{3}{10} - \frac{1}{10}i$ (*vii*) $\frac{35}{53} - \frac{10}{53}i$ **17.** $a = 3, k = 17$

20. $\frac{3}{34} - \frac{29}{34}i$ **22.** (*i*) $-\frac{2}{5}$ (*ii*) 0 **24.** 6

HINTS TO SELECTED QUESTIONS

9. (*vii*) and (*viii*). Rationalise the denominator.

14. $[4 + 3\sqrt{-20}]^{\frac{1}{2}} = [9 + (-5) + 2(3)\sqrt{-5}]^{\frac{1}{2}} = [(3 + \sqrt{-5})^2]^{\frac{1}{2}} = 3 + \sqrt{-5}$

Similarly, $[4 - 3\sqrt{-20}]^{\frac{1}{2}} = 3 - \sqrt{-5}$. Now add.

24. $x = -2 - \sqrt{3}i \Rightarrow x + 2 = -\sqrt{3}i$

$\Rightarrow \quad (x+2)^2 = (-\sqrt{3}i)^2 = -3 \quad \Rightarrow \quad x^2 + 4x + 7 = 0$ ...(1)

Now $2x^4 + 5x^3 + 7x^2 - x + 41 = 2x^2(x^2 + 4x + 7) - 3x^3 - 7x^2 - x + 41$

$= 2x^2(x^2 + 4x + 7) - 3x(x^2 + 4x + 7) + 5x^2 + 20x + 41$

$= 2x^2(x^2 + 4x + 7) - 3x(x^2 + 4x + 7) + 5(x^2 + 4x + 7) + 6$

$= 2x^2 \times 0 - 3x \times 0 + 5 \times 0 + 6 = 6$. Using (1)

25. First find z^2. Then $z^3 = z.\ z^2$ and finally $z^4 = z^3.z$.

9.12. Modulus or absolute value of a complex number

The absolute value or modulus of any complex number z such that $z = a + bi$, where $a, b \in R$, is denoted by $|z|$ or $|a + bi|$ and is given by

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

Thus, the modulus $|a + bi|$ is just the distance from the origin to the point $a + bi$.

For example, $|2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

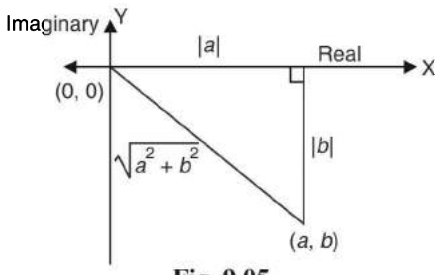


Fig. 9.05

$|-5+12i| = \sqrt{(-5)^2+12^2} = \sqrt{169} = 13$, $|-8-17i| = \sqrt{(-8)^2+(-17)^2} = \sqrt{353}$

$|-3i| = |0-3i| = \sqrt{(0)^2+(-3)^2} = 3$, $|-7| = |-7+0i| = \sqrt{(-7)^2+0^2} = 7$

$|\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$.

Theorem. ***If z_1 and z_2 are complex numbers of two points A and B on the Argand plane then the distance between them is given by***

$$AB = |z_1 - z_2| \ .$$

Proof. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

Let these numbers on the Argand diagram be represented by the points $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively. Then

$$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Also, $z_2 - z_1 = (x_2+y_2 i) - (x_1+y_1 i) = (x_2-x_1) + (y_2-y_1)i$

$\therefore \quad |z_2 - z_1| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = AB$

Ex. 23. Find the modulus of the following complex numbers.

(*i*) $8 - 6i^7$ **(*ii*)** $\dfrac{2+3i}{3+2i}$ **(*iii*)** $\dfrac{(3+2i)^2}{(4-3i)}$

(*iv*) $(3 + 2i)\,(5 - 4i)$ **(*v*)** $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$

Sol. (*i*) $8 - 6i^7 = 8 - 6i^4.i^3 = 8 - 6i^3 = 8 - 6i^2.i = 8 - 6(-1)i = 8 + 6i$.

$\therefore \quad |8-6i^7| = |8+6i| = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = \mathbf{10}$.

(*ii*) $\dfrac{2+3i}{3+2i} = \dfrac{2+3i}{3+2i} \times \dfrac{3-2i}{3-2i} = \dfrac{6+5i-6i^2}{9-4i^2} = \dfrac{12+5i}{13}$ (Putting $i^2 = -1$)

$$= \frac{12}{13} + \frac{5}{13}i$$

$\therefore \quad \left|\dfrac{2+3i}{3+2i}\right| = \left|\dfrac{12}{13} + \dfrac{5}{13}i\right| = \sqrt{\left(\dfrac{12}{13}\right)^2 + \left(\dfrac{5}{13}\right)^2}$

$$= \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = \sqrt{1} = \mathbf{1}.$$

(*iii*) $\dfrac{(3+2i)^2}{4-3i} = \dfrac{9+12i+4i^2}{4-3i} = \dfrac{5+12i}{4-3i} = \dfrac{5+12i}{4-3i} \times \dfrac{4+3i}{4+3i}$

$$= \frac{20+63i+36i^2}{16-9i^2} = \frac{20+63i-36}{16+9} = \frac{-16+63i}{25} = \frac{-16}{25} + \frac{63}{25}i$$

$\therefore \quad \left|\dfrac{(3+2i)^2}{4-3i}\right| = \left|\dfrac{-16}{25} + \dfrac{63}{25}i\right| = \sqrt{\left(\dfrac{-16}{25}\right)^2 + \left(\dfrac{63}{25}\right)^2}$

$$= \sqrt{\frac{256}{625} + \frac{3969}{625}} = \sqrt{\frac{4225}{625}} = \frac{65}{25} = \mathbf{\frac{13}{5}}.$$

(iv) $(3 + 2i)(5 - 4i) = 15 - 2i - 8i^2 = 15 - 2i + 8 = 23 - 2i$

$\therefore \quad |(3+2i)(5-4i)| = |23-2i| = \sqrt{(23)^2+(-2)^2} = \sqrt{529+4} = \sqrt{533}$.

(v) $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i} = \dfrac{(1+i)^2-(1-i)^2}{(1-i)(1+i)} = \dfrac{4i}{1-i^2} = \dfrac{4i}{1+1} = \dfrac{4i}{2} = 2i$

$\therefore \quad \left|\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}\right| = |0+2i| = \sqrt{0^2+2^2} = 2.$

Ex. 24. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, then is it necessary that $z_1 = z_2$.

Sol. Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$.

Then, $|z_1| = \sqrt{a_1^2 + b_1^2}$ and $|z_2| = \sqrt{a_2^2 + b_2^2}$

$\because \quad |z_1| = |z_2| \quad \Rightarrow \quad \sqrt{a_1^2 + b_1^2} = \sqrt{a_2^2 + b_2^2}$

$\Rightarrow \quad a_1^2 + b_1^2 = a_2^2 + b_2^2$ …(1)

(1) does not imply that $a_1 = a_2$ and $b_1 = b_2$.

Hence, z_1 and z_2 need not be equal.

Ex. 25. Show that the points representing the complex numbers $(3 + 3i)$, $(-3 - 3i)$ and $(-3\sqrt{3} + 3\sqrt{3}i)$ on the Argand plane are the vertices of an equilateral triangle.

Sol. Let the given numbers be represented on the Argand plane by the point A, B and C respectively.

Then, $AB = |(3+3i) - (-3-3i)| = |6+6i| = \sqrt{(6)^2+(6)^2} = \sqrt{72} = 6\sqrt{2} \quad (d = |z_1 - z_2|)$

$BC = |(-3-3i) - (-3\sqrt{3}+3\sqrt{3}i)| = |3(\sqrt{3}-1) - 3(1+\sqrt{3})i| = \sqrt{9(\sqrt{3}-1)^2 + 9(1+\sqrt{3})^2}$

$= \sqrt{9\left[(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2\right]} = \sqrt{9(3+1-2\sqrt{3}+3+1+2\sqrt{3})}$

$= \sqrt{9 \times 8} = \sqrt{72} = 6\sqrt{2}$

$AC = |(3+3i) - (-3\sqrt{3}+3\sqrt{3}i)| = |3(1+\sqrt{3}) + 3(1-\sqrt{3})i| = \sqrt{9(1+\sqrt{3})^2 + 9(1-\sqrt{3})^2}$

$= \sqrt{9\left[(1+\sqrt{3})^2 + (1-\sqrt{3})^2\right]} = \sqrt{9 \times 8} = \sqrt{72} = 6\sqrt{2}$

Thus, $AB = BC = AC$. Hence, the given points on the Argand plane are the vertices of an equilateral triangle.

Ex. 26. If $|z| = 1$, then prove that $\dfrac{z-1}{z+1}$ $(z \neq -1)$ is a purely imaginary number. What is the conclusion if $z = 1$? ***(ISC 1996)***

Sol. Let $z = x + yi$. Then $|z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1 \Rightarrow x^2 + y^2 = 1$

$$\frac{z-1}{z+1} = \frac{x+yi-1}{x+yi+1} = \frac{(x-1)+yi}{(x+1)+yi} \times \frac{(x+1)-yi}{(x+1)-yi}$$

$$= \frac{(x^2-1)+y^2+[y(x+1)-y(x-1)]i}{(x+1)^2-(yi)^2} = \frac{(x^2+y^2-1)+2yi}{(x+1)^2+y^2}$$

$$= \frac{0+2yi}{(x+1)^2+y^2} = \frac{2yi}{(x+1)^2+y^2}, \qquad [\because x^2+y^2=1]$$

which is purely an imaginary number.

If $z = 1$, then $x + yi = 1 + 0i$

$\Rightarrow x = 1$ and $y = 0$

$\therefore \quad \frac{z-1}{z+1} = \left(\frac{x+yi-1}{x+yi+1}\right) = \frac{1+0i-1}{1+0i+1} = 0$, which is a real number.

Ex. 27. Solve the equation $2z = |z| + 2i$ is complex numbers. ***(ISC 1999)***

Sol. Let $z = x + yi$. ...(1)

Then $2z = |z| + 2i$ (given)

$\Rightarrow \quad 2(x + yi) = \sqrt{x^2+y^2} + 2i$

$\Rightarrow \quad 2x = \sqrt{x^2+y^2}$ and $2y = 2$ [Equating the real parts and imaginary parts]

$\Rightarrow \quad y = 1$ and $4x^2 = x^2 + y^2 \Rightarrow 3x^2 = y^2 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{3}}$, rejecting the negative value.

Substituting in (1), we get $z = \frac{1}{\sqrt{3}} + i$.

Note. We rejected negative value because then L.H.S. will have real part negative and R.H.S. will have real part positive.

Ex. 28. If $z = x + yi$ and $\frac{|z-1-i|+4}{3|z-1-i|-2} = 1$, show that $x^2 + y^2 - 2x - 2y - 7 = 0$. ***(ISC 2000)***

Sol. $\frac{|z-1-i|+4}{3|z-1-i|-2} = 1 \quad \Rightarrow |z-1-i|+4 = 3|z-1-i|-2$

$\Rightarrow \quad 2|z-1-i| = 6 \quad \Rightarrow |z-1-i| = 3$

$\Rightarrow \quad |x+yi-1-i| = 3 \quad \Rightarrow |(x-1)+(y-1)i| = 3$

$\Rightarrow \quad \sqrt{(x-1)^2+(y-1)^2} = 3 \Rightarrow (x-1)^2 + (y-1)^2 = 9$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y - 7 = 0.$

Ex. 29. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. ***(ISC 1997)***

Sol. Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$.

$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$;

$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$

L.H.S. $= |z_1 + z_2|^2 + |z_1 - z_2|^2 = [(x_1 + x_2)^2 + (y_1 + y_2)^2] + |(x_1 - x_2)^2 + (y_1 - y_2)^2|$

$[\because |x + yi|^2 = x^2 + y^2]$

$= 2(x_1^2 + y_1^2) + 2(x_2^2 + y_2^2) = 2|z_1|^2 + 2|z_2|^2 =$ R.H.S.

Ex. 30. For any two complex numbers z_1 and z_2 and any real numbers a and b, prove that

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)[|z_1|^2 + |z_2|^2] \qquad \textit{(ISC 1998)}$$

Sol. Let $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i$

$$\text{L.H.S.} = \left|a(x_1 + y_1 i) - b(x_2 + y_2 i)\right|^2 + \left|b(x_1 + y_1 i) + a(x_2 + y_2 i)\right|^2$$

$$= \left|(ax_1 - bx_2) + (ay_1 - by_2)i\right|^2 + \left|(bx_1 + ax_2) + (by_1 + ay_2)i\right|^2$$

$$= [(ax_1 - bx_2)^2 + (ay_1 - by_2)^2] + [(bx_1 + ax_2)^2 + (by_1 + ay_2)^2]$$

$$[\because |x + yi|^2 = x^2 + y^2]$$

$$= [(a^2x_1^2 + b^2x_2^2 - 2ab\, x_1x_2 + a^2y_1^2 + b^2y_2^2 - 2aby_1y_2]$$

$$+ [(b^2x_1^2 + a^2x_2^2 + 2ab\, x_1x_2 + b^2y_1^2 + a^2y_2^2 + 2aby_1y_2]$$

$$= (a^2x_1^2 + b^2x_2^2 + a^2y_1^2 + b^2y_2^2) + (b^2x_1^2 + a^2x_2^2 + b^2y_1^2 + a^2y_2^2)$$

$$= (a^2 + b^2)(x_1^2 + x_2^2) + (a^2 + b^2)(y_1^2 + y_2^2)$$

$$= (a^2 + b^2)[(x_1^2 + y_1^2) + (x_2^2 + y_2^2)] = (a^2 + b^2)[|z_1|^2 + |z_2|^2]$$

= R.H.S.

Collinear points

Method: ***Use the fact that three points P, Q, R are collinear if PQ + QR = PR.***

Ex. 31. Prove that the representative points of the complex numbers

$1 + 4i$, $2 + 7i$, $3 + 10i$ are collinear.

Sol. Let P, Q, R represent the complex numbers $1 + 4i$, $2 + 7i$, $3 + 10i$ respectively. Then

$$PQ = \left|(1 + 4i) - (2 + 7i)\right| = \left|-1 - 3i\right| \qquad [d = |z_1 - z_2|]$$

$$= \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$$

$$QR = \left|(2 + 7i) - (3 + 10i)\right| = \left|-1 - 3i\right| = \sqrt{10}$$

$$PR = \left|(1 + 4i) - (3 + 10i)\right| = \left|-2 - 6i\right|$$

$$= \sqrt{(-2)^2 + (-6)^2} = 2\sqrt{10}$$

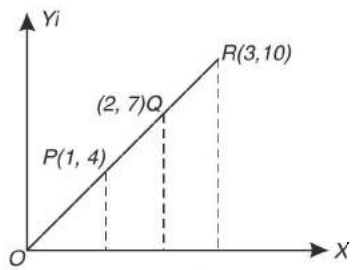


Fig. 9.06

$\because PQ + QR = PR \Rightarrow P, Q, R$ are collinear.

Ex. 32. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, zi and $z + zi$ is $\frac{1}{2}|z|^2$. ***(UPSEAT 2002, AMU 2000, MP PET 1997, IIT)***

Sol. Let $z = x + yi$. Then

$zi = (x + yi)i = -y + xi$ and $z + zi = (x + yi) + (x + yi)i = x + yi + xi + yi^2$

$= x + yi + xi - y = (x - y) + (x + y)i$

$\therefore$ The coordinates of the points representing the given complex numbers are (x, y), $(-y, x)$ and $(x - y, x + y)$.

$\therefore$ Area of the required triangle $= \frac{1}{2}[x\{x - (x + y)\} - y\{x + y - y\} + (x - y)(y - x)]$

$$\left[\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right]$$

$$= -\frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2, \text{ neglecting the –ve sign.}$$

Ex. 33. If z_1, z_2, z_3 are three complex numbers such that there exists a complex number z with $|z_1 - z| = |z_2 - z| = |z_3 - z|$, show that z_1, z_2, z_3 lie on a circle in the Argand diagram.

Sol. Let the numbers z_1, z_2, z_3 and z be represented on the Argand diagram by the points A, B, C and P respectively. Then,

$$PA = |z_1 - z|,\ PB = |z_2 - z|,\ \text{and}\ PC = |z_3 - z|$$

$$\therefore \quad |z_1 - z| = |z_2 - z| = |z_3 - z| \Rightarrow PA = PB = PC.$$

This shows that A, B, C lie on a circle whose centre is P and whose radius is $PA = PB = PC$.

Hence, the complex numbers z_1, z_2, z_3 lie on a circle in the Argand diagram.

Ex. 34. The complex numbers z_1, z_2 and the origin are the vertices of an equilateral triangle in the Argand plane.

Prove that $z_1^2 + z_2^2 = z_1 . z_2$. ***(ISC 1995)***

Sol. Let ABC be an equilateral triangle; let A be the origin and AB the x-axis. Let $AB = a = BC = CA$, and let CD be perpendicular to AB. A, B, C represent the complex numbers $0, z_1, z_2$, respectively.

$$CD = a\sin 60° = \frac{a\sqrt{3}}{2};$$

A is $(0, 0)$, B is $(a, 0)$ and C is $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

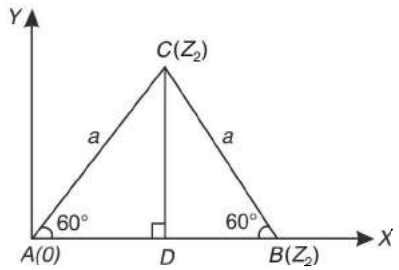


Fig. 9.07

$$\therefore \quad z_1 = a + 0i = a;\ z_2 = \frac{a}{2} + \frac{a\sqrt{3}}{2}i$$

$$z_1^2 + z_2^2 - z_1 z_2 = a^2 + \left(\frac{a}{2} + \frac{a\sqrt{3}}{2}i\right)^2 - a\left(\frac{a}{2} + \frac{a\sqrt{3}}{2}i\right)$$

$$= a^2 + \frac{a^2}{4} + \frac{a^2\sqrt{3}}{2}i - \frac{3}{4}a^2 - \frac{a^2}{2} - \frac{a^2\sqrt{3}}{2}i = 0$$

$$\Rightarrow \quad z_1^2 + z_2^2 = z_1 z_2$$

Ex. 35. (*i*) If z_1 and z_2 are complex numbers of two points, then prove that the complex number of the point, which divides the distance between them internally in the ratio $l : m$ is given by $\frac{lz_2 + mz_1}{l + m}$.

(*ii*) If z_1, z_2, z_3 are three complex numbers representing three vertices of a triangle, then centroid of the triangle be $\frac{z_1 + z_2 + z_3}{3}$.

Sol. (*i*) Let $z_1 = x_1 + y_1 i = A(x_1, y_1)$; $z_2 = x_2 + y_2 i = B(x_2, y_2)$.

Let $z_3 = x_3 + iy_3 = C(x_3, y_3)$, which divides the segment AB in the ratio $l : m$ is given by

$$C\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right).$$

Thus, C is represented by the complex number

$$z_3 = \frac{lx_2 + mx_1}{l + m} + i\frac{ly_2 + my_1}{l + m}$$

$$= \frac{l(x_2 + iy_2) + m(x_1 + iy_1)}{l + m} = \frac{lz_2 + mz_1}{l + m}$$

Hence, $z_3 = \dfrac{lz_2 + mz_1}{l+m}$.

Cor. If $l = 1 = m$, then the mid-point of z_1 and z_2 is given by $\dfrac{z_1 + z_2}{2}$.

(ii) We know that the centroid G divides the median CD in the ratio 2 : 1,

i.e., $CG : GD = 2 : 1$.

The mid-point D of AB is given by

$\dfrac{z_1 + z_2}{2}$ [Cor., Part (*a*)]

The point G is given by

$\dfrac{2\left(\dfrac{z_1 + z_2}{2}\right) + 1 \times z_3}{2+1}$, *i.e.* $\dfrac{z_1 + z_2 + z_3}{3}$

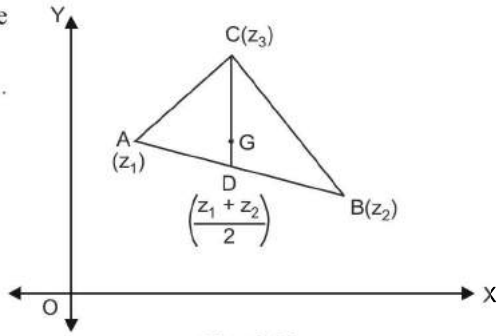


Fig. 9.08

Hence, the centroid of the triangle is given by $\dfrac{z_1 + z_2 + z_3}{3}$.

Cor. If we had taken the medians passing through A and B we would have obtained the same complex number $\dfrac{z_1 + z_2 + z_3}{3}$ for G_1 and G_2 also. In other words, G, G_1, G_2 will coincide. This proves that the three medians will be concurrent.

Ex. 36. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle, and $|z_1| = |z_2| = |z_3|$, prove that $z_1 + z_2 + z_3 = 0$. ***(WB JEE 2003, IIT)***

Sol. Since $|z_1| = |z_2| = |z_3|$, therefore, the origin is the centre of the circle circumscribing the equilateral triangle.

Since centre is also the centroid of the equilateral triangle, therefore, $\dfrac{z_1 + z_2 + z_3}{3} = 0$

$\Rightarrow \quad z_1 + z_2 + z_3 = 0$.

Ex. 37. If z_1, z_2, z_3, z_4 are complex numbers, show that they are vertices of a parallelogram in the Argand diagram if and only if $z_1 + z_3 = z_2 + z_4$. ***(IIT)***

Sol. Let the complex numbers z_1, z_2, z_3 and z_4 be represented on the Argand plane by the points A, B, C and D respectively.

If $ABCD$ is a parallelogram, then the diagonal AC and BD would bisect each other at point O, *i.e.*, the coordinate of the middle point of AC and BD will be the same, *i.e.*,

$\dfrac{1}{2}(z_1 + z_3) = \dfrac{1}{2}(z_2 + z_4) \quad \Rightarrow z_1 + z_3 = z_2 + z_4$.

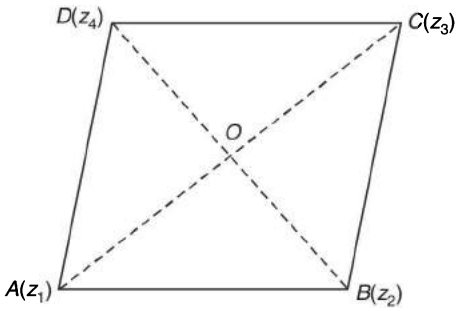


Fig. 9.09

Conversely. Let z_1, z_2, z_3, z_4 be the complex numbers represented by the points A, B, C, D respectively on the Argand plane and let $z_1 + z_3 = z_1 + z_4$. Then

$z_1 + z_3 = z_2 + z_4 \Rightarrow z_2 - z_1 = z_3 - z_4$ and $z_2 - z_3 = z_1 - z_4$

$\Rightarrow \quad |z_2 - z_1| = |z_3 - z_4|$ and $|z_2 - z_3| = |z_1 - z_4|$

$\Rightarrow \quad AB = CD$ and $BC = AD$

$\Rightarrow \quad ABCD$ is a parallelogram.

9.13. Properties of moduli

If z, z_1, z_2 are complex numbers, then

Property 1. ***The modulus of a complex number is zero if and only if the complex number is zero, i.e.*** $|z| = 0 \Leftrightarrow z = 0$, ***i.e., Re(z) = Im(z) = 0.***

Proof. Let $z = a + bi$, then $|z| = 0 \Leftrightarrow |a + bi| = 0$

$\Leftrightarrow \sqrt{a^2 + b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow \text{Re}(z) = 0$ and $\text{Im}(z) = 0 \Leftrightarrow z = 0$.

Property 2. $z\bar{z} = |z|^2$.

Proof. Let $z = a + bi$, Then $\bar{z} = a - bi$

$\therefore \quad z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2 = |z|^2$.

Property 3. ***The modulus of a complex number and that of its conjugate are equal, i.e.*** $|z| = |\bar{z}|$.

Proof. Then Let $z = a + bi$. Then $\bar{z} = a - bi$

$\therefore \quad |z| = |a + bi| = \sqrt{a^2 + b^2}$, $|\bar{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Hence, $|z| = |\bar{z}|$.

Property 4. (*i*) $-|z| \le \text{Re}(z) \le |z|$ **(*ii*)** $-|z| \le \text{Im}(z) \le |z|$

Proof. Let $z = a + bi \Rightarrow |z|^2 = a^2 + b^2$. Now

(*i*) $a^2 \le a^2 + b^2 \quad (\because b^2 \ge 0) \Rightarrow |z|^2 \ge a^2 \Rightarrow |z| \ge \pm a$

$\Rightarrow |z| \ge a$ or $|z| \ge -a \Rightarrow |z| \ge a$ or $-|z| \le a$

$\Rightarrow -|z| \le a \le |z|$, *i.e.*, $-|z| \le \text{Re}(z) \le |z|$.

(*ii*) $b^2 \le a^2 + b^2 \quad (\because a^2 \ge 0) \Rightarrow b^2 \le |z|^2$

$\Rightarrow -|z| \le b \le |z|$, *i.e.* $-|z| \le \text{Im}(z) \le |z|$.

Property 5. $z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

Proof. We have $z\bar{z} = |z|^2$ (Property 2)

Dividing by $|z|^2$, we get $\dfrac{z\bar{z}}{|z|^2} = 1$

Dividing by z, we get $\dfrac{\bar{z}}{|z|^2} = \dfrac{1}{z} \Rightarrow z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

Property 6. ***The modulus of the product of two complex numbers is equal to the product of their moduli, i.e.*** $|z_1 z_2| = |z_1| \cdot |z_2|$.

Proof. By property 2 above,

$$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} \qquad \left[|z|^2 = z\bar{z}\right]$$

$$= (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 \quad \text{(Using Property 2, } i.e.\ z\bar{z} = |z|^2\text{)}$$

$$\Rightarrow \qquad |z_1 z_2| = |z_1||z_2|$$

In general, $|z_1.z_2....z_n| = |z_1||z_2|...|z_n|$, *i.e.*, the modulus of the product of n complex numbers is the product of their moduli.

Cor. If $z_1 = z_2 = z$, then $|z^2| = |z|.|z| = |z|^2$

i.e., **Modulus of the square of a complex number**

= square of the modulus of that complex number

Similarly, if $z_1 = z_2 = \ldots = z_n$, then we have $|z^n| = |z|^n$, $n \in N$.

Remember
$|z^2| = |z|^2$

Property 7. ***The moduli of the quotient of two complex numbers is the quotient of their moduli, i.e.*** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

Proof. By property 2, we have $\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{z_1}{z_2}\right)\left(\frac{\overline{z}_1}{\overline{z}_2}\right) = \left(\frac{z_1\overline{z}_1}{z_2\overline{z}_2}\right) = \frac{|z_1|^2}{|z_2|^2}$

$\Rightarrow$ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

Property 8. $|z_1 + z_2| \le |z_1| + |z_2|$. $\le$ **(*Triangle inequality*)** **(*ISC 2001*)**

Proof. $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$

$= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2$

$= z_1\overline{z}_1 + z_2\overline{z}_2 + (z_1\overline{z}_2 + z_2\overline{z}_1)$

$= |z_1|^2 + |z_2|^2 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2})$ [Since $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$ and $\overline{(\overline{z})} = z$]

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2)$ [Since $z + \overline{z} = 2\,\mathrm{Re}(z)$]

$\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z}_2|$

[Since $a \le \sqrt{a^2 + b^2}$, therefore, the real part of a complex number $a + bi$ can never exceed its absolute value, *i.e.* $\mathrm{Re}(z) \le |z|$]

$\Rightarrow$ $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z}_2|$

$\Rightarrow$ $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$ $(\because |\overline{z}| = |z|)$

$\Rightarrow$ $|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$

Taking positive square roots, we get

$$|z_1 + z_2| \le |z_1| + |z_2| \ .$$

Note. The above inequality can be extended to n complex numbers (by induction), *i.e.*, for any n complex numbers $z_1, z_2, \ldots, z_n$, we have

$$|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n| \ .$$

Alternatively,

Let $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i$; then, $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$

$|z_1| = \sqrt{x_1^2 + y_1^2}$; $|z_2| = \sqrt{x_2^2 + y_2^2}$; $|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$

We have to prove that $|z_1 + z_2| \le |z_1| + |z_2|$

i.e. $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

or $(x_1 + x_2)^2 + (y_1 + y_2)^2 \le (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$

or $2x_1x_2 + 2y_1y_2 \le 2\sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$

or, $(x_1x_2 + y_1y_2)^2 \le (x_1^2 + y_1^2)(x_2^2 + y_2^2)$

or $x_1^2y_2^2 + y_1^2x_2^2 - 2x_1x_2y_1y_2 \ge 0$

or $(x_1y_2 - y_1x_2)^2 \ge 0$, which is true.

Hence, $|z_1 + z_2| \le |z_1| + |z_2|$.

Cor $|z_1 - z_2| \le |z_1| + |z_2|$ **(*ISC 2001*)**

Proof. $|z_1 - z_2| = |z_1 + (-z_2)| \Rightarrow |z_1 - z_2| \le |z_1| + |-z_2|$ (Property 8)

$\Rightarrow |z_1 - z_2| \le |z_1| + |z_2|$ $(\because |-z| = |z|)$

Property 9. $|z_1 - z_2| \ge ||z_1| - |z_2||$, *i.e.* $|z_1| \sim |z_2|$ **(*ISC 1993*)**

Proof. Let $|z_1| > |z_2|$. Now, $z_1 = (z_1 - z_2) + z_2$

$\Rightarrow |z_1| = |(z_1 - z_2) + z_2| \Rightarrow |z_1| \le |z_1 - z_2| + |z_2|$ (Property 8)

$\Rightarrow |z_1| - |z_2| \le |z_1 - z_2| \Rightarrow |z_1 - z_2| \ge |z_1| - |z_2|$ …(1)

If $|z_2| > |z_1|$, then $|z_2| = |(z_2 - z_1) + z_1|$

$\Rightarrow |z_2| \le |z_2 - z_1| + |z_1|$ (Property 8)

$\Rightarrow |z_2| - |z_1| \le |z_2 - z_1| \Rightarrow |z_2| - |z_1| \le |z_1 - z_2|$

$\Rightarrow |z_1 - z_2| \ge |z_2| - |z_1|$ …(2)

From (1) and (2), we get

$|z_1 - z_2| \ge ||z_1| - |z_2||$

$\Rightarrow |z_1 - z_2| \ge |z_1| \sim |z_2|$.

Cor. $|z_1 + z_2| \ge |z_1| - |z_2|$.

Proof. $|z_1 + z_2| = |z_1 - (-z_2)|$

$\Rightarrow |z_1 + z_2| \ge |z_1| - |-z_2|$ (Property 9)

$\Rightarrow |z_1 + z_2| \ge |z_1| - |z_2|$ $(\because |-z_2| = |z_2|)$

9.14. Geometrical interpretation of $|z_1 + z_2| \le |z_1| + |z_2|$.

Let the two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be represented by the points P and Q repectively in the Argand diagram in the complex plane as shown in Fig. 9.10. Join the origin O to the point P and Q and complete the parallelogram $OPRQ$. Draw perpendiculars as shown. Then, $\Delta ONQ \cong \Delta PKR$ (AAS)

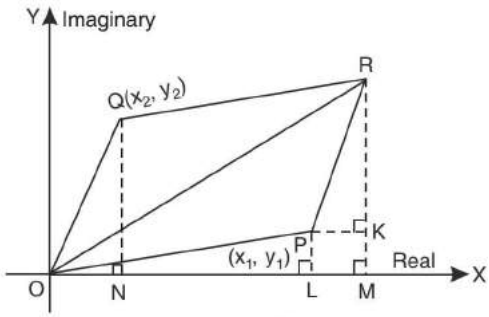


Fig. 9.10

$\Rightarrow$ $ON = PK = LM = x_2$ and $NQ = KR = y_2$

$\Rightarrow$ $OM = OL + LM = x_1 + x_2$ and $MR = MK + KR = y_1 + y_2$

$\Rightarrow$ The coordinates of R are $(x_1 + x_2, y_1 + y_2)$

Hence, the point R represent the complex number $(x_1 + x_2) + (y_1 + y_2)i = z_1 + z_2$.

Now, from the triangle OPR, we know that $OR < OP + PR$, (***In Δ, sum of two sides > third side***) therefore $|z_1 + z_2| < |z_1| + |z_2|$ ...(1)

Also, if O, P, R are collinear, then $OR = OP + PR$, therefore $|z_1 + z_2| = |z_1| + |z_2|$ ...(2)

From (1) and (2) it follows that $|z_1 + z_2| \le |z_1| + |z_2|$. (***Triangle Inequality***)

Ex. 38. If $z = 4 + 3i$, then verify that

(*i*) $|z| = |\bar{z}|$ **(*ii*)** $-|z| < \text{Re}(z) < |z|$ **(*iii*)** $-|z| < \text{Im}(z) \le |z|$ **(*iv*)** $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

Sol. $z = 4 + 3i \Rightarrow \bar{z} = 4 - 3i$

(*i*) $|z| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$; $|\bar{z}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$ $\therefore$ $|z| = |\bar{z}|$.

(*ii*) $-|z| = -5$, $\text{Re}(z) = 4$, $|z| = 5$

Obviously, $-5 < 4 < 5$. Hence, $-|z| < \text{Re}(z) < |z|$.

(*iii*) $-|z| = -5$, $\text{Im}(z) = 3$, $|z| = 5$

Obviously, $-5 < 3 < 5$. Hence, $-|z| < \text{Im}(z) < |z|$.

(*iv*) $z^{-1} = \dfrac{1}{z} = \dfrac{1}{4+3i} = \dfrac{1}{4+3i} \times \dfrac{4-3i}{4-3i} = \dfrac{4-3i}{4^2 - 3^2 i^2} = \dfrac{4-3i}{16+9} = \dfrac{4-3i}{25}$ ...(1)

$\bar{z} = 4 - 3i$, $|z|^2 = 4^2 + 3^2 = 25$ $\therefore$ $\dfrac{\bar{z}}{|z|^2} = \dfrac{4-3i}{25}$ ...(2)

(1) and (2) $\Rightarrow$ $z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

Ex. 39. If $z_1 = 2 + 7i$ and $z_2 = 1 - 5i$, then verify that

(*i*) $|z_1 z_2| = |z_1||z_2|$ **(*ii*)** $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(*iii*) $|z_1 + z_2| \le |z_1| + |z_2|$ **(*iv*)** $|z_1 - z_2| > |z_1| - |z_2|$

Sol. (*i*) $|z_1| = \sqrt{2^2 + 7^2} = \sqrt{53}$, $|z_2| = \sqrt{1^2 + (-5)^2}$ $\sqrt{26}$

$\therefore \quad |z_1||z_2| = \sqrt{53}\sqrt{26} = \sqrt{53 \times 26} = \sqrt{1378}$ ...(1)

$z_1 z_2 = (2 + 7i)(1 - 5i) = 2 - 3i - 35i^2 = 2 - 3i + 35 = 37 - 3i$

$\therefore \quad |z_1 z_2| = \sqrt{(37)^2 + (-3)^2} = \sqrt{1369 + 9} = \sqrt{1378}$ ...(2)

(1) and (2) $\Rightarrow |z_1 z_2| = |z_1||z_2|$.

(*ii*) $\dfrac{z_1}{z_2} = \dfrac{2+7i}{1-5i} = \dfrac{2+7i}{1-5i} \times \dfrac{1+5i}{1+5i} = \dfrac{2+17i+35i^2}{1-5^2 i^2} = \dfrac{2+17i-35}{1+25} = \dfrac{-33+17i}{26} = \dfrac{-33}{26} + \dfrac{17}{26}i$

$\therefore \quad \left|\dfrac{z_1}{z_2}\right| = \sqrt{\left(\dfrac{-33}{26}\right)^2 + \left(\dfrac{17}{26}\right)^2} = \sqrt{\dfrac{1089}{676} + \dfrac{289}{676}} = \sqrt{\dfrac{1378}{676}} = \sqrt{\dfrac{53}{26}}$ ...(1)

$|z_1| = \sqrt{2^2 + 7^2} = \sqrt{53}, \quad |z_2| = \sqrt{1^2 + (-5)^2} = \sqrt{26} \Rightarrow \dfrac{|z_1|}{|z_2|} = \sqrt{\dfrac{53}{26}}$ ...(2)

(1) and (2) $\Rightarrow \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

(*iii*) $z_1 + z_2 = 2 + 7i + 1 - 5i = 3 + 2i \qquad \therefore |z_1 + z_2| = \sqrt{3^2 + 2^2} = \sqrt{13}$ ...(1)

$|z_1| = \sqrt{2^2 + 7^2} = \sqrt{53}, \quad |z_2| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

$\sqrt{13} = 3.61$ (to 2 d.p.), $\sqrt{53} = 7.28$, $\sqrt{26} = 5.10$

Obviously, $\sqrt{13} < \sqrt{53} + \sqrt{26}$. Hence, $|z_1 + z_2| < |z_1| + |z_2|$.

(*iv*) $z_1 - z_2 = (2 + 7i) - (1 - 5i) = 1 + 12i$

$\therefore \quad |z_1 - z_2| = \sqrt{1^2 + 12^2} = \sqrt{145} = 12.04$ (2 d.p)

$|z_1| - |z_2| = 7.28 - 5.10 = 2.18$ (found in *iii*)

Obviously, $12.04 > 2.18$. Hence, $|z_1 - z_2| > |z_1| - |z_2|$.

Ex. 40. If $z_1 = 2 + 3i$ and $z_2 = 3 + i$, plot the number $z_1 + z_2$. Also show that $|z_1| + |z_2| > |z_1 + z_2|$.

Sol. (*i*) We have, $z_1 = 2 + 3i$, $z_2 = 3 + i$

$\therefore \quad z_1 + z_2 = (2 + 3i) + (3 + i) = 5 + 4i$.

Now represent $z_1 = 2 + 3i$ by the point $A(2, 3)$, $z_2 = 3 + i$ by the point $B(3, 1)$ and $z_1 + z_2 = 5 + 4i$ by the point $C(5, 4)$.

(*ii*) $|z_1| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = OA$,

$|z_2| = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10} = OB$

and $|z_1 + z_2| = \sqrt{5^2 + 4^2}$

$= \sqrt{25 + 16} = \sqrt{41} = OC$

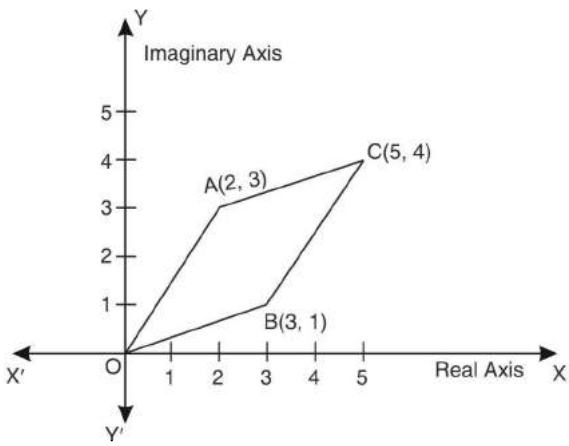


Fig. 9.11

Since $\sqrt{13} + \sqrt{10} > \sqrt{41}$ (You may verify it)

$\therefore \quad OA + OB > OC$. Hence, $|z_1| + |z_2| > |z_1 + z_2|$.

Ex. 41. Find the modulus of $\dfrac{(3+2i)(1+i)(2+3i)}{(3+4i)(4+5i)}$.

Sol. $\left|\dfrac{(3+2i)(1+i)(2+3i)}{(3+4i)(4+5i)}\right| = \dfrac{|(3+2i)(1+i)(2+3i)|}{|(3+4i)(4+5i)|}$ $\left(\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right)$

$= \dfrac{|3+2i|.|1+i|.|2+3i|}{|3+4i|.|4+5i|}$ $\left(\because |z_1 z_2| = |z_1|\cdot|z_2|\right)$

$= \dfrac{\sqrt{3^2+2^2}.\sqrt{1^2+1^2}.\sqrt{2^2+3^2}}{\sqrt{3^2+4^2}.\sqrt{4^2+5^2}} = \dfrac{\sqrt{13}.\sqrt{2}.\sqrt{13}}{\sqrt{25}.\sqrt{41}} = \mathbf{\dfrac{13\sqrt{2}}{5\sqrt{41}}}$.

Ex. 42. (*i*) If $x+yi = \dfrac{a+bi}{c+di}$, prove that $x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$. ***(ISC 1995)***

(*ii*) If $x-yi = \dfrac{a-bi}{c-di}$, prove that $x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

(*iii*) If $x+yi = \sqrt{\dfrac{a+bi}{c+di}}$, prove that $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$. ***(ISC, IIT)***

Sol. (*i*) $x+yi = \dfrac{a+bi}{c+di}$

$\Rightarrow \quad |x+yi| = \left|\dfrac{a+bi}{c+di}\right| = \dfrac{|a+bi|}{|c+di|}$ $\left[\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

$\Rightarrow \quad \sqrt{x^2+y^2} = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \Rightarrow x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

(*ii*) $x-yi = \dfrac{a-bi}{c-di}$

$\Rightarrow \quad |x-yi| = \left|\dfrac{a-bi}{c-di}\right| = \dfrac{|a-bi|}{|c-di|}$ $\left[\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

$\Rightarrow \quad \sqrt{x^2+y^2} = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \Rightarrow x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$

From (*i*) and (*ii*), it follows that

if $x+yi = \dfrac{a+bi}{c+di}$, then $x-yi = \dfrac{a-bi}{c-di}$.

Note. If we multiply these two relations, we get

$(x+yi)(x-yi) = \dfrac{(a+bi)(a-bi)}{(c+di)(c-di)} \Rightarrow x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

(*iii*) $x+yi = \sqrt{\dfrac{a+bi}{c+di}} \Rightarrow (x+yi)^2 = \dfrac{a+bi}{c+di}$

$\Rightarrow \quad \left|(x+yi)^2\right| = \left|\dfrac{a+bi}{c+di}\right|$

$\Rightarrow \quad |x+yi|^2 = \dfrac{|a+bi|}{|c+di|} \qquad \left[\because |z^2| = |z|^2 \text{ and } \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

$\Rightarrow \quad \left(\sqrt{x^2+y^2}\right)^2 = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \Rightarrow x^2+y^2 = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow \quad (x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}.$

Note. In the same way it can be proved that if $x - yi = \sqrt{\dfrac{a-bi}{c-di}}$, then $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

If follows that

if $x + iy = \sqrt{\dfrac{a+bi}{c+di}}$, then $x - yi = \sqrt{\dfrac{a-bi}{c-di}}$.

Alternative Method:

(*i*) If $x + yi = \dfrac{a+bi}{c+di}$, then to prove $x - yi = \dfrac{a-bi}{c-di}$.

$x + yi = \dfrac{a+bi}{c+di} \Rightarrow \overline{x+yi} = \overline{\left(\dfrac{a+bi}{c+di}\right)} = \dfrac{\overline{a+bi}}{\overline{c+di}} \qquad \left[\because \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}\right]$

$\Rightarrow \quad x - yi = \dfrac{a-bi}{c-di}$

(*ii*) If $x + yi = \sqrt{\dfrac{a+bi}{c+di}}$, then to prove $x - yi = \sqrt{\dfrac{a-bi}{c-di}}$

$x + yi = \sqrt{\dfrac{a+bi}{c+di}} \Rightarrow (x+yi)^2 = \dfrac{a+bi}{c+di}$

$\Rightarrow \quad \overline{(x+yi)^2} = \overline{\left(\dfrac{a+bi}{c+di}\right)} = \dfrac{\overline{a+bi}}{\overline{c+di}} = \dfrac{a-bi}{c-di} \qquad \left[\because \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}\right]$

$\Rightarrow \quad \overline{(x+yi)}^2 = \dfrac{a-bi}{c-di} \qquad [\because \ (\overline{z}^2) = (\overline{z})^2 \ ]$

$\Rightarrow \quad (x-yi)^2 = \dfrac{a-bi}{c-di}.$

Ex. 43. Given that $(1 + i)(1 + 2i)(1 + 3i)\ldots(1 + ni) = x + yi$, show that

$2.5.10\ldots(1 + n^2) = x^2 + y^2$. ***(ISC 1997)***

Sol. $(1 + i)(1 + 2i)(1 + 3i)\ldots(1 + ni) = x + yi$

$\Rightarrow \quad |(1 + i)(1 + 2i)(1 + 3i)\ldots(1 + ni)| = |x + yi|$

$\Rightarrow \quad |1 + i| \,.\, |1 + 2i| \,.\, |1 + 3i| \ldots |1 + ni| = |x + yi|$

$\Rightarrow \quad \sqrt{(1+1)} \times \sqrt{(1+2^2)} \times \sqrt{(1+3^2)} \ldots \sqrt{(1+n^2)} = \sqrt{x^2+y^2} \qquad \left[\because \begin{matrix} |z_1 z_2 \ldots z_n| \\ = |z_1| \cdot |z_2| \ldots |z_n| \end{matrix}\right]$

$\Rightarrow \quad \sqrt{2}\cdot\sqrt{5}\cdot\sqrt{10}\ldots\sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow \quad 2 \,.\, 5 \,.\, 10 \ldots (1+n^2) = x^2 + y^2.$

Method II. $(1+i)(1+2i)(1+3i)\ldots(1+ni) = x + yi$

$\therefore \quad (1-i)(1-2i)(1-3i)\ldots(1-ni) = x - yi$

Multiplying together, we get

$[(1+i)(1-i)]\,[(1+2i)(1-2i)]\,[(1+3i)(1-3i)]\ldots[(1+ni)(1-ni)] = (x+yi)(x-yi)$

$\Rightarrow \quad (1-i^2)(1-4i^2)(1-9i^2)\ldots[1-n^2i^2] = x^2+y^2$

$\Rightarrow \quad (1+1)(1+4)(1+9)\ldots(1+n^2) = x^2+y^2$

$\Rightarrow \quad 2 \,.\, 5 \,.\, 10\ldots(1+n^2) = x^2+y^2.$

Ex. 44. Given that $\dfrac{1+i}{1+2^2i}\times\dfrac{1+3^2i}{1+4^2i}\times\ldots\times\dfrac{1+(2n-1)^2i}{1+(2n)^2i} = \dfrac{a+bi}{c+di}$, **show that:**

$$\frac{2}{17}\times\frac{82}{257}\times\ldots\times\frac{(2n-1)^4+1}{(2n)^4+1} = \frac{a^2+b^2}{c^2+d^2}.$$ ***(ISC 1999)***

Sol. $\dfrac{1+i}{1+2^2i}\times\dfrac{1+3^2i}{1+4^2i}\times\ldots\times\dfrac{1+(2n-1)^2i}{1+(2n)^2i} = \dfrac{a+bi}{c+di}$

$\Rightarrow \quad \left|\dfrac{1+i}{1+2^2i}\times\dfrac{1+3^2i}{1+4^2i}\times\ldots\times\dfrac{1+(2n-1)^2i}{1+(2n)^2i}\right| = \left|\dfrac{a+bi}{c+di}\right|$

$\Rightarrow \quad \left|\dfrac{1+i}{1+2^2i}\right|\times\left|\dfrac{1+3^2i}{1+4^2i}\right|\times\ldots\times\left|\dfrac{1+(2n-1)^2i}{1+(2n)^2i}\right| = \left|\dfrac{a+bi}{c+di}\right|$

$\Rightarrow \quad \dfrac{|1+i|}{|1+2^2i|}\cdot\dfrac{|1+3^2i|}{|1+4^2i|}\times\ldots\dfrac{|1+(2n-1)^2i|}{|1+(2n)^2i|} = \dfrac{|a+bi|}{|c+di|}$

$\Rightarrow \quad \dfrac{\sqrt{1+1}}{\sqrt{1+16}}\cdot\dfrac{\sqrt{1+81}}{\sqrt{1+256}}\ldots\dfrac{\sqrt{(2n-1)^4+1}}{\sqrt{(2n)^4+1}} = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow \quad \dfrac{2}{17}\cdot\dfrac{82}{257}\ldots\dfrac{(2n-1)^4+1}{(2n)^4+1} = \dfrac{a^2+b^2}{c^2+d^2}$.

Method II. $\dfrac{1+i}{1+4i}\times\dfrac{1+9i}{1+16i}\times\ldots\times\dfrac{1+(2n-1)^2i}{1+(2n)^2i} = \dfrac{a+bi}{c+di}$

$\Rightarrow \quad \dfrac{1-i}{1-4i}\times\dfrac{1-9i}{1-16i}\times\ldots\dfrac{1-(2n-1)^2i}{1-(2n)^2i} = \dfrac{a-bi}{c-di}$

Multiplying the two equations, we get

$$\frac{(1+i)(1-i)}{(1+4i)(1-4i)}\times\frac{(1+9i)(1-9i)}{(1+16i)(1-16i)}\times\ldots\frac{[1+(2n-1)^2i][1-(2n-1)^2i]}{[1+(2n)^2i][1-(2n)^2i]} = \frac{(a+bi)(a-bi)}{(c+di)(c-di)}$$

$$\Rightarrow \quad \frac{1+1}{1+16}\times\frac{1+81}{1+256}\times\ldots\times\frac{1+(2n-1)^4}{1+(2n)^4} = \frac{a^2+b^2}{c^2+d^2} \quad \Rightarrow \quad \frac{2}{17}\times\frac{82}{257}\times\ldots\frac{1+(2n-1)^4}{1+(2n^4)} = \frac{a^2+b^2}{c^2+d^2}.$$

Ex. 45. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, prove that

$$|z_1 + z_2 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots \frac{1}{z_n}\right|. \qquad \textit{(ISC 1997, IIT (S) 2000)}$$

Sol. $|z_1| = |z_2| = \ldots = |z_n| = 1$

$$\Rightarrow \quad \frac{1}{z_1} = \overline{z}_1, \frac{1}{z_2} = \overline{z}_2 \ldots \frac{1}{z_n} = \overline{z}_n \qquad \left(\text{Since } z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{1} = \overline{z}\right)$$

Now $\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right| = |\overline{z}_1 + \overline{z}_2 + \ldots + \overline{z}_n|$

$= \left|\overline{z_1 + z_2 + \ldots + z_n}\right|$ (Using the property $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$)

$= |z_1 + z_2 + \ldots + z_n|$. (Using the property $|\overline{z}| = |z|$)

Hence proved.

Method II. Let $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i, \ldots z_n = x_n + y_n i$

It is given that $x_1^2 + y_1^2 = x_2^2 + y_2^2 = \ldots\ldots = x_n^2 + y_n^2 = 1$

$$|z_1 + z_2 + \ldots + z_n| = \sqrt{(x_1 + x_2 + \ldots + x_n)^2 + (y_1 + y_2 + \ldots + y_n)^2} \qquad \ldots(1)$$

$$\frac{1}{z_1} = \frac{1}{x_1 + y_1 i} = \frac{1}{x_1 + y_1 i} \times \frac{x_1 - y_1 i}{x_1 - y_1 i} = \frac{x_1 - y_1 i}{x_1^2 + y_1^2} = x_1 - y_1 i \qquad [\because \ x_1^2 + y_1^2 = 1 \ ]$$

Similarly, $\frac{1}{z_2} = x_2 - y_2 i, \frac{1}{z_3} = x_3 - y_3 i, \ldots \frac{1}{z_n} = x_n - y_n i$

$$\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n} = (x_1 + x_2 + \ldots + x_n) - (y_1 + y_2 + \ldots + y_n)i$$

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right| = \sqrt{(x_1 + x_2 + \ldots + x_n)^2 + (y_1 + y_2 + \ldots + y_n)^2} \qquad \ldots(2)$$

From (1) and (2), we get

$$|z_1 + z_2 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|.$$

Ex. 46. Find the greatest value of the moduli of complex numbers z satisfying the equation $\left|z - \frac{4}{z}\right| = 2$. What is the minimum value? ***(ISC 1996)***

Sol. Since, $|z_1 - z_2| \geq ||z_1| - |z_2||$ therefore, (Property 9)

It follows that $\left|z - \frac{4}{z}\right| \geq \left||z| - \left|\frac{4}{z}\right|\right|$ [Here outer vertical lines mean absolute value]

$$\Rightarrow \qquad 2 \geq \left||z| - \left|\frac{4}{z}\right|\right| \qquad \left[\because \left|z - \frac{4}{z}\right| = 2, \text{given}\right]$$

$\Rightarrow \quad \left| |z| - \frac{4}{|z|} \right| \le 2$

$\Rightarrow \quad -2 \le |z| - \frac{4}{|z|} \le 2 \qquad [\because \ |x| \le \Rightarrow \ -2 \le x \le 2]$

$\Rightarrow \quad -2|z| \le |z|^2 - 4 \le 2|z|$

$\Rightarrow \quad |z|^2 + 2|z| - 4 \ge 0 \quad \ldots(1) \quad \text{and} \quad |z|^2 - 2|z| - 4 \le 0 \quad \ldots(2)$

From (1), we get $|z| \ge \frac{-2 \pm \sqrt{4+16}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$

$\Rightarrow \quad |z| \ge -1 + \sqrt{5} \qquad [\because |z| \ge 0, \text{ therefore } -1 - \sqrt{5} \text{ is rejected}]$

From (2), we get $|z| \le \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$

$\Rightarrow \quad |z| \le 1 + \sqrt{5} \qquad [\because |z| \ge 0]$

Combining, $\sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$.

Hence, the maximum value of $|z|$ is $\sqrt{5} + 1$, and the minimum value is $\sqrt{5} - 1$.

Ex. 47. If $z = x + yi$ and $\omega = \frac{1 - zi}{z - i}$ show that $|\omega| = 1 \Rightarrow z$ is purely real.

Sol. $|\omega| = 1$ (given) $\Rightarrow \quad \left| \frac{1 - zi}{z - i} \right| = 1$

$\Rightarrow \quad \frac{|1 - zi|}{|z - i|} = 1 \quad \Rightarrow \quad |1 - zi| = |z - i| \qquad \left(\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right)$

$\Rightarrow \quad |1 - (x + yi)i| = |x + yi - i|, \quad (z = x + yi, \text{ given})$

$\Rightarrow \quad |(1 + y) - xi| = |x + (y - 1)i| \ \Rightarrow \ \sqrt{(1+y)^2 + (-x)^2} = \sqrt{x^2 + (y-1)^2}$

$\Rightarrow \quad (1 + y)^2 = (y - 1)^2 \Rightarrow y = 0 \ \Rightarrow \ z = x + 0i$, which is real.

EXERCISE 9 (*c*)

1. If $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + bi$, find the real numbers a and b with values of a and b, also find the modulus of $a + bi$.

2. Find the modulus of $(1 - i)^{-2} + (1 + i)^{-2}$. ***(ISC)***

3. If $z = 6 + 8i$, verify that

(*i*) $|z| = |\bar{z}|$ (*ii*) $-|z| \le \text{Re}(z) \le |z|$ (*iii*) $-|z| \le \text{Im}(z) \le |z|$ (*iv*) $z^{-1} = \frac{\bar{z}}{|z|^2}$

4. If $z_1 = 3 + 4i$, $z_2 = 8 - 15i$, verify that

(*i*) $|-z_1| = |z_1|$ (*ii*) $|z_1^2| = |z_1|^2$ (*iii*) $|z_1 z_2| = |z_1||z_2|$ (*iv*) $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$

(*v*) $|z_1 + z_2| < |z_1| + |z_2|$ (*vi*) $|z_2 - z_1| > \| z_2 | - | z_1 \|$

(*vii*) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

5. Find the modulus of the following using the property of modulus.

(*i*) $(3 + 4i)(8 - 6i)$ (*ii*) $\dfrac{8+15i}{8-6i}$ (*iii*) $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i}$ (*iv*) $\dfrac{(2-3i)(4+5i)}{(1-4i)(2-i)}$

6. Let z be a complex number such that $\left|\dfrac{z-5i}{z+5i}\right| = 1$, then show that z is purely real.

7. Find the complex number z satisfying the equation $\left|\dfrac{z-12}{z-8i}\right| = \dfrac{5}{3}$, $\left|\dfrac{z-4}{z-8}\right| = 1$.

8. If z is a complex number such that $|z-1| = |z+1|$, show that Re(z) = 0.

9. Solve: $|z| + z = 2 + i$, where z is a complex number.

ANSWERS

1. $a = 0, b = -7\sqrt{3}$, Modulus = $7\sqrt{3}$ **2.** 0

5. (*i*) 50 (*ii*) $\dfrac{17}{10}$ (*iii*) $\dfrac{8}{29}$ (*iv*) $\sqrt{\dfrac{533}{85}}$ **7.** $6 + 8i, 6 + 17i$ **9.** $z = \dfrac{3}{4} + i$

HINT

9 $\sqrt{x^2 + y^2} + (x + yi) = 2 + i \Rightarrow \sqrt{x^2 + y^2} + x = 2$ and $y = 1$

9.15. Trigonometric or polar form of a complex number

The graphical representation of a complex number as a line segment with one end point at the origin suggests that there should be a close relationship between complex numbers and the trigonometric functions. The relationship becomes obvious if we consider the line segment as the terminal side of an angle in standard position.

For example, if we plot the point corresponding to the number 3 + 4*i*, join *OP* and draw *PM* perpendicular to the real axis (Fig. 9.12 (*i*)), then an angle θ is determined $\angle(POM)$ so that $\tan\theta = \dfrac{4}{3}$ and the length *OP*, which we shall call $r = \sqrt{3^2 + 4^2} = 5$. The polar coordinates of *P* are (r, θ).

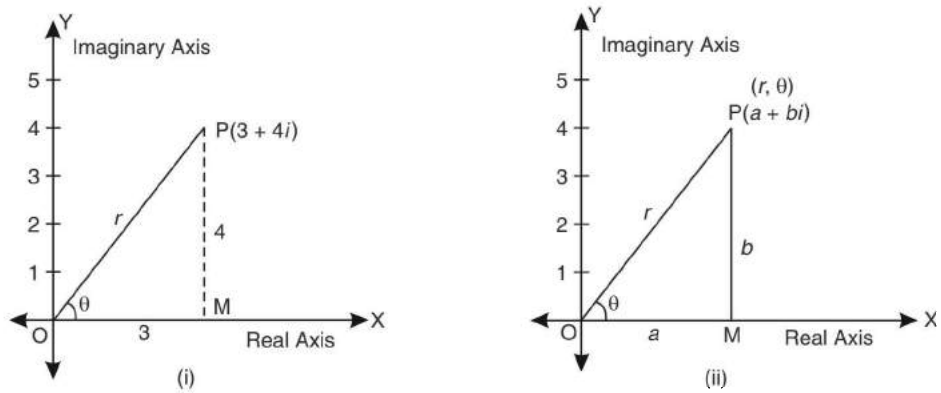


Fig. 9.12

By considering any complex number $a + bi$, the following relationships between the real numbers a and b and the values of θ and r, become apparent

$$a = r\cos\theta \quad \ldots(i)$$

$$b = r\sin\theta \quad \ldots(ii)$$

Since, $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$,

by substituting the values of a and b in $a + bi$, we get $r\cos\theta + i(r\sin\theta)$, which we usually write in the form $\boldsymbol{r(\cos\theta + i\sin\theta)}$.

The above expression is called the trigonometric form or **polar form** or **polar representation** of a complex number $z = a + bi$. The form $a + bi$ is called **rectangular form**. Sometimes the trigonometric form is abbreviated as $\boldsymbol{r}$ **cis** θ.

Given r and θ we can use equations (i) and (ii), to convert from trigonometric to rectangular form. Conversely, given a and b, we can use the following to convert to trigonometric form.

$$\textbf{1. } \tan\theta = \frac{b}{a} \text{ or } \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

$$\textbf{2. } r = \sqrt{a^2 + b^2} = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$$

Both the relationships are easily seen from the drawing of Fig. 9.12 (ii).

It may be noted that r is a positive number. In trigonometric form r is the absolute value or modulus of the complex number $r\text{cis}\,\theta = |z| = +\sqrt{a^2 + b^2}$

Remark. The angle θ is measured in radians and $0 \le \theta < 2\pi$. For every positive value of r and each value of θ between 0 and 2π, we get a unique point in the complex plane with **polar co-ordinates** as (r, θ) and conversely, every point in the plane has polar co-ordinates (r, θ), where ≥ 0 and $0 \le \theta < 2\pi$. The segment OP is called the radius vector of the point P. The number $r = 0$ only if $z = 0$.

9.16. Argument or amplitude of a complex number

The angle θ is called the **amplitude** or **argument** of the complex number $z = a + bi$ and is denoted by θ **= arg** z **or am** z**.**

Thus, θ = am z, or arg z $\tan^{-1}\frac{b}{a}$.

Definition. ***An argument, or amplitude of the complex number*** $z = a + bi$ ***is an angle*** θ***, which OP makes with positive direction of x-axis in anticlockwise sense.***

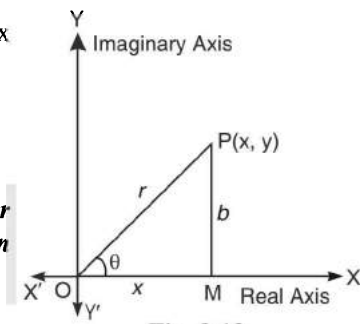


Fig. 9.13

Remarks: 1. The value of θ, where $-\pi < \theta \le \pi$, is called the principal value of the argument, and is written as pr. arg. z. By amplitude we shall mean the principal value. The argument of z depends upon the quadrant in which the point P lies.

2. The argument of 0, *i.e.*, of $0 + 0i$ is not defined as $\tan^{-1}\theta = \frac{0}{0}$ is not defined.

3. The amplitude of a positive real number is zero.

4. The amplitude of a positive imaginary number is $\frac{\pi}{2}$.

5. The amplitude of a negative imaginary number is $-\frac{\pi}{2}$.

6. $am\,(z_1.z_2) = am\,z_1 + am\,z_2$.
   In general, $am\,(z_1.\,z_2.\,z_3\,\ldots\,z_n) = am\,z_1 + am\,z_2 + am\,z_3 + \ldots + am\,z_n$.

7. $am\left(\dfrac{z_1}{z_2}\right) = am\,z_1 - am\,z_2$.

The diagrams given below show how principal value of the argument of a complex number z lying in different quadrants is determined.

Method. Step 1. ***Find the value of*** $\tan^{-1}\left|\dfrac{b}{a}\right|$ ***lying between 0 and*** $\dfrac{\pi}{2}$. Suppose it is α.

Step 2. ***Find out the quadrant in which the given complex number P(a, b) lies.*** Then

(*i*) If $a > 0$, $b > 0$, *i.e.*, $P(a, b)$ lies in the first quadrant, then $arg(z) = \alpha$.

(*ii*) If $a < 0$, $b > 0$, *i.e.*, $P(a, b)$ lies in the second quadrant, then $arg(z) = \pi - \alpha$.

(*iii*) If $a < 0$, $b < 0$, *i.e.*, $P(a, b)$ lies in the third quadrant, then $arg\,z = -(\pi - \alpha)$ or $\alpha - \pi$.

(*iv*) If $a > 0$, $b < 0$, *i.e.*, $P(a, b)$ lies in the fourth quadrant, then $\arg z = -\alpha$.

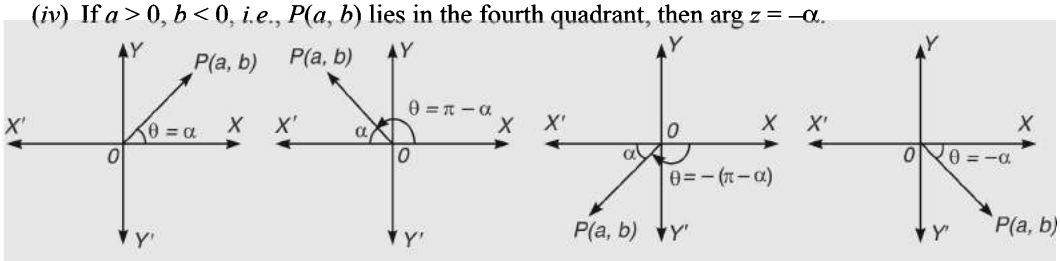


Fig. 9.14

Illustrations

For the complex number z,

(*i*) $\mathbf{1+\sqrt{3}i}$ $\quad\because\ a > 0, b > 0$, so the point lies in 1st quad.

$$\alpha = \tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \tan^{-1}\sqrt{3} \quad \Rightarrow \quad \alpha = \frac{\pi}{3}$$

$$\therefore \qquad \arg(z) = \alpha = \frac{\pi}{3}.$$

(*ii*) $\mathbf{-4+4\sqrt{3}i}$ $\quad\because\ a < 0,\ b > 0$, so the point lies in the 2nd quad.

$$\alpha = \tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \tan^{-1}\left|\frac{4\sqrt{3}}{-4}\right| = \tan^{-1}\sqrt{3} \ \Rightarrow\ \alpha = \frac{\pi}{3}$$

$$\therefore\ \arg z = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

(*iii*) $\mathbf{-2\sqrt{3}-2i}$ $\quad\because\ a < 0, b < 0$, so the point lies in the 3rd quad.

$$\alpha = \tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \tan^{-1}\left|\frac{-2}{-2\sqrt{3}}\right| = \tan^{-1}\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \frac{\pi}{6}$$

$$\therefore\ \arg z = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = \frac{-5\pi}{6}.$$

(*iv*) $\mathbf{5\sqrt{3}-5i,}$ $\because$ $a > 0$, $b < 0$, so the point lies in the 4th quadrant.

$$\alpha = \tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{-5}{5\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \frac{\pi}{6}$$

$\therefore$ $\arg(z) = -\alpha = \dfrac{-\pi}{6}$.

Ex. 48. Find the modulus and amplitude of (*i*) $\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}i$ (*ii*) $-4i$.

Sol. (*i*) Here, $r\cos\theta = \dfrac{-1}{2}$, $r\sin\theta = \dfrac{\sqrt{3}}{2}$

Squaring and adding, we get

$$r^2\cos^2\theta + r^2\sin^2\theta = \frac{1}{4}+\frac{3}{4} = 1 \Rightarrow r^2(\cos^2\theta+\sin^2\theta) = 1$$

$\Rightarrow$ $r^2 = 1$ $\therefore$ $r = 1$ (Taking only positive value)

$\because$ $a < 0$, $b > 0$, the point lies in the second quadrant. If α is the acute angle, then

$$\alpha = \tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}\right| = \sqrt{3} \quad \Rightarrow \quad \alpha = \frac{\pi}{3}$$

$\therefore$ Amplitude $= \pi - \alpha = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

Hence, the modulus is equal to 1, the amplitude is equal to $\dfrac{2\pi}{3}$.

(*ii*) Let $z = -4i = 0 + (-4)i$.

$\therefore$ Then $|z| = \sqrt{0+(-4)^2} = 4$

Clearly, the point $(0, -4)$ representing $z = -4i$ lies on the negative side of imaginary axis.

Therefore, principal amplitude of z is $-\dfrac{\pi}{2}$.

Ex. 49. Find the modulus and amplitude of $\dfrac{2+3i}{3+2i}$. **(*ISC 2007*)**

Sol. $\dfrac{2+3i}{3+2i} = \dfrac{(2+3i)(3-2i)}{(3+2i)(3-2i)} = \dfrac{6-4i+9i-6i^2}{3^2-(2i)^2} = \dfrac{12+5i}{13} = \dfrac{12}{13}+\dfrac{5}{13}i$

$\therefore$ Modulus $= \sqrt{\left(\dfrac{12}{13}\right)^2+\left(\dfrac{5}{13}\right)^2} = \sqrt{\dfrac{144+25}{169}} = \sqrt{\dfrac{169}{169}} = 1$

Since the point $\left(\frac{12}{13}, \frac{5}{13}\right)$ representing the complex number z lies in the first quadrant, therefore, amplitude is the acute angle given by

$$\text{Amplitude} = \tan^{-1}\left(\frac{\frac{5}{13}}{\frac{12}{13}}\right) = \tan^{-1}\frac{5}{12}.$$

Method II. $\left|\frac{2+3i}{3+2i}\right| = \frac{|2+3i|}{|3+2i|} = \frac{\sqrt{2^2+3^2}}{\sqrt{3^2+2^2}} = 1$ $\qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

and $\text{am}\left(\frac{2+3i}{3+2i}\right) = \text{am}(2+3i) - \text{am}(3+2i) = \tan^{-1}\frac{3}{2} - \tan^{-1}\frac{2}{3}$ $\qquad \left(\text{am}\left|\frac{z_1}{z_2}\right| = \text{am}\, z_1 - \text{am}\, z_2\right)$

$$= \tan^{-1}\left(\frac{\frac{3}{2}-\frac{2}{3}}{1+\frac{3}{2}\cdot\frac{2}{3}}\right) = \tan^{-1}\frac{5}{12} \qquad \left[\because \frac{3}{2}\times\frac{2}{3} = 1 > -1\right]$$

Ex. 50. Find the modulus and amplitude of $\frac{2+i}{4i+(1+i)^2}$.

Sol. $\frac{2+i}{4i+(1+i)^2} = \frac{2+i}{4i+1+2i+i^2} = \frac{2+i}{6i+1-1} = \frac{2+i}{6i}\times\frac{i}{i}$

$$= \frac{2i+i^2}{6i^2} = \frac{-1+2i}{-6} = \frac{1}{6} - \frac{1}{3}i = \left(\frac{1}{6}, \frac{-1}{3}\right)$$

$$\text{Modulus} = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{-1}{3}\right)^2} = \sqrt{\frac{1}{36}+\frac{1}{9}} = \frac{\sqrt{5}}{6}.$$

$$\text{Amplitude} = \tan^{-1}\left(\frac{\frac{-1}{3}}{\frac{1}{6}}\right) = \tan^{-1}(-2) = -\tan^{-1}2 = -63°20', \text{ approx.}$$

Since $\left(\frac{1}{6}, \frac{-1}{3}\right)$ lies in the fourth quadrant, the amplitude is $-\tan^{-1}2$.

Ex. 51. Represent the complex numbers z = (*i*) $1+\sqrt{3}i$ (*ii*) $2-2i$ (*iii*) $\frac{1+7i}{(2-i)^2}$ in polar form.

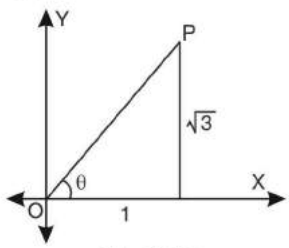


Fig. 9.15

Sol. (*i*) Given, $z = x + yi = 1 + \sqrt{3}i$

Equating real and imaginary parts, we have

$x = 1, y = \sqrt{3}$ [Point $(1, \sqrt{3})$ lies in I quadrant]

$\Rightarrow$ $r = \sqrt{x^2+y^2} = 2$ and $\tan\theta = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$.

Hence, the polar coordinates of $P(1+\sqrt{3}i\ )$ are $(2, \frac{\pi}{3})$ and its polar form is $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

(*ii*) Given $z = x + yi = 2 - 2i$

Equating real and imaginary parts, we have $x = 2, y = -2$

Point $(2, -2)$ lies in the IV quadrant.

$$r = \sqrt{x^2+y^2} = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\therefore \quad z = 2(1-i) = 2\sqrt{2}\left[\frac{1}{\sqrt{2}}+i\left(-\frac{1}{\sqrt{2}}\right)\right]$$

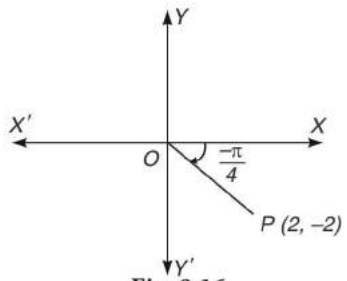


Fig. 9.16

Since cos θ is +ve and sin θ is –ve, so θ lies in 4th quadrant and we take $\theta = -\frac{\pi}{4}$.

Hence, polar form of

$$z = 2\sqrt{2}\ \left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right].$$

(*iii*) $$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i} = \frac{1+7i}{3-4i}$$

$$= \frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i} = \frac{3+25i+28i^2}{9-16i^2}$$

$$= \frac{3+25i-28}{9+16} = \frac{-25+25i}{25} = -1+i \equiv \text{Point } (-1, 1)$$

Let $-1 + i = r(\cos\theta + i\sin\theta)$. Then $r\cos\theta = -1$, $r\sin\theta = 1 \Rightarrow \theta$ lies in 2nd quadrant.

$$\therefore \qquad r^2 = (-1)^2 + 1^2 \Rightarrow r = \sqrt{2}.$$

$$\tan\theta = \left|\frac{1}{-1}\right| = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}.$$

Since point $(-1, 1)$ lies in II quadrant, therefore $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, $\frac{1+7i}{(2-i)^2} = \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

Ex. 52. Change the complex number $4\ (\cos 300° + i\sin 300°)$ to cartesian form.

Sol. Let $4\cos 300° = 4\cos(360° - 60°) = x + yi.$

Equating real and imaginary parts, we have

$$x = 4\cos 300° = 4\cos(360° - 60°) = 4\cos 60° = 4\times\frac{1}{2} = 2$$

$$y = 4\sin 300° = 4\sin(360° - 60°) = 4(-\sin 60°) = 4\times\frac{-\sqrt{3}}{2} = -2\sqrt{3}$$

Hence, the cartesian form of the given complex number is $x + yi = 2 - 2\sqrt{3}i$.

Ex. 53. Represent on complex plane the complex numbers $w = 3 + 4i$ and $z = 6 - 3i$ together with $w + z$ and $w - z$. Obtain the modulus and argument of w and z.

If w and z are known as P and Q obtain the angle between OP and OQ where O is the origin of the coordinates. (*ISC*)

Sol. $w = 3 + 4i \equiv P(3, 4)$; $z = 6 - 3i \equiv Q(6, -3)$

$w + z = (3 + 4i) + (6 - 3i) = 9 + i \equiv R(9, 1)$

$w - z = (3 + 4i) - (6 - 3i) = -3 + 7i \equiv S(-3, 7)$

Argument of $w = \tan^{-1}\dfrac{4}{3} = \tan^{-1} 1.3333 = 53°8'$, approx.

$P(3, 4)$ lies in 1st quadrant.

Modulus of $w = \sqrt{3^2 + 4^2} = 5$

Argument of $z = \tan^{-1}\left|\dfrac{-3}{6}\right|$

$= \tan^{-1}(0.5000)$

$= 26°34'$, approx.

Since, $Q(6, -3)$ lies in 4th quadrant, therefore, arg $z = -26°34'$.

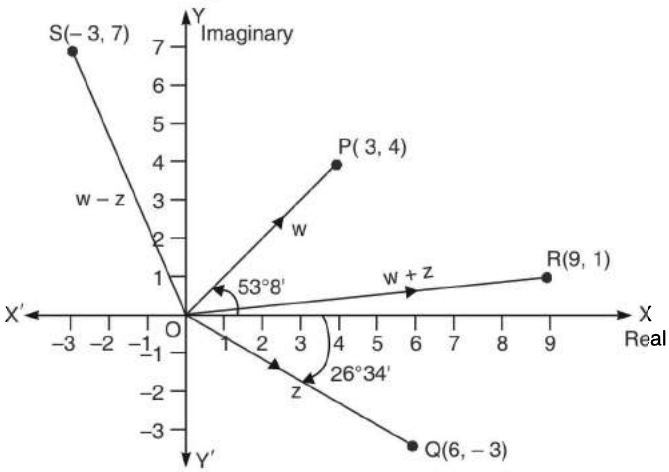


Fig. 9.17

Modulus of $z = \sqrt{6^2 + (-3)^2} = \sqrt{45} = 3\sqrt{5} = 6.708$ approx.

The angle between OP and $OQ = 53°8' + 26°34' = 79°42'$, approx.

EXERCISE 9 (*d*)

Find the modulus and amplitude of the following complex numbers and hence express them into polar form.

1. $\sqrt{3} + i$ **2.** $-\sqrt{3} + i$ **3.** $-2 + 2\sqrt{3}i$ **4.** $-1 - i$

5. $-2i$ **6.** $-1 - \sqrt{3}i$ **7.** -2 **8.** $\dfrac{(1+i)^{13}}{(1-i)^7}$

9. $(3 + i)(4 + i)$ **10.** $\dfrac{(1+i)(2+i)}{(3+i)}$ **11.** $\dfrac{5-i}{2-3i}$ **12.** $\dfrac{(3+4i)(4+5i)}{(4+3i)(6+7i)}$

13. Change the following complex numbers into polar form.

(*i*) $-4 + 4\sqrt{3}i$ (*ii*) $\dfrac{1+3i}{1-2i}$ (*iii*) $\dfrac{1+2i}{1-(1-i)^2}$ (*iv*) $\dfrac{1+7i}{(2-i)^2}$

14. Given the complex number $z = \dfrac{-1+\sqrt{3}i}{2}$ and $w = \dfrac{-1-\sqrt{3}i}{2}$ (where $i = \sqrt{-1}$)

(*i*) Prove that each of these complex numbers is the square of the other.

(*ii*) Calculate the modulus and argument of w and z.

(*iii*) Calculate the modulus and argument of $\dfrac{w}{z}$.

(*iv*) Represent z and w accurately on the complex plane.

ANSWERS

1. Modulus = 2, amplitude = $\frac{\pi}{6}$; $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

2. Modulus = 2, amplitude = $\frac{5\pi}{6}$; $z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

3. Modulus = 4, amplitude = $\frac{2\pi}{3}$; $z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

4. Modulus = $\sqrt{2}$, amplitude = $-\frac{3\pi}{4}$; $z = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

5. Modulus = 2, amplitude = $-\frac{\pi}{2}$; $z = 2\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

6. Modulus = 2, amplitude = $-\frac{2\pi}{3}$; $z = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

7. Modulus = 2, amplitude = π; $z = 2[\cos\pi + i\sin\pi]$

8. Modulus = 8, argument = π

9. Modulus = $\sqrt{170}$, argument = 32°29′

10. Modulus = 1, argument = 53°8′

11. Modulus = $\sqrt{2}$, argument = $\frac{\pi}{4}$

12. Modulus = $\sqrt{\frac{41}{85}}$, argument = 18°12′

13. (*i*) $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ (*ii*) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (*iii*) $\cos 0° + i\sin 0°$

(*iv*) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

14. (*i*) See Art. 9.20.

(*ii*) Modulus of w is 1 and argument of w is –120°. Modulus of z is 1, and argument of z is 120°.

(*iii*) Modulus of $\frac{w}{z}$ is 1 and argument of $\frac{w}{z}$ is 120°.

(*iv*) w and z are represented by $P\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$ and $Q\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively.

HINTS TO SELECTED QUESTIONS

7. $z = -2 + 0i \therefore |z| = \sqrt{(-2)^2 + 0} = 2$. The point (–2, 0) lies on the negative side of real axis. therefore, principal amplitude of z is π.

8. $z = \frac{(1+i)^{13}}{(1-i)^7} \times \frac{(1+i)^7}{(1+i)^7} = \frac{(1+i)^{20}}{(1-i^2)^7} = \frac{1}{2^7}(1+i)^{20} = \frac{1}{2^7}(1+2i+i^2)^{10} = \frac{1}{2^7}(2i)^{10} = \frac{2^{10}}{2^7} \cdot i^8 \cdot i^2 = -8$

9.17. Locus problems

Ex. 54. If z_1 and z_2 are two fixed points in the Argand plane, then find the locus of a point z in each of the following:

(*i*) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ **(*ii*)** $|z - z_1| = |z - z_2|$

(iii) $|z-z_1| = k|z-z_2|, \quad k \in R^+, k \neq 1$ ***(iv)*** $|z-z_1|+|z-z_2| = \text{constant} \neq (|z_1-z_2|)$

(v) $|z-z_1|-|z-z_2| = \text{constant} (\neq |z_1-z_2|)$ ***(vi)*** $|z-z_1|-|z-z_2| = |z_1-z_2|$

Sol. Let A and B be two fixed points representing complex numbers z_1 and z_2 respectively in the Argand plane and let P be a variable point representing complex number z. Then,

$AP = |z-z_1|$, $BP = |z-z_2|$ and $AB = |z_1-z_2|$

(*i*) $|z-z_1|+|z-z_2| = |z_1-z_2|$ (Given)

$\Rightarrow$ $AP + BP = AB$

$\Rightarrow$ P lies on the line segment joining $A(z_1)$ and $B(z_2)$.

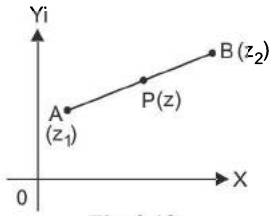


Fig. 9.18

(*ii*) Given: $|z-z_1| = |z-z_2|$

$\Rightarrow$ $AP = BP$

$\Rightarrow$ P is equidistant from A and B.

$\Rightarrow$ P lies on the perpendicular bisector of the line segment AB.

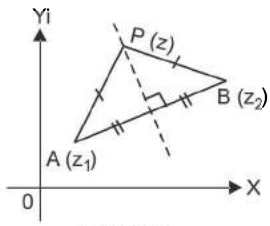


Fig. 9.19

(*iii*) Given: $|z-z_1| = k|z-z_2|$

$\Rightarrow$ $AP = k.BP$

$\Rightarrow$ P lies on a circle

(A circle is also defined as the locus of a point such that the ratio of its distances from two fixed points is always constant.)

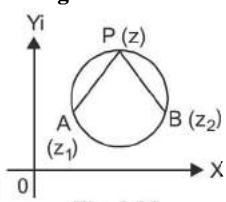


Fig. 9.20

(*iv*) Given: $|z-z_1|+|z-z_2|$ = constant

$\Rightarrow$ $AP + BP$ = constant

i.e., P moves such that the sum of its distances, from two fixed points A and B is always the same.

$\Rightarrow$ P lies on an ellipse having its foci A and B. (Definition of ellipse)

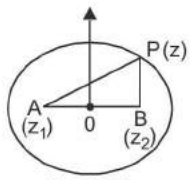


Fig. 9.21

(*v*) Given: $|z-z_1|-|z-z_2|$ = constant $(\neq |z_1-z_2|)$

$\Rightarrow$ $AP - BP$ = constant ($\neq$ AB)

$\Rightarrow$ P lies on a hyperbola having its foci at A and B respectively. (Definition of hyperbola)

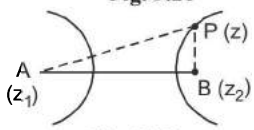


Fig. 9.22

(*vi*) Given: $|z-z_1|-|z-z_2| = |z_1-z_2|$

$\Rightarrow$ $AP - BP = AB$

$\Rightarrow$ P lies on the line joining A and B but does not lie between A and B.

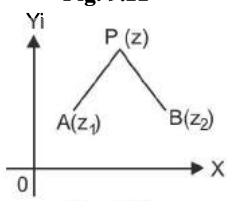


Fig. 9.23

Ex. 55. Illustrate in the complex plane the following set of points and explain your answer:

(i) $|Z| = 3$ ***(ii)*** $|Z| < 5$ ***(iii)*** $|Z-4| < 1$ ***(iv)*** $\arg(Z) = \frac{\pi}{6}$

Sol. Let $Z = x + yi$.

(*i*) $|Z| = \sqrt{x^2 + y^2} = 3$ or $x^2 + y^2 = 9$,

which is a circle with centre at (0, 0) and radius equal to 3.

Hence, $|Z| = 3$ represents a set of points of the circle whose centre is the origin and radius equal to 3. (Fig. 9.24).

(*ii*) $|Z| < 5$, *i.e.*, $x^2 + y^2 < 5$, which represents the set of points in the interior of the circle, whose centre is the origin and radius equal to 5. (Fig. 9.25)

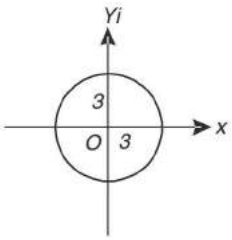


Fig. 9.24

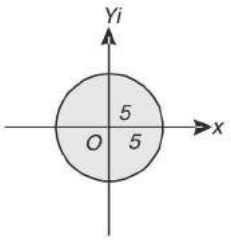


Fig. 9.25

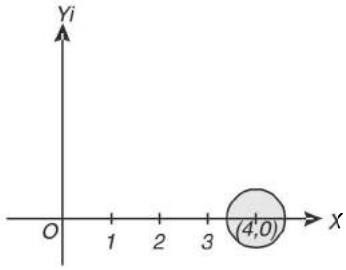


Fig. 9.26

(*iii*) $Z = x + yi$

$$|Z - 4| = |x + yi - 4| = |(x - 4) + yi| = \sqrt{(x-4)^2 + y^2}$$

$$= \sqrt{x^2 + y^2 - 8x + 16}$$

$$\therefore \ |Z - 4| < 1 \Rightarrow \sqrt{x^2 + y^2 - 8x + 16} < 1 \Rightarrow x^2 + y^2 - 8x + 16 < 1$$

$$\Rightarrow x^2 + y^2 - 8x + 15 < 0.$$

The centre of the circle $x^2 + y^2 - 8x + 15 = 0$ is (4, 0) and radius is equal to $\sqrt{16 + 0 - 15} = 1$.

Hence, the set of points satisfying $|Z - 4| < 1$ consists of all points in the interior of the circle as shown in Fig. 9.26.

(*iv*) $\arg(Z) = \frac{\pi}{6}$ or $\frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ or $y = \frac{1}{\sqrt{3}}x$,

which is a straight line passing through the origin and inclined at 30° with the positive direction of the *x*-axis (Fig. 9.27).

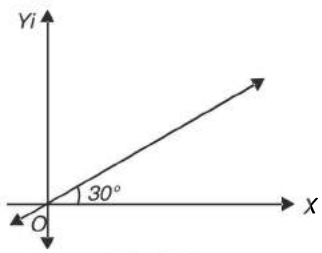


Fig. 9.27

Ex. 56. Illustrate in the complex plane the set of points *z* satisfying $|z + i - 21 \le 2$.

(*ISC 2000*)

Sol. Let $z = x + iy$. Then $|z + i - 2| \le 2 \Rightarrow |x + iy + i - 2| \le 2$

$\Rightarrow |(x - 2) + i(y + 1)| \le 2 \Rightarrow \sqrt{(x-2)^2 + (y+1)^2} \le 2.$

$\Rightarrow (x - 2)^2 + (y + 1)^2 \le 4 \Rightarrow x^2 + y^2 - 4x + 2y + 1 \le 0.$

Now $x^2 + y^2 - 4x + 2y + 1 = 0$ is a circle with centre (2, –1) and radius equal to $\sqrt{4 + 1 - 1}$, *i.e.*, 2.

This circle cuts the x-axis where $y = 0$. Then, we get

$$x^2 - 4x + 1 = 0 \quad \Rightarrow \quad x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}.$$

$\Rightarrow \quad x = 2 \pm 1.732 = 3.732, 0.268 = 3.7, 0.27$, approx.

The circle cuts the y-axis where $x = 0$.

Then $y^2 + 2y + 1 = 0 \quad \Rightarrow \quad (y+1)^2 = 0$

$\Rightarrow \quad y = -1, -1$. Thus, the circle touches the y-axis at $(0, -1)$.

The circle is drawn in the adjoining figure.

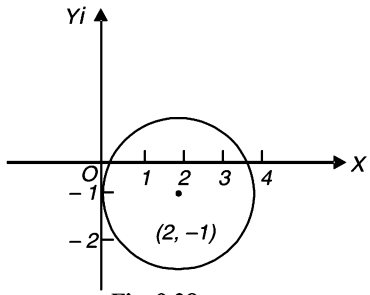


Fig. 9.28

The inequality $x^2 + y^2 - 4x + 2y + 1 \leq 0$ is represented by all points lying inside the circle and on the circle.

Ex. 57. (*i*) A variable complex number z is such that the amplitude of $\frac{z-1}{z+1}$ is always equal to $\frac{\pi}{4}$.

Illustrate the locus of z in the Argand plane. ***(ISC 1995)***

(*ii*) Find the locus of a complex number $z = x + yi$ satisfying the relation $\arg(z - a) = \frac{\pi}{4}$, $a \in R$.

Illustrate the locus of z in the Argand plane. ***(ISC 1998)***

Sol. (*i*) Let $z = x + iy$, then

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

Using the property, $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2$, we have,

$$\text{amp}\left(\frac{\overline{x-1}+iy}{\overline{x+1}+iy}\right) = \frac{\pi}{4} \text{ (given)}$$

$$\Rightarrow \quad \text{amp}\,(\overline{x-1}+iy) - \text{amp}\,(\overline{x+1}+iy) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1}\cdot\frac{y}{x+1}} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{y(x+1) - y(x-1)}{(x^2-1)+y^2} = \tan\frac{\pi}{4} = 1$$

$\Rightarrow \quad 2y = x^2 - 1 + y^2 \Rightarrow x^2 + y^2 - 2y - 1 = 0.$

which is a circle with centre (0, 1), and radius equal to $\sqrt{2}$.

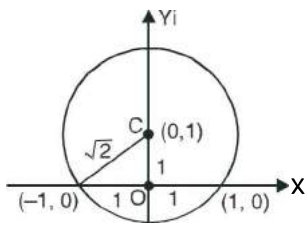


Fig. 9.29

(*ii*) $z = x + iy$

$$\arg(z - a) = \frac{\pi}{4} \quad \Rightarrow \quad \arg(x + iy - a) = \frac{\pi}{4}$$

$$\Rightarrow \arg[(x-a) + iy] = \frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1}\frac{y}{x-a} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{y}{x-a} = \tan\frac{\pi}{4} \Rightarrow \frac{y}{x-a} = 1 \Rightarrow y = x - a,$$

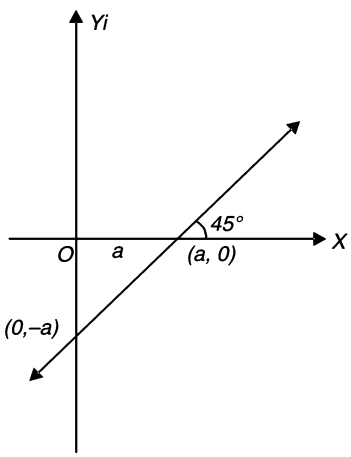


Fig. 9.30

$\Rightarrow$ $y = 1.\ x - a \Rightarrow$ slope $(m) = 1 = \tan 45°$

which is a straight line that intersects the x-axis at $(a, 0)$ and is inclined at an angle of 45° to the positive direction of the x-axis.

Ex. 58. Given $z_1 = 1 + 2i$. Determine the region in the complex plane represented by $1 < |z - z_1| \leq 3$. Represent it with the help of an Argand diagram. ***(ISC 2004, 1994)***

Sol. Let $z = x + iy$. Then

$1 < |z - z_1| \leq 3 \Rightarrow 1 < |x + iy - 1 - 2i| \leq 3$

$\Rightarrow 1 < |(x - 1) + (y - 2)\, i| \leq 3$

$\Rightarrow 1 < \sqrt{(x-1)^2 + (y-2)^2} \leq 3$

$\Rightarrow 1 < x^2 + y^2 - 2x - 4y + 5 \leq 9$

(squaring the three sides)

$\Rightarrow x^2 + y^2 - 2x - 4y + 4 > 0$

and $x^2 + y^2 - 2x - 4y - 4 \leq 0$

We draw the two circles, *viz.*,

$x^2 + y^2 - 2x - 4y + 4 = 0$ with centre (1, 2) and radius $= \sqrt{1 + 4 - 4} = 1$ and $x^2 + y^2 - 2x - 4y - 4 = 0$ with centre (1, 2) and radius $= \sqrt{1 + 4 + 4} = 3$.

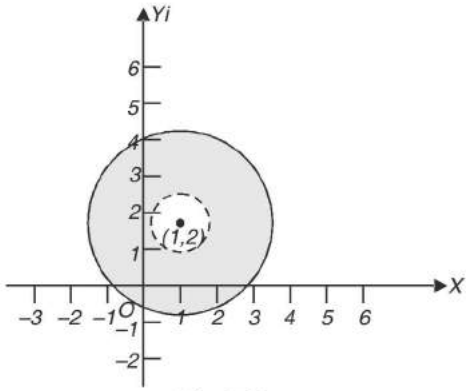


Fig. 9.31

Draw these two concentric circles.

The area between the two circles, *i.e.*, the shaded area is the required region including the circumference of the bigger circle and excluding the circumference of the smaller circle.

Ex. 59. Find the radius and centre of the circle

$$z\bar{z} - (2 + 3i)\, z - (2 - 3i)\, \bar{z} + 9 = 0$$

where z is a complex variable.

Sol. Let $z = x + iy$. Then $\bar{z} = x - iy$

$\therefore \quad z\bar{z} = |z|^2 = x^2 + y^2$

Now, $z\bar{z} - (2 + 3i)z - (2 - 3i)\, \bar{z} + 9 = 0$

$\Rightarrow (x^2 + y^2) - (2 + 3i)\,(x + iy) - (2 - 3i)\,(x - iy) + 9 = 0$

$\Rightarrow x^2 + y^2 - 4x + 6y + 9 = 0.$

This represents a circle whose centre is $(2, -3)$, *i.e.*, $(2 - 3i)$ and radius 2.

Ex. 60. If $z \neq 1$ and $\dfrac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(*a*) either on the real axis or on a circle passing through the origin

(*b*) on a circle with centre at the origin

(*c*) either on the real axis or on a circle not passing through the origin

(*d*) on the imaginary axis ***(AIEEE 2012)***

Sol. Let $z = x + iy$, then

$$\frac{z^2}{z-1} = \frac{(x+iy)^2}{(x+iy)-1} = \frac{(x^2 - y^2) + i(2xy)}{(x-1) + iy}$$

$$= \frac{(x^2 - y^2) + i(2xy)}{(x-1) + iy} \times \frac{(x-1) - iy}{(x-1) - iy}$$

$$= \frac{[(x^2 - y^2) + i(2xy)][(x-1) - iy]}{(x-1)^2 + y^2}$$

Since $\frac{z^2}{z-1}$ $(z \neq 1)$ is purely real, therefore, its imaginary part should be equal to zero.

$\therefore \quad (x^2 - y^2)(-y) + (2xy)(x-1) = 0$

$\Rightarrow \quad y(x^2 - y^2 + 2x - 2x^2) = 0 \quad \Rightarrow \quad y(x^2 + y^2 - 2x) = 0$

Either $y = 0$ or $x^2 + y^2 - 2x = 0$.

Now, $y = 0 \Rightarrow$ Locus of 'z' is x-axis, *i.e.*, real axis and

$x^2 + y^2 - 2x = 0 \Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow (x-1)^2 + y^2 = 1^2 \Rightarrow$ Locus of 'z' is either a circle passing through the origin with centre (1, 0) and radius 1 or real axis. **Ans. (*a*)**

EXERCISE 9 (*e*)

1. Illustrate in the complex plane, the set of points satisfying the following conditions. Explain your answer:

(*i*) $|z| \leq 3$ (*ii*) $\text{agr}(z-2) = \frac{\pi}{3}$. **(*Type ISC 1990, 1998*)**

(*iii*) $|i - 1 - 2z| > 9$

2. Illustrate and explain the region of the Argand's plane represented by the inequality $|z + i| \geq |z + 2|$. **(*ISC*)**

3. Illustrate and explain the set of points z in the Argand diagram, which represents $|z - z_1| \leq 3$ where $z_1 = 3 - 2i$. **(*ISC 1991*)**

4. If $z = x + yi$ and $\omega = \frac{(1 - zi)}{z - i}$, then $|\omega| = 1$ implies that in the complex plane

(*a*) z lies on the imaginary axis (*b*) z lies on the real axis

(*c*) z lies on the unit circle (*d*) None of these.

(*UPSEAT 2003 DCE 2000, 01, IIT Raj. PET 1997*)

5. Find the locus of a complex number z such that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$.

(*DCE, Orissa JEE, Raj PET, MP PET*)

6. If the amplitude of $z - 2 - 3i$ is $\frac{\pi}{4}$, then find the locus of $z = x + yi$. **(*EAMCET 2003*)**

7. Find the locus of z if $\omega = \frac{z}{z - \frac{1}{3}i}$, $|\omega| = 1$.

8. A variable complex number z is such that the amplitude of $\frac{z-1}{z+1}$ is always equal to $\frac{\pi}{4}$.

Illustrate the locus of z in the Argand plane. **(*Sample Paper 2010, Roorkee*)**

9. Find the radius and centre of the circle $z\bar{z} + (1 - i)z + (1 + i)\bar{z} - 7 = 0$.

10. What is the region represented by the inequality $3 < |z - 2 - 3i| < 4$ in the Argand plane.

ANSWERS

1. (*i*) All points on the circumference as well as in the interior of the circle with centre (0, 0) and radius = 3.

(*ii*) Straight line $y = (x-2)\sqrt{3}$ which intersects the x-axis at (2, 0) and is inclined at 60° to the +ve direction of the x-axis.

(*iii*) Exterior of the circle having centre at $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{9}{2}$.

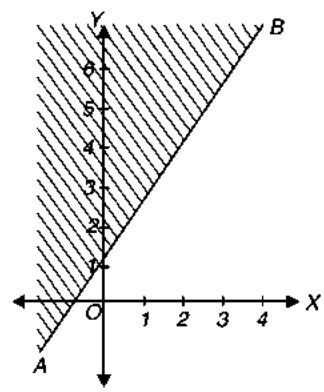


Fig. 9.32

2. We draw the straight line $y = 2x + \frac{3}{2}$ by joining the points $\left(0, \frac{3}{2}\right)$ and $\left(1, \frac{7}{2}\right)$

The region above the line AB, including the line AB, represents the given inequality, *i.e.*, the shaded part.

3. $x^2 + y^2 - 6x + 4y + 4 \le 0$, all points inside and on this circle whose centre is (3, –2) and radius is 3.

4. (*b*)

5. $x^2 + y^2 + 8x + 4 = 0$, which is a circle with centre (–4, 0) and radius $2\sqrt{3}$.

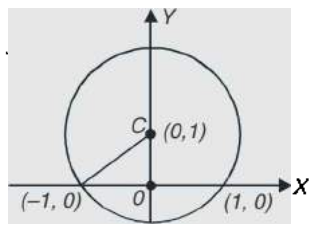


Fig. 9.33

6. $x - y + 1 = 0$

7. $y = \frac{1}{6}$, which is a st.line.

8. A circle with centre (0, 1) and radius = $\sqrt{2}$.

9. Centre is (–1, –1) ; radius = 3

10. Circular region bounded between two concentric circles of radii 4 and 3 respectively having centre at (2, 3).

HINTS TO SELECTED QUESTIONS

1. (*iii*) $|i - 1 - 2z| > 9 \Rightarrow \left|z - 2\left\{z + \frac{1}{2}(1-i)\right\}\right| > 9$

$\Rightarrow 2\left|z - \frac{1}{2}(-1+i)\right| > 9 \Rightarrow \left|x + yi - \frac{1}{2}(-1+i)\right| > \frac{9}{2}$

$\Rightarrow \left|\left(x + \frac{1}{2}\right) + \left(y - \frac{1}{2}\right)i\right| > \frac{9}{2} \Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 > \frac{81}{4}$

2. Let $z = x + yi$. Then, $|z + i| = |x + yi + i| = |x + (y+1)i| = x^2 + (y+1)^2$

$|z + 2| = |x + yi + 2| = |(x+2) + yi| = (x+2)^2 + y^2$

$\therefore |z + i| \ge |z + 2| \Rightarrow x^2 + (y+1)^2 \ge (x+2)^2 + y^2 \Rightarrow y \ge 2x + \frac{3}{2}$.

5. Put $z = x + yi$. Then $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3} \Rightarrow \left|\frac{(x-2)+yi}{(x+2)+yi}\right| = \tan\frac{\pi}{3}$.

6. $z - 2 - 3i = x + yi - 2 - 3i = (x-2) + (y-3)i \Rightarrow \tan^{-1}\left(\frac{y-3}{x-2}\right) = \frac{\pi}{4}$.

7. $\omega = \frac{z}{z - \frac{1}{3}i} \Rightarrow |\omega| = \left|\frac{x + yi}{x + yi - \frac{1}{3}i}\right| \Rightarrow 1 = \left|\frac{x + yi}{x + yi - \frac{1}{3}i}\right|$.

8. Let $z = x + yi$, then $\dfrac{z-1}{z+1} = \dfrac{x^2-1+y^2}{(x+1)^2+y^2} + i \cdot \dfrac{2y}{(x+1)^2+y^2}$

$\therefore$ Amplitude $= \tan^{-1}\left(\dfrac{\dfrac{2y}{(x+1)^2+y^2}}{\dfrac{x^2-1+y^2}{(x+1)^2+y^2}}\right) = \dfrac{\pi}{4} \Rightarrow \dfrac{2y}{x^2-1+y^2} = \tan\dfrac{\pi}{4} = 1.$

9. Type Ex. 59 **10.** Type Ex. 58.

9.18. Square root of a complex number $a + bi$

Let the square root of the complex number $a + bi$ be $x + yi$.

i.e., $\sqrt{a+bi} = x + yi$, where x and $y \in R$.

Squaring $a + bi = (x + yi)^2 \Rightarrow a + bi = x^2 - y^2 + 2xyi$

Equating real and imaginary parts from both sides, we get

$x^2 - y^2 = a$ ... (*i*) $2xy = b$ ... (*ii*)

$x^2 + y^2 = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{a^2+b^2}$

The required square root of $a + bi$ is obtained on solving these equations.

Rule 1. *Let the square root of the given complex number* $= x + yi$.

2. *Square both sides.*

3. *Equate the real and imaginary parts.*

4. *Solve the resulting equations and obtain the values of x and y.*

Note 1. If the given complex number is of the form $a - bi$, let its square root be equal to $x - yi$.

2. In case of complex number of the form $a \pm bi$ also, let square root be equal to $x \pm yi$.

3. Great care is to be observed regarding the signs of x and y. If xy is positive, x and y both be of the same sign. If xy is negative, x and y will be of opposite signs.

Ex. 61. Find the square root of the complex number (*i*) $5 + 12i$ (*ii*) $-4 - 3i$ (*iii*) $18i$

Sol. (*i*) Let $\sqrt{5+12i} = x + yi$ where $x, y \in R$.

Squaring both sides, we get $5 + 12i = x^2 - y^2 + 2xyi$.

Equating real and imaginary parts from both sides, we get

$x^2 - y^2 = 5$ ... (1) and $2xy = 12$ ... (2)

Now, $x^2 + y^2 = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{25+144} = \sqrt{169} = 13$ ... (3)

Adding (1) and (3), we get $2x^2 = 18 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

Subtracting (1) from (3), we get $2y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$.

Since xy is +ve, so x and y are of the same sign. Hence, $x = \pm 3, y = \pm 2$.

$\sqrt{(5+12i)} = \pm 3 \pm 2i$, *i.e.*, **$(3 + 2i)$ or $-(3 + 2i)$**

(*ii*) Let $\sqrt{-4-3i} = x - yi$. Then $-4 - 3i = x^2 - y^2 - 2xyi$

$\Rightarrow$ $x^2 - y^2 = -4$ ... (1) $2xy = 3$ ... (2)

$\therefore$ $x^2 + y^2 = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{16+9} = \sqrt{25} = 5$ ... (3)

On solving the equations (1) and (3), we get $x^2 = \frac{1}{2}, y^2 = \frac{9}{2}$

$\therefore \quad x = \pm\sqrt{\frac{1}{2}}, y = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}}$

Since xy is +ve, $x = \sqrt{\frac{1}{2}}, y = \frac{3}{\sqrt{2}}$ or $x = -\sqrt{\frac{1}{2}}, y = -\frac{3}{\sqrt{2}}$

$\therefore$ Square root of $-4 - 3i = \pm\left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}i\right)$.

(*iii*) The given complex number can be written as $0 + 18i$.

Let $\sqrt{18i} = x + yi$

Squaring both sides $18i = x^2 - y^2 + 2xyi$ ... (1)

Equating real and imaginary parts, we get $x^2 - y^2 = 0$ ... (*i*) $xy = 9$ ... (2)

Now, $x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{0 + 4 \times 81} = 18$ ... (3)

(1) and (3) $\Rightarrow 2x^2 = 18 \Rightarrow x = \pm 3 \therefore y = \pm 3$. Since xy is + ve, so x and y will be of the same sign. Therefore, $\sqrt{18i} = \pm 3 \pm 3i = \pm\mathbf{3\,(1 + i)}$.

9.19. Cube roots of unity

To find all the cube roots of unity

Let $x = (1)^{\frac{1}{3}}$. Then $x^3 = 1 \Rightarrow x^3 - 1 = 0$

$\Rightarrow (x - 1)(x^2 + x + 1) = 0 \Rightarrow x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow x = 1, x = \frac{-1 \pm \sqrt{1-4}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2} \qquad \therefore x = 1, \frac{-1 \pm i\sqrt{3}}{2}$.

$\therefore$ The three cube roots of unity are

$$\mathbf{1, \frac{-1 + i\sqrt{3}}{2}\ and\ \frac{-1 - i\sqrt{3}}{2}}$$

Out to these roots, one root is real and the remaining two roots are complex numbers.

9.20. Properties of cube roots of unity

Property 1. ***Each complex cube root of unity is the square of other.***

The complex roots of unity are $\frac{-1 + i\sqrt{3}}{2}$ and $\frac{-1 - i\sqrt{3}}{2}$

Now $\left(\frac{-1 + i\sqrt{3}}{2}\right)^2 = \frac{1 + (-3) - 2i\sqrt{3}}{4} = \frac{-2 - 2i\sqrt{3}}{4} = \frac{-1 - i\sqrt{3}}{2}$

and $\left(\frac{-1 - i\sqrt{3}}{2}\right)^2 = \frac{1 + (-3) + 2i\sqrt{3}}{4} = \frac{-2 + 2i\sqrt{3}}{4} = \frac{-1 + i\sqrt{3}}{2}$

Hence, each complex cube root of unity is the square of the other.

Remark. If one complex cube root be denoted by ω(omega), then the other complex cube root would be ω^2. Thus, the three cube roots may be denoted by $1, \omega, \omega^2$.

Property 2. ***The sum of the three cube roots of unity is zero.***

Sum of the cube roots of unity $= 1 + \omega + \omega^2$

$$= 1 + \frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2} = \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2} = \frac{0}{2} = 0$$

$\therefore$ $$\mathbf{1+\omega+\omega^2 = 0}$$

Property 3. ***The product of the three cube roots of unity is 1.***

Product of cube roots of unity $= 1.\omega.\omega^2 = 1 \cdot \left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right)$

$$= \left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right) = \frac{(-1)^2-(i\sqrt{3})^2}{4} = \frac{1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$$

$\therefore$ $$\omega^3 = \mathbf{1}$$

Property 4. ***Each complex cube root of unity is the reciprocal of the other.***

If $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $\beta = \frac{-1-i\sqrt{3}}{2}$ then $\alpha\beta = \left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right)$

$$= \frac{(-1)^2-(i\sqrt{3})^2}{4} = \frac{1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1 \Rightarrow \alpha = \frac{1}{\beta} \text{ and } \beta = \frac{1}{\alpha}$$

Hence, each complex cube root of unity is the reciprocal of the other.

9.21. Powers of ω

$$\omega^3 = 1, \quad \omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega, \quad \omega^5 = \omega^3 \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$$

$$\omega^6 = (\omega^3)^2 = (1)^2 = 1, \quad \omega^{3n} = (\omega^3)^n = (1)^n = 1$$

Therefore,

(i) $\omega^k = 1$ **or** ω **or** ω^2 **where** $k \in I$, ***(ii)*** $\omega^{\text{multiple of 3}} = 1$

For example, $\omega^{27} = (\omega^3)^9 = (1)^9 = 1, \quad \omega^{37} = (\omega^{36}) \cdot \omega = (\omega^3)^{12} \cdot \omega = 1 \cdot \omega = \omega$

$$\omega^{50} = \omega^{48} \cdot \omega^2 = (\omega^3)^{16} \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$$

$$\omega^{-20} = \omega^{-21} \cdot \omega = (\omega^3)^{-7} \cdot \omega = 1 \cdot \omega = \omega$$

Remember: $\omega^3 = 1$, $\quad 1+\omega+\omega^2 = 0$

$1 + \omega = -\omega^2$, $\quad 1 + \omega^2 = -\omega$

$\omega + \omega^2 = -1$,

Ex. 62. If 1, ω, ω^2 are the cube roots of unity, prove that

(i) $(1 + \omega)^3 - (1 + \omega^2)^3 = 0$

(ii) $(x - y)(x\omega - y)(x\omega^2 - y) = x^3 - y^3$ ***(CBSE)***

Sol. (*i*) L.H.S. $= (1+\omega)^3 - (1+\omega^2)^3 = (-\omega^2)^3 - (-\omega)^3 = -\omega^6 + \omega^3$

$= -(\omega^3)^2 + \omega^3 = -(1)^2 + 1 = -1 + 1 = 0 =$ R.H.S.

(*ii*) L.H.S. $= (x-y)(x\omega - y)(x\omega^2 - y)$

$= (x-y)[x^2\omega^3 - xy\omega - xy\omega^2 + y^2] = (x-y)[x^2 - xy(\omega + \omega^2) + y^2]$

$= (x-y)\,[x^2 - xy\,(-1) + y^2] = (x-y)\,[x^2 + xy + y^2] = x^3 - y^3 =$ R.H.S.

Ex. 63. If 1, ω, ω² are three cube roots of unity, show that

$$(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^2 + b^2 + c^2 - ab - bc - ca. \qquad \textbf{(CBSE)}$$

Sol. L.H.S. $= (a + \omega b + \omega^2 c)\,(a + \omega^2 b + \omega c)$

$= a^2 + \omega^2 ab + \omega ac + \omega ab + \omega^3 b^2 + \omega^2 bc + \omega^2 ac + \omega^4 bc + \omega^3 c^2$

$= a^2 + \omega^2 ab + \omega ac + \omega ab + b^2 + \omega^2 bc + \omega^2 ac + \omega \cdot bc + c^2$

$[\because\ \omega^3 = 1,\ \omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega\ ]$

$= a^2 + b^2 + c^2 + (\omega^2 + \omega)\,ab + (\omega + \omega^2)\,ac + (\omega^2 + \omega)\,bc$

$= a^2 + b^2 + c^2 + (-1)\,ab + (-1)\,ac + (-1)\,bc\ \ [\because\ 1 + \omega + \omega^2 = 0 \Rightarrow \omega + \omega^2 = -1\ ]$

$= a^2 + b^2 + c^2 - ab - bc - ac =$ R.H.S.

Ex. 64. (*i*) If α and β are the complex cube roots of unity, then prove that

$$(1 + \alpha)\,(1 + \beta)\ (1 + \alpha^2)\,(1 + \beta^2) = 1. \qquad \textbf{(ISC 1994)}$$

(*ii*) If, 1, ω, ω² are cube roots of unity, show that $\dfrac{p + q\omega + r\omega^2}{r + p\omega + q\omega^2} = \omega^2$. **(ISC 1995)**

(*iii*) Given that 1, ω, ω² are cube roots of unity. Show that

$$(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32. \qquad \textbf{(IIT, ISC 1997, Raj PET 1997)}$$

(*iv*) If 1, ω, ω² are cube roots of unity, prove that

$$(x + y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2 = 6xy. \qquad \textbf{(ISC 1999)}$$

Sol. (*i*) Let $\alpha = \omega, \beta = \omega^2$

Then $(1 + \alpha)\,(1 + \beta)\,(1 + \alpha^2)\,(1 + \beta^2) = (1 + \omega)\,(1 + \omega^2)\,(1 + \omega^2)\,(1 + \omega^4)$

$= (-\omega^2)\,(-\omega)\,(-\omega)\,(1 + \omega)$ $\qquad [\because\ 1 + \omega + \omega^2 = 0;\ \omega^3 = 1]$

$= \omega^3 \cdot (-\omega)\,(-\omega^2) = \omega^3 \,.\, \omega^3 = 1 \times 1 = 1.$

(*ii*) $\dfrac{p + q\omega + r\omega^2}{r + p\omega + q\omega^2} = \dfrac{\omega^2\,(p + q\omega + r\omega^2)}{\omega^2\,(r + p\omega + q\omega^2)} = \dfrac{\omega^2\,(p + q\omega + r\omega^2)}{r\omega^2 + p\omega^3 + q\omega^4}$

$= \dfrac{\omega^2\,(p + q\omega + r\omega^2)}{r\omega^2 + p + q\omega}$ $\qquad [\because\ \omega^3 = 1, \omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega]$

$= \dfrac{\omega^2\,(p + q\omega + r\omega^2)}{(p + q\omega + r\omega^2)} = \omega^2.$

(*iii*) We know that $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

$\therefore\ \ (1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = [(1 + \omega^2) - \omega]^5 + [(1 + \omega) - \omega^2]^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5 = -32\omega^5 - 32\omega^{10}$

$= -32\omega^3 \cdot \omega^2 - 32\,(\omega^3)^3 \cdot \omega = -32\omega^2 - 32\omega = -32\,(\omega + \omega^2)$

$= (-32)(-1) = 32.$

(*d*) $(x+y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2$

$= (x^2 + 2xy + y^2) + (x^2\omega^2 + 2xy\omega^3 + y^2\omega^4) + (x^2\omega^4 + 2xy\omega^3 + y^2\omega^2)$

$= (x^2 + 2xy + y^2) + (x^2\omega^2 + 2xy + y^2\omega) + (x^2\omega + 2xy + y^2\omega^2)$ [$\because\ \omega^3 = 1$]

$= x^2(1 + \omega^2 + \omega) + 6xy + y^2(1 + \omega + \omega^2)$

$= x^2 \times 0 + 6xy + y^2 \times 0 = 6xy.$

Ex. 65. If $x = a + b, y = a\alpha + b\beta, z = a\beta + b\alpha$, where α and β are complex cube roots of unity, then show that $xyz = a^3 + b^3$. ***(ISC 1998, IIT Roorkee, Raj PET 1997)***

Sol. Let $\alpha = \omega, \beta = \omega^2$. Then $\beta = \alpha^2, \alpha^3 = 1, \alpha + \alpha^2 = -1$

$xyz = (a+b)(a\alpha + b\beta)(a\beta + b\alpha)$

$= (a+b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a+b)(a^2\omega^3 + ab\omega^2 + ab\omega^4 + b^2\omega^3)$

$= (a+b)(a^2 + ab\omega^2 + ab\omega + b^2)$ [$\because\ \omega^3 = 1, \omega^4 = \omega^3.\omega = \omega$]

$= (a+b)\{a^2 + ab(\omega^2 + \omega) + b^2\}$

$= (a+b)(a^2 - ab + b^2)$ [$\therefore\ \omega^2 + \omega = -1$]

$= a^3 + b^3.$

Hence, $xyz = a^3 + b^3.$

Ex. 66. If $x = a + b, y = a\omega^2 + b\omega, z = a\omega + b\omega^2$, then show that $x^3 + y^3 + z^3 = 3(a^3 + b^3)$. ***(IIT)***

Sol. $x + y + z = (a+b) + (a\omega^2 + b\omega) + (a\omega + b\omega^2)$

$= a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2) = a \times 0 + b \times 0 = 0.$

Now, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$ [$\because\ x + y + z = 0$]

$\Rightarrow\ x^3 + y^3 + z^3 = 3xyz = 3(a^3 + b^3)$ [Using the result found in Ex. 65]

Hence, $x^3 + y^3 + z^3 = 3(a^3 + b^3).$

Ex. 67. Show that $\sqrt{\left[-1\sqrt{\left\{-1-\sqrt{-1+\ldots\ldots \text{ to } \infty}\right.}\right.} = \omega$, or ω^2. ***(IIT)***

Sol. Let $x = \sqrt{\left[-1-\sqrt{\left\{-1-\sqrt{-1+\ldots \text{ to } \infty}\right.}\right.}$, then

$x = \sqrt{-1-x} \ \Rightarrow\ x^2 = -1 - x \ \Rightarrow\ x^2 + x + 1 = 0.$

$\Rightarrow\ x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2} \Rightarrow x = \omega \text{ or } \omega^2.$

EXERCISE 9 (*f*)

1. Find the square root of the following complex numbers.

(*i*) $3 + 4i$ (*ii*) $-8 + 6i$ (*iii*) $-40 - 42i$ (*iv*) i

(*v*) $\left(\dfrac{2+3i}{5-4i} + \dfrac{2-3i}{5+4i}\right)$ ***(CBSE, J & K)***

2. If ω is a cube root of unity, then

(*i*) $\omega + \omega^2 = \ldots\ldots$ (*ii*) $1 + \omega = \ldots\ldots$ (*iii*) $1 + \omega^2 = \ldots\ldots$ (*iv*) $\omega^3 = \ldots\ldots$ ***(CBSE)***

If 1, ω, ω² are three cube roots of unity, prove that:

3. (*i*) $(1+\omega^2)^4 = \omega$ (*ii*) $(1+\omega-\omega^2)^3 = (1-\omega+\omega^2)^3 = -8$

(*iii*) $(1-\omega)(1-\omega^2) = 3$ (*iv*) $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2} = 1$

4. (*i*) $(1-\omega-\omega^2)^6 = 64$ **(*CBSE*)** (*ii*) $(1+\omega-\omega^2)(1-\omega+\omega^2) = 4$ **(*MNR*)**

5. $(3+5\omega+3\omega^2)^6 = (3+5\omega^2+3\omega)^6 = 64$ **(*HB, Pb*)**

6. $\omega^{28} + \omega^{29} + 1 = 0$

7. Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^n + \left(\dfrac{-1-i\sqrt{3}}{2}\right)^n$ is equal to 2 if *n* be a multiple of 3 and is equal to – 1 if *n* be any other integer.

Or

If 1, ω, ω² are the cube roots of unity, prove that $\omega^n + \omega^{2n} = 2$ or – 1 according as *n* is a multiple of 3 or any other integer.

Prove the following:

8. $(1-\omega+\omega^2)(1+\omega-\omega^2)(1-\omega-\omega^2) = 8$.

9. (*i*) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\ldots$ to $2n$ factors = 1 **(*JEE WB 1992, AMU 2000*)**

(*ii*) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ ..... to $2n$ factors = 2^{2n}

10. $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \omega$ **11.** $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = -1$

12. If ω is a cube root of unity and *n* is a positive integer which is not a multiple of 3, then show that $(1+\omega^n+\omega^{2n}) = 0$

13. Show that $(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = x^2+y^2+z^2-yz-zx-xy$.

14. Show that $x^3+y^3 = (x+y)(\omega x+\omega^2 y)(\omega^2 x+\omega y)$. **(*IIT, ISC*)**

15. If 1, ω, ω² are cube roots of unity, prove that 1, ω, ω² are vertices of an equilateral triangle.

ANSWERS

1. (*i*) ± (2 + *i*) (*ii*) ± (1 + 3*i*) (*iii*) ± (3 – 7*i*) (*iv*) $\pm \dfrac{1}{\sqrt{2}}(1+i)$ (*v*) $\pm \dfrac{2}{\sqrt{41}}i$

2. (*i*) – 1 (*ii*) $-\omega^2$ (*iii*) $-\omega$ (*iv*) 1

HINTS TO SELECTED QUESTIONS

6. $\omega^{28} = \omega \times \omega^{27} = \omega \times (\omega^3)^9 = \omega \times 1 = \omega,$ $(\because\ \omega^3 = 1)$

$\omega^{29} = \omega^2 \times \omega^{27} = \omega^2 \times (\omega^3)^9 = \omega^2 \times 1 = \omega^2 \ \therefore\ \omega^{28} + \omega^{29} + 1 = \omega + \omega^2 + 1 = 1 + \omega + \omega^2 = 0.$

7. Let $n = 3p$, where *p* is an integer, *i.e.*, when *n* is a multiple of 3. Then

$\omega^n + \omega^{2n} = \omega^{3p} + \omega^{6p} = (\omega^3)^p + (\omega^3)^{2p} = 1 + 1 = 2$

If *n* is not a multiple of 3, then when *n* is divided by 3 it has a remainder 1 or 2, *i.e.*, then

$n = 3p + 1$, or $3p + 2$

$\therefore\ \omega^n + \omega^{2n} = \omega^{3p+1} + \omega^{6p+2}$ or $\omega^{3p+2} + \omega^{6p+4} = \omega^2 + \omega = -1.$

9. (*ii*) $\omega^4 = \omega^3 \cdot \omega$, $\omega^8 = (\omega^3)^2 \cdot \omega^2 = \omega^2$ etc.

$\therefore$ L.H.S. $= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)\ldots\ldots$ to $2n$ factors

$= \{(-2\omega)(-2\omega^2)\}\{(-2\omega)(-2\omega^2)\}\ldots\ldots$ to $2n$ factors

$= (2^2\omega^3)(2^2\omega^3)(2^2\omega^3)\ldots\ldots$ to n factors.

10. $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \dfrac{a\omega^3+b\omega+c\omega^2}{b+c\omega+a\omega^2}$

11. $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = \dfrac{a\omega^3+b\omega^4+c\omega^2}{c+a\omega+b\omega^2} = \omega^2$, $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \dfrac{a\omega^3+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \omega$

12. $1+\omega^n+\omega^{2n}$ is a G.P. whose first term is 1 and common ratio ω^n

$\therefore\ 1+\omega^n+\omega^{2n} = \dfrac{1\cdot[(\omega^n)^3-1]}{\omega^n-1} = \dfrac{(\omega^3)^n-1}{\omega^n-1} = \dfrac{1-1}{\omega^n-1} = 0.$

15. Let A, B, C be represented by 1, ω, ω^2 respectively, where $\omega = \dfrac{-1+i\sqrt{3}}{2}$, $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$.

Then, $AB = |1-\omega| = \left|1-\dfrac{-1+\sqrt{-3}}{2}\right| = \left|\dfrac{3}{2}-\dfrac{\sqrt{3}}{2}i\right| = \sqrt{\left(\dfrac{3}{2}\right)^2+\dfrac{3}{4}} = \sqrt{3}.$

Similarly, $BC = |\omega-\omega^2| = \sqrt{3}$, $CA = |1-\omega^2| = \sqrt{3}$.

HISTORICAL NOTE (THE STORY OF $\sqrt{-1}$)

Perhaps the earliest encounter with the square root of a negative number is in the expression $\sqrt{81-144}$, which appears in the *Stereo-metrica* of Heron of Alexandria (*c*A.D. 50); the next known encounter is in Diophantus's attempt to solve the equation $336x^2 + 24 = 172x$ (as we would not write it), in whose solution the quantity $\sqrt{1849-2016}$ appears (again using modern notation).

The first clear statement of difficulty with the square root of a negative number was given in India by Mahavir (*c*. 850), who wrote "As in the nature of things, a negative is not a square, it has no square root. "Nicolas Chuquet (1484) and Luca Pacioli (1494) in Europe were among those who continued to reject imaginaries.

Girolamo Cardano (1545), who is also known as Jerome Cardan, is credited with some progress in introducing complex numbers in his solution of the cubic equation, even though he regarded them "fictitious". He used the symbol "*Rx... m*", meaning "radix minus", for the square root of the negative number. Albert Girard (1629) included symbolism such as "$\sqrt{-2}$ ". Rene Descartes (1637) contributed the terms "real" and "imaginary". It was Leonhard Euler (1748) who called $\sqrt{-1}$ by the symbol '*i*' which is still in use. Caspar Wessel (1797) used "$\sqrt{-1} = \varepsilon$". Carl Friedrich Gauss (1832) introduced the term "complex number". Willam Rowar Hamilton (1832) expressed the complex number in the form of number-couple.

Bombelli, continued Cardanos' work. From the equation $x^2 + a = 0$, he spoke of "$+\sqrt{-a}$ " and "$-\sqrt{-a}$ ". Wallis, Wessel (1798); Jean Robert Argand (1806), Gauss (1813), and others made significant contributions to the understanding of complex numbers through graphical representation, and in 1831 Gauss defined complex numbers as ordered pairs of real numbers for which (a, b). $(c, d) = (ac - bd, ad + bc)$, and so forth.

CHAPTER SUMMARY

1. The number i such that $\boldsymbol{i^2 = -1 \Rightarrow i = \sqrt{-1}}$ is an imaginary number. The numbers $\sqrt{-3}$, $\sqrt{-7}$, $\sqrt{-16}$ etc., are imaginary numbers.

   In general, $\boldsymbol{\sqrt{-a} = \sqrt{a}\,i}$ where 'a' is any positive real number.

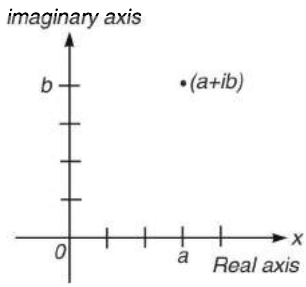
2. **Powers of i.** $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$

   In general, $i^p\ (p > 4) = i^{4q+r} = (i^4)^q . i^r = i^r$ since $i^4 = 1$.

3. A complex number is the sum of a real number and a pure imaginary number. The standard from of a complex number is $\boldsymbol{a + bi}$, where a and b are real numbers.
4. In the complex number $a + bi$, a is the real part and b is the imaginary part.
5. $a + ib = 0$ if $a = 0$ and $b = 0$.
6. **Equality of complex numbers:**

   $a + bi = c + di \Rightarrow a = c$ and $b = d$

   i.e., two complex numbers are equal when their real parts are equal and their imaginary parts are equal.



7. The complex number $a + bi$ is written as an ordered pair (a, b) and can be represented on a complex plane known as **Argand plane**.
8. **Conjugate of a complex number.** The conjugate of a complex number $z = a + ib$ is defined as $a - bi$ and is denoted by $\overline{z}$. Thus, $z = a + bi \Rightarrow \overline{z} = a - bi$.

   Note that the conjugate of a given complex number is obtained **by reversing the sign of the imaginary part.**

9. **Properties of conjugate numbers**

   1. $\boldsymbol{(\overline{z}) = z}$, *i.e., **the conjugate of the conjugate of a complex number is the number itself.***

   2. $\boldsymbol{(\overline{z})^2 = (\overline{z^2})}$, ***i.e., the square of the conjugate of a complex number is equal to the conjugate of the square of that complex number.***

10. **Modulus or absolute value of a complex number.** The modulus of a complex number $z = a + ib$ is given by $\boldsymbol{|z| = |a + bi| = \sqrt{a^2 + b^2}}$.

    e.g., $|3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

11. **Distance between two points on an Argand plane.** If z_1 and z_2 are complex numbers of two points A and B respectively on the Argand plane. Then, the distance between them is given by

    $$\boldsymbol{AB = |z_1 - z_2|}$$

    e.g., if $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then

    $$AB = |(a_1 + b_1 i) - (a_2 + b_2 i)| = |(a_1 - a_2) + (b_1 - b_2)i|$$

    $$= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

12. Reciprocal of a complex number

If $z = a + bi$, then reciprocal of z is

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

13. Properties of moduli of complex numbers

(1) $|z| = 0 \Leftrightarrow z = 0$

Thus, $|a + bi| = 0 \Rightarrow a + bi = 0 \Rightarrow a = 0, b = 0.$

(2) $z\bar{z} = |z|^2$ (3) $|z| = |\bar{z}|$

(4) $|z_1 z_2| = |z_1| . |z_2|$ (5) $|z^2| = |z|^2$ (6) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$

14. A fractional number $\frac{1}{a+bi}$ in which the complex number appears in the denominator can be put in the standard form $x + yi$ by **multiplying the numerator and denominator by the conjugate of the denominator.**

e.g., $\frac{1}{5+7i} = \frac{1}{5+7i} \cdot \frac{5-7i}{5-7i} = \frac{5-7i}{5^2-(7i)^2} = \frac{5-7i}{25-49i^2} = \frac{5-7i}{25+49} = \frac{5}{74} - i\frac{7}{74}.$

15. Cube roots of unity. The three cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$; one cube root being real and the other two complex.

- Each complex cube root of unity is the square of the other, *i.e.*,

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-1-\sqrt{3}i}{2} \text{ and } \left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \frac{-1+\sqrt{3}i}{2}.$$

- If we denote one of the complex cube roots of unity be ω(omega), then the other one is ω^2.
- The sum of cube roots of unity is zero, i.e., $1 + \omega + \omega^2 = 0$.
- The product of the three cube roots of unity is 1, *i.e.*, $1.\omega.\omega^2 = 1 \Rightarrow \omega^3 = 1$.

CHAPTER TEST

1. Find the square root of $5 - 12i$.

2. Find the locus of a complex number $z = x + yi$, satisfying the relation $|z + i| = |z + 2|$. Illustrate the locus of z in the Argand plane.

3. Express $\frac{13i}{2-3i}$ in the form $A + Bi$.

4. If $z = x + yi$ and $\frac{|z-1-i|+4}{3|z-1-i|-2} = 1$, show that $x^2 + y^2 - 2x - 2y - 7 = 0$.

5. If ω and ω^2 are cube roots of unity, prove that $(2 - \omega + 2\omega^2)(2 + 2\omega - \omega^2) = 9$.

6. If $z_1, z_2 \in C$ (set of complex numbers), prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

7. If $z = x + yi$, $\omega = \frac{2-iz}{2z-i}$ and $|\omega| = 1$, find the locus of z and illustrate it in the complex plane.

8. Simplify: $(1 - 3\omega + \omega^2)(1 + \omega - 3\omega^2)$.

9. Sketch in the complex plane the set of points z satisfying $\left|\frac{z-3}{z+1}\right| = 3$.

10. Given that $\frac{2\sqrt{3}\cos 30° - 2i\sin 30°}{\sqrt{2}(\cos 45° + i\sin 45°)} = A + Bi$, find the values of A and B.

11. Simplify: $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$.

12. Find the locus of a complex number $z = x + yi$, satisfying the relation $|2z + 3i| \geq |2z + 5|$. Illustrate the locus in the Argand plane.

13. Find the real values of x and y satisfying the equality $\frac{x-2+(y-3)i}{1+i} = 1 - 3i$.

14. If $i = \sqrt{-1}$, prove the following: $(x+1+i)(x+1-i)(x-1-i)(x-1+i) = x^4 + 4$.

15. If $z = x + yi$ and $|2z + 1| = |z - 2i|$, show that $3(x^2 + y^2) + 4(x - y) = 3$.

16. Find the amplitude of the complex number $\sin\frac{6\pi}{5} + i\left(1 - \cos\frac{6\pi}{5}\right)$.

17. Express $\frac{1-2i}{2+i} + \frac{3+i}{2-i}$ in the form $a + bi$.

18. Find the value of x and y given that $(x + yi)(2 - 3i) = 4 + i$.

19. If the ratio $\frac{z-i}{z-1}$ is purely imaginary, prove that the point z lies on the circle whose centre is the point $\frac{1}{2}(1+i)$ and radius is $\frac{1}{\sqrt{2}}$.

20. If $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + bi$, find the real numbers a and b. With these values of a and b, also find the modulus of $a + bi$.

21. If 1, ω, ω^2 are the three cube roots of unity, then simplify:

$$(3 + 5\omega + 3\omega^2)^2(1 + 2\omega + \omega^2)$$

22. Find the locus of a complex number $z = x + yi$, satisfying the relation $|3z - 4i| \leq |3z + 2|$. Illustrate the locus in the Argand plane.

23. Find the modulus and argument of the complex number $\frac{2+i}{4i + (1+i)^2}$.

24. If $|z - 3 + i| = 4$, then the locus of z is ***(Raj. PET 2004, Kerala PET 2000)***

(*a*) $x^2 + y^2 - 6 = 0$ (*b*) $x^2 + y^2 - 3x + y - 6 = 0$

(*c*) $x^2 + y^2 - 6x - 2 = 0$ (*d*) $x^2 + y^2 - 6x + 2y - 6 = 0$

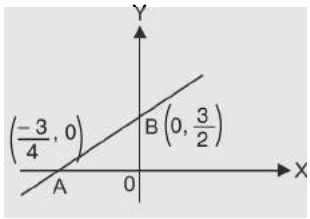
25. The locus of the point z is the Argand plane for which $|z + 1|^2 + |z - 1|^2 = 4$ is a ***(EAMCET 2000)***

(*a*) Straight line (*b*) Pair of straight lines

(*c*) Parabola (*d*) Circle

ANSWERS

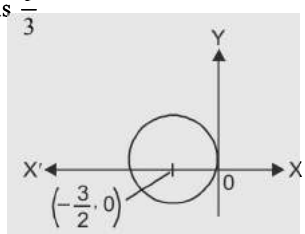
1. $\pm(3 - 2i)$



2. $4x - 2y + 3 = 0$, a straight line.

3. $-3 + 2i$

7. $3x^2 + 3y^2 - 8y - 3 = 0$. This is a circle whose centre is $\left(0, \frac{4}{3}\right)$ and radius is $\frac{5}{3}$

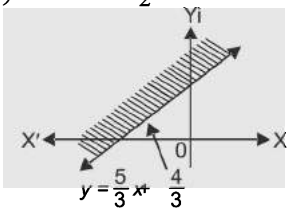


8. 16

9. $x^2 + y^2 + 3x = 0$. A circle with centre $\left(-\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$ units.

10. $A = 1, B = -2$

11. 9



12. $3y \geq 5x + 4$

13. $x = 6, y = 1$

16. $\frac{3\pi}{5}$

17. $1 + 0i$

18. $x = \frac{5}{13}, y = \frac{14}{13}$

20. $a = 0, b = -7\sqrt{3}$, Modulus $= 7\sqrt{3}$

21. 4.

22. $x + 2y = 1$

23. Modulus $= \frac{\sqrt{5}}{6}$; amplitude $= \tan^{-1} 2$.

24. (*d*)

25. (*d*)

HINTS TO SELECTED QUESTIONS

2. $|x + yi + i| = |x + yi + 2| \Rightarrow \sqrt{x^2 + (y+1)^2} = \sqrt{(x+2)^2 + y^2}$

4. $|z - 1 - i| + 4 = 3|z - 1 - i| - 2 \Rightarrow 2|z - 1 - i| = 6$

$\Rightarrow \quad |x + yi - 1 - i| = 3 \Rightarrow \sqrt{(x-1)^2 + (y-1)^2} = 3$

6. Property 8 Ch 9-30.

7. $|\omega| = 1 \Rightarrow \left|\frac{2 - zi}{2z - i}\right| = 1 \Rightarrow \frac{|(2 + y) - xi|}{|2x + i(2y - 1)|} = 1 \Rightarrow |2x + i(2y - 1)| = |(2 + y) - ix|$

$\Rightarrow \qquad \sqrt{(2x)^2 + (2y - 1)^2} = \sqrt{(2 + y)^2 + (-x)^2}$

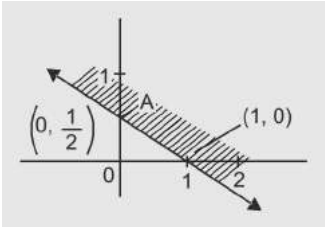
10. Given $\Rightarrow \frac{3 - i}{1 + i} = A + Bi \Rightarrow 3 - i = (A - B) + (A + B)i$

12. $|2x + (2y + 3)i| \geq |(2x + 5) + 2yi| \Rightarrow \sqrt{4x^2 + (2y + 3)^2} \geq \sqrt{(2x + 5)^2 + 4y^2}$.

14. LHS $= [(x + 1)^2 - i^2][(x - 1)^2 - i^2]$

16. Amplitude $\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1-\cos\frac{6\pi}{5}}{\sin\frac{6\pi}{5}} = \tan^{-1}\frac{2\sin^2\frac{3\pi}{5}}{2\sin\frac{3\pi}{5}\cos\frac{3\pi}{5}} = \tan^{-1}\left(\tan\frac{3\pi}{5}\right) = \frac{3\pi}{5}$.

19. $\frac{z-i}{z-1} = \frac{x+yi-i}{x+yi-1} = \frac{x+(y-1)i}{(x-1)+yi} \times \frac{(x-1)-yi}{(x-1)-yi} = \frac{(x^2+y^2-x-y)+(1-x-y)i}{(x-1)^2+y^2}$.



For $\frac{z-i}{z-1}$ to be purely imaginary, $x^2+y^2-x-y=0$. It is a circle whose centre is $\left(\frac{1}{2},\frac{1}{2}\right)$, *i.e.*, the point $\frac{1}{2}(1+i)$ and radius $\sqrt{\frac{1}{4}+\frac{1}{4}-0} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$.

22. $|3z-4i| \le |3z+2| \Rightarrow |3x+(3y-4)i| \le |(3x+2)+3yi|$

$\Rightarrow 9x^2+(3y-4)^2 \le (3x+2)^2+9y^2 \Rightarrow x+2y-1 \ge 0.$

CHAPTER 10

QUADRATIC EQUATIONS

10.01. Quadratic functions—introduction

Quadratic function involves the second power of the variable but no higher power. The kinetic energy (K.E.) of a moving object such as a ball or bullet is given by K.E. = $\frac{1}{2}mv^2$, where m represents the mass and v the velocity of the object. When the mass is 2 units, the equation is K.E. = v^2. The table below contains several pairs of numbers that satisfy the equation, K.E. = v^2.

v	0	1	2	3	4	5
K.E.	0	1	4	9	16	25

The diagram at the right shows the six points yielded by the above table and a smooth curve drawn through them. The graph is a model of a quadratic function where the domain for v is the set of non-negative real numbers since only non-negative velocity was being considered in the problem.

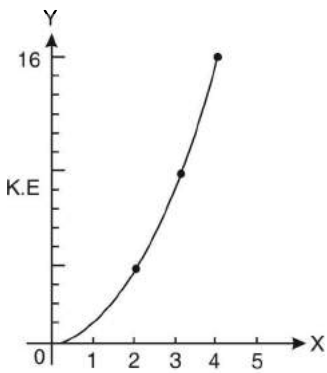


Fig. 10.01

In general, A quadratic function is a function that may be defined by the equation $y = ax^2 + bx + c$, where a, b, and c are complex numbers, constants or parameters, but $a \neq 0$. Notice that the restriction $a \neq 0$ assures that $y = ax^2 + bx + c$ defines a non-linear function.

The quadratic function may be expressed in terms of several models:

The following equations define quadratic functions

$(i)\ y = x^2 \quad (ii)\ y = -5x^2 \quad (iii)\ y = \left(\frac{1}{2}x - 4\right)^2 \quad (iv)\ y = x^2 - 7x + 8$

Note 1: In this book, we shall take the domain of all quadratic functions to be the set of real numbers, unless otherwise specified. We shall show in this chapter that if the domain of a quadratic function is the set of real numbers, the graph is a curve called a parabola. A parabola is an open curve that extends infinitely, becoming wider and wider. The path followed by a bullet or a rocket is a parabola. If you completed the curve formed by the spray of water from the fountain, you would get a parabola.

10.02. Quadratic equation

Def. An equation in which the highest power of the variable is 2 is called a quadratic equation. $ax^2 + bx + c = 0$, where a, b, c are constants, is a general quadratic equation.

10.03. Theorem

Show that a quadratic equation cannot have more than two roots.

If possible, let the quadratic equation $ax^2 + bx + c = 0$ have three different roots, α, β and γ. Then, by definition of a root, we have

$$a\alpha^2 + b\alpha + c = 0 \quad ...(1)$$

$$a\beta^2 + b\beta + c = 0 \quad ...(2)$$

$$a\gamma^2 + b\gamma + c = 0 \quad ...(3)$$

Subtracting (2) from (1), we have

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0 \text{ or } (\alpha - \beta)[a(\alpha + \beta) + b] = 0$$

Since $(\alpha - \beta) \neq 0$ as α and β are different, therefore

$$a(\alpha + \beta) + b = 0 \quad ...(4)$$

Similarly, from (2) and (3) we have

$$a(\beta + \gamma) + b = 0 \quad ...(5)$$

Subtracting (5) from (4), we have

$$a(\alpha - \gamma) = 0 \quad ...(6)$$

which is impossible because neither $a = 0$, in which case the given equation ceases to be quadratic, nor $\alpha = \gamma$ as α and γ are given to be different. Hence, a quadratic equation cannot have more than two roots.

10.04. Solution of the equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$

In earlier classes, you solved quadratic equations with real coefficients and real roots by factorizaton and also by using formula. In this book, our focus will be on quadratic equations with real coefficients and both real and complex roots.

10.05. The quadratic formula for solving quadratic equations

Let the quadratic equation be

$$ax^2 + bx + c = 0 \text{ where } a, b, c, \in R \text{ and } (a \neq 0).$$

$\Rightarrow$ $ax^2 + bx = -c$ (Transposing the constant term)

$\Rightarrow$ $x^2 + \frac{b}{a}x = \frac{-c}{a}$ (Dividing by the coefficient of x^2)

$\Rightarrow$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

(Adding $\frac{b^2}{4a^2}$ to both sides to make L.H.S. a perfect square.)

$\Rightarrow$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ or $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Hence, the roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The roots of the equation are also called the *zeros of the function* defined by $f(x) = ax^2 + bx + c$.

Remark. The roots would be real or complex depending on whether the expression under the root sign, *i.e.*, $\sqrt{b^2 - 4ac}$ is positive or negative.

Historical Note. The first reference to the solution of quadratic equations was found in the Ahmes Papyrus written about 1650 B.C. Certain quadratic equations were solved by the Greeks, but their solutions were geometric and not algebraic. The first algebraic solution of the general equation $ax^2 + bx + c = 0$ ($a \neq 0$), showing the roots to be $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ was given by a Hindu, Sridhara, about 1020 A.D.

Ex. 1. Solve : $2x^2 + 2x - 3 = 0$, giving your answer correct to one decimal place.

Sol. Here, $a = 2, b = 2, c = -3$

$$\text{Using } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times (-3)}}{2 \times 2} = \frac{-2 \pm \sqrt{4 + 24}}{4}$$

$$= \frac{-2 \pm \sqrt{28}}{4} = \frac{-2 \pm 2\sqrt{7}}{4} = \frac{-1 \pm \sqrt{7}}{2} = \frac{-1 \pm 2.645}{2} = \frac{-3.645}{2}, \frac{1.645}{2}$$

$= -1.8225, 0.8225 = -1.8, 0.8$, correct to one decimal place.

Ex. 2. Solve : $x^2 + x + 1 = 0$.

Sol. Comparing the equation with $ax^2 + bx + c = 0$, we see that $a = 1$, $b = 1$, $c = 1$.

$$\text{Using } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ we have}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

Hence, the equation have complex roots $\dfrac{-1 + \sqrt{3}i}{2}$ and $\dfrac{-1 - \sqrt{3}i}{2}$.

EXERCISE 10 (*a*)

Find the roots of the following equations.

1. $2x^2 + x - 3 = 0$ **2.** $6x^2 + 7x - 20 = 0$ **3.** $36x^2 + 23 = 60x$

4. $x^2 - 2x + 5 = 0$ **5.** $3x^2 - 17x + 25 = 0$

6. $x^2 + 3x - 3 = 0$, giving your answer correct to two decimal places.

7. $5x^2 - x + 4 = 0$ **8.** $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$ [***NCERT***]

9. $\dfrac{x^2 + 8}{11} = 5x - x^2 - 5$ **10.** $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = 8\dfrac{1}{3}$

11. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$ is (*a*) 3 (*b*) 4 (*c*) 1 (*d*) 3. [***AIEEE 2003***]

[**Hint.** $x^2 - 3|x| + 2 = 0 \Rightarrow (|x|)^2 - 3|x| + 2 = 0 \Rightarrow (|x| - 2)(|x| - 1) = 0$

$\Rightarrow |x| = 1, 2 \Rightarrow x = -1$ or $+1$ or -2 or 2]

ANSWERS

1. $1, \frac{-3}{2}$ **2.** $1\frac{1}{3}, -2\frac{1}{2}$ **3.** $\frac{5 \pm \sqrt{2}}{6}$

4. $1 \pm 2\sqrt{-1}$ **5.** $\frac{17 \pm \sqrt{-11}}{6}$ **6.** $-3.79, 0.79$ **7.** $\frac{1 \pm \sqrt{79}i}{10}$

8. $\frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$ **9.** $2\frac{1}{3}, 2\frac{1}{4}$ **10.** $6, 3\frac{1}{13}$ **11.** (b)

10.06. Equations reducible to quadratic equations

We assume that the student is quite conversant with the solution of the quadratic equation and that he is quite familiar with the application of the formula obtained in Art. 10.04 above. Now, we give below some of the types of equations which can be reduced to quadratic equations.

10.07. Type I. In the equation of the type $ax^{2n} + bx^n + c = 0$.

If x^n is put equal to y, the equation becomes $ay^2 + by + c = 0$ which is a quadratic equation.

Ex. 3. Solve: $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0$. **[*AI CBSE*]**

Sol. Putting $x^{\frac{1}{3}} = y$, we get $y^2 + y - 2 = 0 \quad \therefore \quad (y + 2)(y - 1) = 0$

or $y = 1$ and -2, or $x^{\frac{1}{3}} = 1$ and -2

Cubing both sides, we get $x = 1$ and -8.

Ex. 4. If $4^{1+x} + 4^{1-x} = 10$, find x.

Sol. $4^1 \times 4^x + 4^1 \times 4^{-x} = 10$

or $4 \times 4^x + \frac{4}{4^x} = 10$, or $4 \times 4^{2x} + 4 = 10 \times 4^x$, or $4 \times 4^{2x} - 10 \times 4^x + 4 = 0$

Putting $4^x = y$, we get $4y^2 - 10y + 4 = 0$

or $2y^2 - 5y + 2 = 0$, or $(y - 2)(2y - 1) = 0$, or $y = 2, \frac{1}{2}$

or $4^x = 2$ and $\frac{1}{2}$; $\Rightarrow \quad 2^{2x} = 2^1$ or $2^{-1} \quad \therefore 2x = 1$ or $-1 \Rightarrow x = \frac{1}{2}, -\frac{1}{2}$.

10.08. Type II. $aZ + \frac{b}{Z} = c$, where a, b, c are constants

Ex. 5. If $\sqrt{\frac{2x^2 + x + 2}{x^2 + 3x + 1}} + 2\sqrt{\frac{x^2 + 3x + 1}{2x^2 + x + 2}} - 3 = 0$, **find x.**

Sol. Let $\sqrt{\frac{2x^2 + x + 2}{x^2 + 3x + 1}} = y$, then $y + 2 \times \frac{1}{y} - 3 = 0$

or $y^2 + 2 - 3y = 0$ or $y^2 - 3y + 2 = 0$ or $(y - 2)(y - 1) = 0$ or $y = 1$ and 2.

$(i) \quad \therefore \sqrt{\frac{2x^2 + x + 2}{x^2 + 3x + 1}} = 1 \Rightarrow \frac{2x^2 + x + 2}{x^2 + 3x + 1} = 1$ (squaring)

$\Rightarrow 2x^2 + x + 2 = x^2 + 3x + 1 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1.$

(*ii*) $\sqrt{\dfrac{2x^2+x+2}{x^2+3x+1}}=2 \Rightarrow \dfrac{2x^2+x+2}{x^2+3x+1}=4$ (squaring) $\Rightarrow 2x^2+11x+2=0$

$\Rightarrow x=\dfrac{-11\pm\sqrt{121-16}}{4}=\dfrac{-11\pm\sqrt{105}}{4}$. Hence, $x=1, \dfrac{-11\pm\sqrt{105}}{4}$.

10.09. Type III. Equations of the type (x + a) (x + b) (x + c) (x + d) + k = 0

When the sum of two of the quantities a, b, c, d is equal to the sum of the other two, it can be solved as shown below.

Ex. 6. Solve: $(x+1)(x+2)(x+3)(x+4)+1=0$.

Sol. $(x+1)(x+2)(x+3)(x+4)+1=0$.

$[(x+1)(x+4)][(x+2)(x+3)]+1=0$ [$\because 1+4=2+3$]

$\therefore \quad (x^2+5x+4)(x^2+5x+6)+1=0$

Let $x^2+5x=y$, then $(y+4)(y+6)+1=0$

or $y^2+10y+25=0$ or $(y+5)^2=0$ or $y=-5$

$\therefore \quad x^2+5x=-5$ or $x^2+5x+5=0$

or $x=\dfrac{-5\pm\sqrt{25-20}}{2}=\dfrac{-5\pm\sqrt{5}}{2}$.

EXERCISE 10 (*b*)

Solve the following equations:

1. $x^4-5x^2+6=0$. **2.** $x^5+242=\dfrac{243}{x^5}$. **3.** $10x^{-2}-9-x^{-4}=0$.

4. $3^{2x}-10\times 3^x+9=0$. **5.** $2^{2x-1}-9\times 2^{x-2}+1=0$. **6.** $3^{2x+1}+3^2=3^{x+3}+3^x$. **[*ISC*]**

7. $\sqrt{x^2-3x}=4x^2-12x-3$ **8.** $\sqrt{\dfrac{x^2+2}{x^2-2}}+6\sqrt{\dfrac{x^2-2}{x^2+2}}=5$. **9.** $\sqrt{\dfrac{2x^2+1}{x^2-1}}+6\sqrt{\dfrac{x^2-1}{2x^2+1}}=5$.

10. $x(x-1)(x+2)(x-3)+8=0$. **11.** $(x-7)(x-3)(x+1)(x+5)=1680$.

12. $(2x-7)(x^2-9)(2x+5)=91$.

13. By substituting $y=2^x$, or otherwise, solve the equation $2^{2x}+2^{x+2}-4\times 2^3=0$. **[*ISC*]**

14. $2^{x^2}:2^x=8:1$. **15.** $2^{2x+3}+2^{x+3}=1+2^x$ **[*ISC*]** **16.** $4^x-3^{x-\frac{1}{2}}=3^{x+\frac{1}{2}}-2^{2x-1}$. **[*ISC*]**

ANSWERS

1. $\pm\sqrt{3}, \pm\sqrt{2}$ **2.** -3 or 1 **3.** $\pm 1, \pm\dfrac{1}{3}$ **4.** $2, 0$

5. 2 or -1 **6.** 2 or -1 **7.** $\dfrac{3\pm\sqrt{13}}{2}, \dfrac{6\pm 3\sqrt{5}}{4}$

8. $\pm\sqrt{\dfrac{5}{2}}, \pm\sqrt{\dfrac{10}{3}}$ **9.** $\pm\sqrt{\dfrac{5}{2}}, \pm\sqrt{\dfrac{10}{7}}$ **10.** $2, -1, \dfrac{1\pm\sqrt{17}}{2}$

11. $9, -7, 1\pm\sqrt{-24}$ **12.** $4, \dfrac{-7}{2}, \dfrac{1\pm\sqrt{65}}{4}$ **13.** $x=2$

14. $\dfrac{1\pm\sqrt{13}}{2}$ **15.** -3 **16.** $\dfrac{3}{2}$

Remark. Before we proceed further, we would like to state the properties of inequalities without proofs. The student can verify the truth of the statements by taking some numerical examples.

Properties of inequalities

1. *An inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same* **Positive** *quantity.*

2. *In an inequality any term may be transposed from one side to the other if its sign is changed.*

3. *Both sides of an inequality can be multiplied or divided by the same* **Negative** *number so long as the inequality sign is* **reversed.**

10.10. Nature or character of the roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Examining the nature of the roots means to see what type of roots the equation has, that is, whether they are real or imaginary, rational or irrational, equal or unequal. The nature of the roots depends entirely on the value of the expression $b^2 - 4ac$ which is called the **discriminant** (Abbrev : disct.) of the equation. Thus : if a, b, c are rational,

1. the roots are real and unequal if $b^2 - 4ac$ is positive.

Further,

(a) The roots are rational if $b^2 - 4ac$ is a perfect square.

(b) The roots are irrational if $b^2 - 4ac$ is not a perfect square.

(c) The roots are equal if $b^2 - 4ac$ is zero, each root being equal to $-\frac{b}{2a}$. It follows that $ax^2 + bx + c$ is a perfect square if $b^2 - 4ac = 0$.

2. The roots are imaginary if $b^2 - 4ac$ is negative, *i.e.*, the roots are complex.

Ex. 7. Examine the nature of the roots of the equations

(i) $2x^2 + 2x + 3 = 0$ **(ii) $2x^2 - 7x + 3 = 0$**

(iii) $x^2 - 5x - 2 = 0$ **(iv) $4x^2 - 4x + 1 = 0$**

Sol. (i) $2x^2 + 2x + 3 = 0$

Disct. $= (2)^2 - 4 \times 2 \times 3 = -20$, is negative. [$a = 2, b = 2, c = 3$]

$\Rightarrow$ The roots are complex.

(ii) $2x^2 - 7x + 3 = 0$

Disct. $= (-7)^2 - 4 \times 2 \times 3 = 25$, is + ve and a perfect square.

$\therefore$ The roots are real and rational.

(iii) $x^2 - 5x - 2 = 0$

Disct. $= (-5)^2 - 4 \times 1 \times (-2)$

$= 25 + 8 = 33$, is +ve and not a perfect square.

$\therefore$ The roots are real and irrational.

(iv) $4x^2 - 4x + 1 = 0$

Disct. $= (-4)^2 - 4 \times 4 \times 1 = 16 - 16 = 0$.

$\therefore$ The roots are real and equal.

Ex. 8. If a, b, c are real, then both the roots of the equation $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are always (A) positive (B) negative (C) real (D) imaginary.

[*WBJEE 2009*]

Sol. The given equation is

$$x^2-(b+c)\,x+bc+x^2-(c+a)\,x+ca+x^2-(a+b)\,x+ab=0$$

i.e., $3x^2-2(a+b+c)\,x+ab+bc+ca=0$

Disct. $(D)=4\,(a+b+c)^2-4.3\,(ab+bc+ca)=4\,(a^2+b^2+c^2-ab-bc-ca)$

$$=2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$\Rightarrow$ $D \geq 0$ (Since the expression inside being the sum of perfect squares is always +ve)

Hence, the roots of the given equation are real.

Ex. 9. For what value of *m*, are the roots of the equation $(3m+1)\,x^2+(11+m)\,x+9=0$ equal? real and unequal?

Sol. $b^2-4ac=(11+m)^2-4\,(3m+1)\,9=m^2-86m+85$

(*a*) For equal roots, $b^2-4ac=0$, *i.e.,* $m^2-86m+85=0$

or $(m-85)\,(m-1)=0 \Rightarrow m=85$ or 1.

(*b*) For real and unequal roots, $b^2-4ac>0$, *i.e.,* $(m-85)\,(m-1)>0$

Putting each factor equal to 0, we get $m=1$ or 85

Make a table showing the signs of the factors in the product $(m-85)\,(m-1)$.

	$m<1$	$m=1$	$1<m<85$	$m=85$	$m>85$
$m-1$	–	0	+	+	+
$m-85$	–	–	–	0	+
$(m-1)\,(m-85)$	+	**0**	–	**0**	+

From the table, we can see that

$(m-85)\,(m-1)>0$ when $m<1$ or $m>85$.

$(m-85)\,(m-1)<0$ when $1<m<85$.

Alternatively, you can find the values by the method of intervals discussed in Art. 10.02.

(*c*) For imaginary roots, $b^2-4ac<0$, *i.e.*, $(m-85)\,(m-1)<0$

It follows from the table that $1<m<85$.

10.11. Sum and product of the roots

If the two roots of the quadratic equation $ax^2+bx+c=0$, obtained in Art. 10.04, be denoted by α and β, we have

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

i.e. $$\alpha+\beta=\frac{-b}{a}$$

$$\boxed{\alpha+\beta=\frac{-b}{a}\qquad \alpha\beta=\frac{c}{a}}$$

$$\alpha\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}\times\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}=\frac{b^2-(b^2-4\,ac)}{4a^2}=\frac{4\,ac}{4\,a^2};\ \textit{i.e.,}\ \alpha\beta=\frac{c}{a}.$$

Aid to Memory : In a given quadratic equation,

(1) Sum of the roots $= -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } x^2}$

(2) Product of the roots $= \dfrac{\text{Absolute term}}{\text{Coeff. of } x^2}$

Thus, if α, β be the roots of the equation $6x^2 - 5x + 7 = 0$,

$$\alpha + \beta = -\left(\frac{-5}{6}\right) = \frac{5}{6}, \quad \alpha\beta = \left(\frac{7}{6}\right).$$

10.12. Values of the symmetric expressions of the roots

If α, β be the roots of a given quadratic equation and we wish to find the value of the function of α and β, we can do so by proceeding as follows:

Procedure : 1. *Write the values of* $\alpha + \beta$ *and* $\alpha\beta$ *from the given equation.*

2. *Express the given function in terms of* $\alpha + \beta$ *and* $\alpha\beta$.

3. *Substitute the values of* $\alpha + \beta$ *and* $\alpha\beta$ *from I.*

Caution. We do not find the values of α and β separately.

We use the following relations :

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \qquad ...(i)$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta \qquad ...(ii)$$

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \qquad ...(iii)$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \qquad ...(iv)$$

$$\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = (\alpha - \beta)[(\alpha - \beta)^2 + 3\alpha\beta]$$
$$= (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta) \qquad ....(v)$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$
$$= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2 \qquad ....(vi)$$

$$\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = (\alpha - \beta)(\alpha + \beta)(\alpha^2 + \beta^2)$$
$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\ (\alpha + \beta)[(\alpha + \beta)^2 - 2\alpha\beta] \qquad ...(vii)$$

Ex. 10. If α, β are roots of $x^2 - px + q = 0$, find the values of

(*i*) $\alpha^2 + \beta^2$ **(*ii*)** $\alpha^3 + \beta^3$ **(*iii*)** $\alpha - \beta$ **(*iv*)** $\alpha^4 + \beta^4$**.**

Sol. $\because$ α, β are the roots of $x^2 - px + q = 0 \Rightarrow \alpha + \beta = p,\ \alpha\beta = q$

(*i*) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q.$

(*ii*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = p^3 - 3pq.$

(*iii*) $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{p^2 - 4q}$, if $\alpha > \beta$; $= -\sqrt{p^2 - 4q}$, if $\alpha < \beta$.

(*iv*) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2$
$= (p^2 - 2q)^2 - 2q^2 = p^4 - 4p^2q + 4q^2 - 2q^2 = p^4 - 4p^2q + 2q^2.$

10.13. Formation of equations

Suppose we have to form the equation whose roots are α and β.

As $x = \alpha$, $x = \beta$, are the roots of the equation, so $x - \alpha = 0$ and $x - \beta = 0$:

$\therefore$ $(x - \alpha)(x - \beta) = 0$ or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e., **x^2 – (sum of the roots) x + product of the roots = 0.**

The **procedure** for forming an equation whose roots are given is as follows:

1. *Find the sum of the given roots.* 2. *Find their product.*

3. *Then, the required equation is* x^2 – *(sum of the roots)*x + *product of the roots* = 0.

Thus, the equation whose roots are 5 and 7 is $x^2 - (7 + 5)\,x + 7 \times 5 = 0$

i.e., $x^2 - 12x + 35 = 0$.

To form an equation whose roots are some functions of the roots of the given equation

Ex. 11. If α and β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Sol. α, β are the roots of $ax^2 + bx + c = 0$,

so, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.

Sum of the roots of the required equation $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$

Product of the roots of the required equation $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{a}{c}$

$\therefore$ The equation required is $x^2 - \left(-\frac{b}{c}\right) x + \frac{a}{c} = 0$ **Note that 'a' and 'c' have interchanged their positions.**

or $cx^2 + bx + a = 0$.

Note. The equation found above has roots which are the reciprocals of the roots of the equation $ax^2 + bx + c = 0$.

Ex. 12. If α, β are the roots of $ax^2 + bx + c = 0$, form that equation whose roots are $(\alpha^2 + \beta^2)$, $\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$.

Sol. $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

Sum of the roots of the required equation $= \alpha^2 + \beta^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} = (\alpha^2 + \beta^2) + \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$$= (\alpha + \beta)^2 - 2\alpha\beta + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = \frac{b^2}{a^2} - \frac{2c}{a} + \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}}$$

$$= \frac{b^2 - 2ac}{a^2} + \frac{b^2 - 2ac}{c^2} = \frac{(a^2 + c^2)(b^2 - 2ac)}{a^2c^2}$$

Product of the roots of the required equation $= (\alpha^2 + \beta^2)\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = (\alpha^2 + \beta^2)\left(\frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}\right)$

$$= \frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2} = \frac{[(\alpha+\beta)^2-2\alpha\beta]^2}{\alpha^2\beta^2} = \frac{\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)^2}{\frac{c^2}{a^2}} = \frac{(b^2-2ac)^2}{a^2c^2}$$

Hence, the required equation is

$$x^2 - \frac{(a^2+c^2)(b^2-2ac)}{a^2c^2}x + \frac{(b^2-2ac)^2}{a^2c^2} = 0$$

or $$a^2c^2x^2 - (a^2+c^2)(b^2-2ac)x + (b^2-2ac)^2 = 0.$$

EXERCISE 10 (*c*)

1. Without solving, find the nature of the roots of the following equations:
(*i*) $3x^2 - 7x + 5 = 0$. (*ii*) $4x^2 + 4x + 1 = 0$. (*iii*) $3x^2 + 7x + 2 = 0$. (*iv*) $x^2 + px - q^2 = 0$.

2. If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, show that $c^2 = a^2(1 + m^2)$.

3. Find the value of m so that the roots of the equation $(4 - m)x^2 + (2m + 4)x + (8m + 1) = 0$ may be equal.

4. If the roots of $ax^2 + x + b = 0$ be real and unequal, show that the roots of $\frac{x^2+1}{x} = 4\sqrt{ab}$ are imaginary.

5. Find a so that the sum of the roots of the equation $ax^2 + 2x + 3a = 0$ may be equal to their product.

6. If α, β are the roots of the equation $x^2 + x + 1 = 0$, find the value of $\alpha^3 + \beta^3$.

7. If α, β are the roots of the equation $x^2 + px + q = 0$, find the value of
(*a*) $\alpha^3\beta + \alpha\beta^3$ (*b*) $\alpha^4 + \alpha^2\beta^2 + \beta^4$.

8. If the roots of the equation $x^2 + px + 7 = 0$ are denoted by α and β, and $\alpha^2 + \beta^2 = 22$, find the possible values of p. **[*SC*]**

9. If α, β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of

$$\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 3\alpha\beta.$$

10. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of

(*i*) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ (*ii*) $\frac{\alpha^3}{\beta}+\frac{\beta^3}{\alpha}$

11. If the sum of the roots of the equation $x^2 - px + q = 0$ be m times their difference, prove that $p^2(m^2 - 1) = 4m^2q$.

12. If one root of the equation $x^2 + ax + 8 = 0$ is 4 while the equation $x^2 + ax + b = 0$ has equal roots, find b.

13. Find the value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other. **[*AIEEE 2003*]**

14. If α, β are the roots of the equation $ax^2 - bx + b = 0$, prove that

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{b}{a}} = 0.$$

15. If α and β are the roots of the equation $x^2 + x - 7 = 0$, form the equation whose roots are α^2 and β^2. **[*SC*]**

16. If α and β are the roots of the equation $2x^2 + 3x + 2 = 0$, find the equation whose roots are $\alpha + 1$ and $\beta + 1$. **[*SC*]**

17. Find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$, where α and β are the roots of the equation $x^2 + 2x + 3 = 0$. **[*SC*]**

18. If α and β are the roots of the equation $2x^2 - 3x + 1 = 0$, form the equation whose roots are $\frac{\alpha}{2\beta + 3}$ and $\frac{\beta}{2\alpha + 3}$.

19. If $a \neq b$ and $a^2 = 5a - 3$, $b^2 = 5b - 3$, then form that equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$.

20. Given that α and β are the roots of the equation $x^2 = x + 7$.

(*i*) Prove that (*a*) $\frac{1}{\alpha} = \frac{\alpha - 1}{7}$ and (*b*) $\alpha^3 = 8\alpha + 7$.

(*ii*) Find the numerical value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$. **[*SC*]**

21. Given that α and β are the roots of the equation $x^2 - x + 7 = 0$, find

(*i*) the numerical value of $\frac{\alpha}{\beta + 3} + \frac{\beta}{\alpha + 3}$;

(*ii*) an equation whose roots are $\frac{\alpha}{\beta + 3}$ and $\frac{\alpha}{\beta + 3}$.

22. Given that α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, find an equation whose roots are $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$. **[*ISC*]**

23. The roots of the quadratic equation $x^2 + px + 8 = 0$ are α and β. Obtain the values of p, if

(*i*) $\alpha = \beta^2$ (*ii*) $\alpha - \beta = 2$. **[*ISC*]**

24. If the roots of $x^2 - bx + c = 0$ be two consecutive integers, then find the value of $b^2 - 4c$.
[*AIEEE 2005*]

25. The roots of the equation $px^2 - 2(p + 2)x + 3p = 0$ are α and β. If $\alpha - \beta = 2$, calculate the value of α, β and p. **[*ISC*]**

26. The roots of the equation $ax^2 + bx + c = 0$ are α and β. Form the quadratic equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.

27. Two candidates attempt to solve a quadratic equation of the form $x^2 + px + q = 0$.
One starts with a wrong value of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots to be 2 and – 9. Find the correct roots and the equation.
[*ISC*]

28. Given that α and β are the roots of the equation $x^2 = 7x + 4$,

(*i*) show that $\alpha^3 = 53\alpha + 28$ (*ii*) find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$. **[*ISC*]**

29. The ratio of the roots of the equation $x^2 + \alpha x + \alpha + 2 = 0$ is 2. Find the values of the parameter α. **[*ISC*]**

30. If $(1 - p)$ is a root of the quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are

(*a*) 0, –1 (*b*) –1, 1 (*c*) 0, 1 (*d*) – 1, 2 **[*AIEEE 2004*]**

ANSWERS

1. (*i*) Imaginary (*ii*) Real and equal (*iii*) Real and unequal and rational.
(*iv*) Real and unequal if $p \neq 0, q \neq 0$; Real and equal if $p = q = 0$. **3.** $m = 0, 3$.

5. $-\frac{2}{3}$ **6.** 2. **7.** $q(p^2 - 2q), p^4 + 3q^2 - 4p^2q$. **8.** ± 6

9. 8. **10.** (*i*) $\frac{b^2(b^2 - 4ac)}{a^2c^2}$ (*ii*) $\frac{b^4 + 2a^2c^2 - 4ab^2c}{a^3c}$.

12. 9. **13.** $\frac{2}{3}$ **15.** $x^2 - 15x + 49 = 0$. **16.** $2x^2 - x + 1 = 0$.

17. $3x^2 + 2x + 3 = 0$. **18.** $40x^2 - 14x + 1 = 0$. **19.** $3x^2 - 19x + 3 = 0$.

20. (*ii*) $\frac{-15}{7}$. **21.** (*i*) $\frac{-10}{19}$. (*ii*) $19x^2 + 10x + 7 = 0$.

22. $8x^2 - 18x + 13 = 0$. **23.** (*i*) $p = -6$, (*ii*) $p = \pm 6$. **24.** 1

25. $\alpha = 3, \beta = 1, p = 2$, or $\alpha = -1, \beta = -3, p = \frac{-2}{3}$. **26.** $acx^2 + b(c + a)x + (c + a)^2 = 0$.

27. $-3, -4$; $x^2 + 7x + 12 = 0$. **28.** (*ii*) $\frac{-57}{4}$ **29.** $\alpha = 6, \frac{-3}{2}$. **30.** (*a*)

HINTS TO SELECTED QUESTIONS

13. $\alpha + 2\alpha = 3\alpha = \frac{1 - 3a}{a^2 - 5a + 3}$, $2\alpha^2 = \frac{2}{a^2 - 5a + 3}$. Eliminate α.

19. a, b are the roots of the equation $x^2 = 5x - 3$.

20. $\alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha$; $\alpha\beta = 7 \Rightarrow \beta = \frac{-7}{\alpha}$ (*a*) $\therefore 1 - \alpha = \frac{-7}{\alpha} \Rightarrow \frac{1}{\alpha} = \frac{1 - \alpha}{-7} = \frac{\alpha - 1}{7}$

(*b*) $\frac{1}{\alpha} = \frac{\alpha - 1}{7} \Rightarrow \alpha^2 = 7 + \alpha$ ...(1)

Also, $\frac{1}{\alpha} = \frac{\alpha - 1}{7} \Rightarrow \alpha^2\left(\frac{1}{\alpha}\right) = \left(\frac{\alpha - 1}{7}\right)\alpha^2 \Rightarrow \alpha = \frac{\alpha^3 - \alpha^2}{7}$

$\Rightarrow \alpha^3 = 7\alpha + \alpha^2 \Rightarrow \alpha^3 = 7\alpha + (7 + \alpha)$. (Using (1)

24. Let the roots be $\alpha, \alpha + 1$. Then $\alpha + (\alpha + 1) = b$, $\alpha.(\alpha + 1) = c$.

27. $(x - 2)(x - 6) \equiv x^2 + px + q \Rightarrow p = -8, \overset{\checkmark}{q} = 12$

$(x - 2)(x + 9) = x^2 + px + q \Rightarrow \overset{\checkmark}{p} = 7, q = -18$

$\Rightarrow$ The values of p and q are 7 and 12.
$\Rightarrow$ The correct equation is $x^2 + 7x + 12 = 0$
$\Rightarrow$ The correct roots are $-3, -4$.

30. Since $(1 - p)$ is a root of the given equation, therefore,
$(1 - p)^2 + p(1 - p) + 1 - p = 0 \Rightarrow (1 - p)(1 - p + p + 1) = 0 \Rightarrow p = 1$
$\therefore$ The equation is $x^2 + x = 0 \Rightarrow x = 0, -1$.

10.14. To find the condition when a relation between the two roots is given

Procedure. 1. *Let one root be* α. *Write the other root using the given relation.*
2. *Write the sum and the product of the roots.*
3. *Eliminate* α *from the two relations so obtained.*
The eliminate is the condition required.

Ex. 13. Find the condition that one root of $ax^2 + bx + c = 0$ may be four times the other.

Sol. Let the roots be α and 4α. Then,

$$\alpha + 4\alpha = 5\alpha = -\frac{b}{a} \quad ...(1) \text{ and } \alpha \times 4\alpha = 4\alpha^2 = \frac{c}{a} \quad ...(2)$$

From (1), $\alpha = -\dfrac{b}{5a}$

Substituting the value of α in (2),

$$4 \times \frac{b^2}{25a^2} = \frac{c}{a}, \text{ } i.e., \; 4b^2 = 25\,ac,$$

which is the condition required.

Ex. 14. Prove that the condition that one root of $ax^2 + bx + c = 0$ may be the square of the other is $b^3 + a^2c + ac^2 = 3abc$.

Sol. Let the roots of the given equation be α and α^2.

Then, $$\alpha + \alpha^2 = -\frac{b}{a} \quad ....(1)$$

$$\alpha \times \alpha^2 = \alpha^3 = \frac{c}{a} \quad ....(2)$$

Cubing both sides of (1), $(\alpha + \alpha^2)^3 = -\dfrac{b^3}{a^3}$

or $$\alpha^3 + \alpha^6 + 3\alpha^3(\alpha + \alpha^2) = -\frac{b^3}{a^3} \qquad [\because (x + y)^3 = x^3 + y^3 + 3xy\,(x + y)]$$

$$i.e., \; \frac{c}{a} + \frac{c^2}{a^2} + 3\frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}, \text{ } i.e., \; b^3 + a^2c + ac^2 = 3abc.$$

10.15. Special roots

(*a*) **Reciprocal roots**

If $\alpha = \dfrac{1}{\beta}, \alpha\beta = 1$, *i.e.*, $\dfrac{c}{a} = 1 \;\; \therefore \; \boldsymbol{a = c.}$

Thus, the roots of a quadratic equation will be reciprocal of each other if **the coeff. of x^2 = the constant term.**

(*b*) **Zero roots**

Case I. *When one root is zero.*

Let $\alpha = 0$, then $\alpha\beta = 0$, *i.e.*, $\dfrac{c}{a} = 0 \qquad \therefore \; \boldsymbol{c = 0.}$

Case II. *When both roots are zero.*

If α, β both are equal to zero, then $\alpha + \beta = 0$ and $\alpha\beta = 0$

i.e., $-\dfrac{b}{a} = 0$ and $\dfrac{c}{a} = 0 \;\; \therefore \; \boldsymbol{b = 0}$ and $\boldsymbol{c = 0.}$

Note. In this case, the equation assumes the form $ax^2 = 0$ whose roots obviously are 0, 0.

(*c*) **Infinite roots**

If α, β be the roots of $ax^2 + bx + c = 0$ ...(1)

The equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ is $cx^2 + bx + a = 0$ ...(2) [Ex. 9]

If one root of (2) is zero which occurs when $a = 0$, the corresponding root of (1), is $\frac{1}{0}$, *i.e.,* **infinitely large.** If both roots of (2) are equal to zero, which occurs when $b = 0$, $a = 0$, both the corresponding roots of (1) will be infinitely large.

Thus, the conditions are

(*a*) For one root to be infinite, $\boldsymbol{a = 0}$ **;**

(*b*) For both roots to be infinite, $\boldsymbol{b = 0, a = 0.}$

10.16. Signs of the roots

(1) **Positive roots.** Both roots will be positive if $\alpha + \beta$ and $\alpha\beta$ both are positive, *i.e.,* $-\frac{b}{a}$ and $\frac{c}{a}$ are positive. It will be so when ***b* and *a* are of opposite signs and *c* and *a* of the same sign.**

(2) **Negative roots.** Both roots will be negative if $\alpha + \beta$ is negative, and $\alpha\beta$ is positive, *i.e.,* $-\frac{b}{a}$ is negative and $\frac{c}{a}$ positive. It will be so when **a, *b, c* are all of the same sign.**

(3) **Roots of opposite signs.** It will occur when $\alpha\beta$ is negative, *i.e., c* **and *a* are of opposite signs.**

(4) **Roots equal in magnitude but opposite in sign.** It will occur when $\alpha + \beta = 0$.

i.e., $-\frac{b}{a} = 0$, *i.e. ,* $\boldsymbol{b = 0}$. For real solutions the signs of *a* and *c* should be opposite.

10.17. Common roots

1. ***To find the condition that two quadratic equations may have one root common.***

Let the two quadratic equations be $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ and α be their common root. Then, $a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$.

Solving by the rule of cross-multiplication,

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - ac'} = \frac{1}{ab' - a'b}$$

From the first two, $\alpha = \dfrac{bc' - b'c}{ca' - ac'}$

From the last two, $\alpha = \dfrac{ca' - ac'}{ab' - a'b}$

Equating the two values of α, $\dfrac{bc' - b'c}{ca' - ac'} = \dfrac{ca' - ac'}{ab' - a'b}$

or $(bc' - b'c)(ab' - ba') = (ca' - ac')^2$, which is the condition required.

2. ***To find the condition that two quadratic equations of (1) may have both roots common.***

Let the common roots be α, β.

Then, from the first equation, $\alpha + \beta = \dfrac{-b}{a}$, $\alpha\beta = \dfrac{c}{a}$

From the second equation, $\alpha + \beta = \frac{-b'}{a'}$, $\alpha\beta = \frac{c'}{a'}$

$\Rightarrow -\frac{b}{a} = \frac{-b'}{a'}$ and $\frac{c}{a} = \frac{c'}{a'} \Rightarrow \boldsymbol{\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}}$.

Ex. 15. For which value of k will the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ have one common root? **[*UPSEE 2005*]**

Sol. Let the common root be α. Then,

$$\alpha^2 - k\alpha - 21 = 0 \quad ...(1)$$

$$\alpha^2 - 3k\alpha + 35 = 0 \quad ...(2)$$

Solving, by the rule of cross-multiplication,

$$\frac{\alpha^2}{-35k - 63k} = \frac{\alpha}{-21 - 35} = \frac{1}{-3k + k}$$

From the first two, $\alpha = \frac{-98k}{-56} = \frac{7k}{4}$

From the last two, $\alpha = \frac{-56}{-2k} = \frac{28}{k}$ $\therefore \frac{7k}{4} = \frac{28}{k} \Rightarrow 7k^2 = 28 \times 4 \Rightarrow k = \pm 4$.

Ex. 16. If $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ have a common root, prove that either $p = q$ or $1 + p + q = 0$. **[*Rajasthan, Kerala PET 2004*]**

Sol. Let α be the common root of the given equations. Then,

$$\alpha^2 + p\alpha + q = 0 \quad ...(1)$$

$$\alpha^2 + q\alpha + p = 0 \quad ...(2)$$

Subtracting (2) from (1), we get

$(p - q)\alpha + q - p = 0 \Rightarrow (p - q)\alpha - (p - q) = 0$

$\Rightarrow (p - q)(\alpha - 1) = 0 \Rightarrow p = q$ or $\alpha = 1$

Putting $\alpha = 1$ in (1), we obtain $1 + p + q = 0$

$\therefore$ Either $p = q$ or $1 + p + q = 0$.

EXERCISE 10 (*d*)

1. Find the condition that one root of $ax^2 + bx + c = 0$ may be
   (*i*) three times the other, (*ii*) n times the other, (*iii*) more than the other by h.
2. Find the condition that the ratio between the roots of the equation $ax^2 + bx + c = 0$ may be $m : n$.
3. If the ratio of the roots of the equation $x^2 + px + q = 0$ is equal to the ratio of the roots of $x^2 + lx + m = 0$, prove that $mp^2 = ql^2$.
4. For what values of a and b, the equation $x^2 + (2a - 3)x = 3b + 4$ should have both the roots zero?
5. Find the values of λ and μ if both the roots of the equation $(3\lambda + 1)x^2 = (2\lambda + 3\mu)x - 3$ are infinite.
6. Find m so that the roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{m - 1}{m + 1}$ may be equal in magnitude and opposite in sign.
7. (*i*) The roots of the quadratic equation $4x^2 - (5a + 1)x + 5a = 0$, are p and q. If $q = 1 + p$, calculate the possible values of a, p and q.

(*ii*) Find the values of p for which the quadratic equation $x^2 - px + p + 3 = 0$ has

(a) coincident roots, (b) real distinct roots, (c) one positive and one negative root. **[*ISC*]**

8. Find the values of m for which the quadratic equation $x^2 - m(2x - 8) - 15 = 0$ has

(*i*) equal roots, (*ii*) both roots positive. **[*ISC*]**

9. If $a + b + c = 0$, prove that the roots of $ax^2 + bx + c = 0$ are rational. Hence, show that the roots of $(p + q)x^2 - 2px + (p - q) = 0$ are rational.

10. Show that the roots of $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are real, and that they cannot be equal unless $a = b = c$.

11. Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

12. Find the value of k, so that the equation $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ may have one root common.

13. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, prove that $a + b + c = 0$ or $a = b = c$.

14. The equations $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$ have a common real root

(a) for no value of a

(b) for exactly one value of a

(c) for exactly two values of a

(d) for exactly three values of a ***[WBJEE 2012]***

ANSWERS

1. (*i*) $3b^2 = 16ac$, (*ii*) $b^2n = ac(1 + n)^2$, (*iii*) $a^2h^2 = b^2 - 4ac$. **2.** $(m + n)^2 ac = mnb^2$.

4. $a = \frac{3}{2}, b = -\frac{4}{3}$. **5.** $\lambda = -\frac{1}{3}, \mu = \frac{2}{9}$. **6.** $\frac{a-b}{a+b}$

7. (*i*) $a = 3, p = \frac{3}{2}, q = \frac{5}{2}$; $a = \frac{-1}{5}, p = \frac{-1}{2}, q = \frac{1}{2}$. (*ii*) ($a$) $6, -2$, (b) $p < -2$ or $p > 6$, (c) $p < -3$.

8. (*i*) $m = 3, 5$, (*ii*) $m > 0, m > \frac{15}{8}$ **11.** $m = \frac{-11}{8}$ or $\frac{7}{4}$ **12.** $k = -3$ or $\frac{-27}{4}$ **14.** (b)

HINTS TO SELECTED QUESTIONS

9. (*i*) $x = 1$ is a root of $ax^2 + bx + c = 0$ since $a + b + c = 0$. Also the irrational or imaginary roots occur in pairs. Therefore, the second root can neither be irrational, nor imaginary.

(*ii*) The sum of the coefficients $= p + q - 2p + p - q = 0$.

Hence, by (*i*) the roots are rational.

10. The given equation is $3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$

The discriminant $= 4(a + b + c)^2 - 12(ab + bc + ca)$

$= 4[a^2 + b^2 + c^2 - ab - bc - ca] = 2[(a - b)^2 + (b - c)^2 + (c - a)^2]$

which is the sum of three non-negative quantities, and therefore must be non-negative. Hence, the roots are real. The roots are equal if the discriminant is zero. This is possible if each of the squares is separately zero, *i.e.* $a - b = 0$, $b - c = 0$, $c - a = 0$, *i.e.* $a = b = c$.

13. Let α be the common root. Then,

$a\alpha^2 + b\alpha + c = 0, \quad b\alpha^2 + c\alpha + a = 0.$

Solving by the rule of cross multiplication, we get

$$\frac{\alpha^2}{ab - c^2} = \frac{\alpha^2}{cb - a^2} = \frac{1}{ac - b^2} \Rightarrow \alpha = \frac{ab - c^2}{cb - a^2} \text{ or } \frac{cb - a^2}{ac - b^2}$$

$\therefore \quad \frac{ab-c^2}{cb-a^2}=\frac{cb-a^2}{ac-b^2} \Rightarrow a(a^3+b^3+c^3-3abc)=0$

$\Rightarrow \quad a=0$ or $a^3+b^3+c^3-3abc=0$

$\Rightarrow \quad (a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)=0$

$\Rightarrow \quad (a+b+c).\ \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$

$\Rightarrow \quad a+b+c=0$ or $(a-b)^2+(b-c)^2+(c-a)^2=0$

The second equation is possible only when $a-b=0,\ b-c=0,\ c-a=0 \Rightarrow a=b=c$.

14. Let α be the common root. Then,

$\alpha^2+\alpha+a=0,\quad \alpha^2+a\alpha+1=0$

$\Rightarrow \qquad \frac{\alpha^2}{1-a^2}=\frac{\alpha}{a-1}=\frac{1}{a-1}$

Now, $\frac{\alpha}{a-1}=\frac{1}{a-1} \Rightarrow \alpha=1$ and $\frac{\alpha^2}{1-a^2}=\frac{\alpha}{a-1} \Rightarrow \alpha^2=-(a+1) \Rightarrow 1=-(a+1) \Rightarrow a=-2$

[Putting $\alpha=1$]

Hence, there is exactly one value for which the given equations have a common root.

10.18. Quadratic functions

An algebraic expression of the form $ax^2 + bx + c$ is called a *quadratic function of x.*

10.19. Graph of the quadratic function $ax^2 + bx + c$ $(a \neq 0)$

Let $f(x) = ax^2 + bx + c$.

Procedure :

1. *Make a table of corresponding values for x and the function.*
2. *Plot these points, on a pair of x, f(x) axes.*
3. *Draw a smooth curve joining the plotted points.*

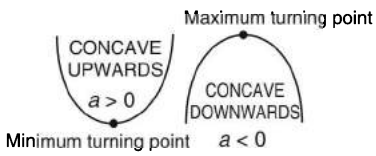


Fig. 10.02

The graph of a quadratic function is a **parabola**. The point at which its direction changes is called its **turning point.** It is more commonly called the **vertex** of the parabola.

The graph of the function is ***concave upwards when a > 0 and concave downwards when a < 0.***

Note. Studying the nature and character of the roots of a quadratic equation by graphing.

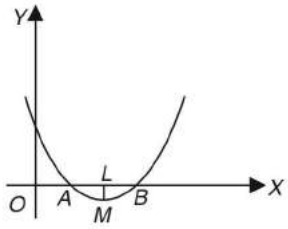
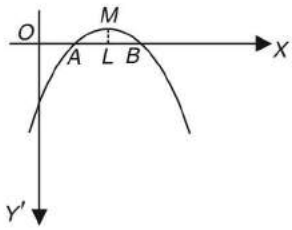
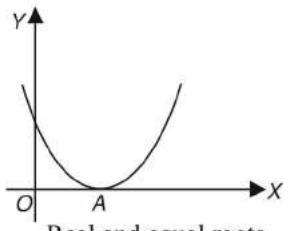
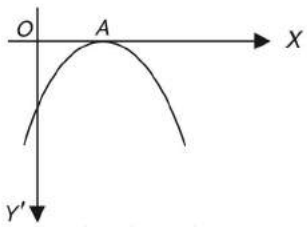
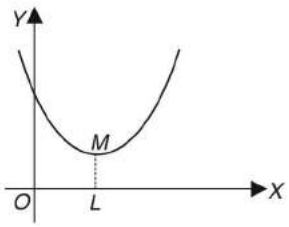
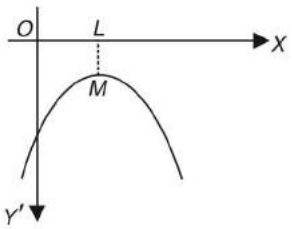
The graphs give below illustrate the following cases.



(*a*) If the graph cuts the *x*-axis, the roots of the equation will be real and unequal. Their values will be given by the abscissae of the points of intersection of the graph and the *x*-axis.

(*b*) If the graph is tangent to the *x*-axis, the roots are real and equal.

(*c*) If the graph has no points in common with the *x*-axis, the roots of the equation are imaginary and cannot be determined from the graph.

Case 1. When a is positive, *i.e.*, $a > 0$	**Case 2. When a is negative, *i.e.*, $a < 0$**
(*i*) $b^2 - 4ac$ is +ve  Real roots OA and OB. Minimum value $LM = \frac{4ac - b^2}{4a}$ at $x = OL = \frac{-b}{2a}$.	 Real roots OA and OB. Maximum value $LM = \frac{4ac - b^2}{4a}$ at $x = OL = \frac{-b}{2a}$.
(*ii*) $b^2 - 4ac = 0$  Real and equal roots. Each root = OA.	 Real and equal roots. Each root = OA.
(*iii*) $b^2 - 4ac$ is –ve  Complex roots. Curve entirely above the x-axis with lowest point M at a height $LM = \frac{4ac - b^2}{4a}$ above the x-axis.	 Complex roots. The curve lies entirely below the x-axis and the vertex will be at a distance $= \frac{4ac - b^2}{4a}$ below OX.

Ex. 17. Graph the expression $x^2 + x - 6$.

Let $y = x^2 + x - 6$.

Table of values:

x	y
3	6
2	0
1	−4
0	−6
−1	−6
−2	−4
−3	0
−4	6

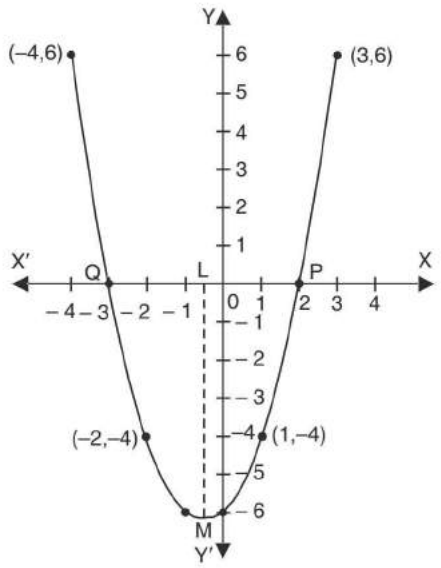


Fig. 10.03

Note. Real roots of the equation $x^2 + x - 6 = 0$ are $x = OP$ and $x = OQ$, *i.e.*, $x = 2$ and $x = -3$. (On reading from the graph).

Minimum value of the function = $LM = -6.25$ and occurs when $x = -0.5$.

Ex. 18. Graph the expression $4 - 5x - x^2$.

Let $y = 4 - 5x - x^2$.

Tables of values :

x	y
−6	−2
−5	4
−4	8
−3	10
−2	10
−1	8
0	4
1	−2

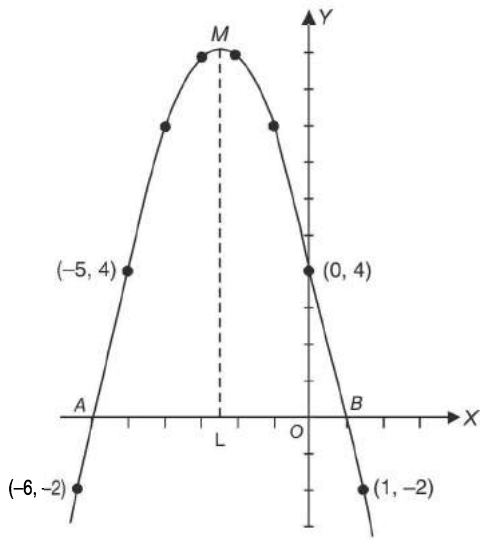


Fig. 10.04

Note. Real roots of the equation $4 - 5x - x^2 = 0$ are $x = OA$ and $x = OB$, *i.e.*, $x = -5.71$, $x = 0.71$

Maximum value of the function = $LM = 10.25$, when $x = OL = -2.5$.

The roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are the x-intercepts of the graph of $y = f(x) = ax^2 + bx + c$.

Ex. 19. The roots $x = 0$ and $x = 3$ of the equation $2x(x - 3) = 0$ are the x-intercepts of the graph of $y = 2x(x - 3)$ as shown in the figure.

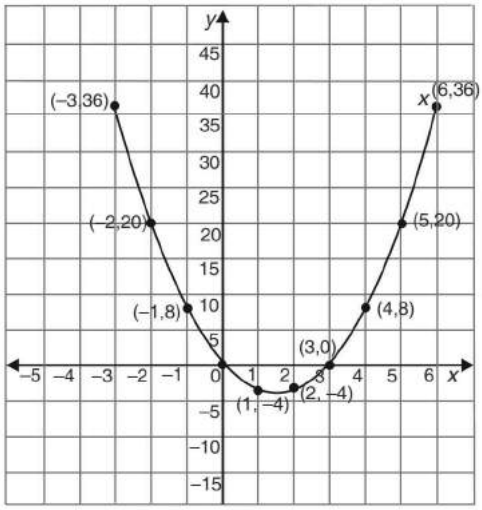


Fig. 10.05

In this case, the graph cuts the x-axis at two points because there are two real and distinct roots. In the formula the part $b^2 - 4ac > 0$.

Ex. 20. The two roots $x = -5, -5$ of the equation $(x + 5)^2 = 0$ is the x-intercept of the graph of $y = (x + 5)^2$ as shown below.

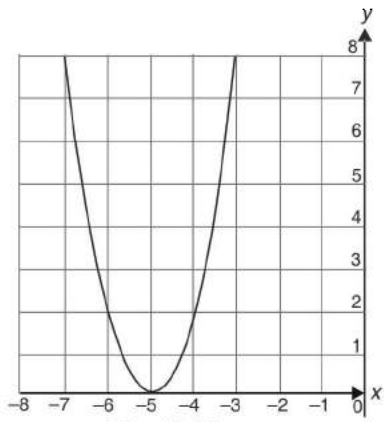


Fig. 10.06

In this case, the vertex of the graph cuts the x-axis at only one point because the roots are equal. In the formula the part $b^2 - 4ac = 0$.

Ex. 21. The two roots of the quadratic equation $x^2 - 2x + 2 = 0$ are $1 + i$ and $1 - i$. Therefore, the graph of the function $f(x) = x^2 - 2x + 2$ does not touch the x-axis as shown below because the roots are imaginary.

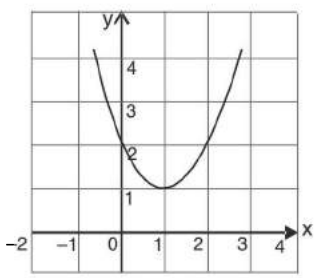


Fig. 10.07

In this case, in the formula the part $b^2 - 4ac < 0$.

Ex. 22. (*i*) Draw a graph of $y = x^2 - 4x + 3$ for $-2 \le x \le 5$.

(*ii*) Use the graph to solve the equation $x^2 - 4x + 3 = 0$.

Sol: The table of values is

x	-2	-1	0	1	2	3	4	5
y	15	8	3	0	-1	0	3	8

To solve the equation it is necessary to find the values of x when $y = 0$, *i.e.*, where the graph crosses the x-axis. These points occur when $x = 1$ and $x = 3$, therefore these are the solutions.

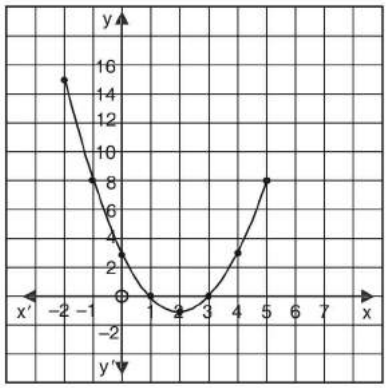


Fig. 10.08

EXERCISE 10 (*e*)

Draw the graph of each of the following quadratic functions.

1. $y = x^2 - 5x + 6$ $\quad 0 \le x \le 5$.

2. $y = -x^2 + 2x + 3$ $\quad -3 \le x \le 5$.

3. $y = x^2 - 4x + 4$ $\quad -1 \le x \le 5$.

4. Solve graphically and compare your answer with algebraic solution either by factorization or formula method:

(*i*) $y = x^2 - 5x + 6$ (*ii*) $y = -x^2 + 2x + 3$

(*iii*) $y = x^2 - 4x + 4$ (*iv*) $y = x^2 - x - 6$

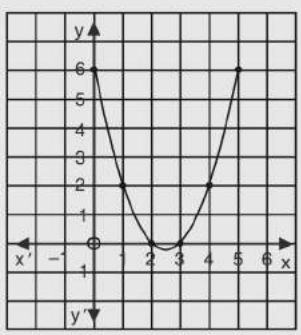
(*v*) $y = x^2 - 6x + 9$ (*vi*) $y = -x^2 - x + 12$

(*vii*) $y = x^2 - 4x + 5 = 0$ (*viii*) $y = x^2 + 2x + 2 = 0$

ANSWERS

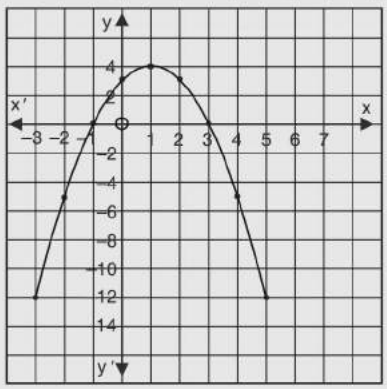
1.

x	0	1	2	3	4	5
y	6	2	0	0	2	6



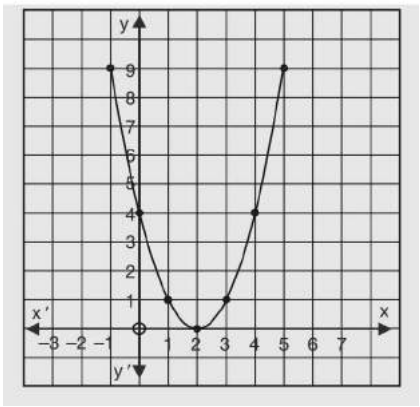
2.

x	−3	−2	−1	0	1	2	3	4	5
y	−12	−5	0	3	4	3	0	−5	−12

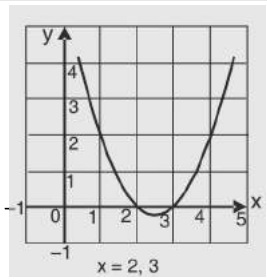


3.

x	−1	0	1	2	3	4	5
y	9	4	1	0	1	4	9

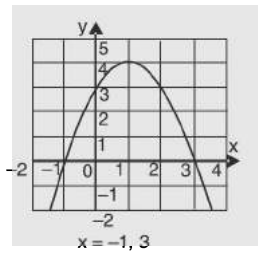


4. (*i*)



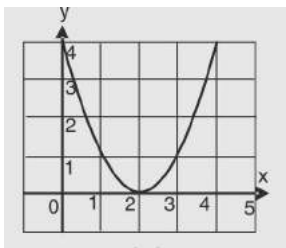
x = 2, 3

(*ii*)



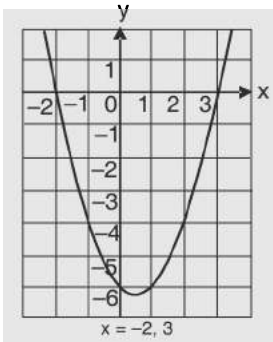
x = −1, 3

(*iii*)



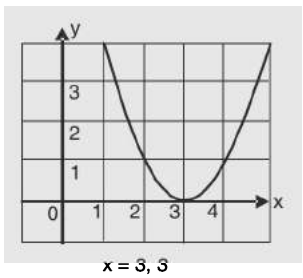
x = 2, 2

(*iv*)



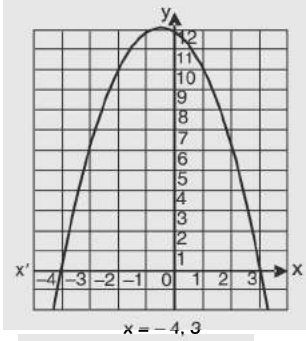
x = −2, 3

(*v*)



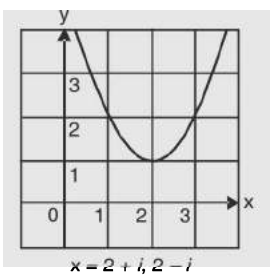
x = 3, 3

(*vi*)



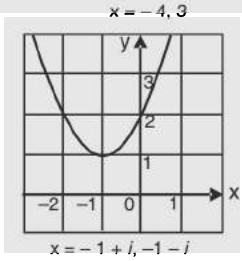
x = −4, 3

(*vii*)



x = 2 + *i*, 2 − *i*

(*viii*)



x = −1 + *i*, −1 − *i*

10.20. Sign of the quadratic function $ax^2 + bx + c$, where $x \in R$.

Since the discriminant $D = b^2 - 4ac$ of $ax^2 + bx + c = 0$ can be positive, zero, or negative, we shall examine what the sign of the expression $ax^2 + bx + c$ would be in these three cases.

Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then $ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$.

Three cases arise:

Case I. ***$b^2 - 4ac$ is negative ($D < 0$), i.e., when the roots are complex.***

We have:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \qquad \textit{Step I. Taking out the coeff. of } x^2.$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \qquad \textit{Step II. Completing the square.}$$

As $b^2 - 4ac$ is negative, so $-\dfrac{b^2 - 4ac}{4a^2}$ is positive and $\left(x + \dfrac{b}{2a}\right)^2$ is also positive for real values of x.

Therefore, the expression has the same sign as a.

Case II. ***$b^2 - 4ac = 0$ ($D = 0$), i.e., when the roots are real and equal*** **($\alpha = \beta$).**

$ax^2 + bx + c = a(x - \alpha)^2 = a \times$ a positive expression .

Hence, the expression has the same sign as a.

Case III. ***$b^2 - 4ac$ is positive ($D > 0$), i.e., when the roots are real and unequal.***

In this case the roots are real and unequal. Let $\alpha > \beta$.

(i) We assume that x does not lie between α and β. Then,

If $x > \alpha$, $(x - \alpha)$ and $(x - \beta)$ both will be positive. [Fig. 10.09 (i)]

If $x < \beta$, $(x - \alpha)$ and $(x - \beta)$ both will be negative. [Fig. 10.09 (ii)]

$\therefore$ The product $(x - \alpha)(x - \beta)$ will always be positive.

$\therefore$ $ax^2 + bx + c \equiv a\ (x - \alpha)(x - \beta) \equiv a \times$ a positive expression.



(i) (ii)

Fig. 10.09

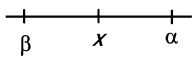


Fig. 10.10

Hence, the expression has the same sign as a.

(ii) If x lies between α and β, then $(x - \alpha)$ and $(x - \beta)$ will be of opposite signs.

$\therefore$ The product $(x - \alpha)(x - \beta)$ will be negative.

$\therefore$ $ax^2 + bx + c = a \times$ a negative expression.

$\therefore$ $ax^2 + bx + c$ and a will be of opposite signs.

From the above discussion, it follows that the sign of the expression $ax^2 + bx + c$ is always the same as that of a , except when the roots of the equation $ax^2 + bx + c = 0$ are real and distinct and x lies between them.

Value of x	less than β	between α and β	greater than α
Sign of $f(x)$	same as that of a	opposite to that of a	same as that of a

Tests:

(*i*) A quadratic $ax^2 + bx + c$ is positive for all real x of $a > 0$ and $D < 0$.

(*ii*) A quadratic $ax^2 + bx + c$ is negative if $a < 0$ and $D < 0$.

Illustrations

1. The sign of $x^2 + 6x + 12$ is positive for all real x since $D = b^2 - 4ac = 36 - 48 < 0$ and $a = 1 > 0$.
2. The sign of $-3x^2 + 5x - 12$ is –ve for all real x since $D = b^2 - 4ac = 25 - 4(-3)(-12) = -119 < 0$ and $a = -3 < 0$.

Ex. 23. Find the sign of $6x^2 - 5x + 1$ for all real values of x.

Sol. $D = b^2 - 4ac = 5^2 - 4 \times 6 \times 1 = 1$, is positive.

$\therefore$ The roots are real and different.

By solving the equation, the roots are $\frac{5 \pm \sqrt{25-24}}{12}$, *i.e.*, $\frac{1}{2}$ and $\frac{1}{3}$.

$\therefore$ The given expression has the same sign as the coefficient of x^2,

i.e., positive for all real values of x except for those which lie between $\frac{1}{3}$ and $\frac{1}{2}$.

Ex. 24. Determine the sign of the function $3x^2 - 2x + 1$ for real values of x.

Sol. Method I. $3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right) = 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{1}{3} - \frac{1}{9}\right\}$ (Completing the square)

$$= 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right\}.$$

As x is real, the term $\left(x - \frac{1}{3}\right)^2$ is always non-negative, and hence the function is positive for all real values of x.

Method II. $D = b^2 - 4ac = (-2)^2 - 4(3)(1) = -8$. Therefore, the roots of $3x^2 - 2x + 1 = 0$ are imaginary. Hence, $3x^2 - 2x + 1$ has the same sign as 'a', *i.e.* 3, *i.e.* positive. **(Case I)**

Ex. 25. Find the ranges of the values of x for which $x^2 - 4x + 2$ lies between – 1 and + 1. **[*GCE*]**

Sol. $x^2 - 4x + 2 > -1 \Rightarrow x^2 - 4x + 3 > 0$.

Since the coefficient of x^2 is positive and the roots of $x^2 - 4x + 3 = 0$ are 1 and 3, therefore,

$x^2 - 4x + 3 > 0$ for $x < 1$ or $x > 3$* ...(1)

Again, $x^2 - 4x + 2 < 1 \Rightarrow x^2 - 4x + 1 < 0$

*Since the coefficient of x^2 is positive and the roots of $x^2 - 4x + 1 = 0$ are $2 - \sqrt{3}$ and $2 + \sqrt{3}$, therefore $x^2 - 4x + 1 < 0$, for $2 - \sqrt{3} < x < 2 + \sqrt{3}$. ...(2)

Combining (1) and (2), we get $2 - \sqrt{3} < x < 1$ or $3 < x < 2 + \sqrt{3}$,

which are the required ranges of the values of x.

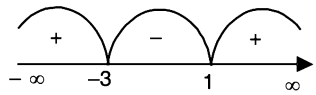
Ex. 26. Determine the values of 'a' so that the expression $x^2 - 2(a+1)x + 4$, $x \in R$ is always positive.

Sol. Given : $x^2 - 2(a+1)x + 4 > 0$ for all x.

---

* Alternatively use Art. 10.21

Since coefft. of $x^2 = 1$ is +ve, therefore, sign of $x^2 - 2(a+1)x + 4$ is positive if $D < 0$.



i.e., if $4(a+1)^2 - 16 < 0$,

$\Rightarrow (a+1)^2 - 2^2 < 0$

$\Rightarrow [(a+1)+2][(a+1)-2] < 0$

$\Rightarrow (a+3)(a-1) < 0$

$\Rightarrow \{a-(-3)\}(a-1) < 0$

$\Rightarrow -3 < a < 1 \Rightarrow a \in (-3, 1)$.

Help Box

Sign of	$a < -3$	$a \in (-3, 1)$	$a > 1$
$(a+3)$	−	+	+
$(a-1)$	−	−	+
$(a+3)(a-1)$	+	−	+

$-\infty$ $\quad -3 \quad 5 \quad 1 \quad \infty$

$\therefore \quad (a+3)(a-1) < 0$ when $a \in (-3, 1)$

Note on use of brackets for open and closed intervals

Sometimes the domains of variation, *i.e.*, the intervals are denoted as follows:

(*i*) '$a < x < b$' is denoted by (a, b) and is called an open interval of the variable x.

(*ii*) '$a \le x \le b$' is denoted by $[a, b]$ and is called a closed interval of the variable x, as x can take up values 'a' and 'b' also.

(*iii*) Any 'x' is denoted by $(-\infty, \infty)$. Here, it should be noted that the symbols $-\infty$, ∞ are not numbers in any sense whatsoever.

(*iv*) '$x \ge a$' is denoted by $[a, \infty)$, '$x \le b$' is denoted by $(-\infty, b]$.

(*v*) '$a \le x < b$' is denoted by $[a, b)$, '$a < x \le b$' is denoted by $(a, b]$. These are called semi-closed intervals.

10.21. Method of intervals

Suppose we have to solve the inequality

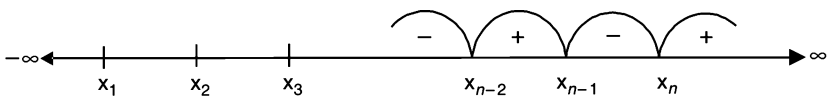
$$(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-2})(x-x_{n-1})(x-x_n) < 0 \qquad \ldots(1)$$

where $x_1, x_2, x_3, \ldots x_{n-2}, x_{n-1}, x_n$ are *distinct real* numbers.

We will assume that $(x_1 < x_2 < x_3 < \ldots\ldots < x_{n-2} < x_{n-1} < x_n)$.

Plot these points on the real number line (Fig. 10.10) and consider the polynomial

$$F(x) = (x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-2})(x-x_{n-1})(x-x_n) \qquad \ldots(2)$$



Always start with + sign from the right towards the left.

It is clear that for all $x > x_n$ all the parenthetic expressions in (2) are positive and hence, for $x > x_n$ we have $F(x) > 0$, *i.e.* positive. Since $x_{n-1} < x < x_n$, the last parenthesis in the expression $F(x)$ is negative, and all other parentheses are positive, it follows that for $x_{n-1} < x < x_n$ we have $F(x) < 0$. Similarly, we obtain $F(x) > 0$ for $x_{n-2} < x < x_{n-1}$, and so on, alternately positive and negative, as is shown in Fig. 10.10. Before putting the signs +, –, ensure that the highest degree term is +ve.

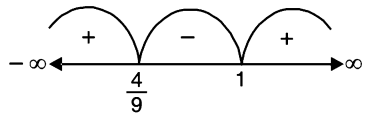
This is the underlying idea of the method of intervals.

On the number line, the numbers $x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n$ must be arranged in order of increasing magnitude. Then place the plus sign in the interval to the right of the largest number x_n. In the next interval (from right to left) place the minus sign, then plus sign, then minus sign, etc. The solution of the inequality (1), *i.e.*, $F(x) < 0$ will then consist of intervals having the minus sign, and the solution of the inequality $F(x) > 0$ will consist of intervals having the plus sign.

10.22. The quadratic fraction

Just like $ax^2 + bx + c$, a fraction of the form $\dfrac{ax^2+bx+c}{px^2+qx+r}$ also varies in magnitude and sign. We can determine this variation in numerical case as illustrated in the following examples.

Ex. 27. If x is real, prove that the value of the expression $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ cannot be between $\dfrac{4}{9}$ and 1.



Sol. Let $\dfrac{(x-1)(x+3)}{(x-2)(x+4)} = y$.

Then $x^2 + 2x - 3 = y\,(x^2 + 2x - 8)$

or $x^2(1-y) + 2x\,(1-y) + (8y-3) = 0$.

For x to be real, its discriminant ≥ 0

or $[2(1-y)]^2 - 4\,(1-y)(8y-3) \geq 0$

$\Rightarrow 9y^2 - 13y + 4 \geq 0$

$\Rightarrow (9y-4)\,(y-1) \geq 0$

$\Rightarrow 9\left(y-\dfrac{4}{9}\right)(y-1) \geq 0$

$\Rightarrow \left(y-\dfrac{4}{9}\right)(y-1) \geq 0$

Sign of	$y < \frac{4}{9}$	$y \in \left(\frac{4}{9}, 1\right)$	$y > 1$
$\left(y-\frac{4}{9}\right)$	−	+	+
$(y-1)$	−	−	+
$\left(y-\frac{4}{9}\right)(y-1)$	+	−	+

$-\infty$ $\quad \frac{4}{9} \quad$ 1 $\quad \infty$

so, $\left(y-\dfrac{4}{9}\right)(y-1)$ is + ve when $y < \dfrac{4}{9}$ or $y > 1$.

The critical points are $\dfrac{4}{9}$ and 1. By the Method of Intervals (Art. 3.19), y cannot lie between $\dfrac{4}{9}$ and 1 as $\left(y-\dfrac{4}{9}\right)(y-1)$ has to be positive. In other words, the given expression cannot lie between $\dfrac{4}{9}$ and 1. Its range is $(-\infty, \dfrac{4}{9}] \cup [1, \infty)$.

Procedure. (Method of intervals explained in Art. 10.21)

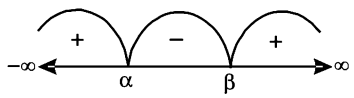
Step 1. *Make the coefficient of x^2 positive, if it is not so, by multiplying both sides of the inequation by – 1.*

On doing so, the inequality will be reversed, i.e., > will change to < and vice-versa.

Step 2. *Factorise the quadratic expression and express the L.H.S. of the inequality in the form $(x-\alpha)\,(x-\beta)$, where $\alpha < \beta$.*

Step 3. *Plot the points α and β on the number line thus, dividing the number line into three parts. Starting from the rightmost region put +, –, + signs as shown as the expression $(x-\alpha)\,(x-\beta)$ is non-negative in the region on the right of β.*

Step 4. Now reason out as under:



(*i*) ***If*** $\boldsymbol{(x-\alpha)\,(x-\beta) > 0}$, then x, lies outside the interval $[\alpha, \beta]$, *i.e.*, $x \in (-\infty, \alpha) \cup (\beta, \infty)$.

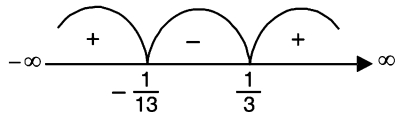
(*ii*) ***If*** $\boldsymbol{(x-\alpha)\,(x-\beta) \geq 0}$, then x lies on and outside the interval (α, β), *i.e.*, $x \in (-\infty, \alpha] \cup [\beta, \infty)$.

(*iii*) ***If*** $\boldsymbol{(x-\alpha)\,(x-\beta) < 0}$, then x lies in the interval (α, β).

(*iv*) ***If*** $\boldsymbol{(x-\alpha)\,(x-\beta) \leq 0}$, then x lies inside the interval $[\alpha, \beta]$, *i.e.*, $x \in [\alpha, \beta]$.

Ex. 28. If x be real, find the maximum value of $\frac{(x+2)}{2x^2+3x+6}$.

Sol. Let $\frac{x+2}{2x^2+3x+6} = y$.



Then, $2x^2y + (3y - 1)x + 6y - 2 = 0$.

For x to be real, $(3y-1)^2 - 8y(6y-2) \geq 0$ or $(1+13y)(1-3y) \geq 0$.

$\Rightarrow \left(y+\frac{1}{13}\right)\left(\frac{1}{3}-y\right) \geq 0$

$\Rightarrow \left(y+\frac{1}{13}\right)\left(y-\frac{1}{3}\right) \leq 0 \quad \ldots (i)$

By the Method of Intervals, y lies in the interval $\left[\frac{-1}{13}, \frac{1}{3}\right]$ as $\left(y+\frac{1}{13}\right)\left(y-\frac{1}{3}\right)$ is negative. The range of y, *i.e.*, the given expression is $\left[\frac{-1}{13}, \frac{1}{3}\right]$. Hence, the maximum value of the given expression is $\frac{1}{3}$.

Help Box

sign of	$y < -\frac{1}{13}$	$y \in \left(-\frac{1}{13}, \frac{1}{3}\right)$	$y > \frac{1}{3}$
$\left(y+\frac{1}{13}\right)$	−	+	+
$\left(y-\frac{1}{3}\right)$	−	−	+
$\left(y+\frac{1}{13}\right)\left(y-\frac{1}{3}\right)$	+	−	+

$-\infty$ $-\frac{1}{13}$ $\frac{1}{13}$ ∞

Ex. 29. If x is real, then find the minimum value of $\frac{x^2-x+1}{x^2+x+1}$. ***[EAMCET 2005]***

Sol. Let $y = \frac{x^2-x+1}{x^2+x+1} \Rightarrow (y-1)x^2 + (y+1)x + y - 1 = 0$

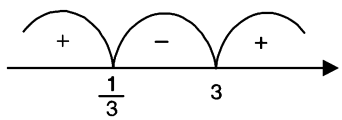
For x to be real, $(y+1)^2 - 4(y-1)^2 \geq 0$ Using $b^2 - 4ac \geq 0$

$\Rightarrow -3y^2 + 10y - 3 \geq 0 \Rightarrow 3y^2 - 10y + 3 \leq 0$ (If $a > b$, then $-a < -b$)

$\Rightarrow (3y-1)(y-3) \leq 0 \Rightarrow (y-\frac{1}{3})(y-3) \leq 0$

Plot points $\frac{1}{3}$ and 3 on the number line and place the signs in the intervals.

$\Rightarrow y \in [\frac{1}{3}, 3]$. $\Rightarrow$ Minimum value of $\frac{x^2-x+1}{x^2+x+1}$ is $\frac{1}{3}$.



Ex. 30. If x is real, the maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is

(*a*) $\frac{17}{7}$ (*b*) $\frac{1}{4}$ (*c*) 41 (*d*) 1 ***[AIEEE 2006]***

Sol. $y = \frac{3x^2+9x+17}{3x^2+9x+7} = 1 + \frac{10}{3x^2+9x+7} = 1 + \frac{10}{3\left[(x+\frac{3}{2})^2 + \frac{1}{12}\right]}$

Obviously, y will be maximum when the denominator is minimum.

Now, minimum value of $3\left[(x+\frac{3}{2})^2 + \frac{1}{12}\right]$ is $3\left(\frac{1}{12}\right) = \frac{1}{4}$ when $x = \frac{-3}{2}$.

$\therefore$ The max. value of $\frac{10}{3x^2+9x+7}$ is $\frac{10}{1/4} = 40$ and max. value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is $1 + 40 = \mathbf{41}$.

EXERCISE 10 (*f*)

1. Show that

(*a*) $x^2 - 3x + 6 > 0$ for all x (*b*) $4x - x^2 - 6 < 0$ for all x. (*c*) $2x^2 - 4x + 7$ is always +ve.

(*d*) $-2x^2 + 3x - 4$ is always –ve. (*e*) $-x^2 + 3x - 3$ is always –ve.

2. Explain why $3x^2 + kx - 1$ is never always positive for any value of k.

3. Under what conditions is $2x^2 + kx + 2$ always positive?

4. Find the values of a so that the expression $x^2 - (a + 2)x + 4$ is always positive.

5. Find the range of values of x for which the expression $12x^2 + 7x - 10$ is negative.

6. (*i*) Find the values of 'a' for which the expression

$x^2 - (3a - 1)x + 2a^2 + 2a - 11$ is always positive. **[*WBJEE 2009*]**

(*ii*) If $x^2 + 4ax + 2 > 0$ for all values of x, then a lies in the interval

(*a*) (–2, 4) (*b*) (1, 2) (*c*) $\left(-\sqrt{2}, \sqrt{2}\right)$

(*d*) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (*e*) (–4, 2) **[*Kerala PET 2009*]**

7. Find the greatest value of $3 + 5x - 2x^2$ for all real values of x.

8. Find the least value of $\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$ for real values of x.

9. If x be real, prove that the value of $\frac{11x^2 + 12x + 6}{x^2 + 4x + 2}$ cannot lie between – 5 and 3.

ANSWERS

2. $a = 3$ which is > 0 and $D = k^2 + 12$ which is always > 0 (as $k^2 > 0$ for all k)

3. $a = 2$ which is > 0 and $D = k^2 - 16$ $\therefore$ The given quadratic will always be positive when $k^2 - 16 < 0$, *i.e.*, $k^2 < 16$, *i.e.*, $-4 < k < 4$.

4. $-6 < a < 2$. **5.** $\frac{-5}{4} < x < \frac{2}{3}$. **6.** (*i*) $5 < a < 9$ (*ii*) (*d*)

7. $6\frac{1}{8}$. **8.** 1

HINTS TO SELECTED QUESTIONS

6. (*i*) Since $x^2 - (3a - 1)x + 2a^2 + 2a - 11 > 0$, therefore, $D < 0$.

(*ii*) Since coefft. of $x^2 = 1$ is + ve, therefore, $x^2 + 4ax + 2$ would be + ve if $D = 16a^2 - 8 < 0$.

7. If x is real, then $D \geq 0$.

8. Let $y = \frac{6x^2 - 22x + 21}{5x^2 - 18x + 17} \Rightarrow (5y - 6)x^2 + (-18y + 22)x + (17y - 21) = 0$

For values of x to be real, $(-18y + 22)^2 - 4(5y - 6)(17y - 21) \geq 0$

$\Rightarrow 4y^2 - 9y + 5 \leq 0 \Rightarrow (4y - 5)(y - 1) \leq 0$.

CHAPTER SUMMARY

1. The roots of the quadratic equation $ax^2 + bx + c = 0$ when $a, b, c \in R$ and $a \neq 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression '$b^2 - 4ac$' is called the **discriminant** and is denoted by D or Δ.

2. **Nature or character of the roots.**
   (*i*) The roots are real and distinct if $b^2 - 4ac > 0$.
   (*ii*) The roots are real and equal if $b^2 - 4ac = 0$.
   (*iii*) The roots are real and rational if $b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square.
   (*iv*) The roots are real and irrational if $b^2 - 4ac > 0$ and $b^2 - 4ac$ is not a perfect square.
   (*v*) The roots are complex, if $b^2 - 4ac < 0$.

3. **Sum and product of the roots.** If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then,

$$\alpha + \beta = -\frac{b}{a}\left(= -\frac{\text{coefft. of } x}{\text{coefft. of } x^2}\right),$$

$$\alpha\beta = \frac{c}{a}\left(= \frac{\text{constant term}}{\text{coefft. of } x^2}\right).$$

4. **Formation of equation when roots are given.** The quadratic equation whose roots are α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$, *i.e.*, $x^2 -$ (sum of the roots) $x +$ product of the roots $= 0$.
5. The quadratic expression $ax^2 + bx + c, x \in R$ is
   (*i*) always positive if $a > 0, D < 0$.
   (*ii*) always negative if $a < 0, D < 0$.

CHAPTER TEST

Solve the following equations:

1. $5^{x+1} + 5^{2-x} = 5^3 + 1$ **[*AICBSE*]**
2. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$ **[*CBSE, CA*]**
3. $(x+1)(x+2)(x+3)(x+4) = 120$
4. Prove that both the roots of the equation $x^2 - x - 3 = 0$ are irrational. **[*AICBSE*]**
5. For what values of m will the equation $x^2 - 2mx + 7m - 12 = 0$ have (*i*) equal roots, (*ii*) reciprocal roots ? **[*AICBSE, CBSE*]**
6. If one root of $2x^2 - 5x + k = 0$ be double the other, find the value of k. **[*WB; AICBSE*]**
7. If α, β be the roots of the equation $x^2 - x - 1 = 0$, determine the value of (*i*) $\alpha^2 + \beta^2$ and (*ii*) $\alpha^3 + \beta^3$.
8. If the roots of the equation $ax^2 + bx + c = 0$ be in the ratio 3 : 4, show that $12b^2 = 49ac$.
9. If x is real, prove that the quadratic expression (*i*) $(x - 2)(x + 3) + 7$ is always positive.
   (*ii*) $4x - 3x^2 - 2$ is always negative.

10. Draw the graph of the quadratic function $x^2 - 4x + 3$ and hence find the roots of the equation $x^2 - 4x + 3 = 0$. What is the minimum value of the function?

11. For what real values of a, will the expression $x^2 - ax + 1 - 2a^2$, for the real x, be always positive?

12. If x be real, prove that the value of $\dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$ cannot lie between –7 and 1. **[*WB, HS*]**

13. If the roots of the equation $qx^2 + 2px + 2q = 0$ are real and unequal, prove that the roots of the equation $(p + q)x^2 + 2qx + (p - q) = 0$ are imaginary.

14. If α, β be the roots of $x^2 - px + q = 0$, find the value of $\alpha^5\beta^7 + \alpha^7\beta^5$ in terms of p and q.

15. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is

(*a*) $(3, \infty)$ (*b*) $(-\infty, -3)$ (*c*) $(-3, 3)$ (*d*) $(-3, \infty)$ **[*AIEEE 2007*]**

16. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2$, 2β be the roots of the equation $x^2 - qx + r = 0$, then the value of r is

(*a*) $\dfrac{2}{9}(p - q)(2q - p)$ (*b*) $\dfrac{2}{9}(q - p)(2p - q)$

(*c*) $\dfrac{2}{9}(q - 2p)(2q - p)$ (*d*) $\dfrac{2}{9}(2p - q)(2q - p)$ **[*IIT JEE 2007*]**

17. α, β are the roots of $ax^2 + 2bx + c = 0$ and $\alpha + \delta$, $\beta + \delta$ are the roots of $Ax^2 + 2Bx + C = 0$, then what is $(b^2 - ac)/(B^2 - AC)$ equal to ?

(*a*) $(b/B)^2$ (*b*) $(a/A)^2$ (*c*) $(a^2 b^2)/(A^2B^2)$ (*d*) ab / AB **[*NDA April 2007*]**

18. If α, β are the roots of the equation $x^2 - 2x - 1 = 0$, then what is the value of $\alpha^2\beta^{-2} + \alpha^{-2}\beta^2$?

(*a*) –2 (*b*) 0 (*c*) 30 (*d*) 34 **[*NDA April 2007*]**

19. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then value of $2 + q - p$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0 **[*AIEEE 2006*]**

20. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval

(*a*) (5, 6] (*b*) (6, ∞) (*c*) (–∞, 4) (*d*) [4, 5] **[*AIEEE 2005*]**

21. If α and β are the roots of $ax^2 + bx + c = 0$ and if $px^2 + qx + r = 0$ has roots $\dfrac{1-\alpha}{\alpha}$ and $\dfrac{1-\beta}{\beta}$ then r =

(*a*) $a + 2b$ (*b*) $a + b + c$ (*c*) $ab + bc + ca$ (*d*) abc **[*EAMCET 2007*]**

22. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then the common root is

(*a*) 1 (*b*) 4 (*c*) 3 (*d*) 2 **[*AIEEE 2008*]**

23. If α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$ and if λ_1 and λ_2 are two values of λ obtained from $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{5}$, then $\dfrac{\lambda_1}{\lambda_2^2} + \dfrac{\lambda_2}{\lambda_1^2}$ equals

(*a*) 4192 (*b*) 4144 (*c*) 4096 (*d*) 4048 **[*DCE 2009*]**

24. If α, β be the roots of $x^2 - a(x - 1) + b = 0$, then the value of

$$\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} + \frac{2}{a+b} \text{ is}$$

(*a*) $\frac{4}{a+b}$ (*b*) $\frac{1}{a+b}$ (*c*) 0 (*d*) – 1 **[*WB JEE 2009*]**

ANSWERS

1. –1, 2 **2.** $\frac{4}{13}, \frac{9}{13}$ **3.** $1, -6, \frac{-5 \pm i\sqrt{39}}{2}$ **5.** (*i*) $m = 3$ or 4 (*ii*) $m = \frac{12}{7}$

6. $k = 2\frac{7}{9}$ **7.** (*i*) 3 (*ii*) 4 **10.** Roots are 1, 3. Minimum value = –1

11. $-\frac{2}{3} < a < \frac{2}{3}$ **14.** $q^5(p^2 - 2q)$ **15.** (*c*) **16.** (*d*) **17.** (*b*)

18. (*d*) **19.** (*c*) **20.** (*c*) **21.** (*b*) **22.** (*d*)

23. (*d*) **24.** (*c*)

HINTS TO SELECTED QUESTIONS

1. Let $5^x = y$, then given eqn. becomes $5y + \frac{25}{y} = 126$.

5. (*ii*) Roots are reciprocal to each other if $\alpha\beta = 1$.

11. $D < 0 \Rightarrow 9a^2 - 4 < 0 \Rightarrow 9a^2 < 4 \Rightarrow |3a| < 2 \Rightarrow -2 < 3a < 2$.

15. $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{a^2 - 4}$

Since $|\alpha - \beta| < \sqrt{5} \Rightarrow \sqrt{a^2 - 4} < \sqrt{5} \Rightarrow a^2 < 9 \Rightarrow -3 < a < 3$.

16. $\alpha + \beta = p, \alpha\beta = r, \frac{\alpha}{2} + 2\beta = q, \alpha\beta = r$

$\Rightarrow \alpha + \beta = p, \alpha + 4\beta = 2q$. Solve for α and β.

17. $\alpha + \beta = -\frac{2b}{a}, \alpha\beta = \frac{c}{a}, (\alpha + \delta) + (\beta + \delta) = -\frac{2B}{A}, (\alpha + \delta)(\beta + \delta) = \frac{C}{A}$

Now, $\alpha - \beta = (\alpha + \delta) - (\beta + \delta) \Rightarrow (\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\overline{\alpha + \delta} + \overline{\beta + \delta})^2 - 4(\alpha + \delta)(\beta + \delta)$.

19. $\frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°} = \tan(30° + 15°) = \tan 45° \Rightarrow \frac{-p}{1 - q} = 1$

20. Sum of the roots $< 10 \Rightarrow 2k < 10 \Rightarrow k < 5$.

Solving, $x = k + \sqrt{5 - k}, k - \sqrt{5 - k}$

Since each root is less than 5, $k + \sqrt{5 - k} < 5 \Rightarrow \sqrt{5 - k} < 5 - k$

Since $k < 5, 5 - k < 25 - 10k + k^2 \Rightarrow k^2 - 9k + 20 > 0$

$\Rightarrow (k - 4)(k - 5) > 0 \Rightarrow k \in (-\infty, 4)$.

21. $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$. ....(*i*)

The equation whose roots are $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ is

$$x^2 - \left(\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta}\right)x + \frac{1-\alpha}{\alpha}\cdot\frac{1-\beta}{\beta} = 0.$$

$\Rightarrow\ cx^2 + (b+2c)x + (a+b+c) = 0$ [Using (i)]

Comparing with $px^2 + qx + r = 0$, we get $r = a + b + c$.

22. Let the roots of the equation $x^2 - 6x + a = 0$ be α and 4β and those of the equation $x^2 - cx + 6 = 0$ be α and 3β. Then

$$\left.\begin{aligned}\alpha + 4\beta &= 6 \\ 4\alpha\beta &= a\end{aligned}\right\} \begin{aligned}&...(1)\\&...(2)\end{aligned} \quad \text{and} \quad \left.\begin{aligned}\alpha + 3\beta &= c \\ 3\alpha\beta &= 6\end{aligned}\right\} \begin{aligned}&...(3)\\&...(4)\end{aligned}$$

(2) and (4) $\Rightarrow a = 8$ $\therefore$ The eqn. becomes $x^2 - 6x + 8 = 0$ $\Rightarrow (x-2)(x-4) = 0$

$\Rightarrow$ The roots are 2 and 4 $\Rightarrow \alpha = 2, \beta = 1$.

Hence, the common root is 2.

23. $\alpha + \beta = \frac{\lambda - 1}{\lambda}$, $\alpha\beta = \frac{5}{\lambda}$; $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$ $\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$

$\Rightarrow \frac{(\lambda-1)^2 - 10\lambda}{5\lambda} = \frac{4}{5} \Rightarrow \lambda^2 - 16\lambda + 1 = 0$

$\therefore\ \lambda_1 + \lambda_2 = 16$ and $\lambda_1\lambda_2 = 1$.

24. $\alpha^2 - a\alpha = -(a+b)$, $\beta^2 - a\beta = -(a+b)$

CHAPTER 11

INEQUALITIES

11.01. Inequations in one variable

Mathematical sentences of the type

$5, 7, 7 > 5, -1 < 2, 3y < 15, 5t > 20$

are called **inequalities** or **inequations**. These sentences say that one thing is not equal to another.

11.02. Replacement set or universal set

Consider the sentences (*i*) $3x = 12$, (*ii*) $2x < 3$.

If the variable x in the equation (*i*) is replaced by the number 4, it yields a true statement. We say that the equation is satisfied. Similarly, the inequation (*ii*) is satisfied if the variable x is replaced by the numbers.

$..., -3, -2, -1, 0, 1.$

Sometimes, it is necessary to specify a set from which the replacement for the variable should be chosen. For example, if we write the open sentence. "is a beautiful city", it would be difficult to give any precise answer, unless the set of cities is specified.

The set of elements from which the replacement for the variables is taken is called the **replacement set** or **universal set**.

11.03. Solution set or truth set

Consider the inequation $1 + x > 5$. We can obtain the following solutions for the replacement sets shown.

Replacement set	*Solutions*
$A = \{0, 1, 2, 5, 7, 8\}$	5, 7 and 8
$B = \{0, 1, 2, 3, 4, 6\}$	6
$B = \{0, 1, 2, 3, 4\}$	No solution or ϕ

The above example illustrates the fact that the solution of an inequation depends upon the replacement set used. It also follows from the above that an inequation may have one, many or no solution, depending upon the replacement set. The solution or solutions of a given inequation form a set, which we call the **solution set** or the **truth set**. It is obviously a subset of the replacement set.

11.04. Inequality postulates

1. ***Adding to or subtracting from both sides of an inequality and non-zero number produces an equivalent inequality.***

(*i*) $x - 3 > 5$ is equivalent to $x - 3 + 3 > 5 + 3$ or $x > 5 + 3$ or $x > 8$.

(*ii*) $x + 1 > 3$ is equivalent to $x + 1 - 1 > 3 - 1$, or $x > 3 - 1$ or $x > 2$.

2. ***Multiplying or dividing both sides of an inequality by the same positive number produces an equivalent inequality.***

(*i*) $5x > 15$ is equivalent to $\frac{5x}{5} > \frac{15}{5}$, or $x > \frac{15}{5}$ or $x > 3$.

(*ii*) $\frac{x}{3} > 2$ is equivalent to $\frac{x}{3} \times 3 > 2 \times 3$, or $x > 2 \times 3$, or $x > 6$.

3. ***Multiplying both sides of an inequality by the same negative number produces an inequality with its direction reversed.***

(*i*) $3 < 4$ is equivalent to $(-1)(3) > (-1)(4)$, or $-3 > -4$.

(*ii*) $x > 8$ is equivalent to $(-1)(x) < (-1)(8)$, or $-x < -8$.

(*iii*) $-x > -8$ is equivalent to $(-1)(-x) < (-1)(-8)$, or $x < 8$.

The following **transformation** is also useful. You can **exchange** members by changing the direction of the inequality.

$7 < x + 3$ is equilvalent to $x + 3 > 7$.

11.05. Graph of a number

The graph of a number on the number line is the point named by that number. The Fig. 11.01 drawn below shows the graph of the set {–1, 4, 5}.

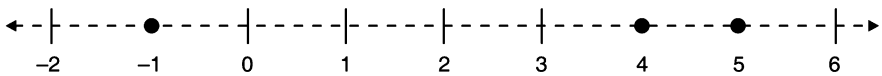
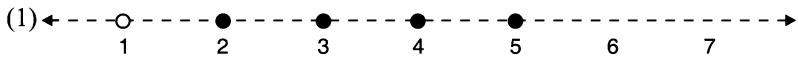


Fig. 11.01

The graph of a number is shown by making the point with a heavy dot. The number line is shown dotted to indicate that it is an integer number line.

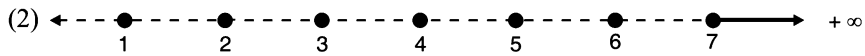
Now look at the following diagrams.



Graph of $\{x : 1 < x \leq 5 ; x \in N\}$

Fig. 11.02

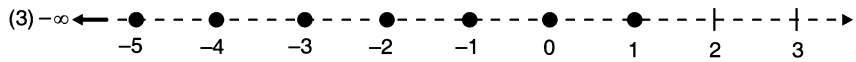
The number line represents a natural number line. The point at 1 is marked with an open circle to indicate that 1 is not included in the graph.



Graph of $\{x : x \geq 2, x \in N\}$

Fig. 11.03

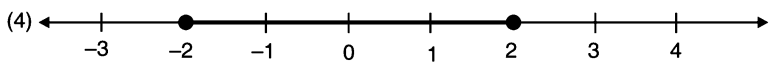
The diagram shows a natural number line. The thick arrow at the extreme right indicates that 7 and all natural numbers greater than 7 are included in the graph.



Graph of $\{x : x \leq 1, x \in Z\}$

Fig. 11.04

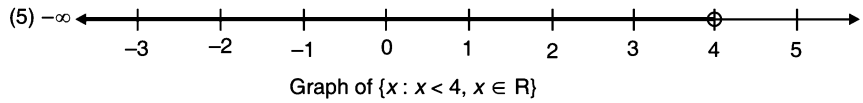
The diagram shows an integer line. The thick arrow at the extreme left indicates that 1 and all integers less than 1 are included in the graph.



Graph of $\{x : -2 \leq x \leq 2, x \in R\}$

Fig. 11.05

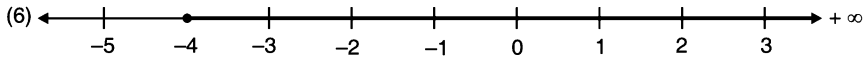
The diagram shows a real number line. The graph consists of the line segment (shown thick) with end-points included in the graph.



Graph of $\{x : x < 4, x \in R\}$

Fig. 11.06

The hollow circle at 4 shows that the number 4 is not included in the graph.



Graph of $\{x : x \geq -4, x \in R\}$

Fig. 11.07

Ex. 1. Solve: $x + 2 > 9$.

Sol. $x + 2 > 9 \Rightarrow x > 9 - 2 \Rightarrow x > 7$.

Therefore, the solution set is $\{x : x > 7\}$.

If $x \in W$, the solution set is {8, 9, 10...} and its graph is as shown below:

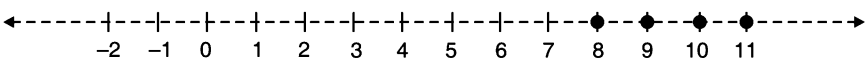


Fig. 11.08

If $x \in R$, the solution set will have to be written in the form $\{x : x > 7, x \in R\}$ and its graph will be as shown below:

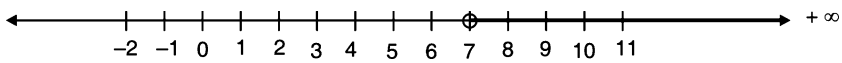


Fig. 11.09

Ex. 2. Solve the inequality $4\left(\frac{1}{2} - p\right) + 7 \geq 57$ over R, and graph the solution set.

Sol. $4\left(\frac{1}{2} - p\right) + 7 \geq 57 \quad \Rightarrow 2 - 4p + 7 \geq 57 \Rightarrow -4p \geq 57 - 2 - 7 \Rightarrow -4p \geq 48$

$\Rightarrow 4p \leq -48 \quad \Rightarrow p \leq -12$

The solution set is $\{p : p \leq -12\}$. The graph is shown below.

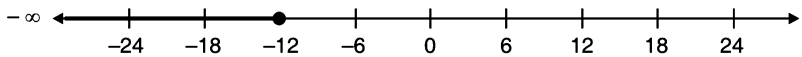


Fig. 11.10

Ex. 3. Find the values of x, which satisfy the inequation:

$$-2 \leq \frac{1}{2} - \frac{2x}{3} \leq 1\frac{5}{6};\ x \in N$$

Graph the solution set on the real number line.

Sol. $-2 \leq \frac{1}{2} - \frac{2x}{3} \leq \frac{11}{6} \Rightarrow -12 \leq 3 - 4x \leq 11$ [Multiplying all terms by the (LCM '12')]

Now, $-12 \leq 3 - 4x \Rightarrow 4x \leq 15 \Rightarrow x \leq \frac{15}{4} \Rightarrow x \leq 3\frac{3}{4}$

and $3 - 4x \leq 11 \Rightarrow -8 \leq 4x \Rightarrow 4x \geq -8 \Rightarrow x \geq -2 \therefore -2 \leq x \leq 3\frac{3}{4}$

As $x \in N$, therefore, the solution set is $\{1, 2, 3\}$. The graph shown below consists of only 3 points at 1, 2, 3 respectively.

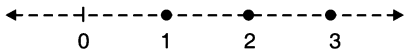


Fig. 11.11

11.06. Solution of system of linear inequations in one variable

Method. To solve a system of inequations, proceed as follows.

Step 1. Find all the values of the variable satisfying each of them separately.

Step 2. Find the values of the variable, which are common to them.

These common values will be the solution of the given system of inequations.

Ex. 4. Solve and graph the solution set of

(*i*) $2x - 9 < 7$ and $3x + 9 \le 25$, $x \in R$; (*ii*) $3x - 2 > 19$ or $3 - 2x \ge -7$; $x \in R$.

Sol. (*i*) Let $A = \{x : 2x - 9 < 7, x \in R\}$

$2x - 9 < 7 \Rightarrow 2x < 16 \Rightarrow x < 8$

Let $B = \{x : 3x + 9 \le 25, x \in R\}$

$3x + 9 \le 25 \Rightarrow 3x \le 16 \Rightarrow x \le \dfrac{16}{3}$

Required solution set = $A \cap B$ ['And' $\Rightarrow \cap$]

$$= \{x : x < 8, x \in R\} \cap \left\{x : x \le \frac{16}{3}, x \in R\right\} = \left\{x : x \le \frac{16}{3}\right\}$$

The graph is shown below.

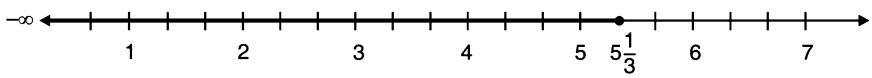


Fig. 11.12

(*ii*) Let $A = \{x : 3x - 2 > 19, x \in R\}$ or $B = \{x : 3 - 2x \ge -7, x \in R\}$

$3x - 2 > 19 \Rightarrow 3x > 21 \Rightarrow x > 7$

$3 - 2x \ge -7 \Rightarrow 10 \ge 2x \Rightarrow 2x \le 10 \Rightarrow x \le 5$

Required solution set = $A \cup B = \{x : x > 7, x \in R\} \cup \{x : x \le 5, x \in R\}$, *i.e.*, $x > 7$ or $x \le 5$

The graph is shown below:

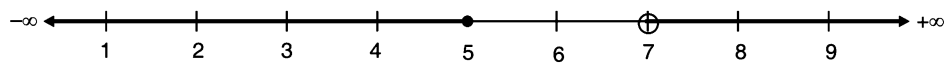


Fig. 11.13

Ex. 5. Given $P = \{x : 5 < 2x - 1 \le 11, x \in R\}$, $Q = \{x : -1 \le 3 + 4x < 23, x \in I\}$ where R = {real numbers}, I = {Integers}.

Represent P and Q on number lines. Write down the elements of $P \cap Q$.

Sol. $P = \{x : 5 < 2x - 1 \le 11, x \in R\}$

$5 < 2x - 1 \Rightarrow 6 < 2x \Rightarrow 3 < x$

and $2x - 1 \le 11 \Rightarrow 2x \le 12 \Rightarrow x \le 6$

$\therefore$ $P = \{x : 3 < x \le 6, x \in R\}$ ... (*i*)

$Q = \{x : -1 \le 3 + 4x < 23, x \in I\}$

Now, $-1 \le 3 + 4x \Rightarrow -4 \le 4x \Rightarrow -1 \le x \Rightarrow x \ge -1$

and $3 + 4x < 23 \Rightarrow 4x < 20 \Rightarrow x < 5$

$\therefore$ $Q = \{x : -1 \le x < 5, x \in I\}$ ... (*ii*)

The graphs of (*i*) and (*ii*) are as follows:

(*i*) ($3 < x \le 6, x \in R$; 3 is not included)

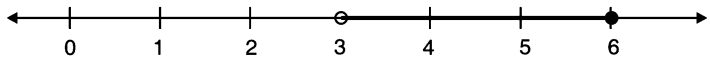


Fig. 11.14

(*ii*) ($-1 \le x < 5, x \in I$; 5 is not included)

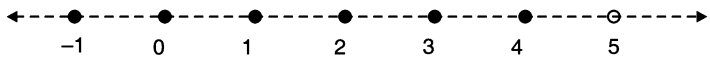


Fig. 11.15

$P \cap Q = \{4\}$.

Ex. 6. The diagram represents two inequations *A* and *B* on real number lines.

(*i*) Write down *A* and *B* in set builder notations.

(*ii*) Reprsent $A \cap B$ and $A \cap B'$ on two different number lines.

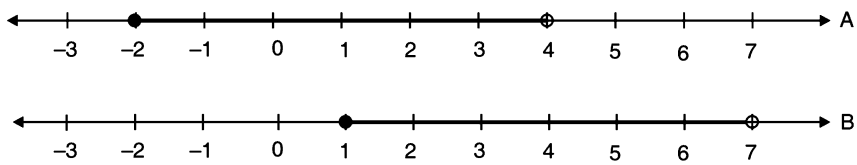


Fig. 11.16

Sol. $A = \{x: -2 \le x < 4\}$

$B = \{x: 1 \le x < 7\}$; $\therefore$ $B' = \{x < 1, x \ge 7\}$

$A \cap B = \{x: 1 \le x < 4\}$ $A \cap B' = \{-2 \le x < 1\}$

The graphs are drawn as follows:

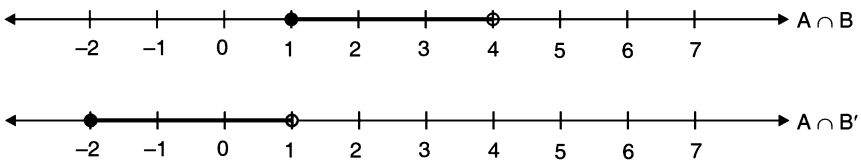


Fig. 11.17

Ex. 7. Solve the following system of inequations

$$\mathbf{2(2x+3) - 10 < 6\,(x-2),\ \frac{2x-3}{4} + 6 \ge 2 + \frac{4x}{3}.}$$

Sol. $2(2x + 3) - 10 < 6\,(x - 2)$

$\Rightarrow \quad 4x + 6 - 10 < 6x - 12 \quad \Rightarrow \quad 4x - 6x < -12 - 6 + 10$

$\Rightarrow \quad -2x < -8 \Rightarrow 2x > 8 \quad \Rightarrow \quad x > 4$ ... (*i*)

Also, $\frac{2x-3}{4} + 6 \ge 2 + \frac{4x}{3} \Rightarrow 3(2x - 3) + 72 \ge 24 + 16x$ [Multiplying all terms by 12]

$\Rightarrow \quad 6x - 9 + 72 \ge 24 + 16x \Rightarrow \quad 6x - 16x \ge 24 + 9 - 72$

$\Rightarrow \quad -10x \ge -39 \Rightarrow 10x \le 39 \Rightarrow x \le \frac{39}{10} \Rightarrow \le 3\frac{9}{10}$ ... (*ii*)

Obviously, there is no value of x common to (*i*) and (*ii*). So, the given system has no solution or we say that the solution set is ϕ.

Inequations involving absolute values.

Ex. 8. Solve $|x| < 7$.

Sol. We know that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$.

Case I. When $x \geq 0$, $|x| < 7 \Rightarrow x < 7$. Hence, in this case we have $0 \leq x < 7$. ... (1)

Case II. When $x < 0$, $|x| < 7 \Rightarrow -x < 7 \Rightarrow x > -7$

Hence, in this case we have $-7 < x < 0$ ... (2)

Combining (1) and (2), the solution of $|x| < 7$ is given by $-7 < x < 7$, *i.e.,* all real numbers x lying between -7 and 7.

Remark. The solutions for $|x| \leq 7$ will be all real numbers x lying between -7 and 7 including the numbers -7 and 7, that is $-7 \leq x \leq 7$.

Thus, $|x| \leq 7$ is equivalent to $-7 \leq x \leq 7$.

Ex. 9. Solve $|x| > 7$.

Sol. When $x \geq 0$, $|x| > 7 \Rightarrow x > 7$ ... (1)

When $x < 0$, $|x| > 7 \Rightarrow -x > 7 \Rightarrow x < -7$ ... (2)

Combining (1) and (2), the solution of $|x| > 7$ is $x < -7$ or $x > 7$.

i.e., the real numbers which are either less than -7 or greater than 7.

Remark. The solution of $|x| \geq 7$ will be given by $x \leq -7$ or $x \geq 7$. Thus, $|x| \geq 7$ is equivalent to $x \leq -7$ or $x \geq 7$.

We may sum up the above results are under and use the same directly in finding solutions of inequations involving absolute values.

Let $a > 0$. Then

(*i*) $|x| < a$ if and only if $-a < x < a$, *i.e.*, $x \in (-a, a)$

(*ii*) $|x| \leq a$ if and only if $-a \leq x \leq a$, *i.e.*, $x \in [-a, a]$

(*iii*) $|x| > a$ if and only if $x < -a$ or $x > a$

(*iv*) $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$

Ex. 10. Solve and graph the solution set

(*i*) $|x - 2| < 3$ (*ii*) $|x - 2| \geq 3$.

Sol. (*i*) $|x - 2| < 3 \Rightarrow -3 < x - 2 < 3$

$\Rightarrow -3 < x - 2$ and $x - 2 < 3$

$\Rightarrow -1 < x$ and $x < 5$

$\Rightarrow x > -1$ and $x < 5$

$\therefore$ The solution set is $\{x = -1 < x < 5\}$.

The graph is shown below,

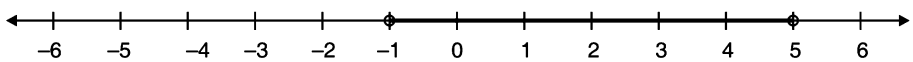


Fig. 11.18

(*ii*) $|x - 2| \geq 3 \Rightarrow x - 2 \leq -3$ or $x - 2 \geq 3$

$\Rightarrow x \leq -1$ or $x \geq 5$

$\therefore$ The solution set is $\{x : x \leq -1$ or $x \geq 5\}$

The graph is shown below

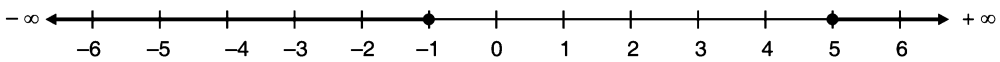


Fig. 11.19

Ex. 11. Graph the set $\{x = |2x + 1| \leq 7\} \cap \{x: 2 - x > 0\}$.

Sol. $|2x + 1| \leq 7$ and $2 - x > 0$

$\Rightarrow -7 \leq 2x + 1 \leq 7$ and $-x > -2$

$\Rightarrow \quad -8 \le 2x \le 6$ and $x < 2$

$\Rightarrow \quad -4 \le x \le 3$

The graph of these solutions are drawn below.

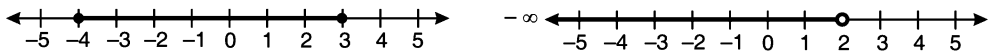


Fig. 11.20

The following number line shows the graph of the given set as the overlapping portion (the intersection) of the preceding graphs.

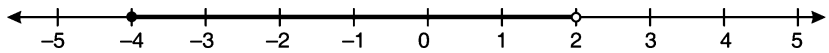


Fig. 11.21

Thus, $\{x : |2x + 1| \le 7\} \cap \{x : 2 - x > 0\} = \{x : -4 \le x < 2\}$.

EXERCISE 11 (*a*)

1. You are given the following numbers:

$-2.6 \quad 5.1 \quad -3 \quad 0.4 \quad 1.2 \quad -3.1 \quad 4.7$

Fill in the blanks.

(*i*) $A = \{x : x \ge -3\} = \{...\}$ (*ii*) $B = \{x : x \le 1\} = \{...\}$

2. If the replacement set is $\{-2, -1, +1, +2, +4, +5, +9\}$, what is the solution set of each of the following mathematical sentences?

(*i*) $x + \frac{3}{2} > \frac{5}{2}$ (*ii*) $2x - 5 \ge 10$ (*iii*) $3y + 2 \le \frac{5}{2}$

3. List the solution set of $30 - 4(2x - 1) < 30$, given that x is a positive integer.

4. (*i*) $x \in \{2, 4, 6, 9\}$ and $y \in \{4, 6, 18, 27, 54\}$. Form all ordered pairs (x, y) such that x is a factor of y and $x < y$.

(*ii*) Find the truth set of the inequality $x > y + 2$ where

$(x, y) \in \{(1, 2), (2, 3), (5, 1), (7, 3), (5, 6), (6, 5)\}$.

5. P is the solution set of $8x - 1 > 5x + 2$ and Q is the solution set of $7x - 2 \ge 3(x + 6)$, where $x \in N$. Find the set $P \cap Q$.

6. Find the solution of the inequality

$2 \le 2p - 3 \le 5, p \in R$

Hence, graph the solution set on the number line.

7. If x is a negative integer, find the solution set of $\frac{2}{3} + \frac{1}{3}(x + 1) > 0$.

8. Solve the inequality: $3 - 2x \ge x - 12$, given that $x \in N$.

9. Find the range of values of x which satisfies

$$-2\frac{2}{3} \le x + \frac{1}{3} < 9\frac{1}{3}, x \in R.$$

Graph these values of x on the number line.

10. $A = \{x: -1 < x \le 5, x \in R\}$, $B = \{x: -4 \le x < 3, x \in R\}$. Represent (*i*) $A \cap B$ (*ii*) $A' \cap B$, on different number lines.

[**Sol.** *A*

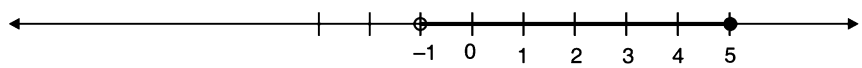


Fig. 11.22

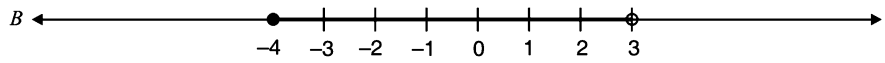


Fig. 11.23

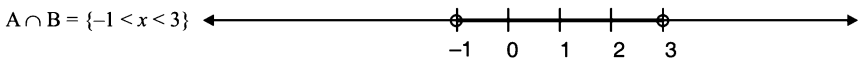


Fig. 11.24

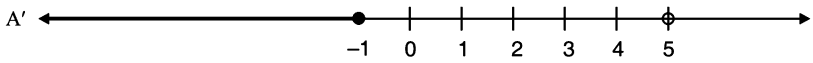


Fig. 11.25

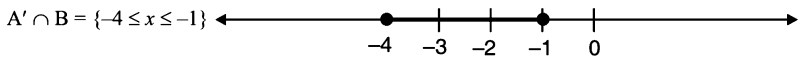


Fig. 11.26

Thus, $A \cap B = \{-1 < x < 3\}$, and $A' \cap B = \{-4 \le x \le -1\}$ and they are represented on different real number lines as shown above].

11. Find the range of values of x, which satisfy the inequality:

$$\frac{-1}{5} \le \frac{3x}{10} + 1 < \frac{2}{5}; x \in R$$

Graph the solution set on the number line.

12. Find the range of values of x which satisfy

$$\frac{-1}{3} \le \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6}; x \in R$$

Graph these values of x on the real number line.

13. Solve the following inequation, and graph the solution set, on the number line.

$2x - 3 < x + 2 \le 3x + 5, x \in R$.

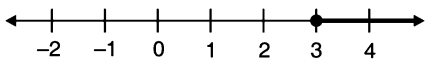
14. Solve the inequation on: $-3 \le 3 - 2x < 9, x \in R$.

15. Find the value of x which satisfies the inequation

$$-2 \le \frac{1}{2} - \frac{2x}{3} \le 1\frac{5}{6}, x \in N$$

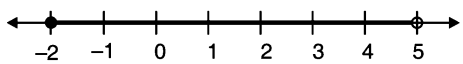
Graph the solution on a number line.

16. Write down the range of real values of x for which the inequation $x > 3$ and $-2 \le x < 5$ are both true. [**Hint:** Clearly, you have to find the common values of the solution sets, which you can easily do by drawing the graphs.



Graph of $\{x : x > 3, x \in R\}$

Fig. 11.27



Graph of $\{x : -2 \le x < 5, x \in R\}$

Fig. 11.28

The common portion of the two graphs is the set $\{x : 3 < x < 5, x \in R\}$.

Hence, the required range is the set $\{x : 3 < x < 5, x \in R\}$.

17. Solve and graph the solution set of $3x - 4 > 11$ or $5 - 2x \ge 7$.

18. The following diagram represents two inequations A and B on number lines.

(*i*) Write down A, B and B' in set builder form.

(*ii*) Represent $A \cap B$ and $A \cap B'$ on two different number lines.

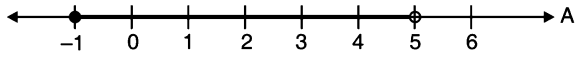


Fig. 11.29

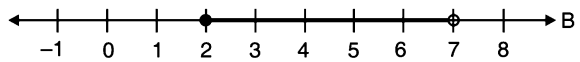


Fig. 11.30

19. P is the solution set of $7x - 2 > 4x + 1$ and Q is the solution set of $(9x - 45) \geq 5(x - 5)$; where $x \in R$. Represent (*i*) $P \cap Q$ (*ii*) $P - Q$, (*iii*) $P \cap Q'$ on different lines.

20. Solve the following system of inequalities ($x \in R$).

(*i*) $2x - 7 > 5 - x$ (*ii*) $2x + 5 \leq 0$ (*iii*) $4x + 3 \geq 2x + 17$ (*iv*) $5x - 7 < 3(x + 3)$

$11 - 5x \leq 1$ $x - 3 \leq 0$ $3x - 5 < -2$ $1 - \frac{3x}{2} \geq x - 4$

(*v*) $\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$ (*vi*) $2(2x + 3) - 10 < 6(x - 2)$ (*vii*) $-11 \leq 4x - 3 \leq 13$

$\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$ $\frac{2x-3}{4} + 6 \geq 2 + \frac{4x}{3}$

21. Solve and graphs the solution set

(*i*) $|x - 3| < 4$ (*ii*) $|x - 3| > 4$ (*iii*) $|x + 3| \geq 4$ (*iv*) $|1 - 3x| < 4$

(*v*) $2 + 3|2y - 1| > 8$

22. $3|x - 6| - 4 \leq 11$ **23.** $2|3p - 5| + 1 > 7$

24. $|5 - (m - 3)| + 8 < 15$

ANSWERS

1. (*i*) $A = \{-3, -2.6, 0.4, 1.2, 4.7, 5.1\}$ (*ii*) $B = \{-3.1, -3, -2.6, 0.4\}$

2. (*i*) $\{+2, +4, +5, +9\}$ (*ii*) $\{+9\}$ (*iii*) $\{-2, -1\}$

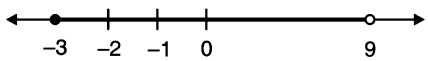
3. $\{1, 2, 3, ...\}$

4. (*i*) $\{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54), (9, 18), (9, 27), (9, 54)\}$ (*ii*) $\{(5, 1), (7, 3)\}$

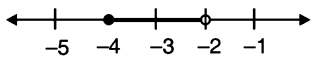
5. $\{5, 6, 7, ...\}$ **6.** $\left\{2\frac{1}{2} \leq x \leq 4, x \in R\right\}$ **7.** $\{-2, -1\}$

8. The solution set is $\{1, 2, 3, 4, 5\}$

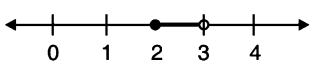
9. $-3 \leq x < 9$



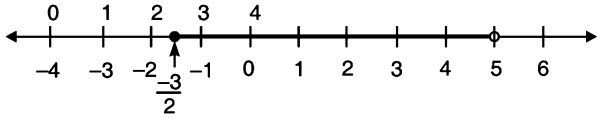
11. $-4 \leq x < -2$



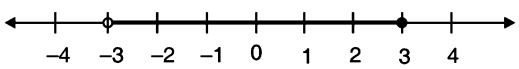
12. $2 \leq x < 3$



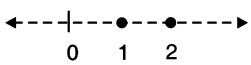
13. $-\frac{3}{2} \leq x < 5, x \in R$



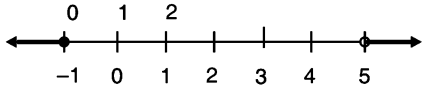
14. $-3 < x \leq 3; x \in R$



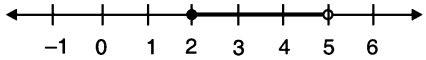
15. $x = 1, 2; x \in N$



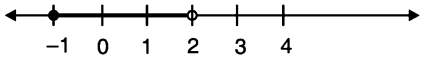
17. $x > 5$ or $x \leq -1; x \in R$



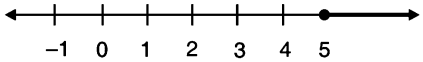
18. $A \cap B = \{x : 2 \le x < 5, x \in R\}$



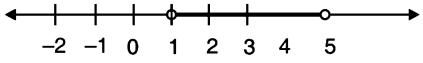
$A \cap B' = \{x : -1 \le x < 2, x \in R\}$



19. (*i*) $P \cap Q = \{x : x \ge 5, x \in R\}$



(*i*) $P - Q = \{x : 1 < x < 5\}$

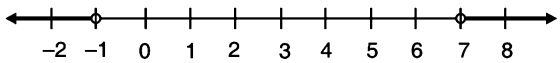


(*iii*) $P \cap Q'$ is the same as (*ii*)

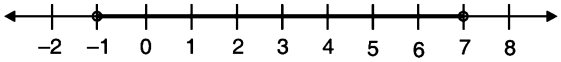
20. (*i*) $x > 4$ (*ii*) $x \le \frac{-5}{2}$ (*iii*) No solution (*iv*) $x \le 2$

(*v*) $(3, \infty)$ (*vi*) No solution (*vii*) $[-2, 4]$

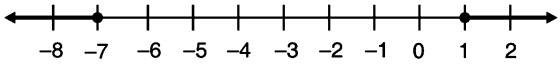
21. (*i*) $\{x : x < -1 \text{ or } x > 7\}$



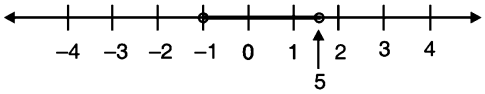
(*ii*) $\{x : -1 < x < 7\}$



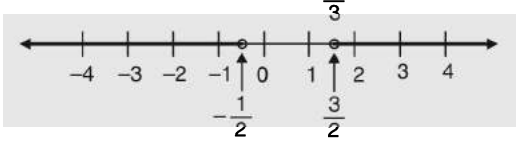
(*iii*) $\{x : x \ge 1 \text{ or } x \le -7\}$



(*iv*) $\left\{x: -1 < x < \frac{5}{3}\right\}$



(*v*) $\left\{y : y > \frac{3}{2} \text{ or } y < -\frac{1}{2}\right\}$



22. $1 \le x \le 11$ **23.** $\left\{p : p < \frac{2}{3} \text{ or } p > -\frac{8}{3}\right\}$ **24.** $1 < m < 15$

11.07. Linear inequalities in two variables

Consider inequalities of the form $ax + by > c$, $ax + by \ge c$, $ax + by < c$ and $ax + by \le c$ (*i.e.*, $4x + 3y > 2$, $3x - 5y \le 13$ etc.). "$ax + by = c$" is called the **associated equation** of each of these inequalities.

The graph of an inequation in two variables is a region of the $x - y$ plane. In Figure (*i*) the area to the right of the y-axis represents the inequation $x > 0$, and the area above the x-axis represents the inequation $y > 0$. In Figure (*ii*) are shown the regions representing inequations

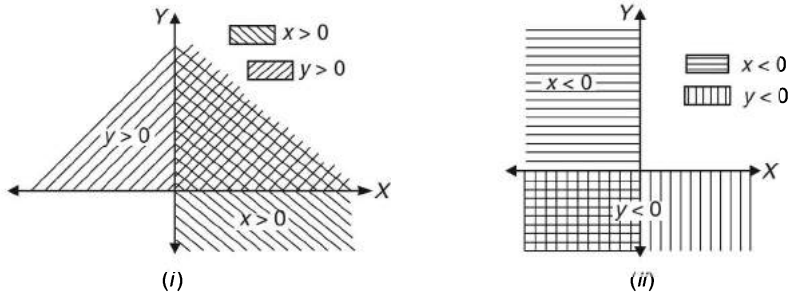


Fig. 11.31

$x < 0$ and $y < 0$. In Figure (*i*) and Figure (*ii*) if we include the y-axis in the graph, the shaded regions will then represent respectively the inequations $x \geq 0$ and $x \leq 0$. Similarly, on including the x-axis in the graph, we will get regions representing the inequations $y \geq 0$ and $y \leq 0$. The double shaded region represents the graph satisfying both the inequations.

Ex. 1. Draw the graph of (*i*) $y > 3$ (*ii*) $y < 3, x, y \in$ R.

Sol. The graphs of both $y > 3$ and $y < 3$ can be constructed by using the graph of $y = 3$. The graph of $y = 3$ is a line as shown in Fig. 11.32 (*i*). This line separates the plane into two regions called **half planes.** As is shown in Fig. 11.32 (*ii*), the ordinate of every point above this line is greater than 3 and ordinate of every point below the line is *less* than 3. The ordinate of every point on the line is 3.

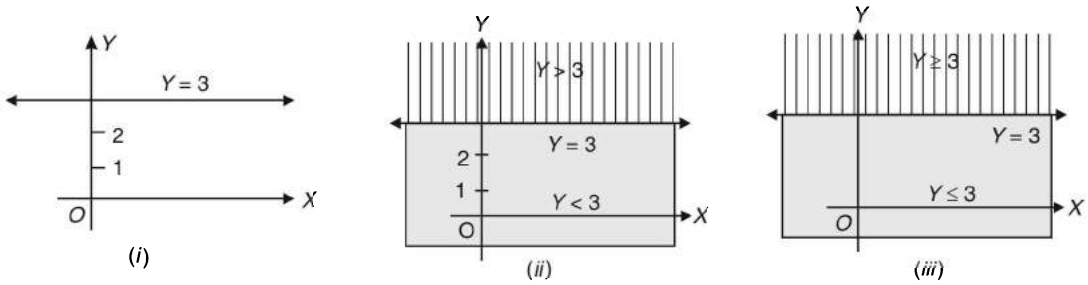


Fig. 11.32

Therefore, the half plane above the line is the graph of $\{(x, y) : y > 3; x, y \in R\}$ and the half plane below the line is the graph of $\{(x, y) : y < 3; x, y \in R\}$.

Each of these regions is called an **open half plane** because the points on boundary line do not belong to the region. In Fig. 11.32 (*i*) we have shown the line $y = 3$ as a dashed line which indicates that the regions are open half planes, *i.e.*, the points on this line lie in neither half plane.

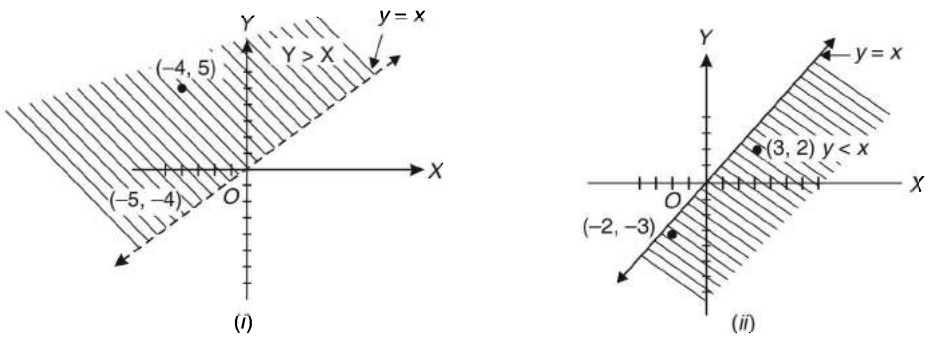
The graph of the inequality "$y \geq 3$" is the union of the half plane above the line "$y = 3$" and the line itself. The union of a half plane and its boundary is called a **closed half plane.** In such a case, we represent the line as a *solid line* to indicate that all points on it belong to the graph. [Fig. 11.32 (*iii*)].

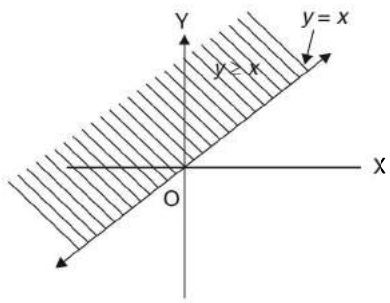
Ex. 2. Draw the graph of (*i*) $y > x$, (*ii*) $y < x$, (*iii*) $y \geq x$, (*iv*) $y \leq x$.

Sol. First graph the line $y = x$. By taking a few ordered pairs satisfying the inequality $y > x$, *i.e.*, (5, 6), (– 4, 5), (– 5, – 4) etc., we can verify that every point of the plane above the line $y = x$ satisfies $y > x$. This region, excluding the line $y = x$ as indicated by the dashed line represents the graph of $y > x$. If we include the line $y = x$ as indicated by solid line in (*iii*), then it represents the graph of $y \geq x$.

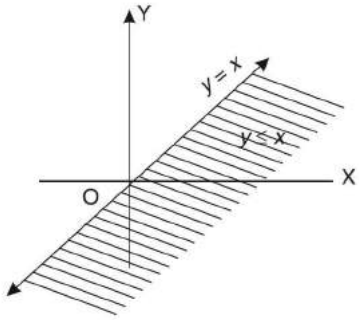
x	y
0	0
2	2
–3	–3

Now, by considering a few ordered pairs satisfying the inequality $y < x$, such as (3, 2), (–2, – 3) etc., we can verify that the half-plane below the line $y = x$ is the graph of the inequality $y < x$. Fig. 11.33 (*iv*) shows the graph of $y \leq x$.





(iii)



(iv)

Fig. 11.33

Ex. 3. Graph the inequality $3x - y \geq 1$.

Sol. Step 1. *Transform the inequality into an equivalent one having y as a member.*

$3x - y \geq 1 \Rightarrow 3x - 1 \geq y \Rightarrow y \leq 3x - 1$

Step 2. *Draw as a solid line the graph of* $y = 3x - 1$. This can be done by plotting a few points on the line say (0, –1), (1, 2) and (–1, – 4).

Step 3. *Shade the half-plane below the line.* The closed half-plane so obtained is the graph of the inequality $3x - y \geq 1$.

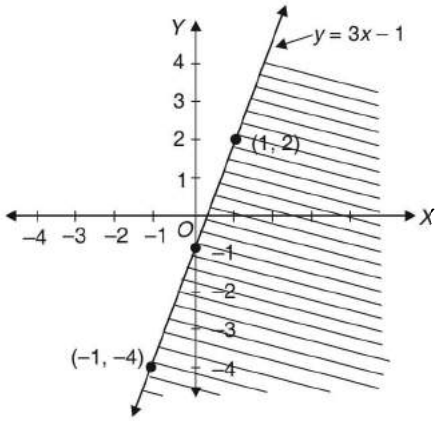


Fig. 11.34

Note 1. Note that the open half plane below the line will represent the graph of $3x - y > 1$. It can be verified by considering a few solutions of this inequality, say (0, – 2), (2, 4), (– 1, – 5), etc. Plot these points and verify whether these lie in the graph or not. If yes, our graph is correct.

x	y
0	–1
1	2
–1	–4

Note 2. The open half-plane above the line will represent the graph of $3x - y < 1$ and the closed half-plane that of $3x - y \leq 1$. This can also be checked by considering a few solutions such as (0, 1), (2, 6), (– 1, – 3), etc.

Ex. 4. Graph the inequation $y \geq |x|$.

Sol. $\because$ $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

$\therefore$ $y \geq |x| \Rightarrow y \geq x$ if $x \geq 0$ and $y \geq -x$ if $x < 0$.

(*i*) To draw the graph of $y \geq x$, where $x \geq 0$, first draw the graph of the equation $y = x$ where $x \geq 0$.

Plot the ordered pairs (0, 0), (1, 1) and (2, 2). Join them with a ruler to get the graph of $y = x$ where $x > 0$.

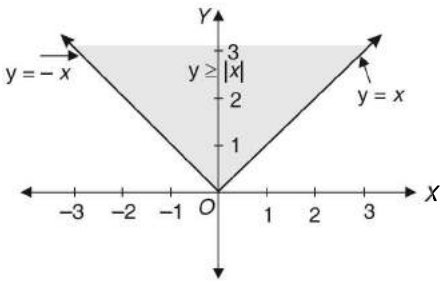


Fig. 11.35

x	y
–1	1
–2	2
–3	–3

For the graph of $y \geq x$

x	y
0	0
1	1
2	2

For the graph of $y \geq -x$

It is a ray with origin as the initial point. Since the points on the ray are included, therefore, the ray should be solid.

Putting the coordinates of any point, say, (3, 1) in the inequation $y > x$, we get $1 > 3$ which is false.

$\therefore$ The graph of the inequation $y \geq x$ is the closed half-plane to the left of the line $y = x$.

(*ii*) To draw the graph of $y \geq -x$, where $x < 0$, first draw the graph of the equation $y = -x$, where $x < 0$.

Plot the ordered pairs (–1, 1), (– 2, 2) and (–3, 3). Join them with a ruler to get graph of $y = -x$ (where $x < 0$). It is again a ray with origin as the initial point. Since the points on the ray are included, the ray should be solid.

Substituting the coordinates of the point (–4, 1) in $y > -x$, we get $1 > 4$ which is false.

Therefore, the graph of the inequation $y \geq -x$ is the closed half plane to the right of the line $y = -x$ and it does not contain the point (–4, 1).

Thus, the required graph of the inequation $y \geq |x|$ is the shaded region in the figure.

11.08. System of inequalities

The solution set of a system of inequalities can be found out graphically in the same manner as that of equations. The given inequations are graphed on the same axes. Then the intersection of the half-planes, i.e, the graphs of the given inequalities represents the solution of the given system of inequalities. Obviously, the solution is an infinite set of ordered pairs.

Ex. 5. Draw the graph of the inequations $3x + 5y \leq 15$ and $x \geq 1$ and shade the common region.

Sol. Step 1. *First draw the graph of the equation $3x + 5y = 15$.*

$3x + 5y = 15 \Rightarrow 5y = 15 - 3x \Rightarrow y = \dfrac{15 - 3x}{5}$.

x	y
5	0
0	3
–5	6

The table of values is as given on the right.

Plot the points (5, 0) (0, 3) and (– 5, 6) and join them to get the graph of the equation $3x + 5y = 15$. Putting the coordiantes of the origin (0, 0) in $3x + 5y < 15$, we get $0 + 0 < 15$, which is true.

$\therefore$ The graph of the inequation $3x + 5y \leq 15$ is the region below the line $3x + 5y = 15$ (containing the origin) and including the line. Shade it by drawing slant lines.

Step 2. Draw the graph of $x = 1$. It is a line parallel to y-axis at a distance of 1 unit from it. Draw a solid line. This line and the region to the right of this line is the graph of the inequation $x \geq 1$. Show it by drawing horizontal line.

Step 3. The crossed area common to both the graphs is the required region.

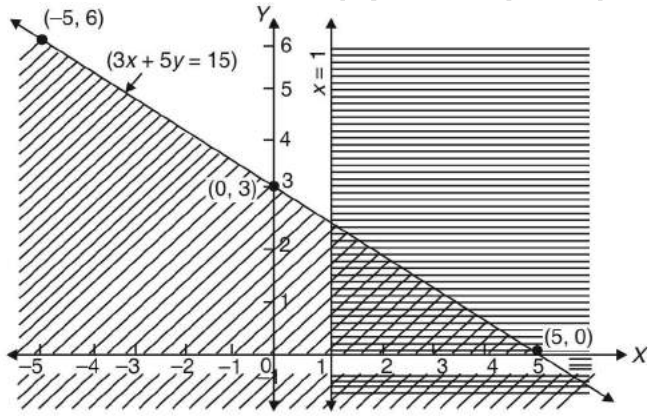


Fig. 11.36

Ex. 6. Graph the solution sets of the following systems.

(*i*) $\begin{cases} 2x - y \le 2 \\ 2x - y \le 0 \end{cases}$ **(*ii*)** $\begin{cases} x + 2y \ge 4 \\ 2x + 4y \le 8 \end{cases}$ **(*iii*)** $\begin{cases} x - 3y \le 6 \\ x - 3y > -6 \end{cases}$ **(*iv*)** $\begin{cases} 3x + y \ge 3 \\ 3x + y \le -3 \end{cases}$

Sol. (*i*) $2x - y \le 2$, $2x - y \ge 0$.

Draw the graph of $2x - y = 2$ or $y = 2x - 2$

x	0	1	2
y	–2	0	2

For $2x - y < 2$,

take $x = -2, y = 0$

$2 \times (-2) - 0 < 2 \Rightarrow -4 < 2$ which is true.

$\therefore$ Shade the region in which the point (–2, 0) lies.

Now, draw the graph of $2x - y = 0$ or $y = 2x$

x	0	1	–1
y	0	2	–2

For $2x - y > 0$, take $x = 3, y = 0$. Now $2 \times 3 - 0 > 0 \Rightarrow 6 > 0$ which is true.

Shade the region which contains the point (3, 0).

Then all the points in the double shaded region are members of the solution set.

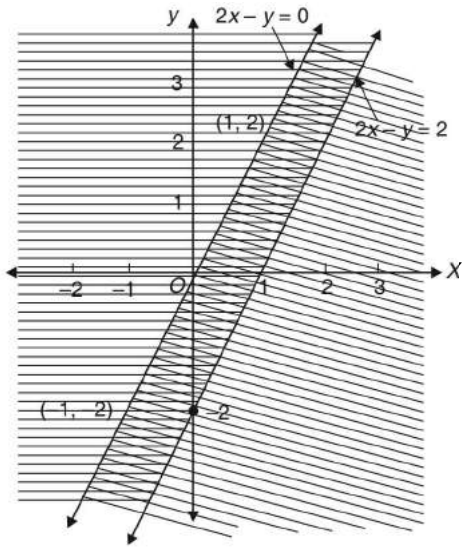


Fig. 11.37

(*ii*) $x + 2y \ge 4$, $2x + 4y \le 8$.

Draw the graph of $x + 2y = 4$ or $y = \dfrac{4 - x}{2}$

x	0	2	4
y	2	1	0

For $x + 2y > 4$, take $x = 0$, $y = 3$ then $0 + 2 \times 3 > 4 \Rightarrow 6 > 4$ which is true.

Shade the region which contains the point (0, 3).

The graph of $2x + 4y = 8$ or $x + 2y = 4$ is the same line l.

For $2x + 4y < 8$, take $x = -1, y = 0$.

$2 \times (-1) + 4 \times 0 < 8 \Rightarrow -2 < 8$, which is true.

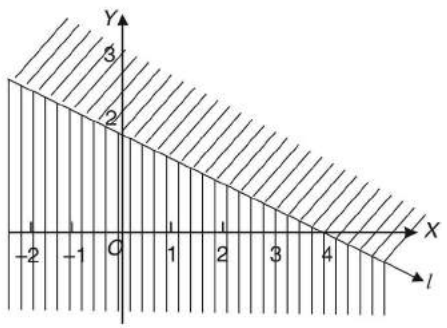


Fig. 11.38

Shade the region in which the point (–1, 0) lies. It is seen from the graph that there are no common shaded regions except the line which is common to both. Therefore, all the points on the line l are the members of the solution set.

***(iii)* $x - 3y \le 6, x - 3y > -6$**

Draw the graph of $x - 3y = 6$

or $$y = \frac{x - 6}{3}$$

x	0	6	3
y	–2	0	–1

For $x - 3y < 6$, take $x = 0, y = 1$. Then $0 - 3 \times 1 < 6 \Rightarrow -3 < 6$ which is true. Shade the region in which the point (0, 1) lies. [Line l_1].

Now, draw the graph of $x - 3y = -6$ or $y = \frac{x+6}{3}$.

The table of values is

x	0	3	–3
y	2	3	1

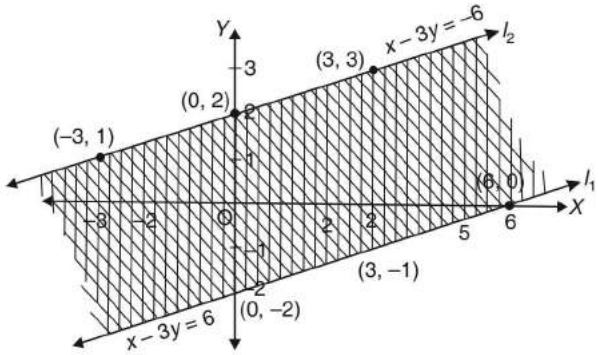


Fig. 11.39

For $x - 3y > -6$, take $x = 1, y = 0$ then $1 - 3 \times 0 > -6 \Rightarrow 1 > -6$, which is true.

Shade the region in which the point (1, 0) lies. Thus, all the points in the double shaded region (excluding the points on the dotted line $x - 3y = -6$ (dotted line l_2) are the members of the solution set.

(*iv*) **$3x + y \geq 3, 3x + y \leq -3$.**

Draw the graph of $3x + y = 3$ or $y = 3 - 3x$ by plotting the points (0, 3), (1, 0) and (2, – 3) and joining them.

x	0	1	2
y	3	0	–3

For $3x + y > 3$, take $x = 2, y = 0$

$3 \times 2 + 0 > 3 \Rightarrow 6 > 3$, which is true.

$\therefore$ Shade the region in which the point (2, 0) lies. [Line l_1]

Now draw the graph of $3x + y = -3$ or $y = -3x - 3$ by plotting the points (0, –3), (–1, 0), (–3, 6) and joining them

x	0	–1	–3
y	–3	0	6

For $3x + y < -3$, take $x = -3, y = 0$

$3x - 3 + 0 < -3 \Rightarrow -9 < -3$, which is true.

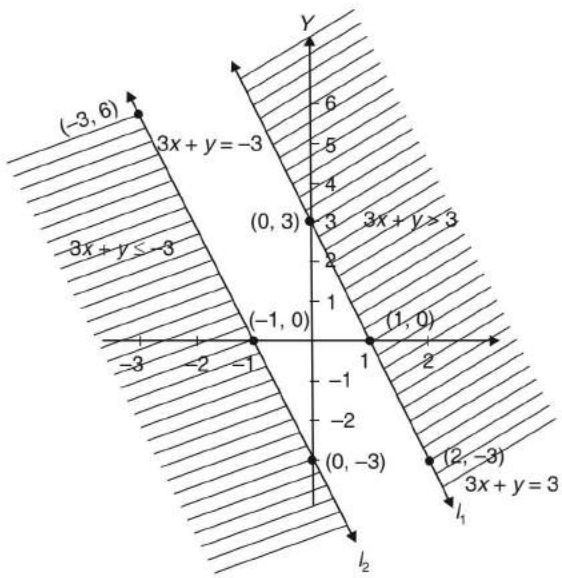


Fig. 11.40

$\therefore$ Shade the region in which the point (–3, 0) lies [line l_2]. The lines l_1 and l_2 are parallel and there is no common shaded region. Therefore, the system has no solution.

11.09. System of three or more inequalities

Ex. 7. Solve graphically: $y - 2 \geq 0$; $x - y \geq 1$; $5y - 2x \leq 10$.

Method 1. *Graph each of the inequalities on the same coordinate plane.*

2. *The solution set is given by the common region of intersection of the graphs of all the inequalities (which are closed half-planes).*

Note. In set notation the solution set can be written as

$\{(x, y \mid y \geq 2)\} \cap \{(x, y) \mid y \leq x - 1\} \cap \left\{(x, y) \mid y \leq 2 - \frac{2}{5}x\right\}$. [This complicated form merely states that the ordered pairs common to the the three inequalities are in the solution set].

Sol. Draw the graph of

(*i*) $y - 2 \geq 0$ (*ii*) $x - y \geq 1$ (*iii*) $5y - 2x \leq 10$ as shown in the figure.

$y = x - 1$ $y = 2 + \frac{2}{5}x$

x	y
1	0
0	–1

x	y
0	2
5	4

Then region bounded by the ΔABC is the intersection of the three graphs (which are 3 closed half-planes).

The solution set is represented by points on the sides and in the interior of ΔABC. Sample ordered pairs in the solution set are: (3, 2), (4, 2.5), (5, 2.1), (4, 3).

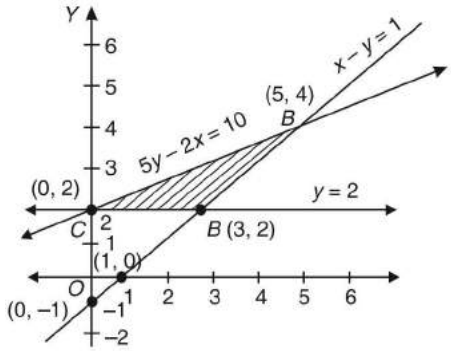


Fig. 11.41

The vertices of the triangle are the corner points of the common region. These can be found by solving the related equations by pairs, *i.e.*, $x - y = 1$ and $5y - 2x = 10$ give (5, 4); $y - 2 = 0$ and $x - y = 1$ give (3, 2); $y - 2 = 0$ and $5y - 2x = 10$ give (0, 2). The corner points thus are $A(3, 2)$, $B(5, 4)$ and C(0, 2).

Note. As has been done in this case the corner points of a figure composed of the intersection of half-planes can always be found by solving the equations of the boundary lines in pairs.

Ex. 8. Solve graphically: $-x + 2y \leq 8$, $2x - y \geq -4$, $y \leq 8$.

Sol. Draw the graph of $-x + 2y = 8$ or $y = \frac{8 + x}{2}$ by plotting the points (0, 4) and (2, 5).

x	0	2	4
y	4	5	6

For $-x + 2y < 8$ take $x = -4$, $y = 0$ then $-(-4) + 2 \times 0 < 8 \Rightarrow 4 < 8$ which is true.

$\therefore$ Shade the region in which the point (–4, 0) lies.

Draw the graph of $2x - y = -4$ by plotting the points (0, 4), (– 2, 0)

x	0	– 2	– 3
y	4	0	– 2

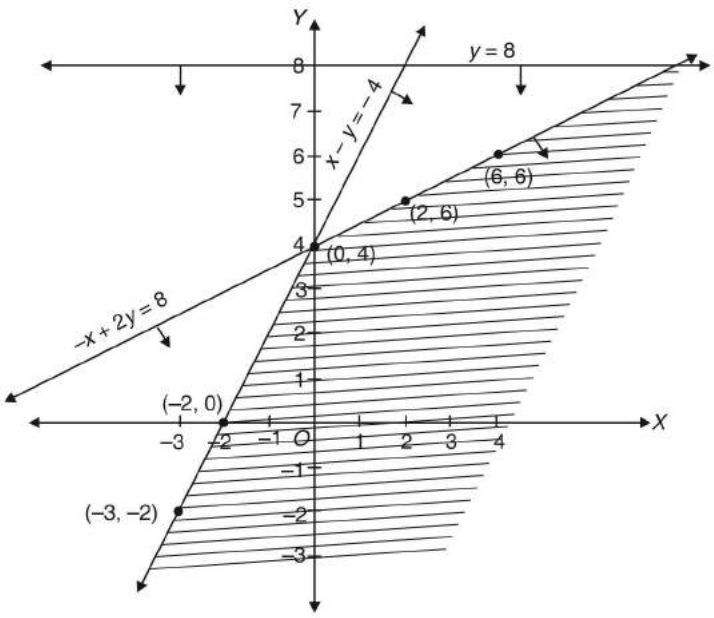


Fig. 11.41

For $2x - y > -4$, take $x = 0, y = 0, 0 > -4$, which is true.

$\therefore$ Shade the region in which the point (0, 0) lies.

Draw the line $y = 8$. It is parallel to x-axis and at a distance of 8 units from it. Shade the region below this line to show $y \leq 8$.

Then the points in the common region of all the three shaded regions *i.e.,* the triple shaded region including the points on the boundary are members of the solution set.

Ex. 9. Draw the graph of the solution set of the following system of inequations.

$x + y \leq 5, 4x + y \geq 4, x + 5y \geq 5, x \leq 4$ and $y \leq 3$

Sol. The graph is drawn below. It is self explanatory.

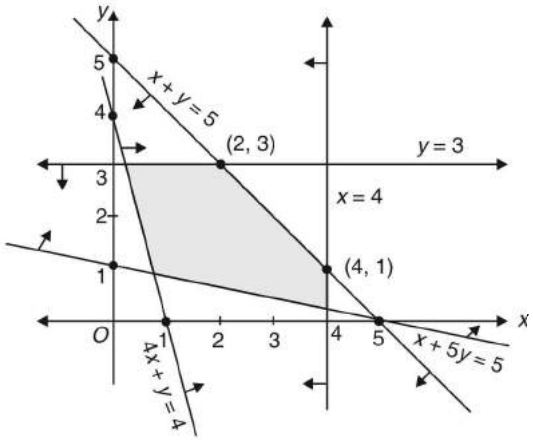


Fig. 11.43

Ex. 10. Find the linear constraints for which the shaded area in the figure given below is the solution set.

Sol. The shaded area is bouned by the following lines.

$x + 2y = 8$ ... (1) $x - y = 1$ ... (2)

$2x + y = 2$ ... (3) $x = 0$ ... (4)

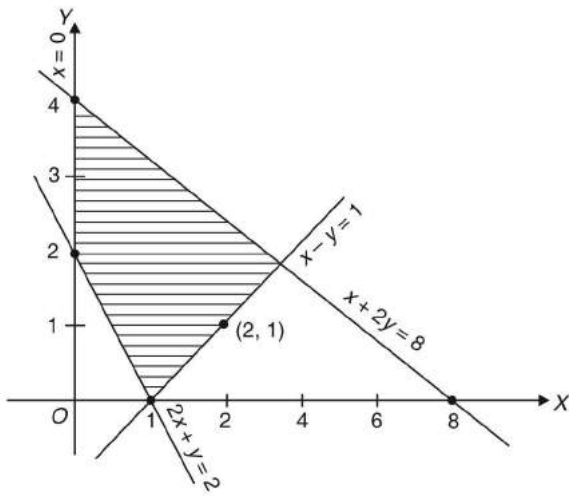


Fig. 11.44

Line (1): The shaded area and the origin lie on the same side of $x + 2y = 8$.

$\therefore$ The corresponding constraint is $x + 2y \leq 8$, because $0 + 2(0) < 8$.

Line (2): The shaded area and the origin lies on the same side of $x - y = 1$

$\therefore$ The corresponding constraint is $x - y \leq 1$, because $0 - 0 = 0 < 1$.

Line (3): The shaded area and the origin lies on the opposite sides of $2x + y = 2$

$\therefore$ The corresponding constraint is $2x + y \geq 2$, because $2(0) + 0 = 0 < 2$.

Line (4): The shaded area lies on the right of $x = 0$

$\therefore$ The required system of constraints is $x + 2y \leq 8$, $x - y \leq 1$, $2x + y \geq 2$, $x \geq 0$.

Ex. 11. Two positive whole numbers are such that their sum is at most 5. Also the difference between the first and twice the second is at most 2. Find the numbers.

Sol. Let the positive numbers be x and y, $x, y \in N$.

[$\because$ +ve whole numbers are natural numbers].

According to the given conditions. $x > 0$, $y > 0$, $x + y \leq 5$, $x - 2y \leq 2$.

The graph of the system is as shown in Fig. 11.45. The points in the region common to all the four inequations are numbers of the solution set of the systems.

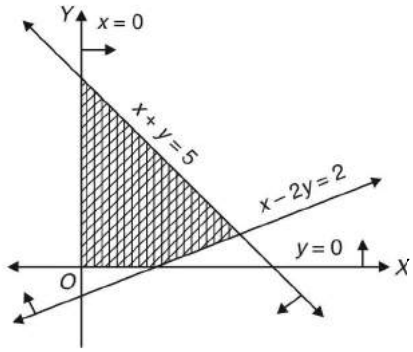


Fig. 11.45

We need to consider those points only coordinates are positive whole numbers.

Hence the solution are

{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}

EXERCISE 11 (*b*)

Graph the solution set of each system in a coordinates plane.

1. $x < 2$
$y > -1$

2. $y > x$
$y \leq 3$

3. $x \leq -y$
$2y < x$

4. $y > 2 - x$
$y > x - 2$

5. $-2y > x - 8$
$4y > x - 12$

6. $3x - 4y < 6$
$2x - y < -1$

7. $x < y < x + 3$

8. $2x + 1 \geq y \geq 2x - 3$

9. $2 < x < 4$
$2y > x - 6$

10. $y \leq x$
$x + y < 4$
$x - 2y < 1$

11. $2x + y < 2$
$x - y > -2$
$x + y > -2$

12. $x + y \geq 3$
$x - y \leq 3$
$x + 5y \geq 15$

13. $y - 2 < 0$
$x + 3 > 0$
$2y + x < 2$
$3y + 3 < 2x$

14. $2y - 6 \leq x$
$2y + 3x \geq -6$
$5y + 15 \geq 2x$
$3y + 5x \leq 15$

15. Find the linear constraints for which the shaded area in the figure given below is the solution set.

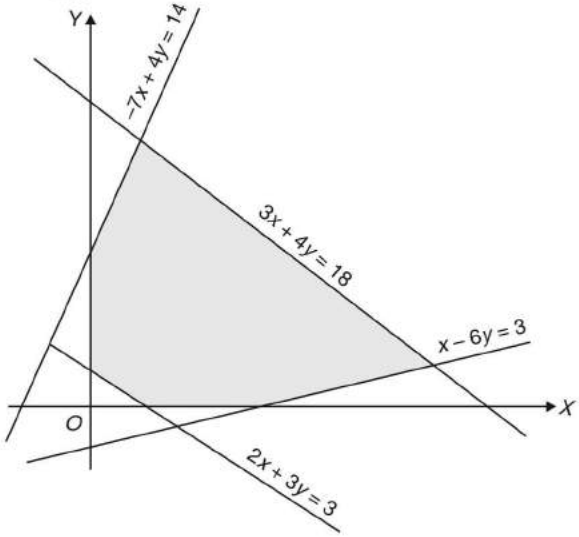


Fig. 11.46

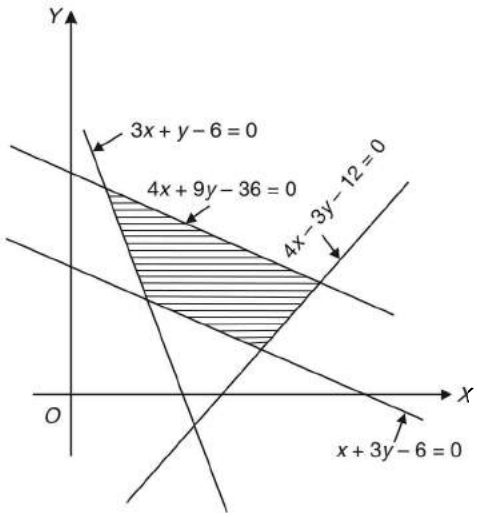


Fig. 11.47

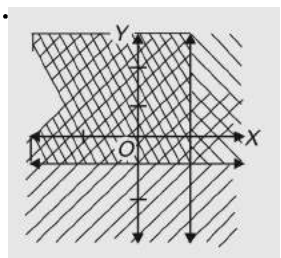
16. $x + y \geq 3,\ 7x + 6y \leq 42,\ x \leq 5,\ y \leq 4\ ,\ x, y > 0$

17. Find the region where all the inequations $x + y \geq 0,\ 2x + y \leq 4,\ x \geq 0$ and $y \leq 2$ hold good. Find the coordinates of the vertices of the region.

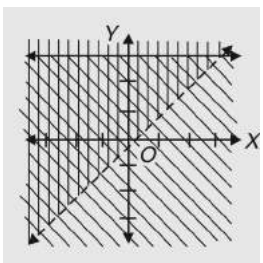
18. Find the region where all the inequations $x + y \leq 6, x \geq y,\ x \geq 0, y \geq 0$ hold good. Find the ordered pairs of the vertices of the region so formed.

ANSWERS

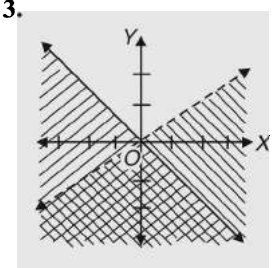
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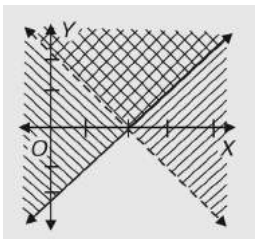
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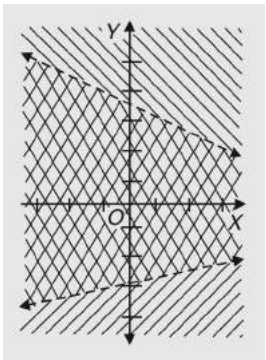
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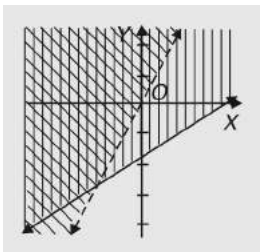
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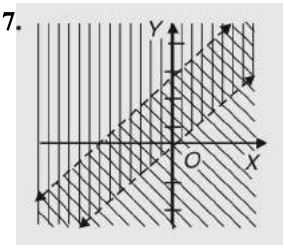


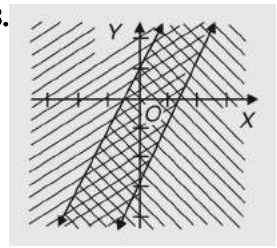
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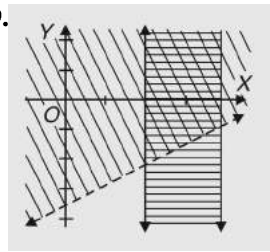


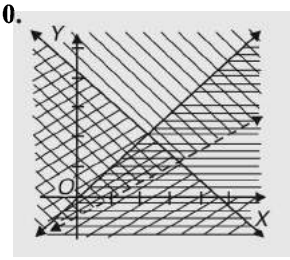
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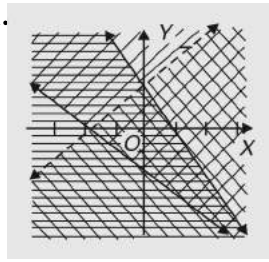


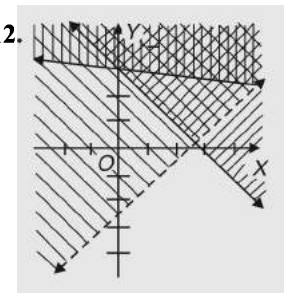
7. 

8. 

9. 

10. 

11. 

12. 

15. $-7x + 4y \le 14,\ 3x + 4y \le 18,\ x - 6y \le 3,\ 2x + 3y \ge 3,\ x \ge 0,\ y \ge 0$
16. $4x + 9y - 36 \le 0,\ 3x + y - 6 \ge 0,\ x + 3y - 6 \ge 0,\ 4x - 3y - 12 \le 0$

QUADRATIC INEQUALITIES

By a quadratic inequality we mean an inequality of one of the following kinds:

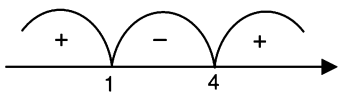
(*i*) $ax^2 + bx + c = 0$ (*ii*) $ax^2 + bx + c < 0$. $(a > 0)$

The range of such an inequality can be found by the method of intervals explained below.

Ex. 12. Solve $x^2 - 5x + 4 > 0$.

Sol. $x^2 - 5x + 4 = (x - 1)(x - 4)$.

Plot 1 and 4 on the number line and place + sign in the interval $x > 4$, – sign in the interval $1 < x < 4$, and + sign in the interval $x < 1$.



sign of $(x - 1)\ (x - 4)$

The required range is, therefore, $x < 1$ or $x > 4$.

[$\because$ $x^2 - 5x + 4$ is positive.]

Note. In case L.H.S, is a perfect square like $x^2 - 6x + 9 > 0$, then the inequality $(x - 3)^2 > 0$ is true for $x \in R$. The range is $(-\infty, \infty)$. If $(x - 3)^2 = 0$, then $x = 3$.

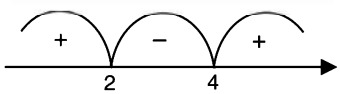
Ex. 13. Solve $-x^2 + 6x - 8 > 0$.

Sol. Multiplying the given inequality by a minus sign and reversing the sign of inequality, we get

Note this step. The highest degree term is made positive.

$x^2 - 6x + 8 < 0$.

Now $x^2 - 6x + 8 = (x - 2)(x - 4)$



Plot 2 and 4 on the number line and place the signs in the intervals.

The required range is $2 < x < 4$. [$\because$ $x^2 - 6x + 8$ is negative.]

Ex. 14. If $\sqrt{9x^2 + 6x + 1} < 2 - x$, then

(*a*) $x \in \left(\frac{3}{2}, \frac{1}{4}\right)$ (*b*) $x \in \left(-\frac{3}{2}, \frac{1}{4}\right)$ (*c*) $x \in \left[-\frac{3}{2}, \frac{1}{4}\right]$ (*d*) $x < \frac{1}{4}$. ***[EAMCET 2006]***

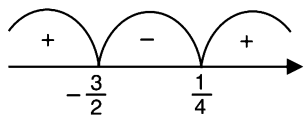
Sol. L.H.S. < R.H.S. and L.H.S. is non-negative,

$\therefore$ R.H.S. > 0

Squaring $9x^2 + 6x + 1 < 4 - 4x + x^2$

$\Rightarrow 8x^2 + 10x - 3 < 0 \Rightarrow (4x - 1)(2x + 3) < 0$

$\Rightarrow 8\left(x - \frac{1}{4}\right)\left(x - \left(-\frac{3}{2}\right)\right) < 0$



Plot $-\frac{3}{2}$ and $\frac{1}{4}$ on the number line and place the signs in the intervals.

$\Rightarrow x \in \left(-\frac{3}{2}, \frac{1}{4}\right)$. **Ans.** (*b*).

11.10. Inequalities involving rational expressions

Inequalities in fractions are different from equations in fractions. In the latter we did not make any mistake in clearing fractions by cross-multiplying, but in the inequalities in fractions we are not justified always if we cross-multiply. Consider the following example :

$\frac{-1}{2} < 0$ or $-1 < 0$, (By cross-multiplying) which is true.

But if we write the given inequality as $\frac{1}{-2} < 0$, then $1 < 0$ (By cross-multiplying)

which is not true.

Now consider the inequality. $\frac{1}{x} < 1$ ...(1) $1 < x$ ...(2)

The inequality (2) gives all *positive* values of x greater than 1, whereas the inequality (1) holds true for all *negative* values of x as well.

From the above it is clear that clearing of fractions by the method of cross-multiplication in equations is quite different from that operation in inequalities.

Actually, clearing of fractions in an equation (or inequality) consists in multiplying both members of the equation (or inequality) by the expression in the denominator. In this operation, equations remain equivalent if they are multiplied by a non-zero expression, but for inequalities this property is more involved : *multiplication of both members of an inequality by a positive expression does not change the sense of the inequality, whereas multiplication by a negative expression reverses the sense of the inequality.*

Therefore, when multiplying both members of the inequality at hand by *x,* one should have taken into account that the x could have assumed negative values as well as positive values, and then one should have reversed the sense of the inequality in the latter case.

Thus, in every case when we wish to multiply both members of an inequality by an expression that is dependent on x and assumes both positive and negative values, the student should examine the two appropriate cases. This rule is often forgotten and is the cause of a lot of trouble.

Let us consider the inequality $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$ ...(1)

The range of the variable x in this inequality consists of all values of x except $x = -2$ and $x = \frac{1}{4}$. Hence, we cannot cross-multiply, *i.e.* cannot multiply both sides by the L.C.M, $(x + 2)(4x - 1)$ unless we are sure that $(x + 2)(4x - 1)$ is a positive quantity. Since we are not sure, we adopt another method. Transpose all terms of the original inequality to the left side and reduce it to the common denominator.

$$\frac{x-2}{x+2}-\frac{2x-3}{4x-1}>0 \Rightarrow \frac{(x-2)(4x-1)-(x+2)(2x-3)}{(x+2)(4x-1)}>0$$

$$\Rightarrow \quad \frac{2(x^2-5x+4)}{(x+2)(4x-1)}>0 \Rightarrow \frac{2(x-1)(x-4)}{4(x+2)\left(x-\frac{1}{4}\right)}>0 \Rightarrow \frac{(x-1)(x-4)}{(x+2)\left(x-\frac{1}{4}\right)}>0 \qquad ...(2)$$

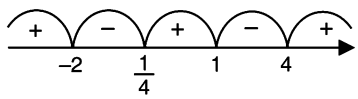
Multiply both sides of (2) by the expression $(x+2)^2\left(x-\frac{1}{4}\right)^2$, which is *positive* for the x under consideration.

$$\therefore \quad \frac{(x-1)(x-4)(x+2)^2\left(x-\frac{1}{4}\right)^2}{(x+2)\left(x-\frac{1}{4}\right)}>0 \Rightarrow (x-1)(x-4)(x+2)\left(x-\frac{1}{4}\right)>0 \qquad ...(3)$$

Putting each factor equal to zero, we get $x=-2, \frac{1}{4}, 1, 4$. Now we use the method of intervals.

Thus the range is

$$x<-2, \text{ or } \frac{1}{4}<x<1, \text{ or } x>4.$$



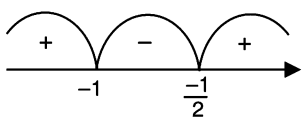
Generally $\frac{F(x)}{f(x)}>0$ and $F(x).f(x)>0$ are equivalent. Therefore to solve the inequality $\frac{F(x)}{f(x)}>0$, where $F(x)$ and $f(x)$ are polynomials, one applies the method of intervals to the inequality $\frac{F(x)}{f(x)}>0$, *i.e.*, $F(x)>0$ which need not even be written out explicitly, it being sufficient to locate the roots of the polynomials $F(x)$ and $f(x)$ on the number line and affix the appropriate sign to each of the resulting intervals.

We now solve a few examples.

Ex. 15. Find the range of values of x for which

$$\frac{x^2+x+1}{x^2+2}<\frac{1}{3}, \; x \text{ being real.}$$ **[*ISC*]**

Sol. $\frac{x^2+x+1}{x^2+2}-\frac{1}{3}<0$ or $\frac{2x^2+3x+1}{3(x^2+2)}<0$



$\Rightarrow$ $2x^2+3x+1<0$, as x^2+2 is positive for all real values of x.

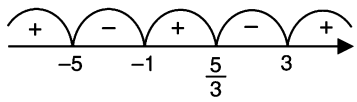
$\Rightarrow$ $(2x+1)(x+1)<0 \Rightarrow 2\left(x+\frac{1}{2}\right)(x+1)<0 \Rightarrow \left(x+\frac{1}{2}\right)(x+1)<0.$

By using the method of intervals, we get the required range as $-1<x<-\frac{1}{2}$.

$\left[\because \left(x+\frac{1}{2}\right)(x+1) \text{ is negative.}\right]$

Ex. 16. Determine the range of values of x for which $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}$. [*ISC*]

Sol. $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}$ or $\dfrac{x^2-2x+5}{3x^2-2x-5} - \dfrac{1}{2} > 0$



or $\dfrac{-x^2-2x+15}{3x^2-2x-5} > 0$ or $\dfrac{x^2+2x-15}{3x^2-2x-5} < 0$

or $\dfrac{(x+5)(x-3)}{(x+1)(3x-5)} < 0$ or $\dfrac{(x+5)(x-3)(x+1)^2(3x-5)^2}{(x+1)(3x-5)} < 0$

(Multiplying by the square of the denominator, which is positive)

or $(x + 5)(x - 3)(x + 1)(3x - 5) < 0$.

Putting each factor equal to zero, we get

$$x = -5, -1, \frac{5}{3}, 3.$$

By using the method of intervals, we get the range as $-5 < x < -1$ or $\dfrac{5}{3} < x < 3$.

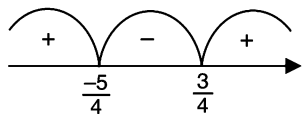
Ex. 17. Find the range of real values of x for which $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$. [*ISC 1990*]

Sol. $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3} \Rightarrow \dfrac{x-1}{4x+5} - \dfrac{x-3}{4x-3} < 0$

$\Rightarrow \dfrac{18}{(4x+5)(4x-3)} < 0 \Rightarrow \dfrac{18(4x+5)(4x-3)}{(4x+5)^2(4x-3)^2} < 0.$

$\Rightarrow (4x+5)(4x-3) < 0$

$\Rightarrow 16\left(x+\dfrac{5}{4}\right)\left(x-\dfrac{3}{4}\right) < 0.$



Putting each factor equal to zero, we get $x = -\dfrac{5}{4}, \dfrac{3}{4}$.

By the method of intervals, we get the required range $-\dfrac{5}{4} < x < \dfrac{3}{4}$.

Ex. 18. For what value of a is the inequality $\dfrac{x^2+ax-2}{x^2-x+1} < 2$ satisfied for all real values of x ? [*ISC 1991*]

Sol. $\dfrac{x^2+ax-2}{x^2-x+1} < 2$ or $\dfrac{x^2+ax-2}{x^2-x+1} - 2 < 0$ or $\dfrac{-x^2+x(a+2)-4}{x^2-x+1} < 0$ ...(*i*)

Now $x^2 - x + 1 = x^2 - 2 \times \dfrac{1}{2} \times x + \left(\dfrac{1}{2}\right)^2 + \dfrac{3}{4} = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$,

which is positive for all real values of x.

$\therefore$ From (*i*) it follows that $-x^2 + x(a + 2) - 4 < 0$

or $x^2 - x(a + 2) + 4 > 0$, *i.e.*, $x^2 - x(a + 2) + 4$ is positive.

We know that if $ax^2 + bx + c$ and 'a' have the same sign, then $b^2 - 4ac < 0$. Here $x^2 - x(a + 2) + 4$ and the coefficient of x^2, namely 1 are positive, *i.e.,* have the same sign, therefore, discriminant < 0.



$\Rightarrow \quad (a + 2)^2 - 4 \times 4 < 0$

or $\quad a^2 + 4a - 12 < 0$

or $\quad (a + 6)(a - 2) < 0.$

Now by the method of intervals we get

$$-6 < a < 2,$$

which is the required range of values of a.

Help-Box

	$a < -6$	$-6 < a < 2$	$a > 2$
$a + 6$	–	+	+
$a - 2$	–	–	+
$(a + 6)(a - 2)$	+	–	+

Ex. 19. Determine the range of values of x for which $\dfrac{3x^2 - 2x - 5}{x^2 - 2x + 5} < 2$. [*ISC 1993*]

Sol. $\dfrac{3x^2 - 2x - 5}{(x-1)^2 + 4} < 2$



$[\because x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 4]$

Since $(x - 1)^2$ is positive, being a square, for all real values of x, therefore the denominator $(x - 1)^2 + 4$ is positive and we can thus cross-multiply making no effect on inequality sign.

$\Rightarrow \quad 3x^2 - 2x - 5 < 2(x^2 - 2x + 5) \Rightarrow 3x^2 - 2x - 5 < 2x^2 - 4x + 10$

$\Rightarrow \quad x^2 + 2x - 15 < 0 \Rightarrow (x + 5)(x - 3) < 0.$

Putting each factor equal to zero,

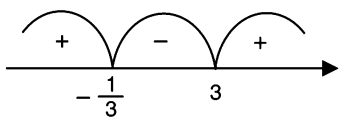
we get $x = -5, 3$.

By using the method of intervals, the required range is $-5 < x < 3$.

Note. Draw help-box to help you.

Ex. 20. Show that the maximum value of $\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$ is 3, for real values of x.

Sol. Let $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$



$\Rightarrow \quad y(x^2 + 2x + 4) = x^2 - 2x + 4 \Rightarrow (y - 1)x^2 + 2(y + 1)x + (4y - 4) = 0.$

For real values of x the discriminant ≥ 0.

$\therefore \quad 4(y + 1)^2 - 4(y - 1)(4y - 4) \geq 0.$

$\Rightarrow \quad (y + 1)^2 - 4(y - 1)^2 \geq 0$

$\Rightarrow \quad -3y^2 + 10y - 3 \geq 0$

$\Rightarrow \quad 3y^2 - 10y + 3 \leq 0$

$\Rightarrow \quad (3y - 1)(y - 3) \leq 0$

$\Rightarrow \left(y - \dfrac{1}{3}\right)(y - 3) \leq 0$

Putting each factor equal to zero,

Help-Box

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$\left(y - \frac{1}{3}\right)$	–	+	+
$(y - 3)$	–	–	+
$\left(y - \frac{1}{3}\right)(y - 3)$	+	–	+

we get $y = \frac{1}{3}, 3.$

By using the method of intervals,

we get $\frac{1}{3} \le y \le 3.$

$\therefore$ The maximum value of y is 3.

In other words, the maximum value of the given fraction is 3.

Ex. 21. Obtain, by graphical method, the solution set of the inequation $x^2 + 3x - 18 \ge 0$. [*ISC 1999*]

Sol. Method I. $x^2 + 3x - 18 \ge 0 \Rightarrow (x + 6)(x - 3) \ge 0$

$x + 6 = 0$ and $x - 3 = 0$ gives $x = -6, x = 3$.

By the method of intervals, we get

$x \ge 3$ and $x \le -6$.



Help-Box

	$x < -6$	$-6 < x < 3$	$x > 3$
$(x + 6)$	–	+	+
$(x - 3)$	–	–	+
$(x + 6)(x - 3)$	+	–	+

Hence, the solution set is

$\{x \le -6 \text{ or } x \ge 3; x \in R\}$.

Method II. $x^2 + 3x - 18 \ge 0$.

Discriminant $= b^2 - 4ac$

$= (3)^2 - 4 \times 1 \times (-18) = 9 + 72 = 81 > 0$.

If we put $y = x^2 + 3x - 18$, then

$y = (x + 6)(x - 3)$.

It cuts the x-axis when $y = 0$, therefore

$x + 6 = 0$ or $x - 3 = 0$, *i.e.*, $x = -6, +3$.

$\therefore$ It cuts the x-axis at $(-6, 0)$, and $(3, 0)$.

Also when $b^2 - 4ac > 0$, the shape of the curve is as drawn here. It is clear that y is positive either to the right of A (3, 0), or to the left of B.

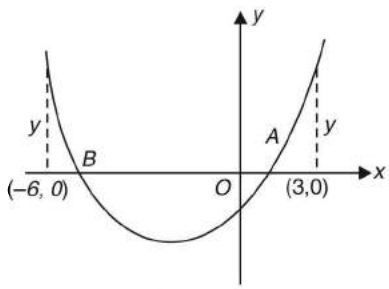


Fig. 11.48

In other words, $x^2 + 3x - 18 \ge 0$

for $x \ge 3$ and $x \le -6$.

Hence, the solution set is

$\{x \le -6 \text{ or } x \ge 3 ; x \in R\}$.

Note. The student will study in the chapter on parabola that $y = x^2 + 3x - 18$ is an *upward parabola*, because

$$y = \left[x^2 + 2 \times \frac{3}{2} \times x + \left(\frac{3}{2}\right)^2\right] - \frac{81}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{81}{4}$$

or $\left(x + \frac{3}{2}\right)^2 = y + \frac{81}{4}$, which is an upward parabola with vertex $\left(\frac{-3}{2}, \frac{-81}{4}\right)$, as is drawn in Fig. 11.48.

Ex. 22. Find the range of values of x which satisfy $x^2 + 6x - 27 > 0$, $-x^2 + 3x + 4 > 0$ simultaneously. [*ISC 1997*]

Sol. $x^2 + 6x - 27 > 0 \Rightarrow (x + 9)(x - 3) > 0$

$(x + 9)(x - 3) = 0$ gives $x = -9, 3$.

By the method of intervals, $x > 3, x < -9$ ....(*i*)

$-x^2 + 3x + 4 > 0 \Rightarrow x^2 - 3x - 4 < 0 \Rightarrow (x-4)(x+1) < 0$

$\Rightarrow (x-4)(x+1) = 0$ gives $x = -1, 4$

By the method of intervals we get $-1 < x < 4$ ....(*ii*)

(*i*) and (*ii*) combined give

$3 < x < 4,$

which is the required range.

Note. You may draw help-box to help you.

Ex. 23. Find the range of real values of x for which $\frac{x-2}{3x+4} < \frac{x-4}{3x-2}$. [*ISC 1996*]

Sol. $\frac{x-2}{3x+4} - \frac{x-4}{3x-2} < 0 \Rightarrow \frac{(x-2)(3x-2)-(x-4)(3x+4)}{(3x+4)(3x-2)} < 0$

$\Rightarrow \frac{(3x^2-8x+4)-(3x^2-8x-16)}{(3x+4)(3x-2)} < 0 \Rightarrow \frac{20}{(3x+4)(3x-2)} < 0$

$\Rightarrow (3x+4)(3x-2) < 0 \Rightarrow 3\left(x+\frac{4}{3}\right) \times 3\left(x-\frac{2}{3}\right) < 0$

$\Rightarrow 9\left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) < 0$

$\Rightarrow \left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) < 0$

$\left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right) = 0$ gives $x = -\frac{4}{3}, \frac{2}{3}$.

By the method of intervals, we get $-\frac{4}{3} < x < \frac{2}{3}$, which is the required range.

Help-Box

	$x < -\frac{4}{3}$	$-\frac{4}{3} < x < \frac{2}{3}$	$x > \frac{2}{3}$
$\left(x+\frac{4}{3}\right)$	–	+	+
$\left(x-\frac{2}{3}\right)$	–	–	+
$\left(x+\frac{4}{3}\right)\left(x-\frac{2}{3}\right)$	+	–	+

Ex. 24. Determine the range of values of x for which $2x^2 + 3x - 9 \le 0$. [*ISC 1995*]

Sol. $2x^2 + 3x - 9 \le 0 \Rightarrow 2x^2 + 6x - 3x - 9 \le 0 \Rightarrow 2x(x+3) - 3(x+3) \le 0.$

$\Rightarrow 2(x+3)\left(x-\frac{3}{2}\right) \le 0$

$\Rightarrow 2(x+3)\left(x-\frac{3}{2}\right) = 0$ gives $x = -3, \frac{3}{2}$

as critical points.

By the method of intervals, we get $-3 \le x \le \frac{3}{2}$, which is the required range.

Help-Box

	$x < -3$	$-3 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$(x+3)$	–	+	+
$\left(x-\frac{3}{2}\right)$	–	–	+
$(x+3)\left(x-\frac{3}{2}\right)$	+	–	+

EXERCISE 11 (*c*)

Find the range of x in each of the following inequalities:

1. $x^2 - 4x + 3 < 0$. **2.** $x^2 + 5x + 4 > 0$. **3.** $x^2 + x - 6 \geq 0$. **4.** $x^2 - 16 < 0$.

5. $x^2 - 6x + 9 \geq 0$. **6.** $-x^2 + 2x + 3 < 0$. **7.** $5x < 2 - 3x^2$. **8.** $-x^2 - 4x - 5 < 0$.

9. $4x^2 + 1 > 4x$. **10.** $-x^2 + x > 0$. **11.** $6 + x < 2x^2$. **[*ISC*]**

12. (*i*) $(x - 4)(x + 6) > 0$; (*ii*) $3 - 2x^2 > 5x$. **[*ISC*]**

13. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$. **[*ISC*]**

14. The set of values of x for which the inequalities $x^2 - 3x - 10 < 0$, $10x - x^2 - 16 > 0$ hold simultaneously is (*a*) (–2, 5) (*b*) (2, 8) (*c*) (–2, 8) (*d*) (2, 5). **[*EAMCET 2007*]**

Solve the following inequalities:

15. $\dfrac{x+3}{x-1} > x$. **16.** $x + 4 < -\dfrac{2}{x+1}$.

17. $\dfrac{x^2 - 2x + 3}{x^2 - 4x + 3} > -3$.

18. $\dfrac{x^2 + 6x - 11}{x + 3} < -1$. **[*ISC*]**

19. $\dfrac{x^2 - 3x + 24}{x^2 - 4x + 3} > -4$. **[*ISC*]**

ANSWERS

1. $1 < x < 3$ **2.** $x < -4$, or $x > -1$ **3.** $x \leq -3$ or $x \geq 2$ **4.** $-4 < x < 4$.

5. All real values of x **6.** $x < -1$, or $x > 3$ **7.** $-2 < x < \dfrac{1}{3}$

8. All real values of x. **9.** $x \in R - \left\{\dfrac{1}{2}\right\}$ **10.** $0 < x < 1$

11. $x < \dfrac{-3}{2}$ or $x > 2$ **12.** (*i*) $x < -6$ or $x > 4$ (*ii*) $-3 < x < \dfrac{1}{2}$ **13.** $-1 \leq x < 1$ or $2 < x \leq 4$

14. (*d*) **15.** $x < -1$ or $1 < x < 3$ **16.** $x < -3$ or $-2 < x < -1$

17. $x < 1$ or $\dfrac{3}{2} < x < 2, x > 3$ **18.** $x < -8$ or $-3 < x < 1$ **19.** $x < 1$ or $x > 3$.

HINTS TO SELECTED QUESTIONS

13. The solution of the first inequality is $x < 1$, or $x > 2$. The solution of the second inequality is $-1 \leq x \leq 4$. The common solution is $-1 \leq x < 1$, or $2 < x \leq 4$.

14. $x^2 - 3x - 10 < 0 \Rightarrow (x + 2)(x - 5) < 0 \Rightarrow x \in (-2, 5)$

$10x - x^2 - 16 > 0 \Rightarrow x^2 - 10x + 16 < 0 \Rightarrow (x - 2)(x - 8) < 0 \Rightarrow x \in (2, 8)$

$\therefore x \in (-2, 5) \cap (2, 8) \Rightarrow x \in (2, 5)$, *i.e.*, (*d*)

CHAPTER TEST

1. Solve: $24x < 100$, when
(i) x is a natural number (ii) x is an integer

2. Solve the inequality $\frac{1}{2}\left(\frac{3}{5}x+4\right) \geq \frac{1}{3}(x-6)$

3. Solve: $-12 \leq 4-\frac{3x}{-5} < 2$

4. Solve the following inequalities graphically
(i) $3x + 4y < 12$ (ii) $x > -3$ (iii) $5y - 3 \leq 12$

5. Solve graphically: $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$

ANSWERS

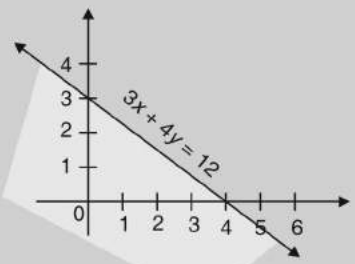
1. (i) $\{1, 2, 3, 4\}$ (ii) $\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4\}$

2. $(-\infty, 120)$

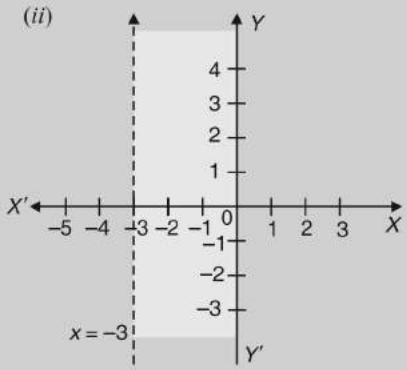


3. $-\frac{80}{3} < x \leq \frac{-10}{3}$

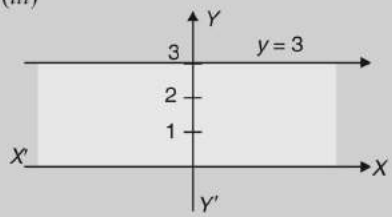
4. (i)



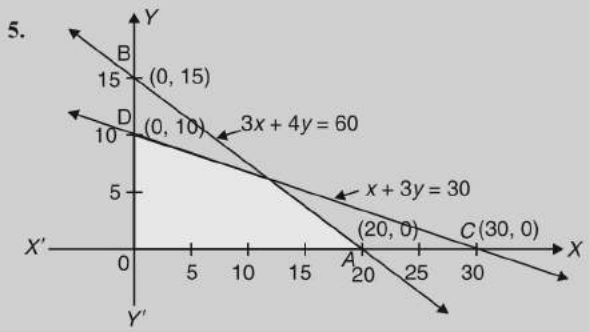
(ii)



(iii)



5.



CHAPTER 12

PERMUTATIONS AND COMBINATIONS

12.01. Introduction

Suppose we have 40 books of different subjects and a shelf which can hold only 20 books. We have to choose 20 books out of these 40 books and then arrange them on the shelf. We can take any 20 books we like and then arrange them on the shelf in any manner we like. Thus, the process of arranging 20 books out of 40 books involves the following two different processes :

1. First, we select 20 books out of 40 books. Suppose these are Algebra, Trigonometry, Calculus, Geometry, etc. It is our sweet will whether we select a particular book or not.

2. After having selected 20 books out of 40 books, we proceed to arrange them on the shelf. We can arrange them on the shelf in any manner we like, for example, it is up to us whether we put a book on trigonometry or physics in the first place.

The above discussion shows that while considering the alternatives of things or acts, we come across two types of problems :

(*a*) ***Selection,*** (*b*) ***Arrangement.***

The process of selecting things is called **combination** and that of arranging things is called **permutation.**

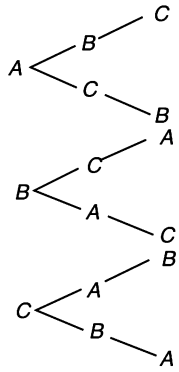
12.02. Difference between a Permutation and a Combination

Combination	*Permutation*
1. Concerns only the selection. 2. Ordering of the selected items is immaterial.	1. Concerns selection as well as arrangement. 2. Ordering is essential.

Thus, if we have 4 objects *A, B, C* and *D* the possible selections (or combinations) and arrangements (or permutations) of 3 objects out of 4 are given below. This will help you to understand clearly the difference between permutation and combination.

Select ↓ **Combination**	**Arrange** ↓ **Permutation**
ABC *ABD* *ACD* *BCD*	**ABC, ACB, BAC, BCA, CAB, CBA* *ABD, ADB, BAD, BDA, DAB, DBA* *ACD, ADC, CAD, CDA, DAC, DCA* *BCD, BDC, CBD, CDB, DBC, DCB*
Total 4 combinations	24 permutations

***Note.** The possible arrangements corresponding to the selection of *A, B* and *C* can be easily written with the help of a tree diagram shown below:



A few problems relating to combination are:

1. Formation of a team from a number of players.
2. Formation of a particular committee from a number of members.

A few problems relating to permutation are:

1. Arrangements of books on a shelf.
2. Formation of numbers with the given digits.
3. Formation of words with the given letters.

12.03. Fundamental principle of counting

To discover the fundamental principle of counting, study the following examples:

1. Suppose you have 3 full-sleeve and 4 half-sleeve shirts. Since you have the choice of wearing any of these shirts, you can wear one shirt in 3 + 4 = 7 ways. If in addition, you have 5 T-shirts, then you can wear one shirt in 3 + 4 + 5 = 12 ways.

The above example illustrates one way of counting, which we may call the sum rule and applies when **one event** has to happen out of given disjoint events.

Here, if wearing a full-sleeve shirt is the event *A* and wearing a half-sleeve shirt is the event *B,* then *A* can happen in 3 ways and *B* can happen in 4 ways. Also, $A \cap B = \phi$, *i.e., A* and *B* can't happen together. We also say that the two events are ***exclusive***.

Now, suppose you have 3 shirts and 4 pairs of pants. In how many possible ways can you dress up by wearing a shirt and a pair of pants?

In the above case, you can wear any of the 3 shirts and after wearing one of these shirts any of these pairs of pants with it. If we label the shirts as S_1, S_2, S_3 and the pants as P_1, P_2, P_3 and P_4, then the different ways of dressing up can be as under :

$S_1 P_1$	$S_2 P_1$	$S_3 P_1$
$S_1 P_2$	$S_2 P_2$	$S_3 P_2$
$S_1 P_3$	$S_2 P_3$	$S_3 P_3$
$S_1 P_4$	$S_2 P_4$	$S_3 P_4$

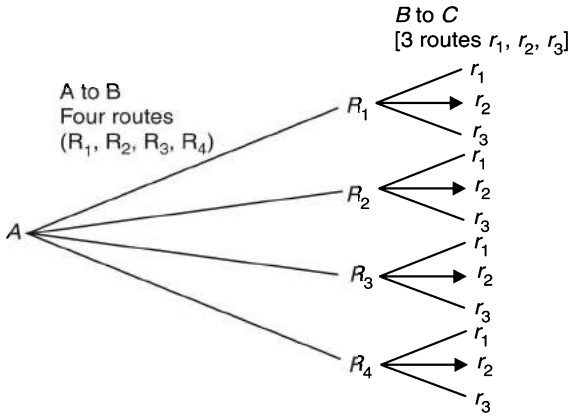
Total number of ways =12 = 3 × 4.

In this illustration, we **multiply** the number of ways in which you can wear a shirt and the number of ways in which you can wear a pair of pants.

2. There are 4 different routes between cities *A* and *B,* and 3 different routes between cities *B* and *C.* How many different routes are there from city *A* to city *C* by way of city *B?*

Discussion: Obviously, one can go from city *A* to city *B* by any of the 4 routes, *i.e.,* in 4 ways. After having gone to *B* by any of the different 4 routes, one can go to city *C* by any of the three routes. Thus, corresponding to one route taken, from *A* to *B,* he has 3 choices from *B* to *C.* Therefore, corresponding to 4 routes, there are 12 choices in all.

Therefore, he can go from *A* to C via *B,* in $4 \times 3 = 12$ ways as depicted in the following tree diagram.



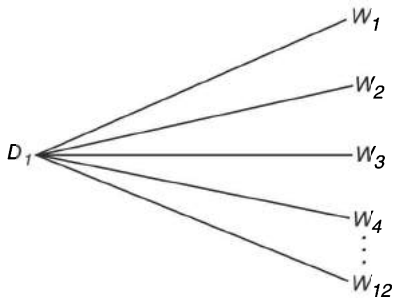
The possible routes taken are :

$R_1 r_1$	$R_2 r_1$	$R_3 r_1$	$R_4 r_1$
$R_1 r_2$	$R_2 r_2$	$R_3 r_2$	$R_4 r_2$
$R_1 r_3$	$R_2 r_3$	$R_3 r_3$	$R_4 r_3$

Total number of ways = 12.

3. (*a*) A house has 5 doors and 12 windows. In how many ways can a person enter the house through a door and exit through a window?

(*b*) In how many ways can a person run out of the house during emergency through a door or a window?



Discussion: (*a*) The person has the choice of entering by any one of the windows or doors. Thus the person can enter the house in $5 + 12 = 17$ ways (sum rule).

(*b*) Since the person has to enter through a door and there are 5 doors, he can do so in 5 ways. After having entered through one of the doors, say, D_1, he can exit by any of the 12 windows, say, W_1, W_2, W_3, ... W_{12}. So, corresponding to entry through a particular door, say, D_1, there are 12 ways of exit as shown below:

$$D_1W_1, D_1W_2, D_1W_3, \ldots D_1W_{12}.$$

Since, for each of the 5 doors D_1 , D_2 , D_3, D_4 and D_5 there are 12 ways of exit, so in all he can exit in $5 \times 12 = 60$ ways.

The examples discussed above illustrate the use of a general principle, called the product rule or the fundamental principle of counting , which is stated below.

12.04. A fundamental principle

If one operation can be performed in m ways, and if corresponding to each of the m ways of performing this operation, there are n ways of performing a second operation, then the number of ways of performing the two operations together is m × n. **(This AND That).**

Suppose that the first operation is performed in any one of the m ways, the second operation can then be performed in n ways and with the particular first operation, we can associate any one of the n ways of performing the second operation. This means that if the first operation could have been performed only in this one way, there would have been $1 \times n$, *i.e.*, n ways of performing both the operations. But it is given that the first operation can be performed in m ways and there are n ways of performing the second operation for every one way of performing the first operation. Therefore, there are $m \times n$ ways of performing both the operations.

Generalisation. The above principle can be extended to the case in which the different operations can be performed in $m, n, p, \ldots$ ways. In this case the number of ways of performing all the operations together would be $m \times n \times p \ldots$

Ex. 1. There are 10 buses running between two towns X and Y. In how many ways can a man go from X to Y and return by a different bus ?

Sol. The man can go from X *to* Y *in* 10 ways and as he is not to return by the same bus that he took while going, corresponding to each of the 10 ways of going, there are 9 ways of returning. Hence the total number of ways in which he can go to Y and be back is $10 \times 9 = 90$.

Ex. 2. How many different numbers of three digits can be formed with the digits 1, 2, 3, 4, 5, no digit is being repeated?

Sol. The unit's place can be filled with either of these 5 digits and so the unit's place can be filled in 5 ways. The ten's place can be filled in 4 ways corresponding to each way of filling up the unit's place, for we can have any digit here except the one used in the unit's place. Similarly, the hundredth's place can be filled in 3 ways as here we have any of the remaining three digits. Therefore, there are $5 \times 4 \times 3 = 60$ ways of forming a number of three digits with the five given digits.

Ex. 3. Each section in first year of plus two course has exactly 30 students. If there are 3 sections, in how many ways can a set of 3 student representatives be selected from each section?

Sol. 1st representative can be selected from first section in 30 ways.

2nd representative can be selected from second section in 30 ways.

3rd representative can be selected from third section in 30 ways.

$\therefore$ Required number of ways $=30 \times 30 \times 30 = 27000$.

Ex. 4. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?

Sol. Any number between 100 and 1000 is of 3 digits. The unit's place can be filled by 2 or 9 in 2 ways.

Similarly ten's place can be filled in 2 ways.

The hundred's place can also be filled in 2 ways.

$\therefore$ Required no. of numbers $=2 \times 2 \times 2 = 8$.

Ex. 5. How many odd numbers less than 1000 can be formed using the digits 0, 2, 5, 7, repetition of digits are allowed?

Sol. Since the required numbers are less than 1000 therefore, they are 1-digit, 2-digit or 3-digit numbers.

One-digit numbers. Only two odd one-digit numbers are possible, namely, 5 and 7.

Two-digit numbers. For two-digit odd numbers the unit place can be filled up by 5 or 7 *i.e.* in two ways and ten's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 2-digit odd numbers $=2 \times 3 = 6$.

Three-digit numbers. For three-digit odd numbers, the unit place can be filled up by 5 or 7 in 2 ways. The ten's place can be filled up by any one of the digits 0, 2, 5, 7 in 4 ways. The hundred's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

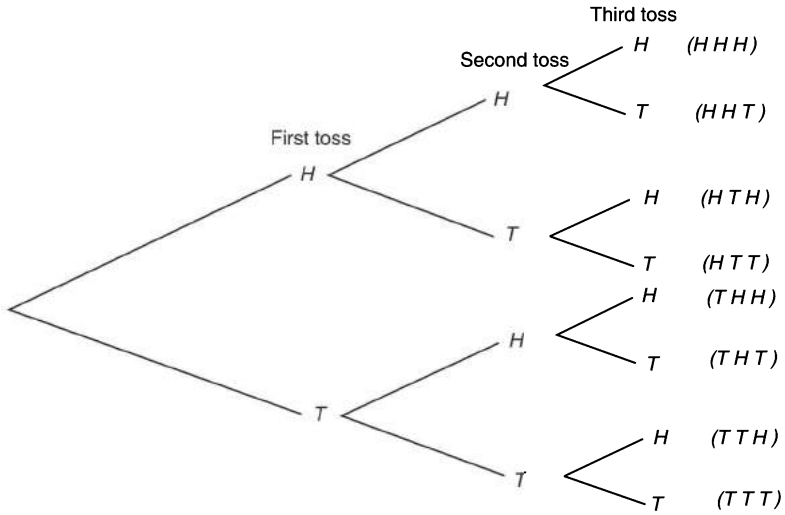
$\therefore$ No. of possible 3-digit numbers = $2 \times 4 \times 3 = 24$

Hence, the total no. of possible odd numbers = $2 + 6 + 24 = 32$.

Ex. 6. A coin is tossed three times and outcomes are recorded. Use the product rule to determine the number of possible outcomes. Then list all the outcomes.

Sol. For each toss of coin we have 2 choices– a head (H) or a tail (T). Therefore, by the product rule, the number of possible outcomes of three tosses is $2 \times 2 \times 2 = 8$.

For a listing of these outcomes, it is convenient to draw a tree diagram



Thus, the possible outcomes are

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

Note. Can you guess how many possible outcomes would be if the coin is tossed four times? Five times? n times?

Arranging in the same manner as above, and applying the product rule, we find that

For 4 tosses, the number of possible outcomes are $2 \times 2 \times 2 \times 2 = 2^4 = 16$

For 5 tosses, the number of possible outcomes are $2^5 = 32$

For n tosses, the number of possible outcomes are 2^n.

EXERCISE 12 (*a*)

1. Two persons go in a railway carriage where there are 6 vacant seats. In how many different ways can they seat themselves?
2. In how many ways can 2 prizes be awarded to 9 contestants provided no contestant gets both the prizes?
3. There are three mathematics teachers in a college in which there are 6 classes. In how many different ways can they choose the classes, provided one teaches one class only?
4. How many words (with or without meaning) of three distinct letters of the English alphabets are there?
5. How many numbers are there between 100 and 1000 such that 7 is in the units place?
6. How many integers of four digits each can be formed with the digits 0, 1, 3, 5, 6 (assuming no repetitions)?
7. How many automobile licence plates can be made if the inscription on each contains two different letters followed by three different digits?
8. Find the number of ways of arranging 6 players to throw the cricket ball so that the oldest player may not throw first.

9. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6?

10. Find the number of even positive integers which have three digits.

11. How many 2-digit numbers can be formed from the digits 8, 1, 3, 5 and 4 assuming
(*a*) repetition of digits is allowed?
(*b*) repetition of digits is not allowed?

12. There are 12 true-false questions in an examination. How many sequences of answers are possible?

13. How many four-digit even integers can be formed using the digits 0, 1, 2, 3, 4, 5?

14. To pass an examination a student has to pass in each of the 3 papers. In how many ways can a student fail in the examination?

15. How many seven-digit phone numbers are possible if 0 and 1 cannot be used as the first digit and the first three digits cannot be 555, 411, or 936?

16. There are five routes for a journey from station *A* to station *B*. In how many different ways can a man go from *A* to *B* and return, if for returning
(*i*) any of the routes is taken (*ii*) the same route is taken
(*iii*) the same route is not taken?

17. How many 9-digit numbers of different digits can be formed?

18. For a set of five true-or false-questions, no student has written the all-correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible?

ANSWERS

1. 30 **2.** 72 **3.** 120 **4.** 15600 **5.** 90

6. 96 **7.** Number of licence plates = $26 \times 25 \times 9 \times 9 \times 8 = 4{,}21{,}200$

8. 600 **9.** 64 **10.** 450 **11.** (*a*) 25, (*b*) 20 **12.** 4096

13. 540 **14.** $2^3 - 1 = 7$ ways **15.** $[8 \times 10 \times 10 - 3] \times 10 \times 10 \times 10 \times 10 = 7970000$

16. (*i*) $5 \times 5 = 25$, (*ii*) $5 \times 1 = 5$, (*iii*) $5 \times 4 = 20$

17. $9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3265920$ **18.** 31.

HINTS TO SELECTED QUESTIONS

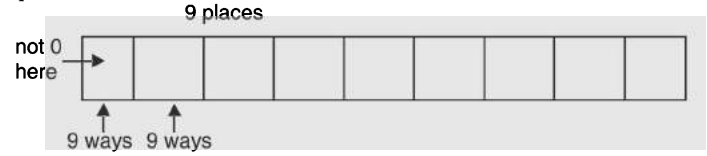
7. There are 26 letters and 10 digits out of which the inscriptions are to be made. Also, the digit 0 cannot be used at the hundred's place.
So the number of license plates that can be made = $26 \times 25 \times 9 \times 9 \times 8$.

8. For first place 5 players (excluding the oldest) and for the remaining places 5 (including the oldest) players are available $\therefore$ The no. of ways = $5 \times 5 \times 4 \times 3 \times 2 \times 1$.

14. For each of the 3 papers, there are two choices – pass (*P*) or fail (*F*). By product rule, there are $2 \times 2 \times 2 = 8$ choices. But he will pass only if he passes in all the papers (*PPP*).
$\therefore$ The number of ways in which a student can fail = $2^3 - 1 = 7$.

15. First three digits can be filled in $(8 \times 10 \times 10 - 3)$ ways. Last four digits can be filled in $(10 \times 10 \times 10 \times 10)$ ways.
$\therefore$ Total number of saven digit phone numbers = $(8 \times 10 \times 10 - 3) \times 10 \times 10 \times 10 \times 10$

17. Number of reqd. numbers = $9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$



18. There is only one sequence of all correct answers, hence the required number = $2 \times 2 \times 2 \times 2 \times 2 - 1$.

12.05. Permutations

Definition: ***Each of the different arrangements which can be made by taking some or all of a number of things at a time is called a permutation.***

Notation. The number of permutations of n things taken r at a time is denoted by ^nP_r ***or*** $P(n, r)$. The letter P is an abbreviation of the word 'permutation'.

Thus, $^6P_4$ denotes the number of permutations or arrangements of 6 things taken 4 at a time.

12.06. The value of ^nP_r

To find the number of permutations of n different things, taken r at a time or to determine ^nP_r

The number of permutations of n things taken r at a time will be the same as the number of ways in which r blank places can be filled up with n given things.

As the first place can be filled in by any one of the n things so there are n ways of filling up the first place.

After having filled in the first place by any one of the n things, there are $(n-1)$ things left. Hence the second place can be filled in $(n-1)$ ways. Now, as for every one way of filling up the first place, there are $(n-1)$ ways of filling up the second place, so the first two places can be filled in $n(n-1)$ ways.

After having filled in the first two places in any one of the above ways, there are $(n-2)$ things left and so the third place can be filled in $(n-2)$ ways. Now for every one way of filling up the first two places, there are $(n-2)$ ways of filling up the third place and so the first three places can be filled up in $n(n-1)(n-2)$ ways.

It may be observed that

(a) At every stage the number of factors is equal to the number of places filled up.

(b) Every factor is by one less than its preceding factor.

Thus, we can conclude that the first $(r-1)$ places can be filled in $n(n-1)(n-2)\ldots\{n-(r-2)\}$ ways. After filling up first $(r-1)$ places the rth place can be filled in $n(n-1)(n-2)\ldots\{n-(r-1)\}$ ways.

Position of the object	1st	2nd	...	$(r-1)$th	rth
Number of ways	n	$n-1$	...	$n-(r-2)$	$n-(r-1)$

Hence, the number of ways of filling up all the r places, *i.e.,* the number of permutations of n different things taken r at a time is $n(n-1)(n-2)\ldots r$ factors

$$= n(n-1)(n-2)\ldots(n-\overline{r-1})$$

Hence, $$\mathbf{^nP_r = n(n-1)(n-2)\ldots(n-r+1)}$$

Thus, $^7P_2 = 7 \times 6$; $^{10}P_4 = 10 \times 9 \times 8 \times 7$, $^{20}P_3 = 20 \times 19 \times 18$.

Cor. The number of permutations of n things taken all at a time is

$$\mathbf{^nP_n = n(n-1)(n-2)\ldots 3.2.1.}$$ [Putting n for r]

Ex. 7. In how many ways can 5 persons occupy 3 vacant seats?

Sol. Total number of ways $= {}^5P_3 = 5 \times 4 \times 3 = 60$.

Ex. 8. If $^{12}P_r = 1320$, find r.

Sol. $^{12}P_r = 12 \times 11 \times \ldots$ to r factors $= 1320 = 12 \times 11 \times 10 \ \therefore \ r = 3$.

12.07. Factorial notation

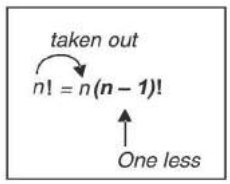
The product of n natural numbers from 1 to n is denoted by n ! or $\lfloor \underline{n}$ and is read as factorial n.

Thus, $n!$ *or* $\lfloor \underline{n} = 1.\ 2.\ 3 \ldots (n-1)\ .\ n$

$4\ ! = 1 \times 2 \times 3 \times 4 = 24;\ 6\ ! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$

$(n-1)! = 1 \times 2 \times 3 \ldots (n-1)$.

It is easily seen that $8\ ! = 8 \times (7\ !)$.



12.08. Values of ^nP_r in terms of factorial notation

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n(n-1)\ldots(n-r+1)}{\lfloor \underline{(n-r)}} \cdot \lfloor \underline{(n-r)}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)(n-r-1)\ldots 3.2.1}{\lfloor \underline{(n-r)}}$$

$$= \frac{n(n-1)(n-2)\ldots 3.2.1}{\lfloor \underline{(n-r)}} = \frac{\lfloor \underline{n}}{\lfloor \underline{(n-r)}}$$

$\therefore$ $\boxed{^nP_r = \dfrac{\lfloor \underline{n}}{\lfloor \underline{(n-r)}}}$

Thus, $^{18}P_5 = \dfrac{\lfloor \underline{18}}{\lfloor \underline{13}}$

Cor. 1. Putting $r = 0$, $^nP_0 = \dfrac{\lfloor \underline{n}}{\lfloor \underline{n}} = 1$

Cor. 2. Value of 0!.

Putting $r = n$, $^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$; But $^nP_n = n!$ $\therefore$ $n! = \dfrac{n!}{0!}$ $\qquad \therefore$ **0 ! = 1**

Note. In fact, 0! is meaningless but in order to avoid contradiction in the results, we suppose that 0! = 1.

Ex. 9. Find the value of n if $^nP_{13} : {}^{n+1}P_{12} = \dfrac{3}{4}$.

Sol. Here, $^nP_{13} = \dfrac{n!}{(n-13)!}$ and $^{n+1}P_{12} = \dfrac{(n+1)!}{(n-11)!}$

$$^nP_{13} : {}^{n+1}P_{12} = \frac{n!}{(n-13)!} \times \frac{(n-11)!}{(n+1)!} = \frac{3}{4} \Rightarrow \frac{n!}{(n-13)!} \times \frac{(n-11)\,(n-12)\cdot(n-13)!}{(n+1).n!} = \frac{3}{4}$$

i.e., $\dfrac{(n-11)(n-12)}{(n+1)} = \dfrac{3}{4} \Rightarrow 4n^2 - 95n + 525 = 0$ or $(n-15)\,(4n-35) = 0$

$\therefore$ $n = 15$ (Rejecting the fractional value of n).

Ex. 10. Show that ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$ where the symbols have their usual meanings. [*IIT*]

Sol. LHS $= {}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$ ...(*i*)

RHS $= {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

$= (n-1)(n-2) \ldots (n-1-r+1) + r.(n-1)(n-2) \ldots (n-1-\overline{r-1}+1)$

$= (n-1)(n-2) \ldots (n-r+1)(n-r) + r(n-1)(n-2) \ldots (n-r+1)$

$= (n-1)(n-2) \ldots (n-r+1)(n-r+r)$

$= n(n-1)(n-2) \ldots (n-r+1) = \text{LHS}$ ...(*ii*)

From (1) and (2), we have ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$.

EXERCISE 12 (*b*)

1. Evaluate $\frac{4!}{2!2!}$.

2. Give the meaning and value of the symbol in the following: **[*SC*]**

(*i*) ${}^5P_2$ (*ii*) ${}^7P_3$ (*iii*) ${}^{10}P_4$.

Find *n* if:

3. ${}^nP_2 = 30$ **4.** ${}^nP_4 : {}^{n-1}P_3 = 9 : 1$

5. ${}^{2n}P_{n+1} : {}^{2n-2}P_n = 56 : 3$ **6.** ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$ **[*Kerala PET 2009*]**

7. ${}^{2n}P_3 = 100.\ {}^nP_2$ **8.** $P(n, 6) = 3P(n, 5)$

9. $2P(n, 3) = P(n+1, 3)$ **10.** Find r if $5P(4, r) = 6P(5, r-1)$, $r \geq 1$.

Prove that:

11. $P(n, n) = 2P(n, n-2)$ **12.** $P(10, 3) = P(9, 3) + 3P(9, 2)$

13. $P(n, r) = (n-r+1)P(n, r-1)$ **14.** $P(n, n) = P(n, n-1)$

15. If $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$, find x. **[*SC*]**

16. If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, find the value of n.

Solve for *n*:

17. $\frac{(2n)!}{3!(2n-3)!} : \frac{n!}{2!(n-2)!} = 44 : 3$ **18.** $(n+1)! = 56.(n-1)!$

19. Prove that $\frac{2n!}{n!} = 1.3.5 \ldots (2n-1)2^n$. **[*ISC*]**

20. Convert into factorial: 7 . 8 . 9 . 10 . 11 . 12 . 13 . 14 . 15.

ANSWERS

1. 6. **2.** (*i*) 20, (*ii*) 210, (*iii*) 5040. **3.** 6. **4.** 9. **5.** 4.
6. 4. **7.** 13. **8.** 8. **9.** 5. **10.** 3. **15.** 121.
16. 5. **17.** $n = 6$. **18.** $n = 7$. **20.** $\frac{15!}{6!}$

HINTS TO SELECTED QUESTIONS

3. ${}^nP_2 = 30 \Rightarrow \frac{n!}{(n-2)!} = 30 \Rightarrow \frac{n(n-1)(n-2)!}{(n-2)!} = 30 \Rightarrow n\,(n-1) = 30$

5. ${}^{2n}P_{n+1} : {}^{2n-2}P_n = 56 : 3 \quad \Rightarrow \quad \frac{2n!}{(2n-n-1)!} : \frac{(2n-2)!}{(2n-2-n)!} = 56 : 3$

$\Rightarrow \frac{2n!}{(n-1)!} \times \frac{(n-2)!}{(2n-2)!} = \frac{56}{3} \quad \Rightarrow \quad \frac{2n(2n-1)\,(2n-2)!}{(n-1)(n-2)!} \times \frac{(n-2)!}{(2n-2)!} = \frac{56}{3}$

$\Rightarrow \frac{2n(2n-1)}{n-1} = \frac{56}{3}.$

9. $2P\,(n, 3) = P\,(n+1, 3) \quad \Rightarrow 2.\,{}^nP_3 = {}^{n+1}P_3 \quad \Rightarrow 2\frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!}$

$\Rightarrow \frac{2n(n-1)(n-2)(n-3)!}{(n-3)!} = \frac{(n+1)n(n-1)(n-2)!}{(n-2)!} \Rightarrow 2n(n-2) = n(n+1)$

10. $5.\,{}^4P_r = 6.\,{}^5P_{r-1} \Rightarrow 5.\,\frac{4!}{(4-r)!} = 6.\,\frac{5!}{(5-r+1)!} \Rightarrow 5.\,\frac{4!}{(4-r)!} = 6.\,\frac{5.4!}{(6-r)(5-r)(4-r)!}$

11. $\text{LHS} = P\,(n, n) = {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

$\text{RHS} = 2.{}^nP_{n-2} = 2.\frac{n!}{(n-n+2)} = 2.\frac{n!}{2!} = n!$

15. $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!} \quad \Rightarrow \quad \frac{11\times10}{11\times10\times9!} + \frac{11}{11\times10!} = \frac{x}{11!} \quad \Rightarrow \quad \frac{110}{11!} + \frac{11}{11!} = \frac{x}{11!}$

19. $\text{LHS} = \frac{1,2,3,4,5,6,7,8 \ldots (2n-2)(2n-1)(2n)}{n!}$

$= \frac{\{1,3,5,7 \ldots (2n-1)\}\{2,4,6 \ldots (2n-2)(2n)\}}{n!}$

$= \frac{\{1,3,5,7 \ldots (2n-1)\}2^n.n!}{n!}$

12.09. Restricted permutations

Type I.

Ex. 11. In how many of the permutations of 10 things taken 4 at a time will (*i*) one thing always occur, (*ii*) never occur?

Sol. (*i*) Keeping aside the particular thing which will always occur, the number of permutations of 9 things taken 3 at a time is ${}^9P_3$. Now this particular thing can take up any one of the four places and so can be arranged in 4 ways. Hence the total number of permutations = ${}^9P_3 \times 4 = 9 \times 8 \times 7 \times 4 = 2016$.

(*ii*) Leaving aside the particular thing which has never to occur, the number of permutations of 9 things taken 4 at a time is ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$.

Ex. 12. In how many of the permutations of *n* things taken *r* at a time will 5 things (*i*) always occur, (*ii*) never occur ?

Sol. (*i*) Keeping aside the 5 things, the number of permutations of $(r - 5)$ things taken out of $(n - 5)$ things is ${}^{n-5}P_{r-5}$. Now these 5 things can be arranged in r places in ${}^rP_5$ ways. Hence, the total number of permutations is ${}^rP_5 \times {}^{n-5}P_{r-5}$.

(*ii*) Total number of permutations = ${}^{n-5}P_r = \frac{(n-5)!}{(n-r-5)!}$

Type II. When certain things are not to occur together

Case I. When the number of things not occurring together is two.

Procedure

1. *Find the total number of permutations when no restriction is imposed on the manner of arrangement.*
2. *Then find the number of permutations when the two things occur together.*
3. *The difference of the two results gives the number of permutations in which the two things do not occur together.*

Ex. 13. Prove that the number of ways in which n books can be placed on a shelf when two particular books are never together is $(n-2) \times (n-1)$!.

Sol. Regarding the two particular books as one book, there are $(n-1)$ books now which can be arranged in ${}^{n-1}P_{n-1}$, *i.e.*, $(n-1)$! ways. Now, these two books can be arranged amongst themselves in 2 ! ways. Hence, the total number of permutations in which these two books are placed together is 2 ! . $(n-1)$!. The number of permutations of n books without any restriction is n !.

Therefore, the number of arrangements in which these two books never occur together

$$= n\,! - 2\,! \,.\, (n-1)\,! = n \,.\, (n-1)\,! - 2 \,.\, (n-1)\,! = (n-2) \,.\, (n-1)\,!$$

Case II. When the number of things not occurring together is more than two.

Ex. 14. In how many ways can 6 boys and 4 girls be arranged in a straight line so that no two girls are ever together ?

Sol. The seating arrangement may be done as desired in two operations.

(*i*) First we fix the positions of 6 boys. Their positions are indicated by $B_1, B_2, ..., B_6$.

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times$$

This can be done in 6! ways.

(*ii*) Now if the positions of girls are fixed at places (including those at the two ends) shown by the crosses, the four girls will never come together. In any one of these arrangements there are 7 places for 4 girls and so the girls can sit in ${}^7P_4$ ways.

Hence, the required number of ways of seating 6 boys and 4 girls under the given condition

$$= {}^7P_4 \times 6! = 7 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{604800.}$$

Type III. Formation of numbers with digits.

Ex. 15. Suppose the six digits 1, 2, 4, 5, 6, 7 are given to us and we have to find the total number of numbers with no repetition of digits which can be formed under different conditions.

Sol. 1. There is no restriction. The number of 6-digit numbers .

$$= {}^6P_6 = 6\,! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

2. Numbers in which a particular digit occupies a particular place.

Suppose we have to form numbers in which 5 always occurs in the ten's place. In this case the ten's place is fixed and the remaining five places can be filled in by the remaining 5 digits in ${}^5P_5$, *i.e.*, 5 ! = 120 ways.

The number of numbers in which 5 occurs in the ten's place = 120.

3. Numbers divisible by a particular number. Suppose we have to form numbers which may be divisible by 2. These numbers will have 2 or 4 or 6 in the unit's place. Thus the unit's place can be filled in 3 ways. After having filled up the unit's place in any one of the above ways, the remaining five places can be filled in ${}^5P_5 = 5! = 120$ ways.

$\therefore$ The total number of numbers divisible by $2 = 120 \times 3 = 360$.

4. Numbers having particular digits in the beginning and the end. Suppose we have to form numbers which begin with 1 and end with 5. Here, the first and the last places are fixed and the remaining four places can be filled in 4!, *i.e.,* 24 ways by the remaining four digits.

Therefore, the total number of numbers beginning with 1 and ending with 5 = 24.

Note. If the numbers could have 1 or 5 in the beginning or the end, the number would have been 2! . 4!, *i.e.* 48.

5. Numbers which are smaller than or greater than a particular number. Suppose we have to form numbers which are greater than 4,00,000. In these numbers there will be 4 or a digit greater than 4, *i.e.* 5, 6 or 7 in the lac's place. Thus this place can be filled in 4 ways. The remaining 5 places can then be filled in 5! = 120 ways.

$\therefore$ The total number of numbers = $4 \times 120 = 480$.

Ex. 16. How many numbers can be formed by using any number of the digits 3, 1, 0, 5, 7, 2, 9 no digit being repeated in any number ?

Sol. The number of single digit numbers is ${}^7P_1$.

The permutations of 7 digits taken 2 at a time are ${}^7P_2$. But ${}^6P_1$ of these have zero in the ten's place and so reduce to one digit numbers.

Hence, the number of two-digit numbers is ${}^7P_2 - {}^6P_1$.

Similarly, the number of the three-digit numbers is ${}^7P_3 - {}^6P_2$ and so on.

$\therefore$ The total number required

$= {}^7P_1 + ({}^7P_2 - {}^6P_1) + ({}^7P_3 - {}^6P_2) + ({}^7P_4 - {}^6P_3) + ({}^7P_5 - {}^6P_4) + ({}^7P_6 - {}^6P_5) + ({}^7P_7 - {}^6P_6) =$ **11743.**

Ex. 17. How many different numbers can be formed with the digits 1, 3, 5, 7, 9, when taken all at a time, and what is their sum?

Sol. The total number of numbers = 5! = 120. Suppose we have 9 in the unit's place. We will have 4! = 24 such numbers. The number of numbers in which we have 1, 3, 5 or 7 in the unit's place is also 4! = 24 in each case.

Hence, the sum of the digits in the unit's place in all the 120 numbers

$= 24 (1 + 3 + 5 + 7 + 9) = 600$.

The number of numbers when we have any one of the given digits in ten's place is also 4 ! = 24 in each case. Hence, the sum of the digits in the ten's place

$= 24 (1 + 3 + 5 + 7 + 9)$ tens = 600 tens = 600×10.

Proceeding similarly, the required sum

= 600 units + 600 tens + 600 hundreds + 600 thousands + 600 ten thousands

$= 600 (1 + 10 + 100 + 1000 + 10000) = 600 \times 11111 =$ **6666600**

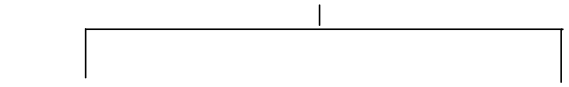
Type IV. Word building

The following cases may arise :—

1. No letter may be repeated.
2. Some letters may be repeated.*
3. There may be a particular letter in the beginning or the end.

* This case is taken up in the next article.

4. Some letters may occur together.



Vowels occur together Consonants occur together

Ex. 18. Suppose the word 'PENCIL' is given to us and we have to form words with the letters of this word.

Sol. 1. There is no restriction on the arrangement of the letters.

The six different letters can be arranged in ${}^6P_6 = 6\,! = 720$ ways.

Hence, the total number of words formed = 720.

2. All words begin with a particular letter

Suppose all words begin with *E*. The remaining 5 places can be filled with remaining 5 letters in 5! = 120 ways.

3. All words begin and end with particular letters

Suppose all words begin with *L*, and end with *P*. The remaining 4 places can then be filled in 4! ways.

$\therefore$ The total number of words formed = 4 ! = 24.

Note. If the words were to begin or end with *E* or *L*, these two positions could have been filled in ${}^2P_2 = 2$ ways. Hence the number of words in this case would have been = 2 × 24 = 48.

4. *N* is always next to *E*.

P	*EN*	*C*	*I*	*L*

Since *N* is always next to *E*, therefore '*EN*' is considered to be one letter.

$\therefore$ Required no. of permutations = 5! = 5 × 4 × 3 × 2 × 1 = 120.

5. Vowels occur together

The vowels are *E* and *I*. Regarding them as one letter, the 5 letters can be arranged in 5 ! = 120 ways. These two vowels can be arranged amongst themselves in 2 ! = 2 ways.

$\therefore$ The total number of words = 2 × 120 = 240.

6. Consonants occur together

Regarding these consonants as one letter the three letters *E*, *I*, (*PNCL*) can be arranged in 3 !, *i.e.,* 6 ways. The letters *PNCL* can be arranged among themselves in 4 ! = 24 ways.

$\therefore$ The number of words in which consonants occur together = 6 × 24 = 144.

7. Vowels occupy even places

×	×	×	×	×	×
1	2	3	4	5	6

There are 6 letters and 3 even places. *E* can be placed in any one of the three even places in ${}^3P_1$, *i.e.,* 3 ways. Having placed *E* in any one of these places, *I* can be placed in any one of the remaining 2 places in ${}^2P_1$, *i.e.,* 2 ways. Thus, the vowels can occupy even places in 2 × 3 = 6 ways. After the vowels have been placed the remaining 4 letters can take up their positions in ${}^4P_4$, *i.e.,* 24 ways.

$\therefore$ The total number of words = 6 × 24 = 144.

Ex. 19. How many ways are there to arrange the letter in the word GARDEN with the vowels in alphabetical order? [*AIEEE 2004*]

Sol. The word *GARDEN* contains 6 letters - 4 consonants (*G*, *R*, *D*, *N*) and 2 vowels *A*, *E*. The 4 consonants can be arranged in 6 places in ${}^6P_4$ ways. In each of these arrangements two places will

remain blank in which the first place will be filled by A and the place after this in only way by E as per the given condition (vowels in alphabetical order)

$\therefore$ Required number of ways = ${}^6P_4 \times 1 = 6 \times 5 \times 4 \times 3 =$ **360.**

12.10. Permutations of alike things

The number of permutations of n things taken all at a time where p of the things are alike and of one kind, q others are alike and of another kind, r others are alike and of another kind, and so on is

$$x = \frac{n!}{p!\,q!\,r!\ldots}.$$

Suppose the things are letters of which p are a's, q are b's, r are c's and so on. In any one of the required x permutations, replace the p a's by p new letters (say $a_1, a_2, \ldots a_p$) different from each other and different from the remaining $(n - p)$ letters. Then, permuting these p new letters among themselves without disturbing the rest, we get $p!$ new permutations. If this change is made in each of the x permutations, we will obtain $x \times p!$ new permutations in which p letters a's are now all different. Now, in any one of the $x \times p!$ permutations, we replace the q b's by q different letters $b_1, b_2, \ldots b_q$. Permuting them without disturbing the rest, we get $q!$ permutations in all. By doing this in each of the $x \times p!$ permutations we would obtain $x \times p!\,q!$ permutations in which all p letters 'a's' and all q letters 'b's' are different. Similarly on replacing the r c's by r different letters and permuting them, we will form $x \times p!\,q!\,r! \ldots$ permutations in which all the n letters are now different. But n different letters can be permuted in $n!$ ways. Hence, $x \,.\, p!\,q!\,r! \ldots = n!$

$$x = \frac{n!}{p!\,q!\,r!\ldots}.$$

Ex. 20. In how many ways can the letters of the word 'INDIA' be arranged ?

Sol. The word contains 5 letters of which 2 are 'I's.

The number of words possible $= \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60.$

Ex. 21. How many signals can be made by hoisting 2 blue, 2 red and 5 yellow flags on a pole at the same time ?

Sol. The number of signals $= \frac{9!}{2!2!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 5 \times 4 \times 3 \times 2 \times 1} =$ **756.**

Ex. 22. A coin is tossed 6 times. In how many different ways can we obtain 4 heads and 2 tails?

Sol. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

$\therefore$ The number of arrangements of 4 heads and 2 tails out of 6 is $\frac{6!}{4!2!} = 15.$

Ex. 23. How many numbers can be formed with digits 1, 2, 3, 4, 3, 2, 1, so that odd digits always occupy the odd places? ***[IIT, ISC]***

Sol. The odd digits having two 1's alike and two 3's alike can be arranged in four odd places in $\frac{4!}{2!2!} = 6$ ways.

The three even digits having two 2's alike can be arranged in the three even places in $\frac{3!}{2!} = 3$ ways.

$\therefore$ The number of numbers = $6 \times 3 = 18$.

Ex. 24. There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf. ***[IIT]***

Sol. Total number of books = $3 \times 4 = 12$

Each of the 4 different titles has 3 copies each

$\therefore$ Required number of ways of arranging them on a shelf = $\frac{12!}{3!3!3!3!} = \frac{12!}{(3!)^4} = 369600.$

Ex. 25. Find the number of arrangements of the letters of the word 'BANANA' in which the two N's do not appear adjacently. ***[IIT Prel. 2002]***

Sol. Considering the two *N's* as one letter, the number of letters to be arranged = 5.

Therefore, the number of arrangements = $\frac{5!}{3!} = 20$ ($\because$ *A* is repeated 3 times)

Total number of arrangements if there were no restriction imposed = $\frac{6!}{3!2!} = 60.$

(*A* repeated 3 times and *N* repeated 2 times)

$\therefore$ Required number of arrangements = 60 – 20 = **40.**

Ex. 26. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Sol. A million is a 7-digit number. So any number greater than 1 million will contain all the seven digits. Since the digit 2 occurs twice and digit 3 occurs thrice and the rest are different, therefore, number of possible numbers which can be formed with the given seven digits = $\frac{7!}{(2!)(3!)} = 420.$

These possible numbers include those which have 0 at the millions place. Keeping 0 fixed at the millions place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are different can be arranged in $\frac{6!}{(2!)(3!)} = 60$ ways.

$\therefore$ Number of numbers greater than 1 million made from the given digits = 420 – 60 = **360.**

Ex. 27. In how many ways can the letters of the word 'ARRANGE' be arranged such that the two *r's* do not occur together?

Sol. There are two *a's*, two *r's* in the word 'arrange', therefore the number of arrangements = $\frac{7!}{2!2!} = 1260.$ ...(1)

The number of arrangements in which the two *r's* occur together = $\frac{6!}{2!} = 360.$ ...(2)

$\therefore$ The number of arrangements in which two *r's* do not occur together = (1) – (2)

= 1260 – 360

= **900**.

Ex. 28. If the letters of the word 'AGAIN' be arranged in a dictionary, what is the fiftieth word?

Sol. In a dictionary, the words are arranged in an alphabetical order.

(*i*) Starting with *A*, the remaining 4 letters *G, A, I, N* can be arranged in 4! = 24 ways. These are the first 24 words.

(*ii*) Then, starting with *G*, the remaining letters *A*, *A*, *I*, *N* can be arranged in $\frac{4!}{2!} = 12$ ways. Thus, there are 12 words starting will *G*.

(*iii*) Now, the words will start with *I*. Starting with *I*, the remaining letters *A*, *G*, *A*, *N* can be arranged in $\frac{4!}{2!} = 12$ ways. So, there are 12 words, which start with *I*.

(*iv*) Thus, so far, we have constructed 24 + 12 + 12, *i.e.,* 48 words. The 49th word will start with *N* and is *NAAGI*. Hence, the 50th word is ***NAAIG***.

Ex. 29. The letters of the word 'RANDOM' are written in all possible ways and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.

Sol. The words in a dictionary are written in an alphabetical order, which, here is *ADMNOR*.

(*i*) Starting with *A*, the remaining letters *D*, *M*, *N*, *O*, *R* can be arranged in 5! = 120 ways. So, there are 120 words starting with *A*.

(*ii*) Similarly, number of words starting with *M* = 120, starting with *N* = 120, and starting with *O* = 120.

(*iii*) Now, the number of words starting with *R* is also 120. Out of these words, one word is *RANDOM*. First, we find the words starting with *RAD* and *RAM*.

Number of starting with *RAD* = 3! = 6

Number of starting with *RAM* = 3! = 6

(*iv*) Thus, so far, 120 + 120 + 120 + 120 + 120 + 6 + 6 = 612 words have been constructed.

(*v*) Now, the words starting with *RAN* will appear. Their number is also = 3! = 6. One of these words is the word *RANDOM* itself.

The first word beginning with *RAN* is *RANDMO*. It is 613th word and so the next word is *RANDOM*. *RANDOM* is the 614th word.

Hence, rank of *RANDOM* = **614**.

12.11. Permutations of repeated things

The number of permutations of n different things taken r at a time, when each thing may occur any number of times is n^r.

Suppose *r* places are to be filled with *n* things. The first place can be filled in *n* ways and when this has been filled up in any one of these ways, the second place can also be filled in *n* ways for the thing occupying the first place may occupy the second place also. Thus, the first two places can be filled in $n \times n = n^2$ ways. Similarly the third place can also be filled in *n* ways.

Arguing in the same manner, we conclude that the *r* places can be filled in $n \times n \times n$.....*r* times, *i.e.* n^r ways.

Ex. 30. In how many ways can 3 prizes be distributed among 4 boys, when

(*i*) no boy gets more than one prize;

(*ii*) a boy may get any number of prizes;

(*iii*) no boy gets all the prizes.

Sol. (*i*) The first prize can be given to any of the four boys. Then, the second prize can be given to any of the three boys. Lastly, the third prize can be given to any one of the remaining 2 boys.

$\therefore$ The number of ways in which all the 3 prizes can be given = $4 \times 3 \times 2 =$ **24**.

(*ii*) In this case, each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given = $4 \times 4 \times 4 = 4^3 =$ **64**.

(*iii*) Since anyone of the 4 boys can get all the prizes, therefore, the number of ways in which a boy gets all the 3 prizes = 4.

$\therefore$ Number of ways in which a boy does not get all the prizes = $64 - 4 =$ **60**.

Ex. 31. How many numbers of 3-digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

Sol. Since repetition is allowed, each of the 3 places in a 3-digit number can be filled in 5 ways.

H	T	O
5 ways	5 ways	5 ways

$\therefore$ Required number of 3-digit numbers
$= 5 \times 5 \times 5 = 5^3 = 125$

Ex. 32. How many numbers each containing four digits can be formed, when a digit may be repeated any number of times?

Sol. There are in all 10 digits, including zero. As the first digit of the number cannot be zero, so it can be chosen in 9 ways. Again, as a digit may occur any number of times in a number, the second, third and fourth digits of the numbers can be any one of the ten digits and so each of the remaining three places can be filled in 10 ways.

Hence, the total number of 4-digit numbers = $9 \times 10^3 = 9000$.

Verification. All the 4-digit numbers will be between 1000 and 9999 and so their number is $9999 - 999 = 9000$.

Ex. 33. Eight different letters of an alphabet are given. Words of 4 letters from these are formed. Find the number of such words with at least one letter repeated. ***[EAMCET 2006]***

Sol. If any letter can be used any number of times, then the number of words of 4 Letters with 8 different letters is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$

Number of words of 4 letters with at least one letter repetition not allowed

$$= {}^8P_4 = 8 \times 7 \times 6 \times 5$$
$$= 1680$$

$\therefore$ Number of 4 letter words with at least one letter repeated is $8^4 - {}^8P_4 = 4096 - 1680 =$ **2416.**

12.12. Circular permutations

Method I. If we have to arrange the five letters *A, B, C, D, E,* two of the arrangements would be *ABCDE, EABCD* (Fig. 12.01) which are two distinct arrangements. Now, if these arrangements are written along the circumference of a circle, the two arrangements are one and the same. Thus, we conclude that circular permutations are different only when the relative order of the objects is changed otherwise they are the same. Thus, the arrangements in Fig.12.02 are different.

As the number of circular permutations depends on the relative positions of objects, we *fix the position of one object and then arrange the remaining* $(n - 1)$ *objects in all possible ways. This can be done in* $(n - 1)!$ *ways.*

Method II. Let the 5 persons be denoted by the letters *A,B,C, D, E* and one of the ways in which they can form a ring be as shown in Fig. 12.01.

Starting with different letters and reading them in the clockwise direction the various arrangements of the letters thus obtained are *ABCDE, BCDEA, CDEAB, DEABC, EABCD.*

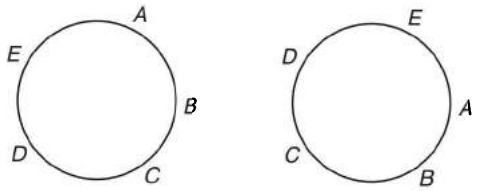


Fig. 12.01

These are all *different linear* arrangements but have no essential difference when regarded as circular arrangements. This shows that a single circular arrangement of the 5 letters gives rise to 5 different linear arrangements.

Hence, if the required number of circular arrangements of the 5 persons be x, the total number of linear arrangements of the same persons will be $5x$.

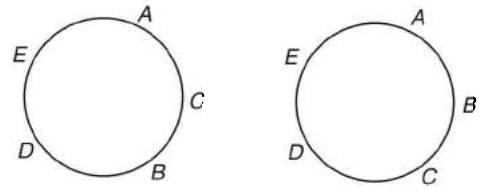


Fig. 12.02

But we know that the total number of linear arrangements of 5 persons is 5!.

$$\therefore \qquad 5x = 5\,!\,\therefore\ x = \frac{5!}{5} = 4!$$

Thus, the number of ways in which *five* persons form a ring $= 4! = (5 - 1)!$.

Similarly, **the number of ways in which *n* persons can form a ring = $(n - 1)!$.**

Ex. 34. 20 persons were invited for a party. In how many ways can they and the host be seated around a circular table? In how many of these ways will two particular persons be seated on either side of the host? **[*IIT*]**

Sol. There is 1 host and 20 guests, they are to be seated around a circular table.

(*i*) Let us fix the seat of one person, say the host, the 20 guests will be seated around the circular table in 20! ways, [or, $(n - 1)! = (21 - 1)\,! = 20!$]

(*ii*) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated around the circular table in 18! ways.

Hence, the number of ways of seating two particular persons on either side of the host $= 2 \times 18!$. [or, $2! \times (19 - 1)!$, considering the host and two particular persons as one entity.]

Ex. 35. *(i)* In how many ways can a party of 4 boys and 4 girls be seated at a circular table so that no 2 boys are adjacent? **[*ISCBM 2004*]**

(*ii*) In how many ways can 10 boys and 5 girls sit around a circular table, so that no two girls sit together?

Sol. (*i*) Let the girls first take up their seats. They can sit in 3! ways. When they have been seated, then there remain 4 places for the boys each between two girls. Therefore the boys can sit in 4! ways. Therefore there are $3! \times 4!$, *i.e.,* 144 ways of seating the party.

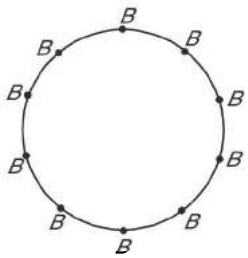


Fig. 12.03

(*ii*) Let B, denote the position of the boys around the table. 10 boys can be seated around the table in 9 ! ways.

There are 10 spaces between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence,

total number of ways $= 9! \times {}^{10}P_5 = \dfrac{\mathbf{9!10!}}{\mathbf{5!}}$.

Ex. 36. A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates are

(*i*) always together, (*ii*) never together?

Sol. (*i*) Let D_1 and D_2 be the two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. 19 delegates can be seated round a circular table in (19 – 1)! = 18! ways. But two particular delegates can seat themselves in 2! (D_1 D_2 or D_2D_1) ways.

Hence, the total number of ways = 18! × 2! = 2 (18!)

(*ii*) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of ways of seating 20 persons *i.e.*, (20 – 1) ! = 19! ways.

Hence, the total number of ways in this case = 19! – 2 (18!) = 19 (18!) – 2 (18!) = **17 (18!).**

12.13. Clockwise and counter-clockwise permutations

We have two types of circular permutations:

(*i*) Those in which counter-clockwise and clockwise are distinguishable. Thus, while seating 4 persons *A, B, C, D* around a table, the following permutations are considered different. (Fig. 12.04)

(*ii*) Those in which counter-clockwise and anti-clockwise are not distinguishable. (Fig. 12.05)

Thus, (*a*) while forming a garland of roses or jasmine, the following arrangements are not disturbed if we turn the garland over.

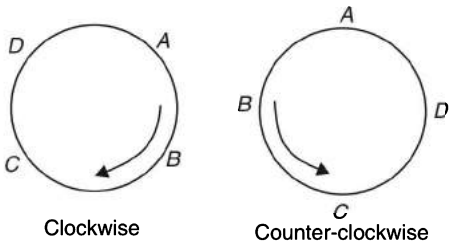


Fig.12.04

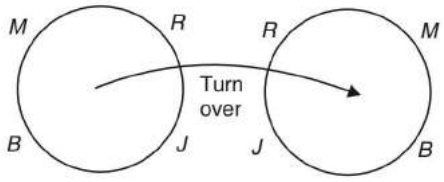


Fig. 12.05

(*b*) The distinction between clockwise and anti-clockwise is ignored when a number of people have to be seated around a table so as not to have the same neighbours.

Points to remember (Circular permutations)

We can **summarise** the above discussion as under:

1. Number of circular arrangements (permutations) of n different things = $(n-1)!$
2. Number of circular arrangements of n different things when clockwise and anti-clockwise arrangements are not different = $\frac{1}{2}(n-1)!$
3. Number of circular permutations of n different things taken r at a time when **clockwise and anti-clockwise arrangements are taken as different** is $\frac{{}^nP_r}{r}$.
4. Number of circular permutations of n different things, taken r at a time, when clockwise and anti-clockwise arrangements are not different = $\frac{{}^nP_r}{2r}$.

Ex. 37. Find the number of ways in which (*i*) *n* different beads, (*ii*) 10 different beads can be arranged to form a necklace.

Sol. Fixing the position of one bead, the remaining beads can be arranged in $(n-1)!$ ways. As there is no distinction between the clockwise and anti-clockwise arrangements, the required number of ways $= \frac{(n-1)!}{2}$.

Number of ways in which 10 different beads can be arranged $= \frac{(10-1)!}{2} = \frac{\mathbf{9!}}{\mathbf{2}}$.

Ex. 38. Find the number of ways in which 10 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Sol. Consider 4 particular flowers as one flower. Then, we have 7 flowers which can be strung to form a garland in $(7-1)! = 6!$ ways. But 4 particular flowers can be arranged in 4! ways.

Hence, the required number of ways $= \frac{1}{2}(6! \times 4!) =$ **8640.**

Ex. 39. In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements.

Sol. 7 persons can sit around a table in 6 ! ways but as each person will have the same neighbours in clockwise and anti-clockwise arrangements, the required number $= \frac{1}{2}\,6! =$ **360.**

EXERCISE 12 (*c*)

1. Of 12 different books a shelf will hold five; how many different arrangements may be made on the shelf ?
2. In how many ways can the letters of the following words be arranged:
   (*i*) *RADIO* (*ii*) *FOREIGN* ?
3. In how many other ways can the letters of the word '*SIMPLETON*' be arranged?
4. How many different words beginning and ending with a consonant can be made out of the letters of the word '*EQUATION*'?
5. How many permutations can be made out of the letters of the word '*TRIANGLE*'? How many of these will begin with T and end with E?
6. How many different words can be formed of the letters of the word '*MALENKOV*' so that
   (*i*) no two vowels are together,
   (*ii*) the vowels may occupy odd places?
7. In how many ways can the letters of the word '*COMBINE*' be arranged so that;
   (*i*) the vowels are never separated ; (*ii*) all the vowels never come together;
   (*iii*) vowels occupy only the odd places?
8. Three persons have 4 coats, 5 waistcoats, and 6 hats. Find in how many ways can they put on the clothes.
9. If out of 6 flags any number of flags can be shown at a time, find how many different signals can be made out of them.
10. In how many ways can 9 things be arranged taken 4 at a time, and in how many of these arrangements will a particular thing be included?
11. How many different numbers of 4 digits each can be formed with the ten digits, 0, 1, 2,...9 , when digits are not repeated?

12. From the digits 1, 2, 3, 4, 5, 6, how many three-digit odd numbers can be formed when the repetition of the digits is not allowed ***[ISC]***

13. (*i*) How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5?
(*ii*) How many of them are divisible by 10?
(*iii*) How many of them will have zero in the ten's place?

14. How many 5-digit telephone numbers can be formed with the digits 0, 1, 2,..., 8, 9 if each number starts with 35 and no digit appears more than once? **[*ISC*]**

15. There are 5 boys and 3 girls. In how many ways can they stand in a row so that no two girls are together?

16. There are 5 red, 4 white and 3 blue marbles in a bag. They are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements. **[*ISC*]**

17. How many 7-digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4? **[*ISC*]**

18. (*i*) How many different words can be formed with the letters of the word '*BHARAT*'?
(*ii*) In how many of these *B* and *H* are never together?
(*iii*) How many of these begin with *B* and end with *T*?

19. (*i*) Find how many arrangements can be made with the letters of the word "*MATHEMATICS*"?
(*ii*) In how many of them the vowels occur together? **[*ISC*]**

20. Ten different books are arranged on a shelf. Find the number of different ways in which this can be done, if two specified books are (*a*) to be together, (*b*) not to be together.

21. In how many ways can 20 books be arranged on a shelf so that a particular pair of books shall not come together?

22. Find the number of permutations of the letters of the words
(*i*) *INDIA* (*ii*) *ALLAHABAD* (*iii*) *CHANDIGARH* (*iv*) *COMMISSION.*

23. Find the number of ways in which five identical balls can be distributed among ten identical boxes, if not more than one can go into a box. **[*IIT*]**

24. How many numbers are there in all which consist of 5 digits?

25. In how many ways can 5 prizes be distributed among 4 students, when each student may receive any number of prizes?

26. In how many ways can 3 letters be posted in four letter boxes in a village? If all the three letters are not posted in the same letter box, find the corresponding number of ways of posting.

27. In how many ways can 8 people sit around a table?

28. (*i*) In how many ways can 10 people sit around a table so that all shall not have the same neighbours in any two arrangements?
(*ii*) In how many ways can 20 persons be seated round a table if there are 9 chairs.

29. A committee of 11 members sits at a round table. In how many ways can they be seated if the 'President' and the 'Secretary' choose to sit together?

30. In how many ways can 30 different pearls be arranged to form a necklace?

31. In how many ways 6 gentlemen and 3 ladies can be seated round a table so that every gentleman may have a lady by his side.

32. The letters of the word *ZENITH* are written in all possible orders. How many words are possible if all these words are written out as in a dictionary? What is the rank of the word *ZENITH*?
[AIEEE 2005 Type]

ANSWERS

1. ${}^{12}P_5 = 95{,}040$ **2.** (*i*) ${}^5P_5 = 5! = 120$ (*ii*) $7! = 5040$ **3.** $(9!) - 1 = 3{,}62{,}879$

4. ${}^3P_2 \times {}^6P_6 = 4320$ **5.** $8! = 40320$, ${}^6P_6 = 6! = 720$ **6.** (*i*) 14400 (*ii*) 2880

7. (*i*) ${}^5P_5 \times {}^3P_3 = 5! \times 3! = 720$

(*ii*) $7! - (3! \times 5!) = 4320$

O, I, E	C, M, B, N
3 vowel	4 consonants

(*iii*) There are 4 odd places which can be filled by 3 vowels in ${}^4P_3$ ways. 4 consonants can be arranged in 4! ways.

1	2	3	4	5	6	7
□	□	□	□	□	□	□

$\therefore$ Required number of ways $= {}^4P_3 \times 4!$

8. ${}^4P_3 \times {}^5P_3 \times {}^4P_3 = 172800$ **9.** ${}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6 = 1956$

10. (*i*) ${}^9P_4 = 3024$, $4 \times {}^8P_3 = 1344$ **11.** 4536 **12.** 60 **13.** (*i*) 600 (*ii*) 120 (*iii*) 120

14. ${}^8P_3 = 336$. First 2 places are fixed and remaining 3 places can be filled in ${}^8P_3$ ways.

15. 14400 **16.** $\dfrac{12!}{5! \times 4! \times 3!} = 27720$ **17.** 360 **18.** (*i*) 360 (*ii*) 240 (*iii*) 12

19. (*i*) $\dfrac{11!}{2!2!2!} = 4989600$ (*ii*) 120960 **20.** (*a*) $9! \times 2! = 725760$ (*b*) $10! - 9! \times 2! = 2903040$

21. $20! - 19! \times 2! = 18(19!)$ **22.** (*i*) 60 (*ii*) 7560 (*iii*) 907200 (*iv*) 226800

23. $\dfrac{10!}{5!5!}$ **24.** $9 \times 10^4 = 90{,}000$ **25.** $2^4 = 1024$ **26.** 64, 60 **27.** 5040

28. (*i*) $\dfrac{1}{2}(9!)$ (*ii*) $\dfrac{{}^{20}P_9}{9}$ **29.** $9! \times 2! = 2(9!)$ **30.** $\dfrac{1}{2}(29!)$ **32.** 616.

HINTS TO SELECTED QUESTIONS

17. Reqd. number of numbers = Total possible arrangements – those arrangements have 0 in the extreme left position. $= \dfrac{7!}{3!2!} - \dfrac{6!}{3!2!}$. **18.** Type Solved Ex. 27

19. (*ii*) *A, A, E, I,* *M, T, H, M, T, C, S*

Imagine 4 vowels written together, Then these 8 letters can be permuted in $\dfrac{8!}{2!2!} = 10080$ ways.

Corresponding to each of these permutations, the 4 vowels can be arranged among themselves in $\dfrac{4!}{2!} = 12$ ways.

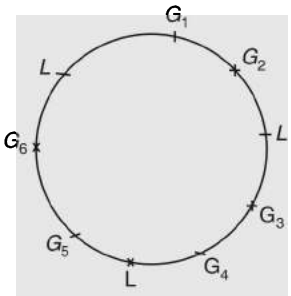
$\therefore$ Reqd. number of words in which vowels occur together

$$\frac{8!}{2!2!} \times \frac{4!}{2!} = 10080 \times 12 = 120960.$$

31. Arrange the seats for 6 gentlemen and 3 ladies as shown. This can be done in 5! (Gentlemen) × 3! (Ladies) = 720 ways.

If arrangements are made in the opposite direction, then the number of arrangements = $5! \times 3! = 720$.

$\therefore$ Total number of required arrangements = 720 + 720 = 1440.



32. Type Solved Ex. 29.

12.14. Combinations

Definition: ***Each of the different groups or selections which can be made by taking some or all of a number of things at a time (irrespective of the order) is called a combination.***

By the number of combinations of n things taken r at a time is meant the number of groups of r things which can be formed from the n things. The same is denoted by the symbol ^nC_r or $C(n, r)$ or $\binom{n}{r}$.

12.15. Value of ^nC_r or $C(n, r)$

Each combination consists of r different things which can be arranged among themselves in $r!$ ways. Therefore, the number of arrangements for all the ^nC_r combinations is $^nC_r \times r!$. This is equal to the permutations of n different things taken r at a time.

$\therefore \quad {}^nC_r \times r! = {}^nP_r$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2).....(n-r+1)}{1.2.3.....r}$$

Cor. 1. $\quad {}^nP_r = \dfrac{n!}{(n-r)!} \therefore {}^nC_r = \dfrac{n!}{r!\,(n-r)!}$

Cor. 2. To prove that $^nC_0 = {}^nC_n = 1$.

Putting $r = 0$ in $^nC_r = \dfrac{n!}{(n-r)!}\cdot\dfrac{1}{r!}$, we have

$${}^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{1\times n!} = 1.$$

Putting $r = n$, we have

$${}^nC_n = \frac{n!}{(n-n)!}\cdot\frac{1}{n!} = \frac{n!}{0!}\cdot\frac{1}{n!} = \frac{n!}{1}\cdot\frac{1}{n!} = 1. \qquad (\because\ 0! = 1)$$

$\therefore \quad \boxed{{}^nC_0 = {}^nC_n = 1}$

12.16. Properties of ^nC_r

Property 1. For non-negative integers r and n, where $0 \le r \le n$, $^nC_r = {}^nC_{n-r}$.

The number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken $(n - r)$ at a time, *i.e.,* $^nC_r = {}^nC_{n-r}$.

Proof.

Method I. Every time we select a group of r things we leave behind another group of $(n - r)$ things. Thus for every combination of $(n - r)$ things there corresponds a combination of r things.

$\therefore \quad {}^nC_r = {}^nC_{n-r}$

Method II. $^nC_r = \dfrac{n!}{r!\times(n-r)!}$ and $^nC_{n-r} = \dfrac{n!}{(n-r)!\,(n-\overline{n-r})!} = \dfrac{n!}{(n-r)!\,r!}$

$${}^nC_r = {}^nC_{n-r}$$

Property 2. If $^nC_x = {}^nC_y$, then, either $x = y$ or $x + y = n$.

Since $^nC_x = {}^nC_y = {}^nC_{n-y} \quad \therefore\ x = y$ or $x = n - y$, *i.e.,* $x + y = n$.

Property 3. An important formula

$\mathbf{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad (1 \le r \le n)}$ **(*Pascal's Rule*)**

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} \\
&= n!\left\{\frac{(n-r+1)+r}{r!\,.(n-r+1)!}\right\} = \frac{(n+1)\,n!}{r!\cdot(n-r+1)!} = \frac{(n+1)!}{r!\cdot(n-r+1)!} = {}^{n+1}\mathrm{C}_r.
\end{aligned}
$$

Another form. $C(n, r) + C(n, r-1) = C(n+1, r)$,

i.e., $C(n, r-1) = C(n+1, r) - C(n, r)$

Alternative Proof. The total number of combinations of $n + 1$ things taken r at a time = combinations that contain a particular thing + combinations that do not contain a particular thing.

$\therefore \quad {}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.

Note. ${}^nC_r$ is greatest if (*i*) $r = \frac{n}{2}$, when n is even, (*ii*) $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ when n is odd.

Property 4. For non-negative integers *n* and *r*, where $1 \le r \le n$, $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$.

Proof.
$$
\begin{aligned}
\frac{{}^nC_r}{{}^nC_{r-1}} &= \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} \\
&= \frac{1}{r.(r-1)!.(n-r)!} \times \frac{(r-1)!(n-r+1)(n-r)!}{1} \\
&= \frac{n-r+1}{r}.
\end{aligned}
$$

Property 5. For non-negative integers *n* and *r*, where $1 \le r \le n$, ${}^nC_r = \dfrac{n}{r} \times {}^{n-1}C_{r-1}$.

Proof.
$$
\begin{aligned}
{}^nC_r &= \frac{n!}{(n-r)!r!} \\
&= n.\frac{(n-1)!}{[(n-1)-(r-1)]!r.(r-1)!} \\
&= \frac{n}{r}.\frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!} = \frac{n}{r}.\,{}^{n-1}C_{r-1}.
\end{aligned}
$$

Property 6. If $1 \le r \le n$, then $n.{}^{n-1}C_{r-1} = (n-r+1)\,.\,{}^nC_{r-1}$

Proof.
$$
\begin{aligned}
n.{}^{n-1}C_{r-1} &= n.\frac{(n-1)!}{[(n-1)-(r-1)]!(r-1)!} \\
&= \frac{n!}{(n-r)!(r-1)!} = \frac{(n-r+1).n!}{(n-r+1).(n-r)!(r-1)!} \\
&= (n-r+1)\left(\frac{n!}{(n-r+1)!(r-1)!}\right) \\
&= (n-r+1).\left(\frac{n!}{\{(n-(r-1)\}!(r-1)!}\right) \\
&= (n-r+1).\,{}^nC_{r-1}.
\end{aligned}
$$

Ex. 40. Find the values of ${}^{6}C_3$ and ${}^{30}C_{28}$.

Sol. $(i)\ {}^{6}C_3 = \dfrac{6\times5\times4}{1\times2\times3} = 20.$

$(ii)\ {}^{30}C_{28} = {}^{30}C_{30-28} = {}^{30}C_2 = \dfrac{30\times29}{1\times2} = 435.$

Ex. 41. If ${}^{18}C_r = {}^{18}C_{r+2}$, find the value of ${}^{r}C_5$.

Sol. As $r \neq r+2$, so $r + (r+2) = 18 \quad \therefore \quad r = 8 \ \therefore\ {}^{r}C_5 = {}^{8}C_5 = {}^{8}C_3 = \dfrac{8\times7\times6}{1\times2\times3} = 56.$

Ex. 42. Find the value of ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$.

[DEC 2009, MPPET 09, IIT, Raj PET 2002, UPSEAT 2000]

Sol. ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3 = {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{48}C_4 \qquad [\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r]$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{49}C_4 = {}^{51}C_3 + {}^{50}C_3 + {}^{50}C_4$

$= {}^{51}C_3 + {}^{51}C_4 = {}^{52}C_4.$

Ex. 43. If ${}^{n}C_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^{n}C_{r+1} + {}^{n}C_{r-1} + 2 \times {}^{n}C_r$ equals

(a) ${}^{n+1}C_{r+1}$ ***(b)*** ${}^{n+2}C_r$ ***(c)*** ${}^{n+2}C_{r+1}$ ***(d)*** ${}^{n+1}C_r$ ***[AIEEE 2003]***

Sol. Given exp. $= {}^{n}C_{r+1} + {}^{n}C_{r-1} + 2\times{}^{n}C_r = {}^{n}C_{r+1} + {}^{n}C_{r-1} + {}^{n}C_r + {}^{n}C_r$ (write $2\times{}^{n}C_r = {}^{n}C_r + {}^{n}C_r$)

$= ({}^{n}C_{r+1} + {}^{n}C_r) + ({}^{n}C_{r-1} + {}^{n}C_r)$

$= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1}$ **Ans.** (c)

Ex. 44. If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, then find the value of r.

[DCE 2000, MP PET 2009, 01, IIT, Pb CET]

Sol. $\dfrac{{}^{n}C_r}{{}^{n}C_{r-1}} = \dfrac{84}{36} \Rightarrow \dfrac{n-r+1}{r} = \dfrac{84}{36} = \dfrac{7}{3}$ ...(i) [Property 4]

Also, $\dfrac{{}^{n}C_{r+1}}{{}^{n}C_r} = \dfrac{126}{84} \Rightarrow \dfrac{n-r}{r+1} = \dfrac{3}{2}$ ...(ii) [Property 4]

From (ii) $2n - 2r = 3r + 3 \Rightarrow n = \dfrac{5r+3}{2}$

Substituting in (i), we get $\dfrac{\frac{5r+3}{2} - r + 1}{r} = \dfrac{7}{3} \Rightarrow \boldsymbol{r = 3.}$

Ex. 45. If ${}^{n}C_3 + {}^{n}C_4 > {}^{n+1}C_3$, then

(a) $n > 6$ ***(b)*** $n > 7$ ***(c)*** $n < 6$ ***(d)*** **none of these** ***[Raj. PET 1999, IIT]***

Sol. ${}^{n}C_3 + {}^{n}C_4 > {}^{n+1}C_3 \Rightarrow {}^{n+1}C_4 > {}^{n+1}C_3 \Rightarrow \dfrac{(n+1)!}{4!\,(n-3)!} > \dfrac{(n+1)!}{3!\,(n-2)!}$

$\Rightarrow \frac{1}{4} > \frac{1}{n-2}$ ($\because$ 4 ! = 4.3 ! and $(n-2)! = (n-2).(n-3)!$)

$\Rightarrow \frac{n-2}{4} > 1 \Rightarrow n-2 > 4 \Rightarrow \boldsymbol{n > 6}$. **Ans.** (*a*).

EXERCISE 12 (*d*)

1. Find the value of:
(*i*) ${}^5C_2$ (*ii*) ${}^{10}C_4$ (*iii*) ${}^{50}C_{47}$

2. Evaluate :
(*i*) $C(15, 14)$ (*ii*) $C(8, 5)$ (*iii*) ${}^{11}C_2$.

3. Evaluate :
(*i*) $C(19, 17) + C(19, 18)$ (*ii*) $C(31, 26) - C(30, 26)$.

4. If ${}^4P_2 = n.{}^4C_2$, find n.

5. If ${}^nC_4 = {}^nC_6$, find n. **[*SC*]**

6. If $C(2n, 3) : C(n, 2) = 12 : 1$, find n.

7. If ${}^nC_r : {}^nC_{r+1} = 1 : 2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2 : 3$, determine the values of n and r. **[*ISC*]**

8. If $C(n, 10) = C(n, 12)$, determine $C(n, 5)$.

9. If $C(2n, r) = C(2n, r+2)$, find r in term of n.

10. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is
(*a*) ${}^{55}C_4$ (*b*) ${}^{55}C_3$ (*c*) ${}^{56}C_3$ (*d*) ${}^{56}C_4$ ***[AIEEE 2005]***

11. ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$ if
(*a*) $n > 7$ (*b*) $n \geq 7$ (*c*) $n > 6$ (*d*) $n \geq 6$ ***[WBJEE 2010, Raj. PET 2000, IIT]***

ANSWERS

1. (*i*) 10 (*ii*) 210 (*iii*) 19600 **2.** (*i*) 15 (*ii*) 56 (*iii*) 55 **3.** (*i*) 190 (*ii*) 142506 **4.** 2
5. 10 **6.** 5 **7.** $n = 14, r = 4$ **8.** 26334
9. $r = n - 1$ **10.** (*d*) **11.** (*a*)

HINTS TO SELECTED QUESTIONS

3. (*ii*) Using $C(n, r) + C(n, r-1) = C(n+1, r)$, we have $C(30, 26) + C(30, 25) = C(31, 26)$
$\Rightarrow C(31, 26) - C(30, 26) = C(30, 25)$

5. Since $4 \neq 6$, therefore, $4 + 6 = n$ [Art. 12.15, cor. 4]

7. $\frac{n!}{r!(n-r)!} : \frac{n!}{(r+1)!(n-r-1)!} = 1 : 2$ and $\frac{n!}{(r+1)!(n-r-1)!} : \frac{n!}{(r+2)!(n-r-2)!} = 2 : 3$

$\Rightarrow \frac{n!}{r!(n-r)(n-r-1)!} \times \frac{(r+1)\ r!(n-r-1)!}{n!} = \frac{1}{2}$

and $\frac{n!}{(r+1)!\ (n-r-1)(n-r-2)!} \times \frac{(r+2)\ (r+1)!(n-r-2)!}{n!} = \frac{2}{3}$

$\Rightarrow \frac{r+1}{n-r}=\frac{1}{2}$ and $\frac{r+2}{n-r-1}=\frac{2}{3}$.

$\Rightarrow n-3r=2$ and $2n-5r=8$. Solve.

10. $^{50}C_4+\sum_{r=1}^{6} {}^{56-r}C_3 = {}^{50}C_4+{}^{55}C_3+{}^{54}C_3+{}^{53}C_3+{}^{52}C_3+{}^{51}C_3+{}^{50}C_3$

$= \left({}^{50}C_4+{}^{50}C_3\right)+{}^{55}C_3+{}^{54}C_3+{}^{53}C_3+{}^{52}C_3+{}^{51}C_3$

$= \left({}^{51}C_4+{}^{51}C_3\right)+{}^{55}C_3+{}^{54}C_3+{}^{53}C_3+{}^{52}C_3$ [Using ${}^{n}C_r+{}^{n}C_{r-1}={}^{n+1}C_r$]

$= \left({}^{52}C_4+{}^{52}C_3\right)+{}^{55}C_3+{}^{54}C_3+{}^{53}C_3$

Continuing the pairwise association, you will obtain in the end $^{56}C_4$.

11. Type Solved Ex. 45. ${}^{n-1}C_3+{}^{n-1}C_4>{}^{n}C_3 \Rightarrow {}^{n}C_4>{}^{n}C_3 \Rightarrow \frac{{}^{n}C_4}{{}^{n}C_3}>1$

12.17. Practical problems on combination

Ex. 46. In how many ways can 4 persons be selected from amongst 9 persons? How many times will a particular person be always selected?

Sol. The number of ways in which 4 persons can be selected from amongst 9 persons

$= {}^{9}C_4=\frac{9\times8\times7\times6}{1\times2\times3\times4}=126.$

The number of ways in which a particular person is always to be selected

$= {}^{8}C_3=\frac{8\times7\times6}{1\times2\times3}=$ **56.**

Ex. 47. (*i*) How many diagonals are there in an n-sided polygon ($n > 3$)?
(*ii*) How many triangles can be formed by joining the vertices of an *n*-sided polygon?

Sol. (*i*) Number of line segments obtained by joining *n* vertices of a polygon.

= Number of ways in which 2 points can be selected from *n* given points

$= {}^{n}C_2=\frac{n(n-1)}{1.2}$

Out of ${}^{n}C_2$ line segments, *n* are the sides of the polygon.

Hence, **the number of diagonals = $^{n}C_2 - n$.** (Remember) $=\frac{n(n-1)}{2}-n=\frac{n(n-3)}{2}$.

(*ii*) Number of triangles formed by joining the *n* vertices of a polygon

= Number of ways in which 3 points can be selected from *n* given points

$= {}^{n}C_3=\frac{n(n-1)(n-2)}{3.2.1}$.

Ex. 48. Find the number of diagonals that can be drawn by joining the angular points of a
(*i*) heptagon
(*ii*) a polygon of 20 sides. ***[J&K CET 2009]***

Sol. (*i*) A heptagon has seven angular points and seven sides.
The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points = ${}^7C_2 = \dfrac{7 \times 6}{1 \times 2} = 21.$

But the number of sides = 7 $\therefore$ The number of diagonals = 21 – 7 = **14.**

(*ii*) Number of diagonals in a polygon of n sides = ${}^nC_2 - n$. Here, $n = 20$

$\therefore$ Required number of diagonals = ${}^{20}C_2 - 20 = \dfrac{20 \times 19}{2 \times 1} - 20 = \mathbf{170}.$

Ex. 49. A committee of 4 is to be selected from amongst 5 boys and 6 girls. In how many ways can this be done so as to include (*i*) exactly one girl, (*ii*) at least one girl?

Sol. (*i*) In this case we have to select one girl out of 6 and 3 boys out of 5.

The number of ways of selecting 3 boys = ${}^5C_3 = {}^5C_2 = 10.$

The number of ways of selecting one girl = ${}^6C_1 = 6.$

$\therefore$ The required committee can be formed in $6 \times 10 = 60$ ways.

(*ii*) The committee can be formed with

(*a*) one boy and three girls, or (*b*) 2 boys and 2 girls,

or (*c*) 3 boys and one girl, or (*d*) 4 girls alone.

The committee can be formed in (*a*) ${}^5C_1 \times {}^6C_3$ ways ;

The committee can be formed in (*b*) ${}^5C_2 \times {}^6C_2$ ways ;

The committee can be formed in (*c*) ${}^5C_3 \times {}^6C_1$ ways ;

The committee can be formed in (*d*) ${}^6C_4$ ways.

Hence the required number of ways of forming the committee

$= {}^5C_1 \times {}^6C_3 + {}^5C_2 \times {}^6C_2 + {}^5C_3 \times {}^6C_1 + {}^6C_4 = 100 + 150 + 60 + 15 = \mathbf{325\ ways.}$

(*ii*) **Method II.** Required ways = (Committees of 4 out of 11 without any restriction) – (Committees in which no girl is included) = ${}^{11}C_4 - {}^5C_4 = 325.$

Ex. 50. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. Find the number of choices available to him.

[AIEEE 2003]

Sol. Two cases are possible :

(*i*) Selecting 4 out of first five questions and 6 out of remaining 8 questions

$\therefore$ Number of choices in this case = ${}^5C_4 \times {}^8C_6 = {}^5C_1 \times {}^8C_2 = \dfrac{5 \times 8 \times 7}{1 \times 2} = 140$

(*ii*) Selecting 5 out of first five questions and 5 out of remaining 8 questions.

$\Rightarrow$ Number of choices = ${}^5C_5 \times {}^8C_5 = 1 \times {}^8C_3 = \dfrac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$

$\therefore$ Total number of choices = 140 + 56 = **196.**

12.18. Mixed problems on permutations and Combination

Type I. Total number of combinations

To find the total number of combinations of n dissimilar things taking* any number *of them at a time.

Case I. When all things are different

Each thing may be disposed of in two ways. It may either be included or rejected.

$\therefore$ The total number of ways of disposing of all the things = $2 \times 2 \times 2 \times ...$ n times = 2^n

But this includes the case in which all the things are rejected.

Hence, the total number of ways in which *one* or *more* things are taken = $2^n - 1$.

Cor. $2^n - 1$ is also the total number of the combinations of n things taken 1, 2, 3, ... or n at a time. Hence, ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = \mathbf{2^n - 1}$.

Ex. 51. There are 5 questions in a question paper. In how many ways can a boy solve one or more questions?

Sol. The boy can dispose of each question in two ways. He may either solve it or leave it. Thus the number of ways of disposing of all the questions = 2^5.

But this includes the case in which he has left all the questions unsolved.

Hence, the total number of ways of solving the paper = $2^5 - 1 = 31$.

Case II. When all things are not different

Suppose, out of $(p + q + r + ...)$ *things, p are alike of one kind, q are alike of a second kind, r alike of a third kind, and the rest different.*

Out of p things we may take 0, 1, 2, 3 ... or p. Hence, they may be disposed of in $(p + 1)$ ways. Similarly, q alike things may be disposed of in $(q + 1)$ and r alike things in $(r + 1)$ ways. The t different things may be disposed of in 2^t ways.

This includes that case in which all are rejected.

$\therefore$ The total number of selections = $(p + 1)(q + 1)(r + 1)\, 2^t - 1$.

Ex. 52. Prove that from the letters of the sentence, 'Daddy did a deadly deed', one or more letters can be selected in 1919 ways.

Sol. In the given sentence, there are 9 *d's* ; 3 *a*'s ; 3 *e's* ; 2 *y*'s ; 1 *i*; and 1 *l*.

$\therefore$ The total number of selections = $(9 + 1)(3 + 1)(3 + 1)(2 + 1)(1 + 1)(1 + 1) - 1$

$= 10 \times 4 \times 4 \times 3 \times 2 \times 2 - 1 = 1919.$

Type II. Permutations and combinations occurring simultaneously

The method is illustrated by the following examples:

Ex. 53. How many different words, each containing 2 vowels and 3 consonants, can be formed with 5 vowels and 17 consonants? **[*ISC 1996 Type*]**

Sol. Two vowels can be selected in ${}^5C_2$ ways.

Three consonants can be selected in ${}^{17}C_3$ ways.

$\therefore$ 2 vowels and 3 consonants can be selected in ${}^5C_2 \times {}^{17}C_3$ ways.

Now, each group of 2 vowels and 3 consonants can be arranged in 5! ways.

$\therefore$ The total number of words = ${}^5C_2 \times {}^{17}C_3 \times 5!$ = **816000.**

Ex. 54. Find the number of (*i*) combinations, (*ii*) permutations of four letters taken from the word EXAMINATION. **[*ISC 1992 Type*]**

Sol. There are 11 letters not all different. They are as: (*AA*) ; (*II*) ; (*N, N*) ; *E* ; *X* ; *M* ; *T* ; 0.

The following combinations are possible:

(*a*) 2 alike, 2 alike $= {}^3C_2 = 3$;

(*b*) 2 alike, 2 different $= {}^3C_1 \times {}^7C_2 = 63$;

(*c*) All 4 different $= {}^8C_4 = 70$.

$\therefore$ The total number of combinations = 3 + 63 + 70 = 136.

$\therefore$ The number of permutations in (*a*) to (*c*)

$$= 3 \times \frac{4!}{2!2!} + 63 \times \frac{4!}{2!} + 70 \times 4! = 18 + 756 + 1680 = \mathbf{2454.}$$

EXERCISE 12 (*e*)

1. In how many ways can a committee of 8 be chosen from 10 individuals?

2. In how many ways can a committee of five persons be formed out of 8 members when a particular member is taken every time?

3. In how many ways can a committee of 4 be selected out of 12 persons so that a particular person may (*i*) always be taken, (*ii*) never be taken?

4. In how many ways can a team of 11 players be selected from 14 players when two of them can play as goalkeepers only?

5. A person has got 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests such that 5 of them may be relatives?

6. How many diagonals are there in a polygon of (*i*) 8 sides, (*ii*) 10 sides? **[*ISC*]**

7. In how many ways can a committee, consisting of a chairman, secretary, treasurer and four ordinary members be chosen from eight persons?
(Committees with different chairmen, secretaries, treasurers count as different committees.)
[*SC*]

8. (*a*) In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student? **[*ISC*]**
(*b*) In how many ways can we select a cricket eleven from 17 players in which 5 players can bowl? Each cricket team must include 2 bowlers. **[*SC*]**

9. How many committees of 5 members each can be formed with 8 officials and 4 non-official members in the following cases:
(*i*) each consists of 3 officials and 2 non-official members;
(*ii*) each contains at least two non-official members;
(*iii*) a particular official member is never included;
(*iv*) a particular non-official member is always included?

10. In a college team there are 15 players of whom 3 are teachers. In how many ways can a team of 11 players be selected so as to include (*i*) only one teacher, (*ii*) at least one teacher?

11. How many different groups can be selected for playing tennis out of 4 ladies and 3 gentlemen, there being one lady and one gentleman on each side? **[*ISC*]**

12. If ${}^{n}C_{10} = {}^{n}C_{14}$, find the value of ${}^{n}C_{20}$ and ${}^{25}C_{n}$.

13. In how many ways can I invite one or more of six friends to a dinner ?

14. In how many ways can 10 marbles be divided between two boys so that one of them may get 2 and the other 8?

15. In how many ways can a selection be made out of 5 oranges, 8 mangoes and 7 plantains ?

16. In how many ways can 20 articles be packed in the three parcels so that the first contains 8 articles, the second 7 and the third 5?

17. Find the number of four letter arrangements of the letters of the word '*SHOOT*'. How many of them begin with *O*? **[*SC*]**

18. (*i*) Find the number of permutations of the letters of the word '*MATHEMATICS*', taken 4 at a time.
[Hint. See Solved Ex. 54]
(*ii*) How many four-letter words can be formed using the letters of the word '*INEFFECTIVE*'?
(*iii*) Find the number of ways in which (*a*) a selection, (*b*) an arrangement of four letters can be made from the letters of the word '*PROPORTION*'?

19. A table has 7 seats, 4 being on one side facing the window and 3 being on the opposite side. In how many ways can 7 people be seated at the table.
(*i*) if 2 people, *X* and *Y*, must sit on the same side; (*ii*) *X* and *Y* must sit on opposite sides;
(*ii*) if 3 people, *X*, *Y* and *Z*, must sit on the side facing the window? **[*ISC*]**

20. Seven cards, each bearing a letter, can be arranged to spell the word "*DOUBLES*". How many three-letter code-words can be formed from these cards ?
How many of these words
(*i*) contain the letter *S* ; (*ii*) do not contain the letter *O*;
(*iii*) consist of a vowel between two consonants? **[*GCE*]**

21. How many triangles may be formed by joining any three of the nine points when
(*i*) no three of them are collinear ; (*ii*) five of them are collinear?

22. A committee of 5 is to be formed from a group of 12 students consisting of 8 boys and 4 girls. In how many ways can the committee be formed if it
(*i*) consists of exactly 3 boys and 2 girls ; (*ii*) contains at least 3 girls? **[*ISC*]**

23. There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?

24. There are 12 points in a plane, of which 5 are collinear. Find **[*ISC 1999*]**
(*i*) the number of triangles that can be formed with vertices at these points;
(*ii*) the number of straight lines obtained by joining these points in pairs. **[*ISC*]**

25. A committee of 5 is to be formed from a group of 10 people, consisting of 4 single men, 4 single women and a married couple. The committee is to consist of a chairman, who must be a single man, 2 other men and 2 women,
(*i*) Find the total number of committees possible,
(*ii*) How many of these would include the married couple? **[*ISC*]**

26. A committee of 5 persons is to be formed from a group of 6 gentlemen and 4 ladies. In how many ways can this be done if the committee is to include at least one lady? **[*ISC*]**

27. Out of 3 books on Economics, 4 books on Political Science and 5 books on Geography, how many collections can be made, if each collection consists of
(*i*) exactly one book on each subject, (*ii*) at least one book on each subject? **[*ISC 1990*]**

28. Find the number of words which can be formed by taking two alike and two different letters from the word "*COMBINATION*". **[*ISC 1992*]**

ANSWERS

1. 45 **2.** 35 **3.** (*i*) 165 (*ii*) 330

4. ${}^2C_1 \times {}^{12}C_{10} = {}^2C_1 \times {}^{12}C_2 = 132$ **5.** 336 **6.** (*i*) ${}^8C_2 - 8 = 20$ (*ii*) ${}^{10}C_2 - 10 = 35$

7. 1680 **8.** (*a*) ${}^7C_3 = 35$ (*b*) ${}^5C_2 \times {}^{12}C_9 = 2200$

9. (*i*) 336 (*ii*) 456 (*iii*) 462 (*iv*) 330 **10.** 198 ; 1353 **11.** 36

12. 10626, 25 **13.** 63 **14.** 90

15. 431 **16.** $\frac{20!}{8!7!5!}$ **17.** 60 ; 24

18. (*i*) 2454, (*ii*) 1422, (*iii*) (*a*) 53 (*b*) 758

19. (*i*) 2160 (*ii*) 2880 (*iii*) 576

20. 210 (*i*) 90 (*ii*) 120 (*iii*) 36 **21.** (*i*) 84 (*ii*) 74 **22.** (*i*) 336 (*ii*) 120

23. 2880 **24.** (*i*) 210 (*ii*) 57 **25.** (*i*) 240 (*ii*) 48

26. 246

HINTS TO SELECTED QUESTIONS

23. 5 gentlemen can seat in $(5-1)\,! = 4\,! = 24$ ways. The four ladies will occupy the places marked L so that no two ladies sit together. They can do so in ${}^5C_4 \times 4! = 5 \times 4! = 5! = 120$ ways.

$\therefore$ Reqd. number of ways $= 24 \times 120$.

26. The no. of committees is equal to

$({}^4C_1 \times {}^6C_4) + ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1)$

$= (4 \times 15) + (6 \times 20) + (4 \times 15) + (1 \times 6) = 60 + 120 + 60 + 6 = 246.$

Method II. ${}^{10}C_5 - {}^6C_5 = 246$. See solved Ex. 49.

27. (*i*) The no. of collections $= {}^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$.

(*ii*) The no. of collections

$= ({}^3C_1 + {}^3C_2 + {}^3C_3)\,({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4)\,({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$

$= (3 + 3 + 1)\,(4 + 6 + 4 + 1)\,(5 + 10 + 10 + 5 + 1) = 7 \times 15 \times 31 = 3255.$

28. There are 11 letters, not all different. They are as (*O, O*), (*I, I*), (*N, N*) ; *C, M, B, A, T.*

The number of combinations taking two alike and two different $= {}^3C_1 \times {}^7C_2 = 3 \times \dfrac{7 \times 6}{2 \times 1} = 63.$

Each of these combinations gives rise to $\dfrac{\lfloor 4}{\lfloor 2}$, *i.e.* 12 words.

Therefore 63 combinations give rise to 63×12, *i.e.* 756 words. (See also solved Ex. 54).

CHAPTER SUMMARY

1. Factorial notation: $n! = n(n-1)\,(n-2)\,...\,3.2.1.$

e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1$, $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

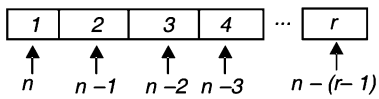
Special cases. $0! = 1$, $1! = 1$.

2. PERMUTATIONS

Case 1: Number of permutations without repetition

(*a*) Arranging n objects taken r at a time is equivalent to filling r places out of n things $= n(n-1)$

$(n-2)\,...\,(n-\overline{r-1})$ ways $= \dfrac{n!}{(n-r)!} = {}^nP_r$.

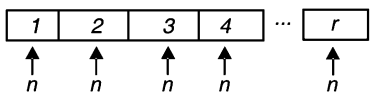


(*b*) **Special cases** (*i*) The number of arrangements of n different objects taken **all** at a time $= {}^nP_n = n!$

(*ii*) ${}^nP_0 = \dfrac{n!}{n!} = 1$; ${}^nP_r = n.\,{}^{n-1}P_{r-1}$.

Case 2: Number of permutations with repetition

(*a*) Number of permutations of n objects taken r at a time when each object may occur any number of times (once, twice, ... r times) is equivalent to the number of ways of filling r places when each place can be filled by any one of the n objects $= n.n.n.$ ... r times $= n^r$. For example, Number of numbers which can be formed out of the digits 1, 2, 5, 7 taken 3 at a time when each digit may be repeated $= 4^3 = 64$.



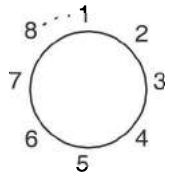
(*b*) **When some of the objects are identical**

The number of possible arrangements of r objects out of n objects when p are of one kind, q of another kind, r of third kind and so on = $\frac{n!}{p!q!r!...}$.

E.g. Number of words formed from the letters of the word 'BANANA' = $\frac{6!}{3!2!}$

[Total = 6 letters; 3A's and 2N'S]

Case 3: Circular permutations



In circular permutations what really matters is the position of an object relative to the others. Therefore, in circular permutations, the position of one of the objects is fixed and then the remaining objects are arranged in all possible ways.

There are two types of circular permutations.

(*i*) The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, *i.e.*, arrangements, *e.g.,* seating arrangements of people round a table.

(*ii*) The circular permutations in which clockwise and anticlockwise do not give rise to different arrangements, *e.g.,* arranging beads to form a necklace.

Theorems on circular permutations

Th. 1. The number of circular permutations of n different objects is $\boldsymbol{(n-1)!}$

Th. 2. The number of ways in which n persons can be seated around a table = $\boldsymbol{(n-1)!}$

Th. 3. The number of ways in which n different beads can be arranged to form a necklace $= \boldsymbol{\frac{1}{2}(n-1)!}$

3. COMBINATIONS

The process of selecting things is called **combination** and that of arranging things is called **permutation.** It is to be noted that while combination concerns only with selection, permutation concerns selection as well as arrangement of the selected things in all possible ways. Thus, ordering is immaterial in combination while it is essential in permutation.

The number of combinations of r different things out of n things is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$ and is given by

$$\boldsymbol{{}^nC_r = \frac{n(n-1)(n-2)\ldots(n-\overline{r-1})}{1.2.3\ldots r} = \frac{n!}{n!(n-r)!}}$$

Thus, ${}^6C_2 = \frac{6\times 5}{1\times 2}$, ${}^{15}C_4 = \frac{15\times 14\times 13\times 12}{1\times 2\times 3\times 4} = \frac{15!}{4!\,11!}$

It may be noted that

- ${}^nC_r$ is always a natural number.
- ${}^nC_0 = {}^nC_n = 1$
- ${}^nC_1 = n$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$. $(1 \le r \le n)$. ***Pascal's rule***

- If ${}^nC_x = {}^nC_y$, then, either $x = y$ or $x + y = n$.
- ${}^nC_r = \frac{n-r+1}{r} . {}^nC_{r-1}$
- ${}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}$
- $n.{}^{n-1}C_{r-1} = (n-r+1). {}^nC_{r-1}$

Conditional cases. Number of selections of r things out of n different things.

- when p particular things are always included $= {}^{n-p}C_{r-p}$
- when p particular things are excluded $= {}^{n-p}C_r$
- when p particular things are not together in any selection $= {}^{n-p}C_{r-p}$.

Geometrical cases.

- If n distinct points are given in the plane such that no three of them are collinear, then the number of line segments formed $= {}^nC_2$.
- If m of these points are collinear $(m \geq 3)$, then the number of line segments is $({}^nC_2 - {}^mC_2)+1$.
- The number of diagonals in an n-sided closed polygon $= {}^nC_2 - n$.
- If n distinct points are given on the circumference of a circle, then
  (a) Number of straight lines $= {}^nC_2$.
  (b) Number of triangles $= {}^nC_3$.
  (c) Number of quadrilaterals $= {}^nC_4$, and so on.

CHAPTER TEST

Short Answer Type Questions

1. Find the value of $P(7, 3)$.
2. How many numbers of 5 digits can be formed with the digits 0, 2, 5, 6, 7 without taking any of these digits more than once.
3. In how many ways can 8 persons sit in a round table. **[*Orissa JEE 2010*]**
4. A man has to post 5 letters and there are 4 letter boxes in the locality, in how many ways can he post the letters?
5. If ${}^4P_2 = n.{}^4C_2$, find the value of n.
6. A man has 6 friends. In how many ways may he invite one or more of them to dinner?
7. In how many ways 5 members forming a committee out of 10 be selected so that (i) 2 particular members must be included, (ii) 2 particular members must not be included.
8. Find the number of different words that can be formed from 12 consonants and 5 vowels by taking 4 consonants and 3 vowels in each word.
9. In an examination there are three multiple choice questions and each question has 4 choices. Find the number of ways in which a student can fail to get all answers correct.
   [*UPSEAT 2001, Pb CET 1992*]
10. Find the number of ways in which 12 apples may be equally divided among 3 childrens.

Multiple Choice Questions (MCQs)

11. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is.

(*a*) 3 (*b*) 36 (*c*) 66 (*d*) 108 **[*AIEEE 2010*]**

12. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups each containing 6 questions. He is not permitted to attempt more than 5 questions from each group. The number of ways in which he can choose the 7 questions is

(*a*) 780 (*b*) 640 (*c*) 820 (*d*) None of these **[*Punjab CET 2010*]**

13. The number of permutations by taking all letters and keeping the vowels of the word *COMBINE* in the odd places is.

(*a*) 96 (*b*) 144 (*c*) 512 (*d*) 576 **[*WBJEE 2010*]**

14. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800$, then, r is

(*a*) 40 (*b*) 51 (*c*) 101 (*d*) 410 (*e*) 41 **[*Kerala PET 2010*]**

15. For $2 \le r \le n$, ${}^{n}C_{r} + 2.{}^{n}C_{r-1} + {}^{n}C_{r-2} =$

(*a*) ${}^{n+1}C_{r-1}$ (*b*) $2.{}^{n+2}C_{r+1}$ (*c*) $2.{}^{n+2}C_{r}$ (*d*) ${}^{n+2}C_{r}$ **[*AMU (Engg.) 2010*]**

16. How many different words can be formed by jumbling the letters in the word *MISSISSIPPI* in which no two *S* are adjacent?

(*a*) $8.{}^{6}C_{4}.{}^{7}C_{4}$ (*b*) $6.7.{}^{8}C_{4}$ (*c*) $6.8.{}^{7}C_{4}$ (*d*) $7.{}^{6}C_{4}.{}^{8}C_{4}$ **[*AIEEE 2008*]**

17. The number of diagonals of a polygon of 20 sides is

(*a*) 210 (*b*) 190 (*c*) 180 (*d*) 170 **[*J & K CET 2009*]**

18. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(*a*) 6! × 5! (*b*) 30 (*c*) 5! × 4! (*d*) 7! × 5! **[*AIEEE 2003*]**

19. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then $n =$

(*a*) 5 (*b*) 7 (*c*) 6 (*d*) 4 **[*IIT 2001, AIEEE 2002*]**

20. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of r is

(*a*) 12 (*b*) 8 (*c*) 6 (*d*) 10 **[*Kerala PET 2002*]**

Long Answer Type Questions

21. A committee of 7 members has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

(*i*) exactly 3 girls (*b*) at least 3 girls (*c*) atmost three girls

22. In how many ways can the letters of the word *'PERMUTATIONS'* be arranged if the

(*i*) words start with *P* and end with *S*

(*ii*) vowels are all together

23. In how may different ways can the letters of the word *'SALOON'* be arranged

(*i*) if the two *O*'s must not come together?

(*ii*) if the consonants and vowels must occupy alternate places? **[*SC*]**

ANSWERS AND HINTS

1. ${}^7P_3 = 7 \times 6 \times 5 = 210$.

2. Since 0 cannot be placed at TTh place, so number of numbers = $4 \times 4 \times 3 \times 2 \times 1$.

3. Keeping 1 boy fixed, the reqd. number of ways = 7! = 5040.

4. $4 \times 4 \times 4 \times 4 \times 4$ ways = 1024.

5. $4 \times 3 = n.\dfrac{4 \times 3}{1 \times 2} \Rightarrow n = 2$

6. ${}^6C_1 + {}^6C_2 + \ldots {}^6C_6$

7. (*i*) ${}^8C_3$ (*ii*) ${}^8C_5$

8. ${}^{12}C_4 \times {}^5C_3 \times 7! = 24948000$.

9. $4^3 - 1 = 63$. Each question can be answered in 4 ways and all questions can be answered correctly in only one way.

10. $\dfrac{12!}{(4!)^3} = 34650$

11. (*d*) $({}^3C_2 \times {}^9C_2)$

12. (*a*) **[Hint.** Reqd. number of ways of choosing of questions from two groups of 6 questions each $= {}^6C_2 \times {}^6C_5 + {}^6C_3 \times {}^6C_4 + {}^6C_4 \times {}^6C_3 + {}^6C_5 \times {}^6C_2$].

13. (*d*) [**Hint.** Vowels are *O*, *I*, *E*. Number of odd places = 4.

$\therefore$ Number of ways $= {}^4P_3 \times 4!$]

14. (*e*) **[Hint.** $1 \times \dfrac{56!}{(50-r)!} = 30800 \times \dfrac{54!}{(51-r)!} \Rightarrow 51 - r = \dfrac{30800}{56 \times 55}$.]

15. (*d*) **[Hint.** Type Solved Ex. 43. ${}^nC_r + 2.{}^nC_{r-1} + {}^nC_{r-2} = ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$.]

16. (*d*) **[Hint.** Number of letters is $M \to 1$, $I \to 4$, $S \to 4$, $P \to 2$. Other than *S*, remaining 7 letters can be arranged in a row in $\dfrac{7!}{2!\,4!}$ ways. $\times M \times I \times I \times I \times I \times P \times P \times$

Since no two *S's* are adjacent, 4*S's* can be placed in 8 spaces in ${}^8C_4$ ways.

$\therefore$ Total number of required ways $= {}^8C_4 \cdot \dfrac{7!}{2!\,4!} = 7 \times {}^6C_4 \times {}^8C_4$.]

17. (*d*)

18. (*a*) **[Hint.** Same type as Q. 23 in Exercise 5 (*e*)]

19. (*b*) **[Hint.** To form a triangle, 3 out of *n* vertices of a regular polygon should be chosen. Hence, the number of triangles that can be formed $= T_n = {}^nC_3$

Given: $T_{n+1} - T_n = 21$. So ${}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow {}^nC_3 + {}^nC_2 - {}^nC_3 = 21 \left[\text{Using } {}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}\right] \Rightarrow {}^nC_2 = {}^7C_2 \Rightarrow n = 7$

20. (*a*)

21. (*i*) ${}^9C_4 \times {}^4C_3 = 504$ ways (*ii*) ${}^4C_3 \times {}^9C_4 + {}^4C_4 \times {}^9C_3 = 588$ ways.

(*iii*) 1632 **[Hint.** The ways of forming a committee in this case are (*a*) 7 boys and no girl (*b*) 6 boys and 1 girl (*c*) 5 boys and 2 girls (*d*) 4 boys and 3 girls].

22. (*i*) There are 10 letters between *P* and *S* with two *T'S*.

$\therefore$ Number of arrangements $= \dfrac{10!}{2!} = 5(9!) = 18,14,400$

(*ii*) The word 'PERMUTATIONS' has 12 letters in all with 5 vowels. Since vowels are to occur together, consider them as a single items (EUAIO).

Now, we have 8 items which include 2*T's*

$\therefore$ Number of arrangements = $\frac{8!}{2!} = 4(7!)$.

Also, the five vowels can be arranged in ${}^5P_5$, *i.e.*, 5! ways

$\therefore$ Total number of arrangements = $4(7!) \times 5! = 2419200$.

23. (*i*) Total number of arrangements = $\frac{6!}{2!} = 360$

Number of arrangements when 2 *O*'s must come together = 5! = 120

$\therefore$ Number of arrangements when 2 *O*'s must not come together = 360 – 120 = 240.

(*ii*) Number of arrangements in which vowels occupy odd places and consonants even places

$= \frac{3!}{2!} \times 3! = 18$

Number of arrangements in which vowels occupy even places and consonants odd places

$= \frac{3!}{2!} \times 3! = 18$

$\therefore$ Total number of arrangements in which consonants and vowels must occupy alternate places

$= 18 + 18 = 36$.

CHAPTER 13

BINOMIAL THEOREM

13.01. Binomial Expression

Definition. An expression consisting of two terms is called a *binomial expression, e.g.,* $x + a$, $2x + 3y$, $5x^2 - 6y^2$, $2x - \frac{1}{3x}$ are all binomial expressions.

13.02. Binomial Theorem

Binomial theorem helps us to expand any power of a given binomial expression. It was first established by Isaac Newton. First we take up the case when the index is a positive integer.

13.03. Development of Binomial Expansion

By actual multiplication, we may obtain the following expansions:

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2 = {}^2C_0\, x^2 + {}^2C_1\, xy + {}^2C_2\, y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = {}^3C_0\, x^3 + {}^3C_1\, x^2y + {}^3C_2 x\, y^2 + {}^3C_3\, y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$= {}^4C_0\, x^4 + {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 + {}^4C_3\, xy^3 + {}^4C_4 y^4$

$(x+y)^5 = x^5 + 5x^4y + 10\, x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$= {}^5C_0\, x^5 + {}^5C_1\, x^4y + {}^5C_2\, x^3y^2 + {}^5C_3\, x^2y^3 + {}^5C_4 x\, y^4 + {}^5C_5\, y^5.$

A careful observation of these expansions shows that $(x + y)^n$ (where n = 1, 2, 3, 4, 5), when expanded, has the following properties :

1. The coefficients form a certain pattern as shown below:

Pascal's Triangle

Number n	*Coefficients in the expansion of* $(x+y)^n$
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Inspection will show that each term in the table is derived by adding together the two terms in the line above, which lie on either side of it. Thus, in the line for $n = 5$, the term 10 is found by adding together the terms 4 and 6 in the line $n = 4$.

2. The first term in the expansion is (first term of the binomial)^{index}, *i.e.*, x^n, *i.e.*, ${}^nC_0\, x^n$.
3. The second term is $nx^{n-1}y$.
4. As the expansion progresses, the exponent of x decreases by one and the exponent of y increases by one.
5. The total number of terms in the expansion is one more than the index *i.e.*, $n + 1$.
6. The last term is the $(n + 1)$th term and is (second term of the binomial)^{index}, *i.e.*, y^n or ${}^nC_n y^n$.
7. The coefficients of the various terms in the expansion from the first term onwards are ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_n$.

Assuming that the above properties hold for all integral values of n, we may write the five terms in the expansion of $(x + y)^n$ as follows:

First term $= {}^nC_0\, x^n = x^n$

Second term $= {}^nC_1 x^{n-1} y = n\, x^{n-1} y$

Third term $= {}^nC_2 x^{n-2} y^2 = \dfrac{n(n-1)}{1\times 2} x^{n-2} y^2$

Fourth term $= {}^nC_3\, x^{n-3}\, y^3 = \dfrac{n(n-1)\,(n-2)}{1\times 2\times 3} x^{n-3} y^3$

Fifth term $= {}^nC_4\, x^{n-4}\, y^4 = \dfrac{n(n-1)\,(n-2)\,(n-3)}{1\times 2\times 3\times 4} x^{n-4} y^4$

Continuing the above process, it is easily seen that last but one term [*i.e.*, nth term] of the expansion $= {}^nC_{n-1} xy^{n-1} = nxy^{n-1}$

Last term [*i.e.*, $(n + 1)$ th term] $= {}^nC_n\, y^n = y^n$.

Thus, the expansion of $(x + y)^n$, where x and y are any two *real* numbers and n is any positive integer, may be stated as follows:

$$\mathbf{(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_r x^{n-r} y^r \ldots + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n y^n} \quad \ldots(1)$$

or $\quad \mathbf{x^n + nx^{n-1} y + \dfrac{n(n-1)}{1\times 2} x^{n-2} y^2 + \dfrac{n(n-1)(n-2)}{1\times 2\times 3} x^{n-3} y^3 + \ldots + y^n}.$

In short form : $\mathbf{(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r}.$

Note. Observe that the **number of terms in the expansion of** $(x + y)^n$ **is** $(n + 1)$, *i.e.*, **one more than the index.**

The numbers ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ etc. are called **binomial coefficients.**

Cor. Putting 1 for x and x for y

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1\times 2} x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3} x^3 + \ldots + x^n.$$

Deductions

1. Replacing y by $-y$, we get

$$(x - y)^n = {}^nC_0 . x^n - {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 - \ldots + (-1)^r\, {}^nC_r\, x^{n-r} y^r + \ldots + (-1)^n . {}^nC_n y^n \quad \ldots(2)$$

i.e., $(x - y)^n = \sum_{r=0}^{n} (-1)^r .{}^nC_r\, x^{n-r} . y^r$

2. Replacing x by 1 and y by x in (1), we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2 \,.\, x^2 + \ldots + {}^nC_r \,.\, x^r + \ldots + {}^nC_n \,.\, x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + x^n$$

$$i.e.,\ (1+x)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^r$$

3. Replacing x by 1 and y by $-x$ in (1), we get

$$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - \ldots + (-1)^r \,.\, {}^nC_r\, x^r + \ldots + (-1)^n \,.\, {}^nC_n\, x^n$$

$$= 1 - nx + \frac{n(n-1)}{2!}x^2 + \ldots + (-1)^n x^n$$

$$i.e.,\ (1-x)^n = \sum_{r=0}^{n} (-1)^r \,.\, {}^nC_r x^r$$

Thus, (1) The coefficient of $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(2) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

4. Adding (1) and (2), we get

$$(x+y)^n + (x-y)^n = 2[{}^nC_0\, x^n\, y^0 + {}^nC_2\, x^{n-2}\, y^2 + {}^nC_4\, x^{n-4}\, y^4 + \ldots]$$

= 2 [sum of terms at odd places]

The last term is ${}^nC_n \,.\, y^n$ or ${}^nC_{n-1} \,.\, xy^{n-1}$ according as n is even or odd respectively.

5. Subtracting (2) from (1), we get

$$(x+y)^n - (x-y)^n = 2[{}^nC_1\, x^{n-1}\, y + {}^nC_3\, x^{n-3}\, y^3 + \ldots] = \textbf{2 [sum of terms at even places]}$$

The last term is ${}^nC_{n-1}\ xy^{n-1}$ or ${}^nC_n\ y^n$ according as n is even or odd respectively.

Note : (*i*) If n is odd, then the number of terms in $[(x+y)^n + (x-y)^n]$ and $[(x+y)^n - (x-y)^n]$ equal to $\dfrac{n+1}{2}$.

(*ii*) If n is even, then the number of terms in $[(x+y)^n + (x-y)^n]$ is $\left(\dfrac{n}{2}+1\right)$ and in $[(x+y)^n - (x-y)^n]$ is $\dfrac{n}{2}$.

Note. Many Mathematicians have done their best work at an early age. One of the most celebrated of these was **Blaise Pascal** (1623 – 1662). He did not invent the triangular array of numbers that has been named after him. It was Pascal, however, who saw the relationship of the triangular array to the expansion of a binomial. He also found connections between the triangle and such problems as finding the number of different combinations possible from a given set of objects. Pascal made significant advances in algebra and was one of the founders of probability theory.

13.04. Proof of Binomial Theorem for Positive Integral Index

To prove that when n is a positive integer

$$\textbf{(}x+a\textbf{)}^n = {}^nC_0 x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + \ldots + {}^nC_r\, x^{n-r}a^r + \ldots + {}^nC_n\, a^n.$$

We shall prove it by the method of induction.

By actual multiplication, we have

$(x + a)^1 = {}^1C_0\, x^1 + {}^1C_1\, x^{1-1}\, a^1$

$(x + a)^2 = x^2 + 2ax + a^2 = {}^2C_0\, x^2 + {}^2C_1\, xa + {}^2C_2\, a^2$

$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 = {}^3C_0\, x^3 + {}^3C_1\, x^2a + {}^3C_2\, xa^2 + {}^3C_3\, a^3$.

Thus, we notice that the theorem is true for $n = 1$, $n = 2$ and $n = 3$.

Let us *assume* that it is true for a particular integral power $n = m$, say. Then,

$$(x + a)^m = {}^mC_0\, x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a^2 + \ldots + {}^mC_r x^{m-r}\, a^r + \ldots + {}^mC_m\, a^m \quad \ldots (1)$$

Multiplying both sides by $(x + a)$, we get

$$(x + a)^{m+1} = {}^mC_0\, x^{m+1} + {}^mC_1 x^m a + {}^mC_2\, x^{m-1}\, a^2 + \ldots + {}^mC_r x^{m-r+1}\, a^r + \ldots + {}^mC_m xa^m$$
$$+ C_0\, x^m a + {}^mC_1\, x^{m-1}\, a^2 + \ldots + {}^mC_{r-1}\, x^{m-r+1} a^r + \ldots + {}^mC_{m-1}\, xa^m + {}^mC_m\, a^{m+1}$$

$$\therefore \quad (x + a)^{m+1} = {}^{m+1}C_0 x^{m+1} + {}^{m+1}C_1\, x^m\, a + {}^{m+1}C_2\, x^{m-1}\, a^2 + \ldots + {}^{m+1}C_r\, x^{m+1-r}\, a^r$$
$$+ \ldots + {}^{m+1}C_m x\, a^m + {}^{m+1}C_{m+1}\, a^{m+1} \quad \ldots (2)$$

$[\because {}^mC_0 = {}^{m+1}C_0,\ {}^mC_m = {}^{m+1}C_{m+1},\ {}^{m+1}C_r = {}^mC_r + {}^mC_{r-1}]$.

We notice that the successive terms in (2) are of the same form as in (1) except that m is changed into $m + 1$.

Therefore, if the theorem is true for $n = m$, it is also true for $n = m + 1$. But the theorem is true for $n = 3$.

Therefore, it is true for $n = 3 + 1$, *i.e.* 4. If it is true for $n = 4$, it is also true for $n = 4 + 1$, *i.e.* 5. Continuing this reasoning it follows that the theorem is true for $n = 6, 7, 8$, etc. *i.e.,* true for all positive integral values of n.

Ex. 1. Find the number of terms in the following expansions.

(*i*) $(2x - 3y)^5$ **(*ii*)** $\left(5x - \frac{1}{x^3}\right)^{17}$ **(*iii*)** $(1 + 6x + 9x^2)^{23}$

(*iv*) $\left(\sqrt{3} + \sqrt{5}\right)^7 + \left(\sqrt{3} - \sqrt{5}\right)^7$ **(*v*)** $(3x + 7)^8 + (3x - 7)^8$

(*vi*) $\left(1 + 3\sqrt{5}\, x\right)^9 - \left(1 - 3\sqrt{5}\, x\right)^9$ **(*vii*)** $(x + y)^{100} + (x - y)^{100}$

[*Karnataka CET 2009*]

Sol. (*i*) The number of terms in the expansion of $(2x - 3y)^5 = 5 + 1 = 6$.

(*ii*) The number of terms in the expansion of $\left(5x - \frac{1}{x^3}\right)^{17} = 17 + 1 = 18$.

(*iii*) $\left(1 + 6x + 9x^2\right)^{23} = \left[(1 + 3x)^2\right]^{23} = (1 + 3x)^{46}$

$\therefore$ Number of terms in the expansion of $(1 + 6x + 9x^2)^{23}$ is $46 + 1 = 47$.

(*iv*) If n is odd, then the expansion of $(a + b)^n + (a - b)^n$ contains $\frac{n+1}{2}$ terms. So, the number of terms in the expansion of $\left(\sqrt{3} + \sqrt{5}\right)^7 + \left(\sqrt{3} - \sqrt{5}\right)^7$ is $\frac{7+1}{2}$, *i.e.*, 4.

(*v*) If n is even, then the expansion of $(a + b)^n + (a - b)^n$ contains $\left(\frac{n}{2} + 1\right)$ terms. So, the number of terms in the expansion of $(3x + 7)^8 + (3x - 7)^8$ is $\frac{8}{2} + 1$, *i.e.,* 5.

(*vi*) If n is odd, then the expansion of $(a+b)^n-(a-b)^n$ contains $\frac{n+1}{2}$ terms. So, the number of terms in the expansion of $(1+3\sqrt{5}\,x)^9-\left(1-3\sqrt{5}\,x\right)^9$ is $\frac{9+1}{2}$, *i.e.,* 5.

(*vii*) Since n is even, the number of terms in the expansion of $(x+y)^{100}+(x-y)^{100}$

$= \frac{n}{2}+1=\frac{100}{2}+1=$ **51**.

Ex. 2. Expand $(1+4x)^5$.

Sol. $(1+4x)^5 = 1+{}^5C_1 4x+{}^5C_2(4x)^2+{}^5C_3(4x)^3+{}^5C_4(4x)^4+(4x)^5$

$$= 1+5\times 4x+\frac{5\times 4}{1\times 2}\times 16x^2+\frac{5\times 4}{1\times 2}\times 64x^3+5\times 256x^4+1024x^5$$

$[\because\ {}^5C_3={}^5C_2;\ {}^5C_4={}^5C_1]$

$$= 1+20x+160x^2+640x^3+1280x^4+1024x^5.$$

Ex. 3. Expand $(3x-2y)^4$.

Sol. $(3x-2y)^4=(3x)^4+{}^4C_1\times(3x)^3\times(-2y)+{}^4C_2\times(3x)^2\times(-2y)^2+{}^4C_3(3x)(-2y)^3+(-2y)^4$

$$= 81x^4-4\times 27\times 2\times x^3y+\frac{4\times 3}{2\times 1}\times 9x^2\times 4y^2-4\times 3x\times 8y^3+16y^4$$

$$= 81x^4-216x^3y+216x^2y^2-96xy^3+16y^4.$$

Ex. 4. Expand $\left(x+\frac{1}{x}\right)^6$. $(x\neq 0)$

Sol. $\left(x+\frac{1}{x}\right)^6 = x^6+{}^6C_1\times x^5\times\frac{1}{x}+{}^6C_2\times x^4\times\frac{1}{x^2}$

$$+{}^6C_3\times x^3\times\frac{1}{x^3}+{}^6C_4\times x^2\times\frac{1}{x^4}+{}^6C_5\times x\times\frac{1}{x^5}+\frac{1}{x^6}$$

$$= x^6+6x^4+\frac{6\times 5}{2\times 1}x^2+\frac{6\times 5\times 4}{1\times 2\times 3}+\frac{6\times 5}{1\times 2}\times\frac{1}{x^2}+6\times\frac{1}{x^4}+\frac{1}{x^6}$$

$[\because\ {}^6C_4={}^6C_2,\ {}^6C_5={}^6C_1]$

$$= x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}.$$

Ex. 5. Find the value of $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$ and show that the value of $(\sqrt{2}+1)^6$ lies between 197 and 198. ***[ISC, IIT]***

Sol. We know that $(x+a)^n+(x-a)^n = 2\left[{}^nC_0\,x^n a^0+{}^nC_2x^{n-2}a^2+\ldots\right]$. Here, two binomials are given to us, which are raised to the same power and whose second terms are numerically equal but of opposite sign. On expanding the two, it is obvious that the second, fourth and sixth terms containing odd powers of 1 and –1 will cancel out and the remaining will be added up. Thus,

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2\left[(\sqrt{2})^6+{}^6C_2\,(\sqrt{2})^4+{}^6C_4(\sqrt{2})^2+1\right]$$

$$= 2\left(8+\frac{6\times 5}{1\times 2}\times 4+\frac{6\times 5}{1\times 2}\times 2+1\right) = 2\,(8+60+30+1) = 2\times 99 = 198.$$

Also, $\because (\sqrt{2}-1)^6 = (1.42-1)^6 = (0.42)^6 < 1$

$\therefore (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 198 \Rightarrow (\sqrt{2}+1)^6 = 198 - (\sqrt{2}-1)^6$

$\Rightarrow (\sqrt{2}+1)^6 = 198 -$ A number between 0 and 1

$\Rightarrow (\sqrt{2}+1)^6 =$ A number between 197 and 198.

$\Rightarrow$ Integral part of $(\sqrt{2}+1)^6$ is 197.

Ex. 6. Use the binomial theorem to find the exact value of $(10.1)^5$. **[*SC*]**

Sol. In such questions, first we express the given quantity as a binomial and then expand using the rules of binomial expansion. Thus, $(10.1)^5 = (10 + 0.1)^5$

$= (10)^5 + {}^5C_1 \times (10)^4 \times (0.1) + {}^5C_2 \times (10^3) \times (0.1)^2 + {}^5C_3 \times (10)^2 \times (0.1)^3 + {}^5C_4 \times 10 \times (0.1)^4 + (0.1)^5$

$= 100000 + 5 \times 10\,000 \times 0.1 + \frac{5\times 4}{1\times 2} \times 1000 \times 0.01 + \frac{5\times 4}{1\times 2} \times 100 \times 0.001 + 5 \times 10 \times 0.0001 + 0.00001$

$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501.$

Ex. 7. Expand $(2 + x + x^2)^3$. **[*SC*]**

Sol. Regarding $(2 + x)$ as one term, let $2 + x = y$. Then,

$[2 + x + x^2]^3 = (y + x^2)^3 = {}^3C_0\, y^3 + {}^3C_1\, y^2 x^2 + {}^3C_2\, y^1 (x^2)^2 + {}^3C_3\, (x^2)^3$

$= y^3 + 3y^2 x^2 + 3yx^4 + x^6$

$= (2 + x)^3 + 3(2 + x)^2 x^2 + 3(2 + x) x^4 + x^6$

$= 2^3 + 3 \times 2^2 \times x + 3 \times 2x^2 + x^3 + 3x^2 (4 + 4x + x^2) + 3x^4 (2 + x) + x^6$

$= 8 + 12x + 18x^2 + 13x^3 + 9x^4 + 3x^5 + x^6.$

Ex. 8. Prove that $\sum_{r=0}^{n} 3^r . {}^nC_r = 4^n$.

Sol. $\sum_{r=0}^{n} 3^r . {}^nC_r = 3^0 . {}^nC_0 + 3^1 . {}^nC_1 + 3^2 . {}^nC_2 + ... + 3^n . {}^nC_n$

$= 1 + {}^nC_1 (3)^1 + {}^nC_2 (3)^2 + {}^nC_3 (3)^3 + ... + {}^nC_n (3)^n$

$= (1 + 3)^n = 4^n.$

Ex. 9. $\frac{{}^8C_0}{6} - {}^8C_1 + {}^8C_2 .6 - {}^8C_3 .6^2 + ... + {}^8C_8 .6^7$ is equal to

(*a*) 0 **(*b*) 6^7** **(*c*) 6^8** **(*d*) $\frac{5^8}{6}$** **[*J&K CET 2009*]**

Sol. $\frac{{}^8C_0}{6} - {}^8C_1 + {}^8C_2 .6 - {}^8C_3 .6^2 + ... + {}^8C_8\, 6^7$

$= \frac{1}{6}\left[{}^8C_0 - 6 . {}^8C_1 + 6^2 . {}^8C_2 - 6^3 . {}^8C_3 + ... + 6^8 . {}^8C_8 \right]$

$= \frac{1}{6}(1-6)^8 = \frac{5^8}{6}.$ **Ans.** (*d*)

EXERCISE 13 (*a*)

1. What is the number of terms in the expansion of each of the following?

(*i*) $[(x-2y)^3]^3$ **[*SC*]** (*ii*) $(5a+7b)^8$ (*iii*) $\left(6x-\frac{1}{x^3}\right)^{17}$

(*iv*) $\left(4x^2+12xy+9y^2\right)^9$ (*v*) $\left(3+2\sqrt{5}\right)^{18}-\left(3-2\sqrt{5}\right)^{18}$

(*vi*) $(5+7x)^{15}+(5-7x)^{15}$ (*vii*) $(a+bx)^{17}-(a-bx)^{17}$

2. Write out the expansions of the following:

(*a*) $(3x-y)^4$ (*b*) $(3+2x^2)^4$ (*c*) $\left(x-\frac{y}{2}\right)^4$

(*d*) $\left(2x+\frac{y}{2}\right)^5$ (*e*) $(1+2x)^7$ (*f*) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $x \neq 0$

3. Using binomial theorem, expand $[(x+y)^5+(x-y)^5]$ and hence find the value of $[(\sqrt{3}+1)^5-(\sqrt{3}-1)^5]$.

4. Expand $(2+x)^5-(2-x)^5$ in ascending powers of x and simplify your result. **[*SC*]**

5. Evaluate the following:

(*i*) $(2+\sqrt{5})^5+(2-\sqrt{5})^5$ (*i*) $(\sqrt{3}+1)^5-(\sqrt{3}-1)^5$

Hence, show in (*ii*), without using tables, that the value of $(\sqrt{3}+1)^5$ lies between 152 and 153. **[*ISC 1989 Type*]**

6. If the first three terms in the expansion of $(1+ax)^n$ in ascending powers of x are $1+12x+64x^2$, find n and a. **[*SC*]**

7. Find the first three terms in the expansion of $[2+x(3+4x)]^5$ in ascending powers of x.

8. Expand $(1+2x+3x^2)^n$ in a series of ascending powers of x up to and including the term in x^2. **[*SC*]**

9. Write down the expansion by the binomial theorem of $\left(3x-\frac{y}{2}\right)^4$. By giving x and y suitable values, deduce the value of $(29.5)^4$ correct to four significant figures. **[*SC*]**

10. Using binomial theorem, evaluate : $(999)^3$. **[*ISC*]**

11. Write down in terms of x and n, the term containing x^3 in the expansion of $\left(1-\frac{x}{n}\right)^n$ by the binomial theorem. If this term equals $\frac{7}{8}$ when $x=-2$, and n is a positive integer, calculate the value of n. **[*ISC*]**

12. (*i*) Obtain the binomial expansion of $(2-\sqrt{3})^6$ in the form $a+b\sqrt{3}$, where a and b are integers. State the corresponding result for the expansion $(2+\sqrt{3})^6$ and (*ii*) show that $(2-\sqrt{3})^6$ is the reciprocal of $(2+\sqrt{3})^6$. **[*ISC*]**

13. Find the coefficient of x^5 in the expansion of $(1+2x)^6(1-x)^7$.

14. If the coefficients of second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in A. P., show that $2n^2-9n+7=0$. **[*MNR 1999, AMU 2001, IIT*]**

15. Let n be a positive integer. If the coefficients of 2nd, 3rd, 4th terms in the expansion of $(1 + x)^n$ are in A.P., then find the value of n. **[*Raj. PET 2002, UPSEAT 2002, IIT 1994*]**

16. In the binomial expansion of $(\sqrt[3]{3} + \sqrt{2})^5$ find the term which does not contain Irrational expression. **[*ISC 2002*]**

ANSWERS

1. (*i*) 10 (*ii*) 9 (*iii*) 18 (*iv*) 19 (*v*) 9 (*vi*) 8 (*vii*) 9

2. (*a*) $81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$ (*b*) $81 + 216x^2 + 216x^4 + 96x^6 + 16x^8$

(*c*) $x^4 - 2x^3y + \frac{3}{2}x^2y^2 - \frac{1}{2}xy^3 + \frac{1}{16}y^4$

(*d*) $32x^5 + 40x^4y + 20x^3y^2 + 5x^2y^3 + \frac{5}{8}xy^4 + \frac{1}{32}y^5$

(*e*) $1 + 14x + 84x^2 + 280x^3 + 560x^4 + 672x^5 + 448x^6 + 128x^7$

(*f*) $32x^{-5} - 40x^{-3} + 20x^{-1} - 5x + \frac{5}{8}x^3 - \frac{1}{32}x^5$ **3.** 152 **4.** $160x + 80x^3 + 2x^5$

5. (*i*) 1364 (*ii*) 152 **6.** 9, $\frac{4}{3}$ **7.** $32 + 240x + 1040x^2$ **8.** $1 + 2nx + x^2(2n^2 + n)$

9. $81x^4 - 54x^3y + \frac{27}{2}x^2y^2 - \frac{3}{2}xy^3 + \frac{y^4}{16}$; 757300 correct to 4 s.f. **10.** 997002999

11. $\frac{-(n-1)(n-2)}{6n^2}.x^3$; $n = 8$ **12.** $1351 - 780\sqrt{3}$, $1351 + 780\sqrt{3}$

13. 171 **15.** 2 **16.** $T_3 = 60$.

HINTS TO SELECTED QUESTIONS

3. $(x + y)^5 + (x - y)^5 = 2\left[{}^5C_0 x^5 y + {}^5C_2 x^{5-2}y^2 + {}^5C_4 x^{5-4}y^4\right]$

Put $x = \sqrt{3}$ and $y = 1$ in the expansion obtained above.

5. Type Solved Ex. 5.

12. (*ii*) Show that $(2+\sqrt{3})^6 (2-\sqrt{3})^6 = 1$.

13. $(1 + 2x)^6 (1 - x)^7 = [1 + {}^6C_1.(2x) + {}^6C_2.(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + (2x)^6]$
$\times [1 - {}^7C_1 x + {}^7C_2 x^2 - {}^7C_3 x^3 + {}^7C_4 x^4 - {}^7C_5 x^5 + {}^7C_6 x^6 - x^7]$
$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)$
$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$

Terms containing x^5 in the product are obtained on multiplying constant term by the term containing x^5, term containing x by the term containing x^4, and so on.

These products are $-21x^5 + 420x^5 - 2100x^5 + 3360x^5 - 1680x^5 + 192x^5$

$\therefore$ Coefft. of $x^5 = (-21 + 420 - 2100 + 3360 - 1680 + 192) = 171$.

14. $T_2 = {}^{2n}C_1 x$, $T_3 = {}^{2n}C_2.x^2$, $T_4 = {}^{2n}C_3 x^3$. These being in A.P.
$2.{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3 \Rightarrow 2n^2 - 9n + 7 = 0$.

15. $n^2 - 9n + 14 = 0 \Rightarrow n = 7$, rejecting the other value $n = 2$ because when $n = 2$, then there are only three terms in the expansion of $(1 + x)^2$.

16. Write the expansion of $(\sqrt[3]{3} + \sqrt{2})^5$ and find out the term which does not contain any irrational term.

13.05. General Term T_{r+1}

Suppose we have to find the $(r+1)^{th}$ term in the expansion of $(x+y)^n$. In its expansion

$$T_3 = {}^nC_2 \times x^{n-2}y^2,\ T_4 = {}^nC_3\, x^{n-3}y^3,\ T_5 = {}^nC_4\, x^{n-4}y^4.$$

We can see that all these terms have the following common properties:

(*i*) The suffix of C is one less than the number of the term.

(*ii*) The power of the first factor x is equal to the difference of the upper and lower suffixes of C.

(*iii*) The power of the second factor y and the suffix of C are the same.

(*iv*) The sum of the powers of x and y is equal to the index of the given binomial.

Hence, the $(r+1)^{th}$ term, T_{r+1} is given by

$$\boldsymbol{T_{r+1} = {}^nC_r\, x^{n-r}\, y^r}$$

Note. In the expansion of $(x-y)^n$, $T_{r+1} = {}^nC_r \times x^{n-r} \times (-y)^r = (-1)^r \times {}^nC_r\, x^{n-r}\, y^r$.

Ex. 10. Find the tenth term in the expansion of $(2x-y)^{11}$.

Sol. $T_{10} = T_{9+1} = {}^{11}C_9 (2x)^{11-9}(-y)^9 = {}^{11}C_2\,(2x)^2(-y)^9$

$$= -\frac{11\times 10}{1\times 2} \times 4x^2y^9 = -220x^2y^9.$$

Middle Term

Ex. 11. Write the middle term or terms in the expansion of

(*i*) $\left(x^2 - \frac{1}{x}\right)^6$ **(*ii*)** $\left(3a - \frac{a^3}{6}\right)^9$

Sol. (*i*) There are 7 terms in the expansion and so the 4th term will be the middle term.

$$T_4 = {}^6C_3 (x^2)^{6-3} \times \left(-\frac{1}{x}\right)^3 = -20x^3.$$

(*ii*) There are 10 terms in the expansion and so the 5th and the 6th terms will be the middle terms.

$$T_5 = {}^9C_4 \times (3a)^5 \times \left(-\frac{a^3}{6}\right)^4 = \frac{189a^{17}}{8};\ T_6 = {}^9C_5 \times (3a)^4 \times \left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}a^{19}.$$

Term from the end

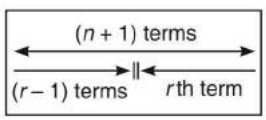
Ex. 12. (*i*) Find the *r*th term from the end in $(x+a)^n$.

(*ii*) Find the 4th term from the end in the expansion of $\left(\frac{3}{x^2} - \frac{x^3}{6}\right)^7$.

Sol. The expansion has $(n+1)$ terms.

*r*th term from the end = {$(n+1) - (r-1)$th term} from the begining

= $(n-r+2)$th term from the begining



Hence, *r*th term from the end = $T_{n-r+2} = {}^nC_{n-r+1}.(x)^{n-(n-r+1)}.a^{n-r+1}$

$$= \boldsymbol{{}^nC_{n-r+1}.x^{r-1}.a^{n-r+1}}.$$

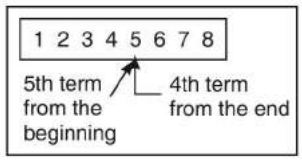
(*ii*) 4th term from the end = $(8-4+1)$th

= 5th term from the beginning

$$T_5 = T_{4+1} = {}^7C_4 \cdot \left(\frac{3}{x^2}\right)^{7-4} \cdot \left(\frac{-x^3}{6}\right)^4$$

$$= {}^7C_3 \cdot \left(\frac{3}{x^2}\right)^3 \left(\frac{x^3}{6}\right)^4 \qquad \left(\because {}^7C_4 = {}^7C_3\right).$$

$$= \frac{7\times6\times5}{3\times2\times1} \cdot \frac{3^3}{x^6} \cdot \frac{x^{12}}{3^4.2^4} = \mathbf{\frac{35}{48}x^6}.$$



Coefficient of a particular power of x:

Procedure. 1. *Let the particular power occur in the* $(r+1)^{th}$ *term.*

2. *Write the* $(r+1)$ *th term of the given binomial.*

3. *Equate the power of x in the* $(r+1)$ *th term and the given power.*

4. *Evaluate r and substitute it in* (2).

Note. The value of r should be a positive integer less than the index of the binomial.

Ex. 13. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

Sol. Let x^{15} occur in the $(r+1)$th term. Then

$$T_{r+1} = {}^{10}C_r\, x^{10-r} \times (-x^2)^r = {}^{10}C_r\, x^{10-r} \times x^{2r} \times (-1)^r = (-1)^r \times {}^{10}C_r x^{10+r}$$

$\therefore \quad x^{10+r} = x^{15} \quad \therefore \quad 10 + r = 15 \quad \therefore r = 5$

$\therefore$ The required coefficient $= {}^{10}C_5\,(-1)^5$

$$= \frac{-10\times9\times8\times7\times6}{1\times2\times3\times4\times5} = -252.$$

Ex. 14. If for positive integers $r > 1$, $n > 2$, the coefficients of the $(3r)$th and $(r+2)$th powers of x in the expansion of $(1+x)^{2n}$ are equal, then prove that $n = 2r + 1$.

[*DCE 2000, AIEEE 2002, UPSEAT 2002, IIT, BITS, Kurushetra CEE*]

Sol. We have, $T_{3r} = {}^{2n}C_{3r-1}\, x^{3r-1}$, $T_{r+2} = {}^{2n}C_{r+1}\, x^{r+1}$

Given, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\therefore$ Either $3r - 1 = r + 1$ or $3r - 1 + (r+1) = 2n$

$\Rightarrow r = 1$ or $r = \frac{n}{2}$ $\therefore r = \frac{n}{2}$. But r is a positive integer greater than 1, so rejecting $r = 1$, we have,

$r = \frac{n}{2}$, provided n is an even integer (> 2), otherwise r has no value.

Term Independent of x:

Procedure. 1. *Let* $(r+1)$*th term be the term independent of x.*

2. *Put the power of x in this term equal to zero and evaluate r.*

Ex. 15. Find the term independent of x in the expansion of

$$\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9.$$

Sol. Let $(r+1)$th term be the term independent *of x.*

Then, $T_{r+1} = {}^9C_r \times \left(\frac{3}{2}x^2\right)^{9-r} \times \left(-\frac{1}{3x}\right)^r = (-1)^r \times {}^9C_r \times \frac{3^{9-2r}}{2^{9-r}} \times x^{18-3r}$

Putting $18 - 3r = 0 \Rightarrow r = 6$.

$\therefore$ The required term $= (-1)^6 \times {}^9C_6 \times \dfrac{3^{-3}}{2^3} = {}^9C_3 \times \dfrac{1}{3^3 \times 2^3} = \dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} \times \dfrac{1}{27 \times 8} = \dfrac{7}{18}$.

Ex. 16. The 2nd, 3rd and 4th terms in the expansion of $(x + y)^n$ are 240, 720 and 1080 respectively; find the values of, x, y and n. ***[DCE 1995, 2001]***

Sol. $nx^{n-1}y = 240$, $\dfrac{n(n-1)}{1.2}x^{n-2}y^2 = 720$ and $\dfrac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 = 1080$

Dividing, $\dfrac{n-2}{3}.\dfrac{y}{x} = \dfrac{3}{2}$ and $\dfrac{n-1}{2}.\dfrac{y}{x} = 3$ $\therefore$ $\dfrac{2(n-2)}{3} = \dfrac{n-1}{2} \Rightarrow n = 5$

Substituting the value of n, we get $x = 2$, $y = 3$.

EXERCISE 13 (*b*)

1. Find the specified term of the expression in each of the following binomials:

(*i*) Fifth term of $(2a + 3b)^{12}$. Evaluate it when $a = \dfrac{1}{3}, b = \dfrac{1}{4}$.

(*ii*) Sixth term of $\left(2x - \dfrac{1}{x^2}\right)^7$. **[*ISC 2000*]**

(*iii*) Middle term of $\left(2x - \dfrac{1}{y}\right)^8$. (*iv*) Middle term of $\left(x^4 - \dfrac{1}{x^3}\right)^{11}$.

(*v*) Middle term of $\left(\dfrac{x^2}{4} - \dfrac{4}{x^2}\right)^{10}$ **[*ISC 2003*]**

2. Find the term independent of x in the expansion of the following binomials:

(*i*) $\left(x - \dfrac{1}{x}\right)^{14}$ (*ii*) $\left(\sqrt{\dfrac{x}{3}} - \dfrac{\sqrt{3}}{2x}\right)^{12}$ **[*ISC 1990*]**

(*iii*) $\left(2x^2 - \dfrac{1}{x}\right)^{12}$. What is its value ? **[*SC*]**

3. Find the coefficient of

(*i*) a^6b^3 in the expansion of $\left(2a - \dfrac{b}{3}\right)^9$ **[*SC*]** (*ii*) x^7 in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{11}$

(*iii*) $\dfrac{1}{x^{17}}$ in the expansion of $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$. (*iv*) x^4 in the expansion of $\left(\dfrac{x}{2} - \dfrac{3}{x^2}\right)^{10}$.

[DCE 2000, Raj. PET 2001, UPSEAT 2002, IIT, EAMCET]

4. If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are the same, find the value of a. **[*SC*]**

5. Write down the fourth term in the binomial expansion of $\left(px + \dfrac{1}{x}\right)^n$. If this term is independent of x, find the value of n. With this value of n, calculate the value of p given that the fourth term is equal to $\dfrac{5}{2}$. **[*ISC*]**

6. The expansion by the binomial theorem of $\left(2x+\frac{1}{8}\right)^{10}$ is $1024x^{10} + 640x^9 + ax^8 + bx^7 + ...$
Calculate
(*i*) the numerical value of a and b; (*ii*) coefficient of x^8 in $(3x - 2)\left(2x+\frac{1}{8}\right)^{10}$;
(*iii*) the value of x, for which the third and the fourth terms in the expansion of $\left(2x+\frac{1}{8}\right)^{10}$ are equal. **[*ISC*]**

7. Find the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in $\left(ax+\frac{1}{bx^2}\right)^{11}$. If these coefficients are equal, find the relation between a and b. **[*AIEEE 2005, ISC*]**

8. In a binomial expansion, $(x + a)^n$, the first three terms are 1, 56 and 1372 respectively. Find values of x and a.

9. Write the 4th term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$.

10. The coefficients of $(2r + 1)$ th and $(r + 2)$th terms in the expansions of $(1 + x)^{43}$ are equal. Find the value of r.

11. The coefficint of the middle term in the binomial expansion in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals
(*a*) $\frac{-3}{10}$ (*b*) $\frac{10}{3}$ (*c*) $\frac{-5}{3}$ (*d*) $\frac{3}{5}$ ***[AIEEE 2004]***

12. Find the sixth term of the expansion of $(y^{1/2} + x^{1/3})^n$, if the binomial coefficient of the third term from the end is 45. **[*ISC 1999 Type, 1992*]**

13. Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

14. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1.3.5 \ldots (2n-1)}{n!} . 2^n x^n$. where $n \in N$.

15. Find the coefficient of x^5 in the expansion of $1 + (1 + x) + (1 + x)^2 + ... + (1 + x)^{10}$.

16. If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, prove that its coefficient is $\frac{(2n)!}{\left[\frac{1}{3}(4n-p)!\right]\left[\frac{1}{3}(2n+p)!\right]}$

17. If P be the sum of odd terms and Q be the sum of even terms in the expansion of $(x + a)^n$, prove that (*i*) $P^2 - Q^2 = (x^2 - a^2)^n$, (*ii*) $4PQ = (x + a)^{2n} - (x - a)^{2n}$ and (*iii*) $2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$. **[*Roorkee*]**

18. If the coeffts. of the rth, $(r + 1)$th and $(r + 2)$th terms in the expansion of $(1 + x)^n$ are in *A.P.*, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$. **[*IIT*]**

19. In the expansion of $\left(x^2+\frac{1}{x}\right)^n$, the coefficient of the fourth term is equal to the coefficient of the ninth term. Find n and the sixth term of the expansion. **[*ISC 2000*]**

20. The cofft. of x^n in the expansion of $(1 + x)(1 - x)^n$ is
(*a*) $(-1)^{n-1} . (n - 1)^2$ (*b*) $(-1)^n (1 - n)$ (*c*) $n - 1$ (*d*) $(-1)^{n-1}.n$ **[*AIEEE 2004*]**

ANSWERS

1. (*i*) $\frac{55}{9}$ (*ii*) $\frac{-84}{x^8}$ (*iii*) $\frac{1120x^4}{y^4}$ (*iv*) $-462x^9, 462x^2$ (*v*) -252

2. (*i*) $T_8 = -3432$, (*ii*) $T_5 = \frac{55}{16}$, (*iii*) $T_9 = 7920$

3. (*i*) $\frac{-1792}{9}$ (*ii*) 462 (*iii*) -1365 (*iv*) $\frac{405}{256}$. **4.** $\frac{9}{7}$

5. ${}^nC_3P^{n-3}\,x^{n-6}, n = 6, p = \frac{1}{2}$ **6.** (*i*) $a = 180, b = 30$, (*ii*) -270, (*iii*) $x = \frac{1}{6}$

7. Hint. ${}^{11}C_5\frac{a^6}{b^5} = {}^{11}C_6\frac{a^5}{b^6} \Rightarrow ab = 1$ **8.** $n = 8, x = 1, a = 7$ **9.** $\frac{672}{x^3}$

10. 14 **11.** (*a*) **12.** $252y^{5/2}\,x^{5/3}$ **15.** 462 **19.** $n = 11$; $T_6 = 462\,x^7$ **20.** (*b*)

HINTS TO SELECTED QUESTIONS

8. Type Solved Ex. 16.

9. Here, 4th term from the end = (10 – 4 + 1)th, *i.e.,* 7th term from the beginning.

11. The expansion of $(1 + ax)^4$ contains 5 terms, so 3rd term will be the middle term. Similarly, in the expansion of $(1 - \alpha x)^6$, 4th term will be the middle term.

Since the coeffts. in the two terms are given to be equal, we have

${}^4C_2 \cdot \alpha^2 = {}^6C_3 \cdot (-\alpha)^3 \Rightarrow 6 = -20\alpha \Rightarrow \alpha = \frac{-3}{10}$.

12. The third term from the end is equal to $\{(n + 1) - 3 + 1\}^{th}$, *i.e.* $(n - 1)^{th}$ term from the beginning.

13. The expansion of $(1 + x)^{2n}$ contains $(2n + 1)$ terms and therefore, the middle term is the $(n + 1)$th term.

$\therefore$ Middle term in the expansion of $(1 + x)^{2n}$ is given by

$$T_{n+1} = {}^{2n}C_n \cdot x^n = \frac{(2n)!}{(n!)\,.\,(n!)} \cdot x^n = \frac{(2n)!}{(n!)^2} \cdot x^n$$

$\therefore$ Coefft.of middle term of $(1 + x)^{2n}$ is $\frac{(2n)!}{(n!)^2}$. ...(1)

Now, the expansion of $(1 + x)^{2n-1}$ contains $2n$ terms. So, it has two middle terms, *viz.* the nth and the $(n + 1)$th terms.

$$T_n = {}^{2n-1}C_{n-1} \cdot x^{n-1},\ T_{n+1} = {}^{2n-1}C_n \cdot x^n$$

$\therefore$ Sum of the coeffts. of middle terms of $(1 + x)^{2n-1} = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$= {}^{2n-1+1}C_n = {}^{2n}C_n = \frac{(2n)!}{n! \times (2n-n)!} = \frac{(2n)!}{(n!)^2}$ (Applying ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

= coefficient of middle term of $(1 + x)^{2n}$ [From (1)].

14. See Q. 13 above. Middle term = $\frac{(2n)!}{(n!) \times (n!)}x^n = \frac{1.2.3.4 \ldots (2n-2)\,.\,(2n-1)\,(2n)}{(n!) \times (n!)} \cdot x^n$

15. Given exp. = $\frac{(1+x)^{11}-1}{(1+x)-1} = \frac{(1+x)^{11}-1}{x}$ [Sum of a G.P.]

$\therefore$ Coefft. of x^5 in the given exp. = Coefft. of x^5 in $\left[\frac{(1+x)^{11}-1}{x}\right]$.

= Coefft. of x^6 in $[(1+x)^{11}-1]$ $= {}^{11}C_6 = {}^{11}C_5 = 462$.

16. $T_{r+1} = {}^{2n}C_r \,.\, x^{(4n-3r)}$. Putting $4n - 3r = p$, we get $r = \frac{1}{3}(4n-p)$

$\therefore$ Coefft. of $x^p = {}^{2n}C_{\frac{1}{3}(4n-p)}$

17. $(x+a)^n = T_1 + T_2 + T_3 + T_4 + ... + T_{n+1}$, $(x-a)^n = T_1 - T_2 + T_3 - T_4 + ... + (-1)^n T_{n+1}$

P = Sum of odd terms $= T_1 + T_3 + T_5 + ...$; Q = Sum of even terms $= T_2 + T_4 + T_6 + ...$

$\therefore$ $(x+a)^n = P + Q$ and $(x-a)^n = P - Q$

18. Coefft. of $T_r = {}^nC_{r-1}$, coefft. of $T_{r+1} = {}^nC_r$, coefft. of $T_{r+2} = {}^nC_{r+1}$

Since they are in A.P., therefore, ${}^nC_{r-1} + {}^nC_{r+1} = 2.\,{}^nC_r$

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = 2.\frac{n!}{r!(n-r)!}$$

Multiply both sides by $(r+1)!\,(n-r+1)!$

20. $(1+x)(1-x)^n = (1+x)\,[{}^nC_0 + {}^nC_1\,.\,(-x) + {}^nC_2\,.\,(-x)^2 + ... + {}^nC_{n-2}\,(-x)^{n-2} + {}^nC_{n-1}\,(-x)^{n-1} + {}^nC_n\,(-x)^n]$

$\therefore$ Coefft. of x^n in the expansion $= (-1)^n\,[1 + n\,(-1)] = (-1)^n\,.\,(1-n)$

CHAPTER SUMMARY

1. Expansion of a binomial

If x and a are real numbers, then for all $n \in N$,

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1}a^1 + {}^nC_2\, x^{n-2}a^2 + ...$$

$$+ {}^nC_r\, x^{n-r}a^r + ... + {}^nC_{n-1}x^1 a^{n-1} + {}^nC_n x^0 a^n \quad i.e.,\ (x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}a^r.$$

2. Properties of binomial expansion

(1) The **total number of terms** in the expansion of $(x+a)^n$ is $\boldsymbol{n+1}$.

(2) The sum of the indices of x and a in each term is n.

(3) The coefficients of terms equidistant from the beginning and the end are equal. These coefficients are known as the binomial coefficients.

i.e., $\boldsymbol{{}^nC_r = {}^nC_{n-r}}$, $r = 0, 1, 2, ..., n$

$\Rightarrow$ ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}$ and so on.

(4) Replacing a by $-a$, in the expansion of $(x+a)^n$, we have

$$(x-a)^n = {}^nC_0 x^n a^0 - {}^nC_1 x^{n-1}a^1 + {}^nC_2 x^{n-2}a^2 - {}^nC_3 x^{n-3}a^3$$

$$+ ... + (-1)^r\, {}^nC_r\, x^{n-r}a^r + ... + (-1)^n\, {}^nC_n x^0 a^n$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r . x^{n-r}a^r.$$

(5) $(x+a)^n + (x-a)^n = 2\,({}^nC_0 x^n a^0 + {}^nC_2\, x^{n-2}a^2 + ...)$

$(x+a)^n - (x-a)^n = 2\,({}^nC_1 x^{n-1}a^1 + {}^nC_3\, x^{n-3}a^3 + ...)$

(6) The sum of the binomial coefficients is 2^n.

i.e., $C_0 + C_1 + C_2 + C_3 + ... + C_n = 2^n$.

(7) $C_1 + C_3 + C_5 + ... = C_0 + C_2 + C_4 + ... = 2^{n-1}$.

(8) The $(r + 1)$th term is called the **general term** and is given by $\boldsymbol{T_{r+1} = {}^nC_r x^{n-r} a^r}$.

In the expansion of $(x - a)^n$, $T_{r+1} = (-1)^r {}^nC_r x^{n-r} a^r$.

(9) **Term from the end.** In the expansion of $(x + a)^n$, the rth term from the end is $\boldsymbol{(\overline{n+1} - r + 1)}$**th,** *i.e.*, $\boldsymbol{(n - r + 2)}$**th term** from the beginning.

(10) **Middle term.** In the expansion of $(x + a)^n$, if *n is an even natural number*, then

middle term $= \left(\frac{n}{2} + 1\right)$ **th term.**

If *n is an odd number,* then the $\left(\frac{n+1}{2}\right)$**th** term and $\left(\frac{n+3}{2}\right)$**th** term are the **two middle terms.**

CHAPTER TEST

1. Expand $\left(\frac{2}{x} - \frac{x}{2}\right)^5$, $x \neq 0$.

2. Using binomial theorem, write the value of $(a + b)^n + (a - b)^n$ and hence find the value of $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$. **[*SC*]**

3. Find the 9th term in the expansion of $\left(3x - \frac{1}{2x}\right)^8$, $x \neq 0$. **[*Karnataka CET 2007*]**

4. Find the term independent of x in $\left(2x^2 - \frac{1}{x}\right)^{12}$.

5. Find the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$. **[*CBSE*]**

6. (*i*) In the binomial expansion of $(1 + a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.
(*ii*) Prove that the coefficients of x^n in $(1 + x)^{2n}$ is twice the coefficient of x^n in $(1 + x)^{2n} - 1$.

7. Use binomial theorem to evaluate $(10.1)^5$. **[*AP*]**

8. Examine whether or not there is any term containing x^9 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$. **[*SC*]**

9. In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ equals

(*a*) $\frac{n-5}{6}$ (*b*) $\frac{n-4}{5}$ (*c*) $\frac{5}{n-4}$ (*d*) $\frac{6}{n-5}$

10. If $m = {}^nC_2$, then ${}^mC_2$ is equal to

(*a*) $3.{}^nC_4$ (*b*) ${}^{n+1}C_4$ (*c*) $3.{}^{n+1}C_4$ (*d*) $3.{}^{n+1}C_3$ **[*Kerala PET 2007*]**

11. If the coefft. of *r*th and $(r + 4)$th terms are equal in the expansion of $(1 + x)^{20}$, then the value of *r* will be

(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 10

[*Kerala Engg. 2001, MP PET 2002*]

12. If $(1 + x - 2x^2)^6 = 1 + a_1 x + a_2x^2 + ... + a_{12} x^{12}$, then find $a_2 + a_4 + ... + a_{12}$.

[*UPSEAT 2003, Raj. PET 1999*]

13. If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ be same, then the value of '*a*' is

(*a*) $\frac{3}{7}$ (*b*) $\frac{7}{3}$ (*c*) $\frac{7}{9}$ (*d*) $\frac{9}{7}$ **[*WB JEE 2009*]**

14. Using binomial theorem, the value of $(0.999)^3$ correct to 3 decimal places is

(*a*) 0.999 (*b*) 0.998 (*c*) 0.997 (*d*) 0.995 **[*WB JEE 2009*]**

ANSWERS

1. $32x^{-5} - 40x^{-3} + 20x^{-4} - 5x + \frac{5}{8}x^3 - \frac{1}{32}x^5$.

2. $(a+b)^n - (a-b)^n = 2[{}^nC_1 a^{n-1}b + {}^nC_3 a^{n-3}b^3 + ...]$; $396\sqrt{6}$

3. $256\,x^8$ **4.** T_9; 7920 **5.** $\frac{-105}{8}x^9$ and $\frac{35}{48}x^{12}$

6. 105101.00501 **9.** (*b*) **10.** (*c*) **11.** (*c*) **12.** 31 **13.** (*d*) **14.** (*c*)

HINTS TO SELECTED QUESTIONS

6. (*ii*) Coefft. of x^n in $(1 + x)^{2n}$ $= {}^{2n}C_n = \frac{(2n)!}{n!\,n!} = \frac{(2n)(2n-1)!}{n(n-1)!n!} = 2\frac{(2n-1)!}{(n-1)!n!}$

8. If possible, let x^9 occur in the $(r + 1)$th term

T_{r+1} is obtained $= (-1)^r.\ {}^{20}C_r.\ 2^{20-r}.x^{40-3r}$.

$40 - 3r = 9$ gives $r = \frac{31}{3}$, which is not an integer. Hence, the expansion of the given binomial does not contain x^9.

9. $T_5 + T_6 = 0 \Rightarrow {}^nC_4\, a^{n-4}(-b)^4 + {}^nC_5\, a^{n-5}(-b)^5 = 0$

$\Rightarrow {}^nC_4\, a^{n-4}(-b)^4 = -\left({}^nC_5\, a^{n-5}(-b)^5\right) \Rightarrow \frac{a}{b} = \frac{n-4}{5}$.

10. $m = {}^nC_2 = \frac{n(n-1)}{1.2}$

$\therefore\ {}^mC_2 = {}^{\frac{n(n-1)}{2}}C_2 = \frac{\frac{n(n-1)}{2}.\left(\frac{n(n-1)}{2}-1\right)}{2\times 1} = 3.\frac{(n+1)n(n-1)(n-2)}{1\times 2\times 3\times 4}$

11. ${}^{20}C_{r-1} = {}^{20}C_{r+3} \Rightarrow {}^{20}C_{20-r+1} = {}^{20}C_{r+3}$

CHAPTER 14

SEQUENCE AND SERIES

14.01. Sequence

A sequence may be thought of as a set of numbers specified in a definite order by some assigned rule or law.

Thus, the set of numbers 5, 9, 13, 17, 21,... in which each succeeding number is obtained on adding 4 to the preceding number forms a sequence.

A sequence such as 2, 4, 8, 16, 32, 64 which has a last term is called a **finite sequence**. A sequence, such as 1, 4, 9, 16, 25, 36,... which has no last term is an **infinite sequence**.

14.02. nth Term of a Sequence

To refer to the terms of any sequence, we often choose some letter as a, or T and then attach to that letter a small numeral, or subscript, to denote a particular term of the sequence. For instance, we call the first term a_1 or T_1, the second term a_2 or T_2 and so on. In general, a_n or T_n is the ***n*th term** of the sequence (also called the general term).

14.03. Series

An expression consisting of the term of a sequence, alternating with the symbol '+' is called a series.

For example, associated with the sequence

we have the series $\frac{3}{5}+\frac{5}{7}+\frac{7}{9}+\ldots$ $\quad \frac{3}{5},\frac{5}{7},\frac{7}{9},\ldots$

14.04. To Find the nth Term of a Sequence when the Sum of the First n Terms of a Sequence is Given

Let $T(n)$ be a sequence. Let us form another sequence (S_n) as follows:

$$S_1 = T_1,\ S_2 = T_1 + T_2,\ S_3 = T_1 + T_2 + T_3,\ldots$$

$$S_{n-1} = T_1 + T_2 + T_3 + \ldots + T_{n-1}, \qquad S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$

Here, S_n is the sum of the first n terms of the sequence. The basic problem in sequence is to find the sum of the first n terms.

It is clear from the above that

$$S_n - S_{n-1} = (T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n) - (T_1 + T_2 + T_3 + \ldots + T_{n-1}) = T_n$$

or $$\boxed{T_n = S_n - S_{n-1}}$$

This result enables us to find the nth term when the sum of n terms is given.

Ex. 1. Find the rule that determines the sequence –2, –1, 0, 1,... Use this rule to find the fifth, sixth, seventh and eighth terms of the sequence.

Sol. $a_1 = -2 = 1 - 3$, $a_2 = -1 = 2 - 3$, $a_3 = 0 = 3 - 3$, $a_4 = 1 = 4 - 3$...

$\therefore\ a_n = n - 3$. This is the required rule.

Thus, $a_5 = 5 - 3 = 2$, $a_6 = 6 - 3 = 3$, $a_7 = 7 - 3 = 4$, $a_8 = 8 - 3 = 5$.

Ex. 2. Find the general term of the sequence 4, 10, 28, 82,...

Sol. $T_1 = 4 = 3^1 + 1$, $T_2 = 10 = 3^2 + 1$, $T_3 = 28 = 3^3 + 1$, $T_4 = 82 = 3^4 + 1$, ...

$\therefore\ T_n = 3^n + 1$.

Ex. 3. If for a sequence (T_n), (*i*) $S_n = 2n^2 + 3n + 1$, (*ii*) $S_n = 2(3^n - 1)$, find T_n and hence T_1 and T_2.

Sol. (*i*) $S_n = 2n^2 + 3n + 1$

$\therefore\ S_{n-1} = 2(n-1)^2 + 3(n-1) + 1 = 2(n^2 - 2n + 1) + 3(n-1) + 1 = 2n^2 - n$

$\therefore\ T_n = S_n - S_{n-1} = (2n^2 + 3n + 1) - (2n^2 - n) = 4n + 1$

$\therefore\ T_1 = 4 \times 1 + 1 = 5$ and $T_2 = 4 \times 2 + 1 = 9$.

(*ii*) $S_n = 2(3^n - 1)$ $\therefore\ S_{n-1} = 2(3^{n-1} - 1)$

$T_n = S_n - S_{n-1} = 2(3^n - 1) - 2(3^{n-1} - 1)$

$= 2(3^n - 3^{n-1}) = 2.3^{n-1}(3 - 1) = 4.3^{n-1}$

$T_1 = 4.3^{1-1} = 4.3^0 = 4 \times 1 = 4$; $T_2 = 4.3^{2-1} = 4.3^1 = 12$.

EXERCISE 14 (*a*)

1. Write the first five terms of the sequence using the given rule. In each case, the initial value of the index is 1.

(*i*) $a_n = 2n$; (*ii*) $a_n = 3n - 2$; (*iii*) $a_n = n^2 + 5$;

(*iv*) $a_n = \dfrac{(-1)^{n-1}}{n^3}$; (*v*) $a_n = n$th prime number for all natural numbers n.

2. Write the first four terms of the sequence whose nth term is given

(*i*) $\dfrac{2n+1}{2n-1}$; (*ii*) $\dfrac{n^2+1}{n}$; (*iii*) $\dfrac{2^n}{n^2}$;

(*iv*) $\sin^n 30°$; (*v*) $(-1)^n \sin\dfrac{n\pi}{2}$; (*vi*) $(-1)^{n-1}\cos\dfrac{n\pi}{4}$.

3. Find the first 4 terms and the 20th term of the sequence whose

$S_n = \dfrac{3}{2}(3^n - 1)$.

4. Find the 10th term of the sequence whose sum to n terms is $6n^2 + 7$.

ANSWERS

1. (*i*) 2, 4, 6, 8, 10 (*ii*) 1, 4, 7, 10, 13 (*iii*) 6, 9, 14, 21, 30

(*iv*) $1, \dfrac{-1}{8}, \dfrac{1}{27}, \dfrac{-1}{64}, \dfrac{1}{125}$ (*v*) 2, 3, 5, 7, 11

2. (*i*) $3, \dfrac{5}{3}, \dfrac{7}{5}, \dfrac{9}{7}$ (*ii*) $2, \dfrac{5}{2}, \dfrac{10}{3}, \dfrac{17}{4}$ (*iii*) $\dfrac{1}{2}, 1, \dfrac{8}{9}, 1$

(*iv*) $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}$ (*v*) –1, 0, 1, 0 (*vi*) $\dfrac{\sqrt{2}}{2}, 0, \dfrac{-\sqrt{2}}{2}, 1$

3. $T_1 = 3$, $T_2 = 9$, $T_3 = 27$, $T_4 = 81$, $T_{20} = 3^{20}$ **4.** 114

14.05. Progressions

A sequence is also called a progression. We now study three important types of sequences:

(1) The Arithmetic Progression, (2) The Geometric Progression,

(3) The Harmonic Progression.

14.06. Arithmetic Progression (A.P.)

Definition: *A sequence is called an arithmetic progression* (Abbrev. A.P.) *if its terms continually increase or decrease by the same number.* The fixed number by which they increase or decrease is called the **common difference**. The following are examples of sequences in **A.P** :

Sequences	*Common difference*
2, 6, 10, 14,...	4
10, 5, 0, –5, –10,...	–5
$a, a + d, a + 2d, a + 3d,...,$	d

The last sequence in the above examples is a standard A.P.

[**Test.** Subtract from each term, the preceding term. If the differences are equal, the progression is arithmetic.]

Thus, three quantities, a, b, c will be in A.P. if $b - a = c - b$, *i.e.,* $\mathbf{2b = a + c.}$

Aid to memory : *Three quantities are in A.P. if twice the middle = sum of the extremes.*

14.07. nth Term of an A.P.

Let us consider the arithmetic progression

$$a, a + d, a + 2d, a + 3d, ...$$

What is the coefficient of d in the 10th term, the 25th term, the 83rd term ?

It is obvious that the coefficient of d in any term is one less than the number indicating the position of the term, and so the nth term of an A.P. is $a + (n - 1)\, d$. If the same is represented by l, we have

$$\boxed{\mathbf{T_n = l = a + (n - 1)\, d.}}$$

Aid to memory : First term + (number of the terms – 1) common difference.

Note 1. The formula $l = a + (n - 1)\, d$ contains four quantities l, a, n, d. Three quantities being given, the fourth can be found out. If only two quantities are given, two conditions of the problem should be given.

Note 2. $d = T_2 - T_1 = T_3 - T_2 = ... = \mathbf{T_n - T_{n-1}}$.

Ex. 4. The *n*th term of an A.P. is $4n - 1$. Write down the first 4 terms and the 18th term of the A.P.

Sol. $T_n = 4n - 1$

Putting $n = 1, 2, 3, 4$ and 18, we get

$T_1 = 4 \times 1 - 1 = 3$, $T_2 = 4 \times 2 - 1 = 7$, $T_3 = 4 \times 3 - 1 = 11$.

$T_4 = 4 \times 4 - 1 = 15$, $T_{18} = 4 \times 18 - 1 = 71$.

Hence, the first four terms of the A.P. are 3, 7, 11, 15 and the 18th term is 71.

Ex. 5. Find the 9th and *p*th terms of the A.P. 2, 5, 8,...

Sol. Here, $a = 2$ $\quad\therefore$ 9th term $= 2 + (9 - 1) \times 3 = 26$

$d = 3$ $\quad$ pth term $= 2 + (p - 1) \times 3 = 3p - 1$.

Ex. 6. Which term of the series 31 + 29 + 27 + ... is 3 ?

Sol. Here, l is given and n is to be determined ; $a = 31, d = -2$

If 3 be the nth term, then $\quad 3 = 31 + (n - 1)(-2) = 31 - 2n + 2$

$\therefore\ 2n = 30 \quad\therefore\ n = 15$

Hence, 3 is the 15th term.

Ex. 7. The 8th term of a series in A.P. is 23 and the 102nd term is 305. Find the series.

Sol. Let a be the first term and d the common difference.

Then, $T_8 = a + 7d = 23$ ...(1)

$T_{102} = a + 101d = 305$ ...(2)

Solving (1) and (2), $a = 2$, $d = 3$.

Hence, the required series is $2 + 5 + 8 + 11 + ...$

Ex. 8. In a certain A.P., the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term. **[*SC*]**

Sol. Let a be the first term and d the common difference. Then,

By the given condition, $a + 23d = 2(a + 9d)$ $\therefore$ $a = 5d$ ...(1)

Now, 72nd term $= a + 71d = 5d + 71d = 76d$ [From (1)] ...(2)

34th term $= a + 33d = 5d + 33d = 38d$ [From (l)] ....(3)

Obviously, (2) is twice (3). Hence proved.

Ex. 9. If p times the pth term of an A.P. is equal to q times the qth term, prove that $(p + q)$th term is zero.

Sol. Let a be the first term and d the common difference. It is given that $p.t_p = q.t_q$.

i.e., $p\,[a + (p - 1)\,d] = q\,[a + (q - 1)\,d]$

$\Rightarrow$ $(p - q)\,a = (q^2 - q - p^2 + p)\,d \Rightarrow -(q - p)\,a = (q - p)\,[(q + p) - 1]\,d$

$\Rightarrow$ $-a = [(q + p) - 1]\,d \Rightarrow a + [(q + p) - 1]\,d = 0 \Rightarrow t_{p+q} = 0.$ $(\because\ q - p \neq 0)$

Ex. 10. If a, b and c be respectively the pth, qth and rth terms of an A.P., prove that

$$a\,(q - r) + b\,(r - p) + c\,(p - q) = 0.$$

Sol. Let A be the first term and D the common difference.

Since $t_p = a$, therefore, $A + (p - 1)\ D = a$ ...(1)

Since $t_q = b$, therefore, $A + (q - 1)\ D = b$ ...(2)

Since $t_r = c$, therefore, $A + (r - 1)\ D = c$ ...(3)

Multiplying (1) by $(q - r)$, (2) by $(r - p)$ and (3) by $(p - q)$, and adding

$A\,[(q - r) + (r - p) + (p - q)] + [(p - 1)\,(q - r) + (q - 1)\,(r - p) + (r - 1)\,(p - q)]\,D$

$= a\,(q - r) + b\,(r - p) + c\,(p - q)$

$\Rightarrow$ $A\,(0) + (0)\,D = a\,(q - r) + b\,(r - p) + c\,(p - q)$. Hence, $a\,(q - r) + b\,(r - p) + c\,(p - q) = 0$.

EXERCISE 14 (*b*)

1. Write the first six terms of an A.P. in which

(*i*) $a = 5$, $d = 4$; (*ii*) $a = 98$, $d = -3$; (*iii*) $a = 7\frac{1}{2}$, $d = 1\frac{1}{2}$; (*iv*) $a = x$, $d = 3x + 2$.

2. Write the 5th and 8th terms of an A.P. whose 10th term is 43 and the common difference is 4.

3. In each of the following find the terms required.

(*a*) The seventh term of 2, 7, 12,..... (*b*) The fifth term of 21, 28, 35, ...

(*c*) The eighteenth term of 9, 5, 1,.....

4. Find the first four terms and the eleventh term of the series whose nth term is

(*a*) $4n - 2$, (*b*) $6n + 5$, (*c*) $101 - 3n$.

5. The 5th term of an A.P. is 11 and the 9th term is 7. Find the 16th term.

6. Which term of the series 5, 8, 11,..... is 320?

7. The fourth term of an A.P. is ten times the first. Prove that the sixth term is four times as great as the second term. **[*SC*]**

8. The fourth term of an A.P. is equal to 3 times the first term, and the seventh term exceeds twice the third term by 1. Find the first term and the common difference. **[*SC*]**

9. Which term of the progression 19, $18\frac{1}{5}$, $17\frac{2}{5}$, ... is the first negative term?

10. (*i*) Find the value of k so that $8k + 4$, $6k - 2$, and $2k + 7$ will form an A.P.
(*ii*) Find a, b such that 7.2, a, b, 3 are in A.P. **[*SC*]**

11. Determine 2nd term and rth term of an A.P. whose 6th term is 12 and 8th term is 22. **[*SC*]**

12. Prove that the product of the 2nd and 3rd terms of an A.P. exceeds the product of the 1st and 4th by twice the square of the difference between the 1st and 2nd.

13. The 2nd, 31st and last term of an A.P. are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. Find the first term and the number of terms.

14. If 7 times the 7th term of an A.P. is equal to 11 times its 11th term, show that the 18th term of the A.P. is zero.

15. Determine k so that $k + 2$, $4k - 6$ and $3k - 2$ are three consecutive terms of an A.P. **[*ISC*]**

16. The pth term of an A.P. is q and the qth term is p, show that the mth term is $p + q - m$.

17. Let T_r be the rth term of an A.P. whose first term is a and common difference is d. If for some positive integers m, n, $T_m = \frac{1}{n}$, $T_n = \frac{1}{m}$, then $(a - d)$ equals

(*a*) $\frac{1}{mn}$ (*b*) 1 (*c*) 0 (*d*) $\frac{1}{m} + \frac{1}{n}$.

[*AIEEE 2004, UP SEE 2007*]

18. Given that the $(p + 1)$th term of an A.P. is twice the $(q + 1)$th term, prove that the $(3p + 1)$th term is twice the $(p + q + 1)$th term. **[*SC*]**

ANSWERS

1. (*i*) 5, 9, 13, 17, 21, 25; (*ii*) 98, 95, 92, 89, 86, 83; (*iii*) $7\frac{1}{2}$, 9, $10\frac{1}{2}$, 12, $13\frac{1}{2}$, 15;
(*iv*) x, $4x + 2$, $7x + 4$, $10x + 6$, $13x + 8$, $16x + 10$ **2.** 23, 35 **3.** (*a*) 32, (*b*) 49, (*c*) –59
4. (*a*) 2, 6, 10, 14, 42, (*b*) 11,17, 23, 29, 71. (*c*) 98, 95, 92, 89; 68
5. 0 **6.** 106th term **8.** $a = 3$, $d = 2$. **9.** 25th term

10. (*i*) $7\frac{1}{2}$ (*ii*) $a = 5.8$, $b = 4.4$ **11.** $-8, 5r - 18$ **13.** 8, 59 **15.** $k = 3$. **16.** (*c*) **17.** (*c*)

HINTS TO SELECTED QUESTIONS

9. $a = 19$, $d = \frac{-4}{5}$. Given $T_n < 0 \Rightarrow 19 + (n-1)\left(\frac{-4}{5}\right) < 0 \Rightarrow n > 24\frac{3}{4}$.

16. $T_m = \frac{1}{n}$, *i.e.*, $a + (m-1)d = \frac{1}{n}$. Also, $T_n = \frac{1}{m}$, *i.e.*, $a + (n-1)d = \frac{1}{m}$

Solving $d = \frac{1}{mn}$, $a = \frac{1}{mn}$

14.08. Sum of a Stated Number of Terms of an Arithmetic Series (A.S.)

More than 200 years ago, a class of German school children was asked to find the sum of all the integers from 1 to 100 inclusive. One boy in the class, an eight-year-old named *Carl Fredrick Gauss* (1777–1855), who later established his reputation as one of the greatest mathematicians, announced the answer almost at once. The teacher overawed at this asked Gauss to explain how he got this answer. Gauss explained that he had added these numbers in pairs as follows:

$1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101$, and so forth.

These are 50 equal pairs. The answer can be obtained by multiplying 101 by 50 to get 5050.

We hope you have understood the method applied by Gauss. You may write the integers in two ways as under:

$$S = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$

and $$S = 100 + 99 + 98 + \ldots + 3 + 2 + 1$$

$\therefore$ $$2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101 \quad \text{(Adding)}$$

$\Rightarrow$ $$2S = 101 \times 100 \Rightarrow S = \frac{101 \times 100}{2} \Rightarrow S = 101 \times 50 = 5050.$$

The argument depends on the fact that in the series

$1 + 2 + 3 + \ldots + 98 + 99 + 100$, the terms can be paired so that the terms equidistant from the ends of the series have the same sum.

14.09. Sum of n Terms of an A.P.

Let the A.P. be $a, a + d, a + 2d, \ldots$ Let l be the last term, and S the required sum.

Then, $$S = a + (a + d) + (a + 2d) + \ldots + (l - d) + l$$

Reversing the right hand terms

$$S = l + (l - d) + (l - 2d) + \ldots + (a + d) + a$$

Adding,

$$2S = (a + l) + (a + l) + \ldots \text{ to } n \text{ terms} = n(a + l)$$

$\therefore$ $$\boxed{S = \frac{n}{2}(a + l)} \qquad \ldots.(1)$$

If we substitute the value of l *viz.*, $l = a + (n - 1)d$, in this formula, we get

$$S = \frac{n}{2}[a + a + (n - 1)d], \text{ i.e., } \boxed{S_n = \frac{n}{2}[2a + (n - 1)d]} \qquad \ldots.(2)$$

Note 1. Each of these formulas contains four quantities. Three being known, the fourth can be found out.

Note 2. If the sum of n terms be a function of n, the sum of $(n - 1)$ terms can be found on putting $(n - 1)$ for n. Thus, if $S_n = 3n^2 + 2n \Rightarrow S_{n-1} = 3(n - 1)^2 + 2(n - 1)$.

Note 3. Also, from (2) above, we have

$$S_n - S_{n-1} = \frac{n}{2}[2a + (n-1)d] - \frac{n-1}{2}[2a + (n-1-1)d]$$

$$= a[n - (n - 1)] + \frac{(n-1)}{2} d\,[n - (n - 2)]$$

$$= a + \frac{(n-1)}{2} d \times 2 = a + (n - 1)d = n\text{th term}$$

Hence, $$\boxed{T_n = S_n - S_{n-1}}$$

Ex. 11. Find the sum of (*i*) 20 terms, (*ii*) *n* terms of the progression 1, 3, 5, 7, 9, ...

Sol. Here, $a = 1, d = 2$

(*i*) $$S_{20} = \frac{20}{2}\{2 \times 1 + (20 - 1) \times 2\} = 10 \times 40 = 400$$

(*ii*) $S_n = \frac{n}{2}\{2a + (n-1)d\} = \frac{n}{2}\{2 \times 1 + (n-1)\,2\} = n^2$.

Ex. 12. Find the sum of the series 72 + 70 + 68 + ... + 40.

Sol. Here, the last term is given. We will first have to find the number of terms.

$a = 72$ $\qquad l = a + (n-l)d$

$d = -2, l = 40 \Rightarrow 40 = 72 + (n-1)(-2) \qquad \therefore \quad n = 17.$

$\therefore \quad S_{17} = \frac{17}{2}(72 + 40) = 952.$ [Using $S_n = \frac{n}{2}(a+l)$]

Ex. 13. The sum of *n* terms of an A.P. is $4n^2 + 5n$. Fi nd the series.

Sol. $S_1 = 4 \times 1^2 + 5 \times 1 = 9$, $S_2 = 4 \times 2^2 + 5 \times 2 = 26$, $S_3 = 4 \times 3^2 + 5 \times 3 = 51$,...

$\therefore \quad T_1 = S_1 = 9, T_2 = S_2 - S_1 = 26 - 9 = 17, T_3 = S_3 - S_2 = 51 - 26 = 25$

Hence, the series is 9 + 17 + 25 + ...

Alternatively, $T_n = S_n - S_{n-1} = (4n^2 + 5n) - [4(n-1)^2 + 5(n-1)] = 8n + 1$.

Now, put $n = 1, 2, 3, ...$

Ex. 14. How many terms of the A.P. 24, 20, 16, ... must be taken so that the sum may be 72 ? Explain the double answer.

Sol. Let n be the number of the terms. Then

$$72 = \frac{n}{2}\{2 \times 24 + (n-1)(-4)\} = n(26 - 2n) = 26n - 2n^2$$

$\therefore \quad n^2 - 13n + 36 = 0 \Rightarrow (n-4)(n-9) = 0 \quad \therefore \quad n = 4, 9.$

The double answer shows that there are 2 sets of numbers satisfying the condition of the problem. Writing the series up to 9 terms, we have

$$24 + 20 + 16 + 12 + 8 + 4 + 0 - 4 - 8$$

As the last 5 terms cancel out, the sum of 4 terms is the same as that of 9 terms.

Ex. 15. On each birthday, John received from his father ₹ 5 for each year of his age. Find the total sum John had received by the time he was 21 years old. **[*SC*]**

Sol. Here $a = 5, d = 5$ and $n = 21$; $S = \frac{21}{2}\{10 + 20 \times 5\} = 55 \times 21 = ₹\ 1155.$

Ex. 16. The sums of *n* terms of two arithmetic series are in the ratio of $2n + 1 : 2n - 1$. Find the ratio of their 10th terms.

Sol. Let the two A.Ps be $a, a + d, a + 2d$,... and $A, A + D, A + 2D$,...

It is given that $\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2A + (n-1)D]} = \frac{2n+1}{2n-1} \Rightarrow \frac{2a + (n-1)d}{2A + (n-1)D} = \frac{2n+1}{2n-1}$ ...(1)

We have to find the ratio $\frac{t_{10}}{T_{10}} = \frac{a+9d}{A+9D} = \frac{2a+18d}{2A+18D}$.

Putting $n = 19$ in (1), we get $\frac{t_{10}}{T_{10}} = \frac{2a+18d}{2A+18D} = \frac{2 \times 19 + 1}{2 \times 19 - 1} = \mathbf{\frac{39}{37}}$.

Ex. 17. The sums of first *p, q, r* terms of an A.P. are *a, b, c* respectively. Prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

Sol. Let A be the first term and D the common difference. It is given that $S_p = a$

i.e., $\frac{p}{2}[2A+(p-1)D]=a \quad \Rightarrow A+\frac{p-1}{2}D=\frac{a}{p}$ ...(*i*)

Similarly, $A+\frac{q-1}{2}D=\frac{b}{q}$ ...(*ii*) and $A+\frac{r-1}{2}D=\frac{c}{r}$ ...(*iii*)

Multiplying (*i*) by $(q-r)$, (*ii*) by $(r-p)$, (*iii*) by $(p-q)$, and adding, we get

$$A\ (q-r+r-p+p-q)+\frac{D}{2}[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$$
$$=\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$$

$$\Rightarrow A(0)+\frac{D}{2}(0)=\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q) \Rightarrow \frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

Practical Problems

Ex. 18. A man saved ₹ 16,500 in 10 years. In each year after the first he saved ₹ 100 more than he did in the preceding year. How much did he save in the first year?

Sol. The yearly savings form an A.P. whose common difference is 100 and the sum of whose 10 terms is ₹ 16,500. If a denotes in rupees the saving of the first year, then

$$S_n=\frac{n}{2}[2a+(n-1)d] \qquad \Rightarrow 16{,}500=\frac{10}{2}[2a+9\times 100]$$

$\Rightarrow \quad 16{,}500 = 10a + 4{,}500 \Rightarrow 10a = 12{,}000 \Rightarrow a = 1{,}200$

$\therefore$ The first year's saving = **₹ 1,200.**

Ex. 19. 80 coins are placed in a st. line on the ground. The distance between any two consecutive coins is 10 metres. How far must a person travel to bring them one by one to a basket placed 10 metres behind the first coin?

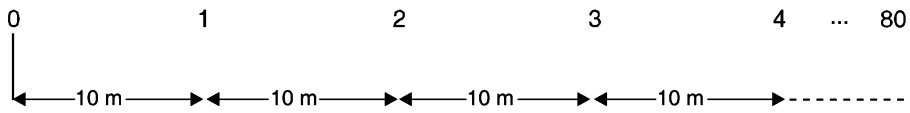


Fig. 14.01

Sol.

Let 1, 2, 3,... 80 represent the positions of the coins and 0 that of the basket.
The distance covered in bringing the first coin = 10 m + 10 m = 20 m
The distance covered in bringing the second coin = 20 m + 20 m = 40 m
The distance covered in bringing the third coin = 30 m + 30 m = 60 m and so on

$\therefore$ The total distance covered = $\frac{n}{2}[2a+(n-1)d]$

$$=\frac{80}{2}[2\times 20+(80-1)\,.\,20]=40\,(40+1580)=40\times 1620=\textbf{64,800 metres.}$$

EXERCISE 14 (*c*)

1. Find the sum of :

(*i*) 10 terms of $5 + 8 + 11 + ...$;

(*ii*) 18 terms of $57 + 49 + 41 + ...$;

(*iii*) n terms of $4 + 7 + 10 + ...$;

(*iv*) 24 terms and n terms of $2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{6}, 5, ...$;

(*v*) $101 + 99 + 97 + ... + 47$.

2. Find the sum of all the numbers between 100 and 200 which are divisible by 7. **[*SC*]**

3. The sum of a series of terms in A.P. is 128. If the first term is 2 and the last term is 14, find the common difference.

4. The sum of 30 terms of a series in A.P., whose last term is 98, is 1635. Find the first term and the common difference.

5. If the sums of the first 8 and 19 terms of an A.P. are 64 and 361 respectively, find (*i*) the common difference and (*ii*) the sum of n terms of the series.

6. Find the number of terms of the series 21, 18, 15, 12,...which must be taken to give a sum of zero. **[*SC*]**

7. The sum of n terms of a series is $(n^2 + 2n)$ for all values of n. Find the first 3 terms of the series. **[*SC*]**

8. The third term of an arithmetical progression is 7, and the seventh term is 2 more than 3 times the third term. Find the first term, the common difference and the sum of the first 20 terms. **[*ISC*]**

9. The interior angles of a polygon are in arithmetic progression. The smallest angle is 52° and the common difference is 8°. Find the number of sides of the polygon. **[*ISC 1990*]**

10. Determine the sum of first 35 terms of an A.P. if $t_2 = 1$ and $t_7 = 22$.

11. Find the sum of all natural numbers between 100 and 1000 which are multiples of 5.

12. How many terms of the A.P. 1,4, 7,.... are needed to give the sum 715? **[*ISC*]**

13. Find the rth term of an A.P., sum of whose first n terms is $2n + 3n^2$.

14. In an arithmetical progression, the sum of p terms is m and the sum of q terms is also m. Find the sum of $(p + q)$ terms. **[*ISC*]**

15. The sum of the first fifteen terms of an arithmetical progression is 105 and the sum of the next fifteen terms is 780. Find the first three terms of the arithmetical progression. **[*ISC*]**

16. The sum of the first six terms of an arithmetic progression is 42. The ratio of the 10th term to the 30th term of the A.P. is $\frac{1}{3}$. Calculate the first term and the 13th term. **[*ISC*]**

17. A sum of ₹ 6240 is paid off in 30 instalments, such that each instalment is ₹ 10 more than the preceding instalment. Calculate the value of the first instalment. **[*ISC*]**

18. The nth term of an A.P. is p and the sum of the first n term is s. Prove that the first term is $\frac{2s - pn}{n}$. **[*SC*]**

19. The sum of the first n terms of the arithmetical progression $3, 5\frac{1}{2}, 8, ...$ is equal to the $2n$th term of the arithmetical progression $16\frac{1}{2}, 28\frac{1}{2}, 40\frac{1}{2}$. Calculate the value of n. **[*ISC*]**

20. If the sum of the first 4 terms of an arithmetic progression is p, the sum of the first 8 terms is q and the sum of the first 12 terms is r, express $3p + r$ in terms of q. **[*ISC*]**

21. The last term of an A.P. 2, 5, 8, 11, ... is x. The sum of the terms of the A.P. is 155. Find the value of x. **[*ISC 1991*]**

22. A gentleman buys every year Banks' certificates of value exceeding the last year's purchase by ₹ 25. After 20 years he finds that the total value of the certificates purchased by him is ₹ 7,250. Find the value of the certificates purchased by him (*i*) in the 1st year (*ii*) in the 13th year.

23. If the sums of the first n terms of two A.P.'s are in the ratio $7n - 5 : 5n + 17$; show that the 6th terms of the two series are equal.

24. (*i*) If the ratio of the sum of m terms and n terms of an A.P. be $m^2 : n^2$, prove that the ratio of its mth and nth terms is $(2m - 1) : (2n - 1)$.

(*ii*) Let $a_1, a_2, a_3, ...$ be terms of an A.P. If $\frac{a_1 + a_2 + ... + a_p}{a_1 + a_2 + ... + a_q} = \frac{p^2}{q^2}, (p \neq q)$ then find $\frac{a_6}{a_{21}}$.

[*AIEEE 2006*]

25. If the sums of n, $2n$, $3n$ terms of an A.S are S_1, S_2, S_3 respectively, prove that $S_3 = 3(S_2 - S_1)$.

26. If the sum of p terms of an A.S is q and the sum of q terms is p, show that the sum of $(p + q)$ terms is $-(p + q)$.

27. The ratio between the sum of n terms of two A.P.'s is $(7n + 1) : (4n + 27)$. Find the ratio of their 11th terms.

ANSWERS

1. (*i*) 185 (*ii*) – 198 (*iii*) $\frac{n}{2}(3n + 5)$ (*iv*) 290, $\frac{5n(n+5)}{12}$ (*v*) 2072

2. 2107 **3.** $\frac{4}{5}$ **4.** First term = 11, Common difference = 3 **5.** $d = 2, S_n = n^2$.

6. 15 **7.** 3, 5, 7 **8.** –1, 4, 740 **9.** 3 **10.** 2387

11. 98450 **12.** 22 **13.** $6r - 1$ **14.** 0 **15.** –14, –11, – 8

16. $T_1 = 2, T_{13} = 26$ **17.** ₹ 63 **19.** $n = 18$ **20.** $3p + r = 3q$

21. $x = 29$ **22.** (*i*) ₹ 125 (*ii*) ₹ 425 **24.** (*ii*) $\frac{11}{41}$ **27.** 148 : 111

HINTS TO SELECTED QUESTIONS

9. $\frac{n}{2}[2 \times 52^\circ + (n - 1)\, 8^\circ] = (n - 2) \times 180^\circ$

$\Rightarrow n = 30$ or $n = 3$. But when $n = 30$, the last angle is $52^\circ + (30 - 1)8^\circ = 284^\circ$ which is not possible, since no interior angle of a polygon is more than 180°

$\therefore$ Number of sides of the given polygon is 3.]

24 (*i*) $\frac{S_m}{S_n} = \frac{\frac{m}{2}[2a + (m-1)d]}{\frac{n}{2}[2a + (n-1)d]} = \frac{m^2}{n^2} \Rightarrow \frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$

$\Rightarrow n[2a + (m - 1)\, d] = m[2a + (n - 1)d]$

Solving, we get $d = 2a$. Substitute in $\frac{a + (m-1)d}{a + (n-1)d}$.

14.10. Arithmetic Mean

The Arithmetic Mean between two numbers is the number which when placed between them forms an arithmetic progression with them. Thus, if a and b are the given numbers and x their arithmetic mean, then *a, x, b are* in A.P.

$$x - a = b - x \Rightarrow 2x = a + b \Rightarrow x = \frac{a+b}{2}$$

Hence, the **arithmetic mean between two numbers is one-half of their sum.**

If we have to find many arithmetic means between two numbers, we can do so by first finding the common difference.

Ex. 20. Find the arithmetic mean between 5 and 9.

Sol. Reqd. A.M. $= \frac{5+9}{2} = 7.$

Ex. 21. Insert 4 arithmetic means between 3 and 23.

Sol. Let A_1, A_2, A_3, A_4 be the 4 arithmetic means between 3 and 23.

Then, 3, A_1, A_2, A_3, A_4, 23 is an A.P., containing 6 terms.

$\therefore$ $23 = 6th \text{ term} = 3 + 5d \therefore d = 4$

$\therefore$ $A_1 = 3 + 4 = 7, A_2 = 3 + 2 \times 4 = 11, A_3 = 3 + 3 \times 4 = 15, A_4 = 3 + 4 \times 4 = 19.$

Hence, the 4 A.M.'s between 3 and 23 are 7, 11, 15 and 19.

Special Types of Questions

Ex. 22. The sum of three numbers in A.P. is 51 and the product of their extremes is 273. Find the numbers.

Sol. Let the numbers be $a - d, a, a + d$.

Then, $(a - d) + a + (a + d) = 51 \Rightarrow a = 17$

and $(a - d)(a + d) = 273 \Rightarrow 17^2 - d^2 = 273 \Rightarrow d = \pm 4.$

Hence, the numbers are 13, 17, 21.

Ex. 23. The sum of four numbers in A.P. is 28 and the sum of their squares is 216. Find the numbers.

Sol. Let the numbers be $a - 3d, a - d, a + d, a + 3d$

Then $(a - 3d) + (a - d) + (a + d) + (a + 3d) = 28 \quad \therefore a = 7$

and $(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 216$

$\Rightarrow$ $2(a^2 + 9d^2) + 2(a^2 + d^2) = 216 \Rightarrow a^2 + 5d^2 = 54 \Rightarrow 5d^2 = 54 - 49 = 5 \therefore d = \pm 1.$

Hence, the numbers are 4, 6, 8 and 10.

Important Note. If we have to find an odd number of terms in an A.P. whose sum is given, it is convenient to take the middle term, a and to take d as the common difference. Thus, three terms are taken as $\boldsymbol{a - d, a, a + d,}$ and five terms are taken as $\boldsymbol{a - 2d, a - d, a, a + d, a + 2d.}$

If we have to find even number of terms, we take $a - d, a + d$ as the middle terms and $2d$ as the common difference.

Thus, four terms are taken as $a - 3d, a - d, a + d, a + 3d.$

14.11. Useful Results

I. Let $a, a + d, a + 2d,....$ be any A.P. ...(1)

(*i*) On adding k to each of its terms, we get the sequence $a + k, a + d + k, a + 2d + k,.....$ which is an A.P with first term $= a + k$, common difference $= d$.

(*ii*) Subtracting k from each term, the new sequence is $a - k, a + d - k, a + 2d - k,.....$ which is an A.P with first term $= a - k$, common difference $= d$.

(*iii*) Multiplying each term by k, the new sequence is $ka, k(a + d), k(a + 2d),.....$ which is an A.P. with first term $= ka$, common difference $= kd$.

(*iv*) Dividing each term by k the new sequence is

$$\frac{a}{k}, \frac{a+d}{k}, \frac{a+2d}{k},..... \text{ which is an A.P. with first term } = \frac{a}{k}, \text{ common difference } = \frac{d}{k}$$

From the above, it is clear that

If each term of a given A.P. is increased or decreased or multiplied or divided by the same number, then the resulting series is also an A.P.

II. If a, b, c are in A.P., then by def.

$$b - a = c - b \Rightarrow \frac{b-a}{c-b} = 1 \Rightarrow \frac{a-b}{b-c} = 1$$

i.e., if three numbers are in A.P. then $\dfrac{\textbf{First term} - \textbf{Second term}}{\textbf{Second term} - \textbf{Third term}} = \mathbf{1}$.

Ex. 24. If a, b, c are in A.P., prove that the following are also in A.P.

(*i*) $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ **(*ii*)** $b + c,\ c + a,\ a + b$

(*iii*) $a^2(b + c),\ b^2(c + a), c^2(a + b)$ **(*iv*)** $\dfrac{a(b+c)}{bc}, \dfrac{b(c+a)}{ca}, \dfrac{c(a+b)}{ab}$.

Sol. (*i*) a, b, c are in A.P.

Dividing each term by abc. $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in A.P.

(*ii*) a, b, c are in A.P.

Subtracting $a + b + c$ from each term

$\Rightarrow$ $a - (a + b + c), b - (a + b + c), c - (a + b + c)$ are in A.P.

$\Rightarrow$ $-(b + c), -(c + a), -(a + b)$ are in A.P.

Multiplying each term by $-1 \Rightarrow b + c, c + a, a + b$ are in A.P.

(*iii*) $a^2(b + c), b^2(c + a), c^2(a + b)$ are in A.P.

if $\dfrac{a^2(b+c) - b^2(c+a)}{b^2(c+a) - c^2(a+b)} = 1$ or $\dfrac{a^2b + a^2c - b^2c - b^2a}{b^2c + b^2a - c^2a - c^2b} = 1$

or $\dfrac{ab(a-b) + c(a^2 - b^2)}{bc(b-c) + a(b^2 - c^2)} = 1$ or $\dfrac{(ab + bc + ca)(a-b)}{(ab + bc + ca)(b-c)} = 1$

or $\dfrac{a-b}{b-c} = 1$, which is true, because a, b, c are given to be in A.P.

$\therefore$ $a^2(b + c), b^2(c + a), c^2(a + b)$ are in A.P.

(*iv*) a, b, c are in A.P.

Dividing each term by abc, $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

Multiplying each term by $ab + bc + ca$,

$\Rightarrow \quad \frac{ab + bc + ca}{bc}, \frac{ab + bc + ca}{ca}, \frac{ab + bc + ca}{ab}$ are in A.P.

$\Rightarrow \quad \frac{a(b+c)+bc}{bc}, \frac{b(c+a)+ca}{ca}, \frac{c(a+b)+ab}{ab}$ are in A.P.

Subtracting 1 from each term, $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are in A.P.

EXERCISE 14 (*d*)

1. Find the A.M. between:

(*i*) 6 and 12 (*ii*) 5 and 22 (*iii*) $(\cos\theta + \sin\theta)^2$ and $(\cos\theta - \sin\theta)^2$

(*iv*) $(x + y)^2$ and $(x - y)^2$.

2. Insert:

(*i*) 3 arithmetic means between 5 and 17.

(*ii*) 4 arithmetic means between 17 and 52.

3. Find two numbers whose product is 91 and whose A.M. is 10.

4. If p arithmetic means are inserted between a and b, prove that $d = \frac{b-a}{p+1}$.

5. The sum of three numbers in A.P. is 33, and the sum of their squares is 461 . Find the numbers.

6. There are four numbers in A. P., the sum of the two extremes is 8, and the product of the middle two is 15. What are the numbers?

7. The sum of the first three terms of an A.P. is 36 while their product is 1620. Find the A.P.

8. The angles of a triangle are in A.P. If the greatest angle is double the least, find the angles.

9. The sum of the first three consecutive terms of an A.P. is 9 and the sum of their squares is 35. Find S_n.

10. a_1, a_2, a_3, a_4, a_5 are first five terms of an A.P. such that $a_1 + a_3 + a_5 = -12$ and $a_1\, a_2\, a_3 = 8$. Find the first term and the common difference. **[*ISC 1992*]**

11. The angles of a quadrilateral are in A.P. and the greatest angle is double the first angle. Find the circular measure of the least angle.

12. If a, b, c are in A.P., show that

$a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

13. If $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are A.P., show that

(*i*) yz, zx, xy are in A.P. (*ii*) xy, zx, yz are in A.P. (*iii*) $\frac{y+z}{x}, \frac{z+x}{y}, \frac{x+y}{z}$ are in A.P.

14. If $(b + c)^{-1}, (c + a)^{-1}, (a + b)^{-1}$ are in A.P., then show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in A.P.

15. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

16. If $x,\ y,\ z$ are in A.P., show that $(xy)^{-1}, (zx)^{-1}, (yz)^{-1}$ are also in A.P.

[Hint. x, y, z are in A.P. Reversing the order of the terms $\Rightarrow z, y, x$ are in A.P.

Dividing each term by xyz, $\frac{1}{xy}, \frac{1}{zx}, \frac{1}{yz}$ are in A.P.]

17. If a^2, b^2, c^2 are in A.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

ANSWERS

1. (*i*) 9 (*ii*) 13.5 (*iii*) 1 (*iv*) $x^2 + y^2$

2. (*i*) 8, 11, 14 (*ii*) 24, 31,38, 45 **3.** 7, 13 **5.** 4, 11, 18

6. 1, 3, 5, 7 **7.** 9, 12, 15 or 15, 12, 9 **8.** 40°, 60°, 80° **9.** n^2 or $n\,(6-n)$

10. First term = 2 and common difference = – 3 **11.** $\frac{\pi}{3}$

GEOMETRIC PROGRESSION

14.12. What is a Geometric Progression?

Consider the following sequences:

(*a*) 2, 4, 8, 16,... (*b*) 2, 6, 18, 54, 162,...

(*c*) 10, 20, 40, 80, 160,... (*d*) 9, 6, 4, $\frac{8}{3}, \frac{16}{9}$,...

These are not arithmetic sequences. What is the speciality of these sequences? Do you see that in each of them, the ratio of the second term to the first is equal to the ratio of the third to its predecessor, the ratio of the fourth term to its predecessor and so on. Such a sequence is called a **geometric progression.**

Definition. ***A Geometric Progression (G.P.) is one in which the ratio of any term to its predecessor is always the same number. This ratio is called the*** **common ratio.**

The following are also examples of geometric progressions:

Progression	***Common ratio***
1, 3, 9, 27, 81,...	$r = 3$
$\frac{16}{27}, \frac{-8}{9}, \frac{4}{3}, -2,...$	$r = \frac{-3}{2}$
$x, x^2, x^3, x^4,...$	$r = x$

[Test: Divide each term by the preceding term, if the quotients are equal the sequence is geometric.]

If a denotes the first term and r the common ratio in a G.P., then the standard G.P. is,

$$a, ar, ar^2, ...$$

Ex. 25. The first three terms of a geometric progression are 48, 24, 12. What are the common ratio and fourth term of this sequence?

Sol. Step 1. *To find the common ratio, divide any term by its predecessor.*

$24 \div 48 = \frac{1}{2}$ or $12 \div 24 = \frac{1}{2}$ $\therefore$ The common ratio is $\frac{1}{2}$.

Step 2. *To find the fourth term, multiply the third term by the common ratio.* $12 \times \frac{1}{2} = 6$

So, the fourth term is 6.

14.13. The *n*th Term

The following table suggests an expression for t_n in terms of n, r, and a.

1st term	*2nd term*	*3rd term*	*4th term*	...	*nth term*
t_1	t_2	t_3	t_4	....	t_n
a	ar	ar^2	ar^3	...	ar^{n-1}

Observing that the exponent of r in each term is one less than the number of the term, you can make the following conjecture:

The nth term of a geometric sequence whose first term is a and common ratio is a non-zero number r is

$$\boxed{t_n = ar^{n-1}}$$

Ex. 26. The *n*th term of a geometric progression is $2(1.5)^{n-1}$ for all values of *n*. Write down the value of *a* (the first term) and the common ratio (*r*). **[*ISC*]**

Sol. $t_n = 2.(1.5)^{n-1}$ $\therefore$ $t_1 = 2.(1.5)^0 = 2$ (Putting $n = 1$)

$t_2 = 2.(1.5)^{2-1} = 3$ $\therefore$ Common ratio $r = \dfrac{t_2}{t_1} = \dfrac{3}{2} = 1.5.$

Ex. 27. Find the (*i*) sixth term, (*ii*) *n*th term of the sequence 2, 6, 18, ...

Sol. Here, $a = 2, r = \dfrac{6}{2} = 3.$

$\therefore$ 6th term $T_6 = ar^5 = 2 \times 3^5 = 486$ and nth term $T_n = ar^{n-1} = 2 \times 3^{n-1}$.

Ex. 28. Write the G .P. whose 4th term is 54 and the 7th term is 1458.

Sol. $t_4 = ar^3 = 54$ ...(1) $t_7 = ar^6 = 1458$ ...(2)

Dividing (2) by (1), we get $r^3 = \dfrac{1458}{54} = 27 \Rightarrow r = 3$ $\therefore$ From (1), $27a = 54 \Rightarrow a = 2$.

Hence, the G.P. is 2, 6, 18, 54, ...

Ex. 29. Which term of the series 8 + 1.6 + 0.32 + ... is 0.00256 ?

Sol. Here, $a = 8$ and $r = \dfrac{1.6}{8} = \dfrac{1}{5}$.

Let 0.00256 be the nth term. Then,

$$0.00256 = ar^{n-1} = 8\left(\frac{1}{5}\right)^{n-1} \quad \therefore \left(\frac{1}{5}\right)^{n-1} = 0.00032 = \frac{32}{100000} = \left(\frac{2}{10}\right)^5 = \left(\frac{1}{5}\right)^5$$

$\therefore$ $n - 1 = 5$ or $n = 6$

$\therefore$ 0.00256 is the 6th term of the series.

Ex. 30. If *a, b, c* are three consecutive terms of an A.P. and *x, y, z* are three consecutive terms of a G.P., then prove that $x^{b-c}.\, y^{c-a}.\, z^{a-b} = 1$.

Sol. Since a, b, c are in A.P. therefore $2b = a + c$ and since x, y, z are in G.P. therefore $y = xr$, $z = xr^2$, where r is the common ratio.

$$x^{b-c}.\, y^{c-a}.\, z^{a-b} = x^{b-c}.\, (xr)^{c-a}.\, (xr^2)^{a-b}$$

$$= x^{b-c+c-a+a-b}.\, r^{c-a+2a-2b} = x^0.\, r^{c+a-2b} = x^0.\, r^{2b-2b} = x^0.\, r^0 = 1.$$

14.14. Geometric Means

The terms between two given numbers of a G.P. are called the **geometric means** between the given numbers. For example, in the sequence 5, 15, 45, 135,

15 and 45 are the two geometric means between 5 and 135.

When three real numbers form a geometric sequence, the middle one is called the single geometric mean of the numbers.

For example, 5, 25, 125 form a G.P., therefore 25 is the geometric mean of 5 and 125.

If G is a geometric mean of two non-zero numbers a and b, then

a, G, b form a geometric sequence, so $\frac{G}{a} = \frac{b}{G} \Rightarrow \boldsymbol{G^2 = ab.}$

Math Alert!

If a and b are positive numbers, the geometric mean of the numbers is usually defined to be the positive root of the equation $G^2 = ab$, whereas if ***a* and *b*** are **negative numbers**, **their geometric mean is the negative root of the equation.**

Thus, $G^2 = ab \Rightarrow \boldsymbol{G = \sqrt{ab}}$ if a and b are positive numbers.

and $\Rightarrow \boldsymbol{G = -\sqrt{ab}}$ if a and b are negative numbers.

Note. If a and b are two numbers of opposite signs, then G.M. between them does not exist. Thus, if –1 and 9 are two numbers, then $G^2 = -9$. This has no real root.

The positive geometric mean between two numbers is the mean proportional between them and is $\boldsymbol{G = \sqrt{ab}}$. This shows that we have a unique G.M. between two numbers.

Note 1. This also shows that if three numbers a, b, c form a G.P., then $\boldsymbol{b^2 = ac.}$
i.e., ***(Square of the middle term) = Product of the extremes.***

Note 2. If there are n positive integers $a_1, a_2, a_3 \ldots\ldots a_n$, then their geometric mean is defined to be equal to $\boldsymbol{(a_1.a_2.a_3 \ldots\ldots a_n)^{1/n}}$.

Ex. 31. Find the geometric mean of 6 and 24.

Sol. Let G be the geometric mean, then $G^2 = 6 \times 24 = 144 \Rightarrow G = 12$ or $G = -12$

$\therefore$ The geometric mean of 6 and 24 is 12.

Note. The G.M. of –6 and –24 is $-\sqrt{-6 \times -24} = -\sqrt{144} = \mathbf{-12.}$

Finding more than 1 Geometric mean between two numbers

Just as you may have more than one arithmetic mean, you may also have more than one geometric mean. The terms between any two given terms of a geometric sequence are geometric means. Thus, in the G.P., 2, 4, 8, 16,..., the terms 4 and 8 are the geometric means between 2 and 16. To insert geometric mean between given numbers, we first find the common ratio.

Ex. 32. Insert 3 real geometric means between 3 and $\frac{3}{16}$.

Sol. Let G_1, G_2, G_3 be the required means.

Then, $3, G_1, G_2, G_3, \frac{3}{16}$ form a G.P. Let r be the common ratio. Then $\frac{3}{16}$ = 5th term = $3r^4$

$\Rightarrow r^4 = \frac{1}{16} \Rightarrow r^4 = \left(\frac{1}{2}\right)^4 \Rightarrow r = \frac{1}{2}.$

$\therefore \quad G_1 = 3\left(\frac{1}{2}\right) = \frac{3}{2}, \; G_2 = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}$ and $G_3 = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$

$\therefore \; \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$ are the required geometric means between 3 and $\frac{3}{16}$.

Ex. 33. Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

Sol. Let the numbers be a and b. Then

$$\text{A.M.} = \frac{a+b}{2} = 34 \text{ (given)} \Rightarrow a + b = 68 \quad \ldots (1)$$

and $\text{G.M.} = \sqrt{ab} = 16$ (given) $\Rightarrow ab = 256$ ... (2)

(1) and (2) $\Rightarrow (a-b)^2 = (a+b)^2 - 4ab = (68)^2 - 4 \times 256 = 3600$

$\Rightarrow a - b = 60$ ... (3)

Solving (1) and (3), we obtain $a = 64$ and $b = 4$.

Hence, the required numbers are **64 and 4.**

4.15. Theorem

***Let A and G be the arithmetic and geometric means of two positive numbers a* and *b, then* $A \geq G$.** [See Theorem 2 after Art. 4.21]

Proof. Case 1. If a and b are two unequal positive numbers, then

$$A = \frac{a+b}{2}, G = \sqrt{ab} \quad \therefore \; A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}.$$

Since $(\sqrt{a} - \sqrt{b})^2$ is always + ve, therefore, $A - G > 0 \Rightarrow A > G$.

Case 2. If $a = b$, then $A = G$. Hence, $A \geq G$.

EXERCISE 14 (*e*)

1. Find: (*i*) the 7th term of 2, 4, 8,... (*ii*) the 9th term of $1, \frac{1}{2}, \frac{1}{2^2}, \ldots$ (*iii*) the *n*th term of $\frac{15}{8}, \frac{3}{8}, \frac{3}{40}, \ldots$

2. The second term of a G.P. is 18 and the fifth term is 486. Find:
(*i*) the first term, (*ii*) the common ratio **[*SC*]**

3. Find the value of x for which $x + 9, x - 6, 4$ are the first three terms of a geometrical progression and calculate the fourth term of progression in each case. **[*SC*]**

4. If $5, x, y, z, 405$ are the first five terms of a geometric progression, find the values of $x, y,$ and z. **[*SC*]**

5. Insert 3 geometric means between 16 and 256.

6. Insert 5 geometric means between $\frac{1}{3}$ and 243.

7. If the A.M. and G.M. between two numbers are respectively 17 and 8, find the numbers.

8. The second, third and sixth terms of an A.P. are consecutive terms of a geometric progression. Find the common ratio of the geometric progression. **[*SC*]**

9. The 5th, 8th and 11th terms of a G.P. are P, Q and S respectively. Show that $Q^2 = PS$.

10. The $(p+q)$th term and $(p-q)$th terms of a G.P. are a and b respectively. Find the pth term.

11. If the pth, qth, rth terms of a G.P. are x, y, z respectively, prove that $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$.

12. In a set of four numbers, the first three are in G.P. and the last three are in A.P. with difference 6. If the first number is the same as the fourth, find the four numbers. **[*IIT*]**

13. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and $a, b, c,$ are in G.P., prove that x, y, z are in A.P. **[*IIT*]**

14. If one G.M., G and two A.M's p and q be inserted between two given numbers, prove that
$$G^2 = (2p - q)(2q - p).$$

15. Construct a quadratic equation in x such that the A.M. of its roots is A and G.M. is G. **[*IIT*]**

16. The fourth term of a G.P. is greater than the first term, which is positive, by 372. The third term is greater than the second by 60. Calculate the common ratio and the first term of the progression. **[*SC*]**

17. The first, eighth and twenty-second terms of an A.P. are three consecutive terms of a G.P. Find the common ratio of the G.P. Given also that the sum of the first twenty-two terms of the A.P. is 275, find its first term. **[*SC*]**

ANSWERS

1. (*i*) 128 (*ii*) $\frac{1}{256}$ (*iii*) $\frac{15}{8}\left(\frac{1}{5}\right)^{n-1}$ **2.** First term = 6, Common ratio = 3

3. $x = 0, 16$; when $x=0$, $t_4 = -2\frac{2}{3}$ when $x = 16$, $t_4 = 1\frac{3}{5}$ **4.** 15, 45, 135 or –15, 45, – 135.

5. 32, 64, 128 **6.** 1, 3, 9, 27, 81 **7.** 32, 2 **8.** 3 **10.** $\sqrt{ab}$

12. 8, –4, 2, 8 **15.** $x^2 - 2Ax + G^2 = 0$. **16.** Common ratio = 5, first term = 3

17. Common ratio = 2, first term = 5 **18.** $T_1 = 2$, Common difference = –3

HINTS TO SELECTED QUESTIONS

8. $(a + 2d)^2 = (a + d).(a + 5d) \Rightarrow d = -2a$, common ratio $= \frac{a+2d}{a+d}$.

10. $T_{p+q} = xr^{p+q-1} = a, T_{p-q} = xr^{p-q-1} \Rightarrow r^{2q} = \frac{a}{b} \Rightarrow r = \left(\frac{a}{b}\right)^{\frac{1}{2q}}$

$\Rightarrow \frac{1}{r} = \left(\frac{b}{a}\right)^{\frac{1}{2q}}$. Now, $T_p = xr^{p-1} = xr^{(p+q-1)}.\left(\frac{1}{r}\right)^q = a\left(\frac{b}{a}\right)^{\frac{q}{2q}} = \sqrt{ab}$.

12. Let the four numbers be x, $a - 6$, a, $a + 6$. Given; x, $a - 6$, a are in G.P.

$\therefore \frac{a-6}{x} = \frac{a}{a-6}$. Also, given that $x = a + 6$.

15. Let the roots be α, β then $\frac{\alpha+\beta}{2} = A$ and $\sqrt{\alpha\beta} = G$.

Quadratic equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

16. $ar^3 - a = 372 \Rightarrow a(r^3 - 1) = 372$ $\Big\}$ $\Rightarrow \frac{r^2+r+1}{r} = \frac{372}{60} = \frac{31}{5}$

$ar^2 - ar = 60 \Rightarrow ar(r - 1) = 60$

17. $(a + 7d)^2 = a(a + 21d) \Rightarrow a = 7d$

$$\text{Common ratio} = \frac{a+7d}{a} = \frac{14d}{7d} = 2$$

$$\frac{22}{2}[2a + 21d] = 275 \quad \Rightarrow \quad 2a + 21 \times \frac{a}{7} = 25 \Rightarrow a = 5$$

14.16. Sum of n Terms of a Geometric Series

Consider the series $a + ar + ar^2 + ... + ar^{n-1}$.

Let S_n denote the sum of n terms of this series. Write the series and subtract from it, term by term, the product of r and the series as shown below:

$$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$$
$$r.\, S_n = ar + ar^2 + ... + ar^{n-2} + ar^{n-1} + ar^n$$

Subtracting, $S_n - rS_n = a - ar^n \Rightarrow S_n(1 - r) = a(1 - r^n)$

$\Rightarrow$ $$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r} \quad (r \neq 1).}$$

We can write and use the above formula in the form $S_n = \frac{a(r^n - 1)}{r - 1}; (r \neq 1)$.

Cor. $\boldsymbol{S_n = \frac{a - lr}{1 - r}}$ **or** $\boldsymbol{\frac{lr - a}{r - 1}}$ if l is the last term.

The corollary can be deduced by writing $l = t_n = ar^{n-1}$ so that $lr = ar^n$.

Ex. 34. Find the sum of each of the following series:

(*i*) 5 – 10 + 20 – ... to 6 terms.

(*ii*) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4}$ + ... to 10 terms

(*iii*) 243 + 324 + 432 + ... to *n* terms.

Sol. (*i*) $S_n = \frac{a(1 - r^n)}{1 - r} (r \neq 1)$. Here, $a = 5$, $r = -2$, and $n = 6$

$$\therefore \quad S_6 = \frac{5[1 - (-2)^6]}{1 - (-2)} = \frac{5 - 5 \times 64}{3} = \frac{-315}{3} = -105.$$

(*ii*) $a = 4, r = \frac{1}{2}$ and $n = 10$

$$\therefore \quad S_{10} = \frac{4\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}} = \frac{4.\left[1 - \frac{1}{1024}\right]}{\frac{1}{2}} = 8 \times \frac{1023}{1024} = 8 \text{ (Approx.)}.$$

(*iii*) $a = 243, r = \frac{324}{243} = \frac{4}{3}$ (which is > 1)

$$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1} = \frac{243\left[\left(\frac{4}{3}\right)^n - 1\right]}{\frac{4}{3} - 1} = \frac{243.\left[\frac{4^n}{3^n} - 1\right]}{\frac{1}{3}} = 3^6.\left(\frac{4^n - 3^n}{3^n}\right) = 3^{6-n}.(4^n - 3^n).$$

Ex. 35. The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the amount of the grains for subsequent squares. How many grains would have to be given to the inventor? (There are 64 squares in the chess board).

Sol. The required number of grains

$$= 1 + 2 + 2^2 + \ldots \text{ to 64 terms} = \frac{1 \times (2^{64}) - 1}{2 - 1} = 2^{64} - 1.$$

Ex. 36. How many terms of the series 2 + 6 + 18 + ... must be taken to make the sum equal to 728?

Sol. Here, $a = 2$, $r = 3$ and $S = 728$.

Let n be the number of terms required, then, $728 = \frac{2(3^n - 1)}{3 - 1}$

$\Rightarrow \quad 3^n - 1 = 728 \Rightarrow 3^n = 729 = 3^6 \Rightarrow n = 6.$

Hence, the number of terms required is 6.

Ex. 37. An insect starts from a point and travels in a straight path one mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point?

Sol. Let the required time be n seconds.

Then $3 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ to n terms. or $3 = \frac{1 \times \left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$

$\Rightarrow \quad \frac{3}{2} = 1 - \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = \frac{-1}{2} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{-1}{2}.$

There is no value of n for which $\left(\frac{1}{2}\right)^n = \frac{-1}{2}$ as exponential functions are always positive.

Hence, the insect would never reach a point 3 mm away from its starting point.

Ex. 38. Sum to *n* terms the series:

(*i*) 7 + 77 + 777 +... **(*ii*) 0.7 + 0.77 + 0.777 +...**

Sol. (*i*) $S_n = 7 + 77 + 777 + \ldots$ to n terms

$= 7\,[1 + 11 + 111 + \ldots \text{ to } n \text{ terms}]$

$= \frac{7}{9}\,[9 + 99 + 999 + \ldots \text{ to } n \text{ terms}]$

$= \frac{7}{9}\,[(10 - 1) + (100 - 1) + (1000 - 1) + \ldots \text{ to } n \text{ terms}]$

$$= \frac{7}{9}\left[\{10 + 10^2 + 10^3 + \ldots \text{ to } n \text{ terms}\} - (1 + 1 + 1 + \ldots \text{ to } n \text{ terms})\right]$$

$$= \frac{7}{9}\left[\frac{10.(10^n - 1)}{10 - 1} - n\right] = \frac{7(10^{n+1} - 10)}{81} - \frac{7}{9}n.$$

(*ii*) $S_n = 0.7 + 0.77 + 0.777 + \ldots$ to n terms

$$= 7\,[0.1 + 0.11 + 0.111 + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\,[0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\,[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\left[n - \frac{0.1\{1 - (0.1)^n\}}{1 - 0.1}\right]$$

$$= \frac{7}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right] = \frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right].$$

14.17. Sum of an Infinite Geometric Progression

We have, $S_n = \dfrac{a - ar^n}{1 - r} = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$.

If $|r| < 1$, $r^n \to 0$ as $n \to \infty$. Therefore, $\boldsymbol{S_\infty = \dfrac{a}{1 - r}}$.

Caution. Sum to infinity exists only when r is numerically less than 1, *i.e.,* $|r| < 1$.
If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

Ex. 39. Sum the following series to *n* terms and to infinity,

$$\mathbf{1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots}$$

Sol. $a = 1,\ r = -\dfrac{1}{2} \quad \therefore\ S_n = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{1 \times \left[1 - \left(-\frac{1}{2}\right)^n\right]}{1 - \left(-\frac{1}{2}\right)} = \dfrac{\left[1 - \left(-\frac{1}{2}\right)^n\right]}{\frac{3}{2}} = \dfrac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]$

$$\Rightarrow S_\infty = \underset{n \to \infty}{Lt}\ \frac{2}{3}\left[1 - \left(\frac{-1}{2}\right)^n\right] = \frac{2}{3}(1 - 0) = \frac{2}{3}.$$

Note. We could straightway use the result $\left[S_\infty = \dfrac{a}{1 - r} = \dfrac{1}{1 - \left(-\frac{1}{2}\right)} = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}\right]$

Ex. 40. The first term of a G.P. is 2 more than the second term and the sum to infinity is 50. Find the G.P.

Sol. Let the first two terms be a, $a-2$.

Then common ratio $= \dfrac{a-2}{a}$ $\quad\therefore\ S_\infty = \dfrac{a}{1-\dfrac{a-2}{a}} = \dfrac{a^2}{2}$ $\qquad \left[S_\infty = \dfrac{a}{1-r}, |r|<1.\right]$

$\therefore\ \dfrac{a^2}{2} = 50 \Rightarrow a^2 = 100 \Rightarrow a = \pm 10.$

If $a = -10$, $\quad r = \dfrac{-10-2}{-10} = \dfrac{12}{10} = \dfrac{6}{5}$, which >1

$\therefore$ The value $a = -10$ is inadmissible $\therefore\ a = 10$.

Hence, the series is $10, 8, \dfrac{32}{5}, \ldots$

Recurring Decimals

Ex. 41. Find the value of $0.4\overline{23}$.

Sol. $0.4\overline{23} = 0.423\ 23\ 23 \ldots$ ad inf.

$= 0.4 + 0.023 + 0.00023 + \ldots$ ad inf.

$= \dfrac{4}{10} + \dfrac{23}{1000} + \dfrac{23}{100000} + \ldots$ ad inf.

$= \dfrac{4}{10} + \dfrac{23}{10^3}\left(1 + \dfrac{1}{10^2} + \dfrac{1}{10^4} + \ldots \text{ ad inf.}\right)$

$= \dfrac{4}{10} + \dfrac{23}{10^3} \times \dfrac{1}{1-\left(\dfrac{1}{10^2}\right)} = \dfrac{4}{10} + \dfrac{23}{1000} \times \dfrac{100}{99} = \dfrac{419}{990}.$ $\qquad \left[S_\infty = \dfrac{a}{1-r}\right]$

Ex. 42. The side of a given square is equal to a. The mid-points of its sides are joined to form a new square. Again, the mid-points of the sides of this new square are joined to form another square. This process is continued indefinitely. Find the sum of the areas of the squares and the sum of the perimeters of the squares. **[*ISC 1991*]**

Sol. Since the diagonal of a square is $\sqrt{2}$ times the side of a square, we get the following series as an infinite G.P. of the areas of the squares:

$$a^2 + \left(\sqrt{2} \times \frac{a}{2}\right)^2 + \left(\sqrt{2} \times \sqrt{2} \times \frac{a}{4}\right)^2 + \left(\sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \frac{a}{8}\right)^2 + \ldots \text{ to } \infty$$

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \frac{a^2}{8} + \ldots \text{ to } \infty, \quad \therefore\ S_\infty = \frac{\text{First term}}{1 - \text{common ratio}} = \frac{a^2}{1-\frac{1}{2}} = 2a^2 \text{ sq units.}$$

Similarly, since the diagonal of a square is $\sqrt{2}$ times the side of a square, we get the following series as an infinite G.P. of the sum of the perimeters of the squares:

$$4\left[a + \left(\sqrt{2} \times \frac{a}{2}\right) + \left(\sqrt{2} \times \sqrt{2} \times \frac{a}{4}\right) + \ldots \text{ to } \infty\right] = 4\left[a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \ldots \text{ to } \infty\right]$$

$$= 4 \times \frac{a}{1-\frac{1}{\sqrt{2}}} = \frac{4\sqrt{2}a}{\sqrt{2}-1} = \frac{4\sqrt{2}(\sqrt{2}+1)a}{(\sqrt{2}-1)(\sqrt{2}+1)} = 4\sqrt{2}(\sqrt{2}+1)a \text{ units.}$$

EXERCISE 14 (*f*)

1. Find the sum to

(*i*) 8 terms of $3 + 6 + 12 + ...$ (*ii*) 20 terms of $2 + 6 + 18 + ...$ **[*SC*]**

(*iii*) 10 terms of $1 + \sqrt{3} + 3 + ...$ (*iv*) n terms of $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + ...$

2. Sum the following series to infinity:

(*i*) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ (*ii*) $16, -8, 4, ...$

(*iii*) $\sqrt{2} - \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + ...$ (*iv*) $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + ...$

3. Find the sum of a geometric series in which $a = 16, r = \frac{1}{4}, l = \frac{1}{64}$.

4. Find the sum of the series $81 - 27 + 9 - ... - \frac{1}{27}$.

5. The first three terms of a G.P. are $x, x + 3, x + 9$. Find the value of x and the sum of first eight terms. **[*SC*]**

6. Of how many terms is $\frac{55}{72}$, the sum of the series $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - ...$?

7. The second term of a G.P. is 2 and the sum of infinite terms is 8. Find the first term.

8. (*i*) Find the value of $0.2\dot{3}\dot{4}$ regarding it as a geometric series.

(*ii*) Evaluate : (*a*) $0.9\overline{7}$ (*b*) $0.\overline{45}$ (*c*) $0.23\dot{4}\dot{5}$

(*iii*) Find a rational number which when expressed as a decimal will have $1.2\overline{56}$ as its expansion.

9. If $a + b + ... + l$ is a G.P., prove that its sum is $\frac{bl - a^2}{b - a}$.

10. The nth term of a geometrical progression is $\frac{2^{2n-1}}{3}$ for all values of n. Write down the numerical values of the first three terms and calculate the sum of the first 10 terms, correct to 3 significant figures. **[*ISC*]**

11. A geometrical progression of positive terms and an arithmetical progression have the same first term. The sum of their first terms is 1, the sum of their second terms is $\frac{1}{2}$ and the sum of their third terms is 2. Calculate the sum of their fourth terms. **[*ISC*]**

12. In a geometric progression, the third term exceeds the second by 6 and the second exceeds the first by 9. Find (*i*) the first term, (*ii*) the common ratio and (*iii*) the sum of the first ten terms. **[*ISC*]**

13. In an infinite geometric progression, the sum of first two terms is 6 and every term is four times the sum of all the terms that follow it. Find:

(*i*) the geometric progression and (*ii*) its sum to infinity. **[*ISC*]**

14. Three numbers are in A.P. and their sum is 15. If 1, 4 and 19 be added to these numbers respectively, the numbers are in G.P. Find the numbers. **[*ISC*]**

15. Calculate the least number of terms of the geometric progression 5 + 10 + 20 + ... whose sum would exceed 10,00,000. **[*ISC*]**

16. If S be the sum, P the product and R the sum of the reciprocals of n terms in G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$.

17. Find the sum of the first n terms of the series:

$0.2 + 0.22 + 0.222 + ...$ **[*WBJEE 2009*]**

18. If $\frac{2}{3} = \left(x - \frac{1}{y}\right) + \left(x^2 - \frac{1}{y^2}\right) + ...$ to ∞

and $xy = 2$, then calculate the values of x and y with the condition that $x < 1$. **[*ISC*]**

19. $S_1, S_2, S_3, ..., S_n$ are sums of n infinite geometric progressions. The first terms of these progressions are $1, 2^2 - 1, 2^3 - 1, ..., 2^n - 1$ and the common ratios are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ..., \frac{1}{2^n}$. Calculate the value of $S_1 + S_2 + ... + S_n$. **[*ISC*]**

20. Find three numbers a, b, c between 2 and 18 such that:

(i) their sum is 25, and

(ii) the numbers 2, a, b are consecutive terms of an arithmetic progression, and

(iii) the numbers b, c, 18 are consecutive terms of a geometric progression. **[*ISC*]**

21. Three numbers, whose sum is 21, are in A.P. If 2, 2, 14 are added to them respectively, the resulting numbers are in G.P. Find the numbers. **[*ISC 1990*]**

22. If $x = 1 + a + a^2 + ... \infty$, $a < 1$ and $y = 1 + b + b^2 + ... \infty$, $b < 1$,

then prove that $1 + ab + a^2b^2 + ... \infty = \frac{xy}{(x + y - 1)}$.

23. If $S_1, S_2, S_3, ..., S_p$ are the sums of infinite geometric series whose first terms are 1, 2, 3,...., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p+1}$ respectively, prove that

$$S_1 + S_2 + S_3 + ... + S_p = \frac{1}{2} p (p + 3).$$ **[*TS Rajendra*]**

ANSWERS

1. (i) 765, (ii) $3^{20} - 1$, (iii) $121(\sqrt{3} + 1)$, (iv) $\frac{81}{8}\left\{1 - \left(\frac{2}{3}\right)^n\right\}$

2. (i) 2, (ii) $\frac{32}{3}$, (iii) $\frac{2\sqrt{2}}{3}$, (iv) $\frac{3\sqrt{3}}{2}$

3. $21\frac{21}{64}$ **4.** $60\frac{20}{27}$ **5.** $x = 3$; $S_8 = 765$ **6.** 5 **7.** 4

8. (i) $\frac{116}{495}$, (ii) (a) $\frac{44}{45}$, (b) $\frac{5}{11}$, (c) $\frac{129}{550}$, (iii) $\frac{622}{495}$

10. First 3 terms are $\frac{2}{3}, \frac{8}{3}, \frac{32}{3}$; $S_{10} = 233000$ to 3 s.f. **11.** $9\frac{1}{2}$

12. (i) -27 (ii) $\frac{2}{3}$ (iii) $-81\left[1 - \left(\frac{2}{3}\right)^{10}\right]$. **13.** ($i$) 5, 1, $\frac{1}{5}$,... (ii) $6\frac{1}{4}$ **14.** 2, 5, 8 or 26, 5, –16

15. 18 **17.** $\frac{2}{9}n - \frac{2}{81} + \frac{2}{81}(0.1)^n$ **18.** $x = \frac{1}{2}, y = 4$ **19.** $2(2^n - 1)$ **20.** $a = 5, b = 8, c = 12$

21. 1, 7, 13 or 25, 7, –11

HINTS TO SELECTED QUESTIONS

5. $\frac{x+3}{x} = \frac{x+9}{x+3} \Rightarrow x = 3 \therefore$ Common ratio $= \frac{x+3}{x} = 2$

8. $1.2\overline{56} = 1.2 + 0.056 + 0.00056 + 0.0000056 + ... = 1.2 + \frac{0.056}{1-0.01}$. $\left\{\text{using } S_\infty = \frac{a}{1-r}\right\}$

11. Let a be the first term of both the A.P. as well as G.P. and d the common difference of A.P. and r the common ratio of G.P. Then,

Given : $a + a = 1 \Rightarrow a = \frac{1}{2}$

$$\left.\begin{aligned} \left(\frac{1}{2} + d\right) + \frac{1}{2}r = \frac{1}{2} \quad &\Rightarrow \quad 2d + r = 0 \\ \left(\frac{1}{2} + 2d\right) + \frac{1}{2}r^2 = 2 \quad &\Rightarrow \quad 4d + r^2 = 3 \end{aligned}\right\} \text{Solve}$$

13. Given $a + ar = 6$, $ar^{n-1} = 4$ $(ar^n + ar^{n+1} + ...)$

14. Let the three numbers in A.P. be $a - d$, a, $a + d$.

15. $S_n = \frac{5.(2^n - 1)}{2-1} = 5.(2^n - 1)$

We have to find the least value of n such that

$5(2^n - 1) > 10{,}00{,}000 \Rightarrow 2^n - 1 > 2{,}00{,}000 \Rightarrow 2^n > 2{,}00{,}001$ ...(1)

If $n = 17$, then $2^n = 2^{17} = 131072$ which is < 2,00, 001.

If $n = 18$, then $2^n = 2^{18} = 262144$ which is > 2,00,0001.

Hence, the least value of $n = 18$.

Alternatively. From (1), we have $n \log 2 > \log 2{,}00{,}001 \Rightarrow n > \frac{\log 200001}{\log 2}$

$\Rightarrow n > \frac{5.3010}{0.3010}$ [From log tables] $\Rightarrow n > 17.61$. Hence, $n = 18$.

17. Type Solved Ex. 35 (*ii*).

18. $\frac{2}{3} = (x + x^2 + ... \infty) - \left(\frac{1}{y} + \frac{1}{y^2} + ... \infty\right)$

$\Rightarrow \frac{2}{3} = \frac{x}{1-x} - \frac{1/y}{1-1/y} \Rightarrow x + y = \frac{9}{2} \Rightarrow \frac{2}{y} + y = \frac{9}{2}$ $(\because xy = 2 \text{ and } x < 1)$

20. $(i) \Rightarrow a + b + c = 25$, ... (1) $(ii) \Rightarrow 2a = 2 + b$ ... (2) $(iii) \Rightarrow c = \sqrt{18b}$ ... (3)

Combining (1), (2) and (3), we have $\frac{2+b}{2} + b + \sqrt{18b} = 25 \Rightarrow 2\sqrt{18b} = 48 - 3b$

Square both sides and solve for b. You get $b = 32, 8$. Since b lies between 2 and 18, therefore, $b = 8$.

14.18. Special Types of Problems

Ex. 43. ***Find three real numbers in G .P. whose sum is 30 and product is 216.***

Sol. Let the three real numbers in G.P. be $\frac{a}{r}$, a, ar.

$\because$ Their product = 216 $\therefore \frac{a}{r}.a.\,ar = 216 \Rightarrow a^3 = 6^3 \Rightarrow a = 6$

Their sum = 30 $\therefore \frac{a}{r} + a + ar = 30 \Rightarrow \frac{6}{r} + 6 + 6r = 30 \Rightarrow 6r^2 - 24r + 6 = 0$

$\Rightarrow r^2 - 4r + 1 = 0$, which is a quadratic equation in r.

(Disct.) $= b^2 - 4ac = (-4)^2 - 4\,(1) = 12 \quad \therefore \quad r = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

when $a = 6$, $r = 2 + \sqrt{3}$, the three numbers are $\frac{6}{2+\sqrt{3}}, 6, 6\,(2+\sqrt{3})$

i.e., $\frac{6}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}, 6, 6\,(2+\sqrt{3})$ *i.e.,* $6(2-\sqrt{3}), 6, 6\,(2+\sqrt{3})$

when $a = 6$, $r = 2 - \sqrt{3}$, the three numbers are $\frac{6}{2-\sqrt{3}}, 6, 6\,(2-\sqrt{3})$

i.e., $\frac{6}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}, 6, 6\,(2-\sqrt{3})$, *i.e.,* $6(2+\sqrt{3}), 6, 6\,(2-\sqrt{3})$

Note. When product is known, suppose the numbers in G.P. are as under:

(*i*) If the odd numbers of terms are to be considered, suppose them as follows by taking a as the mid-term and common ratio r,

$$\frac{a}{r}, a, ar; \quad \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2.$$

(*ii*) If even number of terms are to be considered, suppose them as under by taking as $\frac{a}{r}, ar$ as the two mid-terms and r^2 as the common ratio.

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3 \text{ [4 terms]} \quad \frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5. \text{ [6 terms]}$$

Ex. 44. The sum of the first three terms of G.P. is 7 and the sum of their squares is 21. Determine the first five terms of the G.P.

Sol. Here, the product of the three terms in G.P. is not given, so it is not necessary to suppose them in a special manner.

Let the first three terms of the G.P. be a, ar, ar^2.

$\because$ Their sum = 7 $\therefore a + ar + ar^2 = 7 \Rightarrow a\,(1 + r + r^2) = 7$ ...(1)

$\because$ Sum of the squares of the terms = 21 $\therefore a^2 + a^2r^2 + a^2r^4 = 21$

$\Rightarrow a^2(1 + r^2 + r^4) = 21 \Rightarrow a^2\,(1 + r^2 + r)\,(1 + r^2 - r) = 21$ **[Note this step]** ...(2)

Squaring (1), $a^2\,(1 + r + r^2)^2 = 49$. ...(3)

Dividing (2) by (3), $\frac{1-r+r^2}{1+r+r^2}=\frac{3}{7} \Rightarrow 7-7r+7r^2=3+3r+3r^2$

$\Rightarrow 2r^2-5r+2=0 \Rightarrow (2r-1)(r-2)=0 \Rightarrow r=\frac{1}{2}$ or 2.

when $r=\frac{1}{2}$ from (1), $a\left(1+\frac{1}{2}+\frac{1}{4}\right)=7 \Rightarrow a=4$

$\therefore$ The first five terms of the G. P. are 4, 2, 1, $\frac{1}{2}, \frac{1}{4}$.

when $r = 2$ from (1), $a(1 + 2 + 4) = 7 \Rightarrow a = 1$

$\therefore$ The first five terms of the G.P. are 1, 2, 4, 8, 16.

Ex. 45. If *a, b, c, d* are in G.P., prove that $a^2-b^2, b^2-c^2, c^2-d^2$ are also in G.P.

Sol. Let r be the common ratio of the G. P. Then $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=r$

$\Rightarrow$ $b = ar, c = br = ar^2, d = cr = ar^3$

$a^2-b^2, b^2-c^2, c^2-d^2$ will be in G.P. if $(b^2-c^2)^2=(a^2-b^2)(c^2-d^2)$

i.e., if $(a^2r^2-a^2r^4)^2=(a^2-a^2r^2)(a^2r^4-a^2r^6)$

i.e., if $a^4r^4(1-r^2)^2=a^2(1-r^2)\,a^2r^4(1-r^2)=a^4r^4(1-r^2)^2$, which is true.

Hence, $a^2-b^2, b^2-c^2, c^2-d^2$ **are in G.P.**

Method. To solve questions of this type

(1) *Let the common ratio of the G.P. be r.*

(2) *Write the values of the given terms in terms of r.*

(3) *Apply the condition that the other terms are in G.P. Establish its truth by substituting known values from (2).*

EXERCISE 14 (*g*)

1. Find three numbers in G.P. whose sum is 19 and product is 216.

2. The sum of the first three terms of a G.P. is $\frac{13}{12}$ and their product is –1. Find the G.P.

3. The product of first three terms of a G.P. is 1000. If we add 6 to its second term and 7 to its third term, the resulting three terms form an A.P. Find the terms of the G.P.

4. If *a, b, c* are in G.P. show that the following are also in G.P.

(*i*) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ (*ii*) $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ (*iii*) a^2, b^2, c^2 (*iv*) b^2c^2, c^2a^2, a^2b^2

5. If *a, b, c, d* are in G. P., show that the following are also in G. P.

(*i*) $a + b, b + c, c + d$ (*ii*) $a^2 + b^2, b^2 + c^2, c^2 + d^2$

6. If *a, b, c, d* are in G.P., prove that

(*i*) $(b + c)(b + d) = (c + a)(c + d)$ (*ii*) $(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2$

7. If the *p*th, *q*th and *r*th terms of an A. P. are in G.P., prove that the common ratio of the G.P. is $\frac{q-r}{p-q}$.

8. If $\frac{1}{x+y}, \frac{1}{2y}, \frac{1}{y+z}$ are the three consecutive terms of an A.P. prove that x, y, z are the three consecutive terms of a G.P.

ANSWERS

1. 4, 6, 9 or 9, 6, 4 **2.** $\frac{4}{3}, -1, \frac{3}{4}, \ldots\ldots;$ or $\frac{3}{4}, -1, \frac{4}{3}, \ldots\ldots$ **3.** 20, 10, 5 or 5, 10, 20

HINTS TO SELECTED QUESTIONS

7. Common ratio $= \frac{a+(q-1)d}{a+(p-1)d} = \frac{a+(r-1)d}{a+(q-1)d} = \frac{q-r}{p-q}\left(\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then each} = \frac{a-c}{b-d}\right)$.

8. $\frac{1}{x+y}, \frac{1}{2y}, \frac{1}{y+z}$ are in A.P. $\Rightarrow \frac{1}{2y} - \frac{1}{x+y} = \frac{1}{y+z} - \frac{1}{2y}$

$\Rightarrow \frac{x-y}{x+y} = \frac{y-z}{y+z} \Rightarrow \frac{(x-y)+(x+y)}{(x-y)-(x+y)} = \frac{(y-z)+(y+z)}{(y-z)-(y+z)}$

[If $\frac{a}{b} = \frac{c}{d}$, then by componendo and dividendo $\frac{a+b}{a-b} = \frac{c+d}{c-d}$]

$\Rightarrow \frac{2x}{-2y} = \frac{2y}{-2z} \Rightarrow \frac{y}{x} = \frac{z}{y} \Rightarrow x, y, z$ are in G.P.

14.19. Arithmetico-geometric Series

A series each term of which is found by multiplying the corresponding terms of an A.P. and a G.P. is called an Arithmetico-Geometric Series.

The *General* or *Standard* form of such a series is

$a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \ldots\ldots + \{a + (n - 1)d\}r^{n-1} + \ldots\ldots$ each term of which is formed by multiplying the corresponding terms of the terms of the two series.

$a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + \{a + (n - 1)d\} + \ldots\ldots$ (A.P.)

$1 + r + r^2 + r^3 + \ldots\ldots + r^{n-1} + \ldots\ldots$ (G.P.)

Note. If in the Arithmetico-Geometric series, we put $r = 1$, we get A. P., and if we put $d = 0$, we get G.P.

14.20. To Find the Sum of an Arithmetico-geometric Series

Let S_n be the sum of the n terms of the series $a + (a + d)r + (a + 2d)r^2 + \ldots + \{a + (n - 1)d\} r^{n-1}$

$S_n = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots + \{a + (n - 1)d\}r^{n-1}$

$r \cdot S_n = ar + (a + d)r^2 + \ldots\ldots + \{a + (n - 2)d\}r^{n-1} + \{a + (n - 1)d\}r^n.$

Subtracting, we get

$(1 - r)S_n = (a + dr + dr^2 + \ldots\ldots + dr^{n-1}) - \{a + (n - 1)d\}r^n.$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - \{a + (n-1)d\}r^n$$

$$\therefore \quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a + (n-1)d\}r^n}{1-r}.$$

Cor. If $r < 1$ numerically, $r^{n-1} \to 0$ and $r^n \to 0$ as $n \to \infty$, therefore $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$.

Note : This result is directly applicable to those A-G series whose G.P. is of the form $1, r, r^2, \ldots$ *i.e.*, whose first term is 1.

Ex. 46. Find the nth term of the series

(*i*) $\dfrac{1}{3} + \dfrac{3}{9} + \dfrac{5}{27} + \dfrac{7}{81} + \ldots$ **(*ii*)** $1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots$

Sol. (*i*) This is an arithmetico-geometric series and therefore

$$T_n = (n\text{th term of } 1, 3, 5, 7, \ldots) \times \left(n\text{th term of } \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\right)$$

$$= [1 + (n-1)2]\left[\left(\frac{1}{3}\right)^n\right] = (2n-1)\frac{1}{3^n} = \frac{2n-1}{3^n}.$$

(*ii*) Even if this is not an arithmetico-geometric series, the law of formation reveals that $T_n = (n\text{th term of } 1^2, 2^2, 3^2, 4^2, \ldots) \times (n\text{th term of } 1, x, x^2, x^3, \ldots) = n^2 x^{n-1}$.

Ex. 47. Find the sum of the series

$1 + 4x + 7x^2 + \ldots$ to n terms. Hence write down the sum to infinity when $x < 1$ numerically.

Sol. Let $S_n = 1 + 4x + 7x^2 + \ldots + (3n-5)\,x^{n-2} + (3n-2)\,x^{n-1}$

$x.\, S_n = x + 4x^2 + \ldots + (3n-5)\,x^{n-1} + (3n-2)\,x^n$.

Subtracting, we get

$$\therefore \quad (1-x)S_n = 1 + 3x + 3x^2 + \ldots + 3x^{n-1} - (3n-2)\,x^n = 1 + \frac{3x\,(1-x^{n-1})}{1-x} - (3n-2)\,x^n$$

$$\therefore \quad S_n = \frac{1}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-2)\,x^n}{1-x}.$$

Sum to infinity $= \dfrac{1}{1-x} + \dfrac{3x}{(1-x)^2}$ as $\underset{n\to\infty}{\text{Lt}}\, x^{n-1} = 0$ and $\underset{n\to\infty}{\text{Lt}}\, x^n = 0$ as $-1 < x < 1$.

Ex. 48. Sum to infinity

$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots$ $x < 1$ numerically.

Sol. Though this is not an arithmetico-geometric series, it can be summed by a somewhat similar method.

Let $S = 1 + 4x + 9x^2 + 16x^3 + \ldots$

$\Rightarrow$ $xS = x + 4x^2 + 9x^3 + \ldots$

Subtracting, we get $S(1-x) = 1 + 3x + 5x^2 + 7x^3 + \ldots$

Alternatively, we can obtain the sum by direct application of the formula as under:

For an AGP, $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

$\Rightarrow$ $S(1-x) = \dfrac{1}{1-x} + \dfrac{2x}{(1-x)^2}$, [Applying the formula $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$]

$$\Rightarrow \quad S(1-x) = \frac{1-x+2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

$$\Rightarrow \qquad S = \frac{1+x}{(1-x)^3}.$$

Ex. 49. Sum to infinity

$1 + (1 + b)r + (1 + b + b^2)\, r^2 + (1 + b + b^2 + b^3)r^3 + \ldots$

***r* and *b* being proper fractions.**

Sol. Let $S = 1 + (1 + b)r + (1 + b + b^2)\, r^2 + (1 + b + b^2 + b^3)\, r^3 + \ldots$

$rS = r + (1 + b)\, r^2 + (1 + b + b^2)\, r^3 + \ldots$

Subtracting $(1 - r)S = 1 + br + b^2r^2 + b^3r^3 + \ldots$

$$= 1 + \frac{br}{1-br} = \frac{1-br+br}{1-br} = \frac{1}{1-br} \qquad \therefore S = \frac{1}{(1-r)(1-br)}.$$

Ex. 50. Show that $2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \ldots \infty = 2$. ***[AIEEE 2002]***

Sol. Let $x = 2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \ldots$

$$= 2^{\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots \infty}$$

$$= 2^{\frac{1}{4}\left(1 + 2.\frac{1}{2} + 3.\frac{1}{2^2} + 4.\frac{1}{2^3} + \ldots \infty\right)}$$

$$= 2^{\frac{1}{4}\left[\frac{1}{1-\frac{1}{2}} + \frac{1.\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}\right]}.$$

[Applying the result $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$ to the AGP inside the bracket]

$$= 2^{\frac{1}{4}\left(2 + \frac{1}{2} \times 4\right)}$$

$$= 2^{\frac{1}{4} \times 4} = 2^1 = 2.$$

EXERCISE 14 (*h*)

Sum up to *n* terms the series

1. $1 + 2x + 3x^2 + 4x^3 + \ldots$ **2.** $1 + 3x + 5x^2 + 7x^3 + \ldots$ **3.** $2.1 + 3.2 + 4.4 + 5.8 + \ldots$

Sum up to infinity the series

4. $\frac{1}{2} + \frac{3}{6} + \frac{5}{18} + \ldots$ **5.** $\frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \ldots$ **6.** $1 - \frac{2}{5} + \frac{3}{5^2} - \frac{4}{5^3} + \ldots$

7. Sum up the series

$\frac{2}{3} + \frac{5}{9} + \frac{8}{27} + \frac{11}{81} + \ldots$ to *n* terms and hence find the sum to infinity.

Sum up to infinity, given that $x < 1$ numerically.

8. $1 + 4x^2 + 7x^4 + \ldots$ **9.** $1 - x + 2x^2 - 3x^3 + 4x^4 - \ldots$

10. $1^2 + 3^2x + 5^2x^2 + 7^2x^3 + \ldots$ **[Hint.** Type Solved Ex. 55]

11. Show that the square root of $3^{\frac{1}{2}} \times 9^{\frac{1}{4}} \times 27^{\frac{1}{8}} \times 81^{\frac{1}{16}} \times \ldots$ to infinity is 3.

[Note. When we are asked to find the square root of a number, it is presumed that we have to find the principal square root.]

[Hint. $S = 3^{\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots\infty}$ ... (1)

Find the sum of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{10}+\ldots \infty$ and substitute in (1).]

ANSWERS

1. $\frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$ **2.** $\frac{1+x}{(1-x)^2} - \frac{2x^n}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$ **3.** $n \cdot 2^n$ **4.** $\frac{3}{2}$

5. $\frac{15}{16}$ **6.** $\frac{25}{36}$ **7.** $\frac{7}{4} - \frac{6n+7}{4.3^n}, \frac{7}{4}$ **8.** $\frac{1}{1-x^2} + \frac{3x^2}{(1-x^2)^2}$

9. $\frac{1+x+x^2}{(1+x)^2}$ **10.** $\frac{1+7x}{(1-x)^2} + \frac{8x^2}{(1-x)^3}$

SERIES INVOLVING NATURAL NUMBERS

14.21. To find the Sum of First n Natural Numbers

Let $S = 1 + 2 + 3 + \ldots + n$.

This is an A.P. whose first term is 1 and common difference is 1, and nth term is n.

$$S = \frac{n}{2}(n+1), \text{ using } S = \frac{n}{2}(a+l).$$

Thus, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Using the sigma notation this may be written as $\sum n = \frac{n(n+1)}{2}$.

14.22. To Find the Sum of the Squares of the First *n* Natural Numbers

Let $S = 1^2 + 2^2 + 3^2 + \ldots + n^2$.

We know that $x^3 - (x-1)^3 = 3x^2 - 3x + 1$.

Putting $x = n, n-1, n-2, \ldots, 3, 2, 1$ successively in this identity, we get

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$
$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$
$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$
$$1^3 - 0^3 = 3.1^2 - 3.1 + 1.$$

Adding, we get $n^3 = 3[1^2 + 2^2 + 3^2 + \ldots + n^2] - 3[1 + 2 + 3 + \ldots + n] + n$

$$\Rightarrow \quad n^3 = 3S - 3\frac{n(n+1)}{2} + n \quad \Rightarrow \quad 3S = (n^3 - n) + 3\frac{n(n+1)}{2} = n(n+1)(n-1) + \frac{3n(n+1)}{2}$$

$\Rightarrow \quad 6S = 2n(n+1)(n-1) + 3n(n+1) = n(n+1)(2n-2+3) = n(n+1)(2n+1)$

$\therefore \qquad S = \dfrac{n(n+1)(2n+1)}{6}$

Using the sigma notation this may be written as $\boxed{\sum n^2 = \dfrac{n(n+1)(2n+1)}{6}.}$

14.23. To Find the Sum of the Cubes of the First *n* Natural Numbers

Let $S = 1^3 + 2^3 + 3^3 + ... + n^3$.

We know that $x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1$.

Putting $x = n, n-1, n-2, ..., 3, 2, 1$ successively in this identity, we get

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1$$

.....

$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$
$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$
$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1.$$

Adding, we get

$$n^4 = 4(1^3 + 2^3 + 3^3 + ... + n^3) - 6(1^2 + 2^2 + 3^2 + ... + n^2) + 4(1 + 2 + 3 + ... + n) - n$$

$$\Rightarrow \quad n^4 = 4S - 6\,\frac{n(n+1)(2n+1)}{6} + 4.\frac{n(n+1)}{2} - n$$

$$\therefore \quad 4S = (n^4 + n) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2) = n(n+1)(n^2 + n) = n^2(n+1)^2$$

$$S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Using the Sigma notation, this may be written as $\boxed{\sum n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2 = \left(\sum n\right)^2.}$

It is clear that the sum of the cubes of the first *n* natural numbers is a perfect square, namely, the square of the sum of the first *n* natural numbers. **[*EAMCET 2007*]**

14.24. A very Important Theorem

If $T_n = an^3 + bn^2 + cn + d$, *then* $S_n = a.\Sigma n^3 + b.\Sigma n^2 + c.\Sigma n + dn$.

Let S_n denote the sum.

$$T_n = an^3 + bn^2 + cn + d.1$$

Put $n = 1, 2, 3, ..., n-2, n-1, n.$

$$T_1 = a.1^3 + b.1^2 + c.1 + d.1$$
$$T_2 = a.2^3 + b.2^2 + c.2 + d.1$$
$$T_3 = a.3^3 + b.3^2 + c.3 + d.1$$

.....

$$\therefore \quad T_{n-2} = a.(n-2)^3 + b.(n-2)^2 + c(n-2) + d.1$$

$$T_{n-1} = a.(n-1)^3 + b.(n-1)^2 + c(n-1) + d \quad .1$$
$$T_n = a.n^3 + b.n^2 + cn + d \quad .1$$

Adding, we get

$$S_n = a(1^3 + 2^3 + 3^3 + \ldots + n^3) + b(1^2 + 2^2 + 3^2 + \ldots + n^2) + c\,(1 + 2 + 3 + \ldots + n) + d\,(1 + 1 + 1 + \ldots \text{ } n \text{ times})$$
$$= a.\Sigma n^3 + b.\Sigma n^2 + c.\Sigma n + d.n.$$

Ex. 51. Find the *n*th term and then the sum to *n* terms of the series 3.5 + 4.7 + 5.9 + ...

Sol. Here, each term is the product of two factors. The factors 3, 4, 5, ... are in A.P. having 3 as the first term and 1 as the common difference and therefore the *n*th factor is $3 + (n - 1)$, *i.e.,* $2 + n$. Also the other factors 5, 7, 9, ... are in A.P., having 5 as the first term and 2 as the common difference and therefore the *n*th factor is $5 + (n - 1)\,2$, *i.e.,* $2n + 3$.

$\therefore$ The *n*th term of the series is $(2 + n)(2n + 3)$.

$\Rightarrow \quad T_n = 2n^2 + 7n + 6$

$\therefore \quad \Sigma T_n = 2.\Sigma n^2 + 7.\Sigma n + 6n$

$$\Rightarrow \quad S_n = \frac{2n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + 6n$$
$$= \frac{2n(n+1)(2n+1) + 21n(n+1) + 36n}{6} = \frac{4n^3 + 27n^2 + 59n}{6}$$

Hence, the *n*th term is $2n^2 + 7n + 6$, and the sum of the *n* terms is $\dfrac{4n^3 + 27n^2 + 59n}{6}$.

Ex. 52. Find the *n*th term and the sum to *n* terms of the following series:

(*i*) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$ (*ii*) $1^2 + 3^2 + 5^2 + \ldots$

(*iii*) $1.4.7 + 2.5.8 + 3.6.9 + \ldots$

Sol. (*i*) $$T_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

$\therefore$ $$S_n = \frac{1}{3}.\sum n^3 + \frac{1}{2}.\sum n^2 + \frac{1}{6}.\sum n$$
$$= \frac{1}{3}.\frac{n^2(n+1)^2}{4} + \frac{1}{2}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}.\frac{n(n+1)}{2}$$

or $$S_n = \frac{n(n+1)}{12}\{n(n+1) + (2n+1) + 1\}$$
$$= \frac{n(n+1)(n^2 + 3n + 2)}{12} = \frac{n(n+1)^2(n+2)}{12}$$

(*ii*) $$T_n = \{1 + (n-1)2\}^2 = (2n-1)^2 = 4n^2 - 4n + 1.$$

$\therefore$ $$S_n = 4.\Sigma n^2 - 4.\Sigma n + n$$
$$= 4.\frac{n(n+1)(2n+1)}{6} - 4.\frac{n(n+1)}{2} + n = n\left[\frac{2}{3}(2n^2 + 3n + 1) - 2n - 2 + 1\right]$$
$$= n\left[\frac{4}{3}n^2 + 2n + \frac{2}{3} - 2n - 1\right] = n\left[\frac{4}{3}n^2 - \frac{1}{3}\right] = \frac{n}{3}(4n^2 - 1).$$

(*iii*) $$T_n = n(n+3)(n+6) = n^3 + 9n^2 + 18n.$$

$\therefore$ $$S_n = \Sigma n^3 + 9.\Sigma n^2 + 18.\Sigma n$$

$$= \frac{n^2(n+1)^2}{4} + \frac{9n(n+1)(2n+1)}{6} + \frac{18n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 3(2n+1) + 18\right]$$

$$= \frac{n(n+1)}{4}(n^2 + n + 12n + 6 + 36) = \frac{n(n+1)(n^2 + 13n + 42)}{4}$$

Ex. 53. Sum up the series

$$\mathbf{1.}n + \mathbf{2.}(n-1) + \mathbf{3.}(n-2) + \ldots n\mathbf{.1.}$$

Sol. Here, find the rth term, instead of finding the nth term.

$$T_r = r.\,[n - (r-1)] = r\,(n - r + 1) = (n+1).\,r - r^2.$$

$$\therefore \quad S = (n+1)\,.\sum_1^n r - \sum_1^n r^2$$

$$= (n+1).\frac{n(n+1)}{2} - \frac{n(n+1)\,(2n+1)}{6} = \frac{n(n+1)}{2}\left[(n+1) - \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left(\frac{3n+3-2n-1}{3}\right) = \frac{n(n+1)\,(n+2)}{6}.$$

14.25. Method of Differences

In certain series, which are neither A.P. nor G.P., it may happen that the successive differences like $(T_2 - T_1)$, $(T_3 - T_2)$, $(T_4 - T_3)$, etc. may form an A.P., or G.P. If it so happens, the nth term can be found and then the sum can be deduced.

Ex. 54. The natural numbers are grouped as follows:

(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), ...

Find an expression for the first term of the nth group. **[*ISC*]**

Sol. We have to find the nth term of the progression 1, 2, 4, 7 ...

Let T_n be the nth term and S denote the sum of the first n terms, then

$$S = 1 + 2 + 4 + 7 + 11 + \ldots + T_{n-2} + T_{n-1} + T_n$$

Also $$S = 1 + 2 + 4 + 7 + \ldots + T_{n-2} + T_{n-1} + T_n$$

Subtracting, we get

$$0 = 1 + 1 + 2 + 3 + 4 + \ldots \text{ up to } (n \text{ terms}) - T_n$$

$$\therefore \quad T_n = 1 + 1 + 2 + 3 + 4 + \ldots \text{ up to } (n \text{ terms})$$

$$= 1 + [1 + 2 + 3 + 4 + \ldots \text{ up to } (n-1) \text{ terms}]$$

$$= 1 + \frac{n-1}{2}(1 + n - 1) = \frac{2 + n^2 - n}{2} \quad \Rightarrow \quad T_n = \frac{n^2 - n + 2}{2}$$

Ex. 55. Find the nth term and deduce the sum to n terms of the series

$$\mathbf{4 + 11 + 22 + 37 + 56 + \ldots}.$$

Sol. Let $$S = 4 + 11 + 22 + 37 + 56 + \ldots + T_{n-1} + T_n$$

Also $$S = 4 + 11 + 22 + 37 + \ldots + T_{n-2} + T_{n-1} + T_n$$

Subtracting, we get $$0 = 4 + 7 + 11 + 15 + 19 + \ldots + (T_n - T_{n-1}) - T_n$$

$$T_n = 4 + \{7 + 11 + 15 + 19 + \ldots (n-1) \text{ terms}\} \qquad (\textit{Note this step.})$$

$$= 4 + \frac{n-1}{2}[14 + (n-2)\,4] \qquad \text{(A.P.)}$$

$$\therefore \quad = 4 + \frac{n-1}{2}(4n+6) = 4 + 2n^2 + n - 3 = 2n^2 + n + 1$$

$$S_n = 2.\Sigma n^2 + \Sigma n + \Sigma 1$$

$$= 2\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = n\left[\frac{(n+1)(2n+1)}{3} + \frac{n+1}{2} + 1\right]$$

$$= n\left[\frac{2n^2+3n+1}{3} + \frac{n+1}{2} + 1\right] = \frac{n}{6}[4n^2 + 6n + 2 + 3n + 3 + 6] = \frac{n}{6}(4n^2 + 9n + 11).$$

Ex. 56. Find the *n*th term and deduce the sum to *n* terms of the series 1 + 5 + 13 + 29 + ...

Sol. Let $S = 1 + 5 + 13 + 29 + \ldots + T_{n-1} + T_n$

Also $S = 1 + 5 + 13 + \ldots + T_{n-2} + T_{n-1} + T_n$

Subtracting, we get $0 = 1 + 4 + 8 + 16 + \ldots + (T_n - T_{n-1}) - T_n$

$$\therefore \quad T_n = 1 + \{4 + 8 + 16 + \ldots (n-1) \text{ terms}\}$$

$$= 1 + \frac{4(2^{n-1}-1)}{2-1} = 1 + 4.2^{n-1} - 4 \qquad (\textit{GP})$$

$$\Rightarrow \quad T_n = 4.2^{n-1} - 3$$

$$n = 1, 2, 3, .., n-1$$

$$T_{n-1} = 4.2^{n-2} - 3$$

$$T_{n-2} = 4.2^{n-3} - 3$$

.....

$$T_3 = 4.2^2 - 3$$

$$T_2 = 4.2^1 - 3$$

$$T_1 = 4.2^0 - 3$$

Adding, we get $S = T_1 + T_2 + T_3 + \ldots T_n$

$$S = 4\,(1 + 2^1 + 2^2 + \ldots + 2^{n-3} + 2^{n-2} + 2^{n-1}) - 3n$$

$$= 4.\frac{2^n - 1}{2-1} - 3n = 4.2^n - 4 - 3n = 2^{n+2} - 3n - 4.$$

Ex. 57. Sum up 5 + 55 + 555 + ... to *n* terms.

Sol. Let $S = 5 + 55 + 555 + \ldots + T_{n-1} + T_n$

Also $S = 5 + 55 + \ldots + T_{n-2} + T_{n-1} + T_n$

Subtracting, we get $0 = 5 + 50 + 500 + \ldots + (T_n - T_{n-1}) - T_n.$

$$T_n = 5 + 50 + 500 + \ldots n \text{ terms}$$

$$T_n = \frac{5(10^n - 1)}{10 - 1} = \frac{5}{9}(10^n - 1)$$

$$T_n = \frac{5}{9}(10^n - 1)$$

$\therefore \quad T_{n-1} = \frac{5}{9}(10^{n-1} - 1)$

$T_{n-2} = \frac{5}{9}(10^{n-2} - 1)$

.....

$T_2 = \frac{5}{9}(10^2 - 1)$

$T_1 = \frac{5}{9}(10 - 1)$

Adding, we get $S = \frac{5}{9}\{(10 + 10^2 + 10^3 + ... + 10^{n-1} + 10^n) - \Sigma 1\}$

$= \frac{5}{9}\left\{\frac{10(10^n - 1)}{10 - 1} - n\right\} = \frac{5}{9}\left\{\frac{10}{9}(10^n - 1) - n\right\}$

Ex. 58. Find the *n*th term, sum of *n* terms, and sum to infinity of the series

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + ...$$ **[*Karnataka CET 2007*]**

Sol. $T_n = \frac{1}{(n\text{th term of } 2, 5, 8, ...)\,(n\text{th term of } 5, 8, 11, ...)}$

$= \frac{1}{(3n-1)(3n+2)} = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$

(*Note this step of splitting into partial fractions.*)

$\therefore \quad T_n = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$

Putting $n = 1, 2, 3, ... n$, we get

$T_1 = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$

$T_2 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$

$T_3 = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right).$

.....

$T_n = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right).$

Adding and cancelling the terms, we get

$S_n = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6} - \frac{1}{3}.\frac{1}{3n+2} = \frac{3n+2-2}{6(3n+2)}$

$\Rightarrow \quad S_n = \frac{n}{2(3n+2)} = \frac{1}{2\left(3 + \frac{2}{n}\right)}$

As $n \to \infty, \frac{2}{n} \to 0$. Therefore, $S_n \to \frac{1}{6}$.

Hence, the sum to ∞, *i.e.* $S_\infty = \frac{1}{6}$.

Ex. 59. Sum up to *n* terms the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots$$

Deduce the sum to infinity also.

Sol. $T_n = \frac{1}{1+2+3+\ldots+n}$

$\Rightarrow$ $T_n = \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{(n+1)n} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$ **[Splitting into partial fractions]**

$\therefore$ $T_n = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$

Putting $n = 1, 2, 3, \ldots, n$, we get $T_1 = 2\left(\frac{1}{1} - \frac{1}{2}\right)$

$T_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$

$T_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$

.....

$T_n = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$

Adding and cancelling the terms, we get

$$S_n = 2\left(\frac{1}{1} - \frac{1}{n+1}\right) = \frac{2n}{n+1} = \frac{2}{1+\frac{1}{n}}.$$

For sum to infinity we note that $\frac{1}{n} \to 0$ as $n \to \infty$, and therefore $S_n \to 2$, *i.e.*, $S_\infty = 2$.

EXERCISE 14 (*i*)

1. Find the sum to *n* terms of the series whose *n*th term is

(*i*) $n(n + 2)$ (*ii*) $3n^2 + 2n$ (*iii*) $4n^3 + 6n^2 + 2n$

2. Find the sum of the series

(*i*) $3 \times 5 + 5 \times 7 + 7 \times 9 + \ldots$ to *n* terms

(*ii*) $1^2 + 3^2 + 5^2 + \ldots$ to *n* terms (*iii*) $2^2 + 4^2 + 6^2 + \ldots$ to *n* terms.

3. Find the *n*th term and the sum to *n* terms of the series $1.2 + 2.3 + 3.4 + \ldots$

4. Sum up to *n* terms the series $1.2^2 + 2.3^2 + 3.4^2 + \ldots$

5. Sum up $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n)$.

6. Sum up to n terms the series $1+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2}+\frac{1}{4}\right)+...$

7. Sum up to n terms the series where nth term is $2^n - 1$.

8. Sum up $1 + 4 + 8 + 13 + ...$ to n terms.

9. Sum up $3 + 5 + 11 + 29 + ...$ to n terms.

10. Sum to n terms the series $7 + 77 + 777 + ...$

11. Sum to n terms the series $1 + 3 + 7 + 15 + 31 + ...$

12. Find the sum to n terms of the series $1.2.3 + 2.3.4 + 3.4.5 + ...$

13. Find the sum of the series to n terms and to infinity: $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...$

14. Sum to n terms the series whose nth term is $\frac{1}{4n^2-1}$.

15. Natural numbers are written as

1
2 3
4 5 6
.....
.....

Show that the sum of the numbers in the nth group is $\frac{n}{2}(n^2 + 1)$.

16. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

(*a*) $\frac{n(4n^2-1)c^2}{6}$ (*b*) $\frac{n(4n^2+1)c^2}{3}$

(*c*) $\frac{n(4n^2-1)c^2}{3}$ (*d*) $\frac{n(4n^2+1)c^2}{6}$ **[*IIT 2009*]**

ANSWERS

1. (*i*) $\frac{n}{6}(n+1)(2n+7)$ (*ii*) $\frac{n}{2}(n+1)(2n+3)$ (*iii*) $n(n+1)^2(n+2)$

2. (*i*) $\frac{n(4n^2+18n+23)}{3}$ (*ii*) $\frac{n(2n+1)(2n-1)}{3}$ (*iii*) $\frac{2n(n+1)(2n+1)}{3}$

3. $T_n = n(n+1)$; $S_n = \frac{n}{3}(n+2)(n+1)$

4. $\frac{1}{12}n(n+1)(n+2)(3n+5)$ **5.** $\frac{1}{6}n(n+1)(n+2)$ **6.** $2n-2+\frac{1}{2^{n-1}}$

7. $2^{n+1} - 2 - n$ **8.** $\frac{n}{6}(n^2+6n-1)$ **9.** $\frac{3^n-1}{2}+2n$

10. $\frac{70}{81}(10^n-1)-\frac{7n}{9}$ **11.** $2^{n+1}-2-n$ **12.** $\frac{n(n+1)(n+2)(n+3)}{4}$

13. $S_n = \frac{n}{2n+1}$, $S_\infty = \frac{1}{2}$ **14.** $\frac{n}{2n+1}$ **16.** (*c*)

HINTS TO SELECTED QUESTIONS

14. $\frac{1}{4n^2-1}=\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$ [Using partial fractions]

15. The first term in the nth group is the nth term of the series, 1, 2, 4, 7, ...

This will be found to be $\frac{n(n-1)}{2}+1$, *i.e.*, $\frac{n^2+2-n}{2}$ (See Sol. Ex. 61)

The last term in the nth group is the nth term of the series 1, 3, 6, 10,. ...

This will be found to be $\frac{n(n+1)}{2}$.

The sum of an A.P. of n terms, is $\frac{n}{2}\left[\frac{n(n-1)}{2}+1+\frac{n(n+1)}{2}\right]$, *i.e.*, $\frac{n}{2}(n^2+1)$.

16. Let $S_n = cn^2$, then $S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$

$\therefore\ T_n = S_n - S_{n-1} = 2cn - c$

$\therefore\ T_n^2 = (2cn-c)^2 = 4c^2n^2 + c^2.1 - 4c^2n$

$\therefore$ Reqd. sum $= \sum T_n^2 = 4c^2\Sigma n^2 + c^2.\Sigma 1 - 4c^2.\Sigma n = 4c^2\Sigma n^2 + c^2 n - 4c^2.\Sigma n$

$$= 4c^2.\frac{n(n+1)(2n+1)}{6} + c^2n - 4c^2.\frac{n(n+1)}{2}$$

$$= \frac{2c^2n(n+1)(2n+1) + 3c^2n - 6c^2n(n+1)}{3}$$

$$= \frac{nc^2(4n^2+6n+2+3-6n-6)}{3} = \frac{nc^2(4n^2-1)}{3}.$$

CHAPTER SUMMARY

A. Arithmetic Progression (A.P.)

1. $T_n = a + (n-1)d$, where a is the first term and d is the common difference. Also, $d = T_n - T_{n-1}$
2. $S_n = \frac{n}{2}[2a + (n-1)d]$
3. $S_n = \frac{n}{2}(a+l)$, where l is the last term
4. $T_n = S_n - S_{n-1}$
5. If a, b, c are in A.P., then $2b = a + c \Rightarrow b = \frac{a+c}{2}$.
6. A.M. (A) between two numbers a and b is $\frac{a+b}{2}$.
7. Three numbers in A.P. can be taken as $a-d, a, a+d$.
   Five numbers in A.P. can be taken as $a-2d, a-d, a, a+d, a+2d$.
   Four numbers in A.P. can be taken as $a-3d, a-d, a+d, a+3d$.

B. Geometric Progression (G.P.)

1. $T_n = ar^{n-1}$, where a = first term and r = common ratio
2. $S_n = \dfrac{a(1-r^n)}{1-r}$, if $r < 1$ and $S_n = \dfrac{a(r^n-1)}{r-1}$ if $r > 1$
3. $S_\infty = \dfrac{a}{1-r}$, $|r| < 1$.
4. Geometric mean (G) between two positive numbers a and b is $G = \sqrt{ab}$.
5. Three numbers in G.P. are taken as $\dfrac{a}{r}, a, ar$.

   Four numbers in G.P. are taken as $\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$.

C. Special Series

1. A sequence of the form

   $a, (a+d)\,r, (a+2d)r^2 \ldots (a+\overline{n-1}d)\,r^{n-1}$

   is called an **arithmetico-geometric sequence.**
2. **Series involving natural numbers**

   1. $\Sigma n = \dfrac{n(n+1)}{2}$ 2. $\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$ 3. $\Sigma n^3 = \dfrac{n^2(n+1)^2}{4} = \left[\dfrac{n(n+1)}{2}\right]^2$
3. If $T_n = an^3 + bn^2 + cn + d$, then

   $S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn$.

CHAPTER TEST

1. Write down the first five terms of the sequence, whose nth term is $(-1)^{n-1}.\ 5^{n+1}$.
2. If the 3rd and the 6th terms of an A.P. are 7 and 13 respectively, find the first term and the common difference.
3. Find the sum of all natural numbers between 100 and 1000 which are multiple of 5.
4. How many terms of the A.P., –6, $\dfrac{-11}{2}$, –5, ... are needed to give the sum –25 ?
5. Determine the sum of the first 35 terms of an A.P. if $a_2 = 2$ and $a_7 = 22$.
6. If the first term of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, show that the 20th term is –112.
7. Insert 3 arithmetic means between 2 and 10.
8. Find 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
9. The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.
10. The sum of first three terms of a G.P. is $\dfrac{39}{10}$ and their product is 1. Find the common ratio and the terms.
11. The sum of some terms of a G.P. is 315 and the first term and the common ratio are 5 and 2 respectively. Find the last term and the number of terms.

12. Find the sum of the series $0.6 + 0.66 + 0.666 + ...$ to the n terms.

13. The sum of an infinite series is 15 and the sum of the squares of these terms is 45. Find the series.

14. Insert three geometric means between 1 and 256.

15. In the sum to infinity of the series $3 + (3 + x)\frac{1}{4} + (3 + 2x)\frac{1}{4^2} + ...$ is $\frac{44}{9}$, find x.

16. Find the sum to n terms of the series $3.8 + 6.11 + 9.14 + ...$

17. Find the sum $5^2 + 6^2 + 7^2 + ... + 20^2$.

18. If in a geometric progression consisting of positive terms, each term equals the sum of the next two terms, then the common ratio of this progression equals

$(a)\ \sqrt{5}$ $(b)\ \frac{1}{2}(\sqrt{5}-1)$ $(c)\ \frac{1}{2}(1-\sqrt{5})$ $(d)\ \frac{1}{2}\sqrt{5}$ **[*AIEEE 2007*]**

19. If the first term of an infinite G.P. is 1 and each term is twice the sum of the succeeding terms, then the sum of the series is

$(a)\ 2$ $(b)\ \frac{5}{2}$ $(c)\ \frac{7}{2}$ $(d)\ \frac{3}{2}$ $(e)\ \frac{9}{2}$ **[*Kerala PET 2007*]**

20. If fifth term of a G.P. is 2, then the product of its first 9 terms is

$(a)\ 256$ $(b)\ 512$ $(c)\ 1024$ (d) none of these **[*AIEEE 2002*]**

21. The sum of three decreasing numbers in A.P. is 27. If –1, –1, 3 are added to them respectively, the resulting series is in G.P. The numbers are

$(a)\ 5, 9, 13$ $(b)\ 15, 9, 3$ $(c)\ 13, 9, 5$ $(d)\ 17, 9, 1$ **[*AMU 1999*]**

22. The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is

$(a) -4$ $(b) -12$ $(c)\ 12$ $(d)\ 4$ **[*AIEEE 2008*]**

23. The sum to infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...$ is

$(a)\ 6$ $(b)\ 2$ $(c)\ 3$ $(d)\ 4$ **[*AIEEE 2009*]**

24. The sum of all odd numbers between 1 and 100 which are divisible by 3, is

$(a)\ 83667$ $(b)\ 90000$ $(c)\ 83660$ (d) None of these **[*UPSEE 2009*]**

25. If a, b, c are in G.P. and x, y are arithmetic means of a, b and b, c respectively, then $\frac{1}{x} + \frac{1}{y}$ is equal to

$(a)\ \frac{2}{b}$ $(b)\ \frac{3}{b}$ $(c)\ \frac{b}{3}$ $(d)\ \frac{b}{2}$ $(e)\ \frac{1}{b}$

[*Kerala PET 2009*]

ANSWERS

1. 25, –125, 625, –3125, 15625 **2.** First term = 3, Common difference = 2

3. 98450 **4.** 5 or 20 **5.** 2310 **7.** 4, 6, 8 **8.** 3072 **9.** ± 3

10. $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$ **11.** Number of terms = 6, Last term = 160

12. $\frac{2n}{3} - \frac{2}{27}(1-10^{-n})$ **13.** $5+\frac{10}{3}+\frac{20}{9}+\ldots\infty$

14. 4, 16, 64 **15.** $x = 2$ **16.** $3n\,(n+1)\,(n+3)$

17. 2840 **18.** (*b*) **19.** (*d*) **20.** (*b*) **21.** (*d*)

22. (*b*) **23.** (*b*) **24.** (*a*) **25.** (*a*)

HINTS TO SELECTED QUESTIONS

3. Reqd. sum = (105 + 110 + ... + 995)

6. $a = 2,\ T_1 + T_2 + T_3 + T_4 + T_5 = \frac{1}{4}[T_6 + T_7 + T_8 + T_9 + T_{10}]$

$$\Rightarrow \frac{5}{2}[2a+(5-1)d] = \frac{1}{4} \times \frac{5}{2}[2(a+5d)+(5-1)\,d]$$

10. Let the numbers be $\frac{a}{r}, a, ar$.

17. Reqd. sum = $(\Sigma n^2)_{n=20} - (\Sigma n^2)_{n=4}$

18. $a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0.$

21. Let the numbers be $a + d, a, a - d$. Then $(a + d) + a + (a - d) = 27 \Rightarrow a = 9$ and $(a + d) - 1$, $(a - 1), (a - d) + 3$ are in G..P. $\Rightarrow (a-1)^2 = (a+d-1)(a-d+3)$

22. $\left.\begin{array}{ll} a+ar=12 & \ldots(1) \\ ar^2+ar^3=48 & \ldots(2) \end{array}\right\} \Rightarrow \frac{ar^2(1+r)}{a(1+r)} = \frac{48}{12} \Rightarrow r^2 = 4$

$\Rightarrow r = -2$ (since the terms are +ve and –ve alternately)

$a = -12.$

23. $$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots\ldots$$

$$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \ldots\ldots$$

Subtracting, $\frac{2}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \ldots\ldots = \frac{4}{3} + \frac{4}{3^2}\left(\frac{1}{1-\frac{1}{3}}\right)$

24. The numbers are 3, 9, 15, ..., 999. First, find n by using $l = a + (n - 1)\,d$

25. a, b, c are in G.P. $\Rightarrow b = ar, c = ar^2,\ x = \frac{a+b}{2},\ y = \frac{b+c}{2}$

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c} = 2\left\{\frac{1}{a+ar} + \frac{1}{ar+ar^2}\right\} \text{ etc.}$$

CHAPTER 15

BASIC CONCEPTS OF POINTS AND THEIR COORDINATES

In Two–Space

15.01. Historical note

Analytical geometry was invented by the French philosopher Rene Descartes (1596–1650) and the year of invention is generally set as 1637, when he published his La Geometric, which provided a new tool for unifying the two branches of mathematics, algebra and geometry. Prior to this, mathematicians confined themselves to Euclidean geometry and did not know how to apply algebra advantageously to the study of geometrical relationships.

15.02. What is analytical geometry?

Analytical geometry is that branch of mathematics which treats geometry algebraically, *i.e.,* in which geometric figures are studied by means of equations. Thus, we have equations of geometric curves which reveal to us their nature and properties.

15.03. The name coordinate geometry

The equation of a curve represents the fundamental properties of different points on the curve which we locate by means of real numbers, called *coordinates* and so, instead of giving this new type of geometry the name analytical geometry, we can call it *coordinate geometry* or *plane Cartesian geometry,* the former name being more commonly used, and the latter is after the name of the inventor, Descartes.

Definition. Coordinate Geometry *is that branch of geometry in which two numbers, called coordinates, are used to indicate the position of a point in a plane and which makes use of algebraic methods in the study of geometric figures.*

15.04. Rectangular axes

The position of a point in a plane is fixed by selecting the *axes of reference* which are formed by combining the two number scales at right angles to each other so that their zero points coincide. The horizontal number scale is called the ***x*-axis** and the vertical number scale the ***y*-axis**. The point where the two scales cross each other is called the **origin**. The two together are called the *rectangular axes.**

15.05. Coordinates-definition and notation

The position of each point of the plane is determined with reference to the rectangular axes by means of a pair of numbers called *co-ordinates* which are the distances of the point from the axes. The distance of the point from the *y*-axis is called the

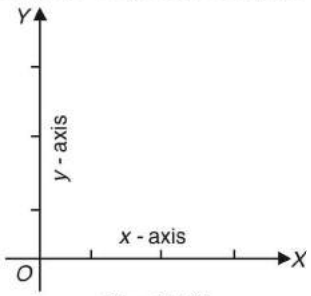


Fig. 15.01

* So called, because they are at right angles to each other.

x-coordinate or **abscissa**** and the distance of the point from the x-axis is called the *y-coordinate* or **ordinate.**

Notations. 1. In stating the coordinates of a point, the abscissa precedes the ordinate. The two are separated by a comma and enclosed in a bracket. Thus a point whose abscissa is x and whose ordinate is y is designated by the notation (x, y).

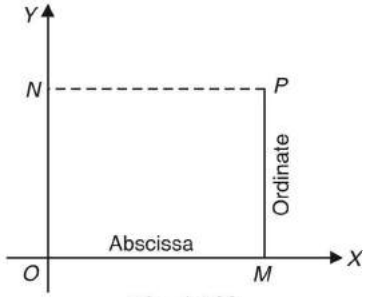


Fig. 15.02

2. When the coordinates of a point are known they are denoted

(i) by the first letters of the alphabet as (a, b), (c, d), (h, k), etc.

(ii) by attaching suffixes to x and y as (x_1, y_1) [read "x one, y one"], $(x_2\ y_2)$ [read "x two, y two"], etc.

(iii) by attaching dashes to x and y as (x', y') [read "x dash, y dash'], (x'', y'') [read "x double dash, y double dash"] etc.

When the coordinates of a point are unknown, they are denoted by (x, y) by using the notation x and y for the variable quantities.

Note 1. The convention of using the first letters of the alphabet $a, b, c,...$ for the knowns and the last letters $x, y, z,...$ for the unknowns is due to Descartes.

Note 2. Coordinates of the origin obviously are (0, 0). The coordinates of any point on the x-axis would be $(x, 0)$ and the coordinates of any point on the y-axis would be $(0, y)$.

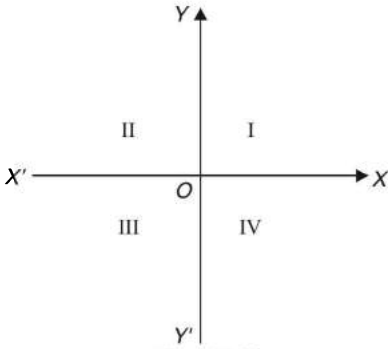


Fig. 15.03

15.06. Quadrants

The coordinate axes separate the plane into four regions. called *quadrants.* By custom, the quadrants are numbered I, II, III and IV in the counter-clockwise direction, as shown in Fig. 15.03.

15.07. Convention of signs

A point of the plane may lie in any quadrant and falling back to our horizontal and vertical number scales, it is easily understood that–

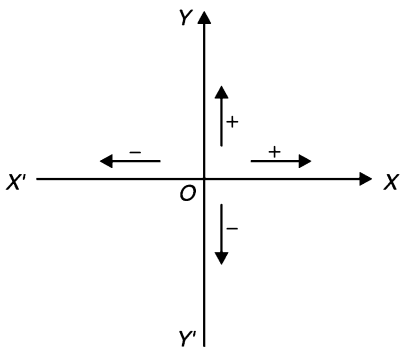


Fig. 15.04

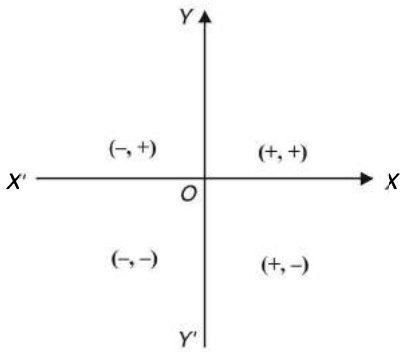


Fig. 15.05

(i) For distances along the x-axis, positive values are measured to the right of the origin, and negative values to its left;

(ii) For distances along the y-axis, positive values are measured upward and negative values downward.

** The words 'abscissa' and 'ordinate' cannot be said to be well chosen words. We could have simpler words like first coordinate and second coordinate. There is no particular reason for having the distance OM as the first coordinate. One is to be taken before the other and we have agreed to take abscissa as the first coordinate.

In accordance with the above convention the signs of the coordinates of a point in different quadrants will be as shown in Fig. 15.05.

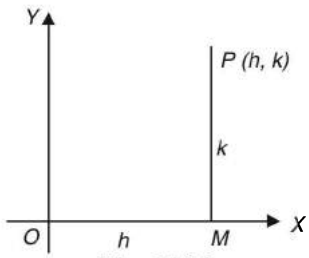


Fig. 15.06

15.08. Plotting of a point whose coordinates are known

The point can be plotted by measuring its proper distances from the axes. Thus, any point (h, k) can be plotted as follows:

1. Measure OM equal to h along the x-axis.
2. Now measure MP perpendicular to OM and equal to k.

Observe the rule of signs in both the cases.

Ex. 1. Points $P(2, -4)$ and $Q(-4, 5)$ would be plotted as shown in Fig. 15.07.

Points can be more easily and correctly plotted by the use of *coordinate paper* which is ruled into small squares and commonly known as *graph paper*.

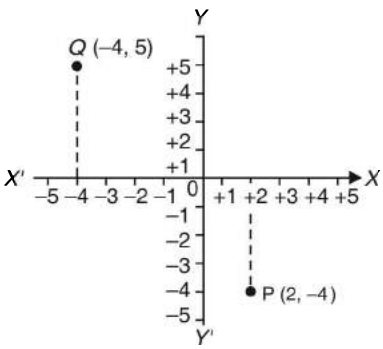


Fig. 15.07

EXERCISE 15 (*a*)

1. Find the coordinates of the points shown on the graph in Fig. 15.08.
   (*i*) P (*ii*) Q (*iii*) O (*iv*) R
   (*v*) S (*vi*) A (*vii*) L (*viii*) M
2. Plot the points (2, 3), (–5, –7). (–4, 6), (0, 8) (–5, 0) on the graph paper.
3. Where will a point lie if (*i*) its ordinate is zero, (*ii*) its abscissa is zero?
4. Where will a point lie if (*i*) the abscissa equals the ordinate; (*ii*) the positive abscissa equals the negative of the positive ordinate?

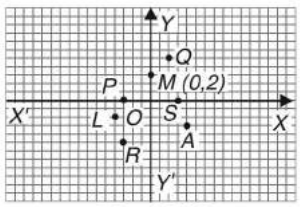


Fig. 15.08

ANSWERS

1. (*i*) (–3, 0) (*ii*) (2, 5) (*iii*) (0, 0) (*iv*) (–3, –5)
(*v*) (3, 0) (*vi*) (4, –3) (*vii*) (–4, –2) (*viii*) (0, 2)

3. (*i*) The x-axis, (*ii*) The y-axis.

4. (*i*) In any quadrant, on the angular bisectors of the axes.
(*ii*) In the fourth quadrant, on the angular bisector of the axes.

15.09. Mid-point formula

The coordinates of the midpoint of the join of (x_1, y_1) and (x_2, y_2) are

$$x_m = \frac{x_1 + x_2}{2}, \quad y_m = \frac{y_1 + y_2}{2}.$$

Let P, Q be the points (x_1, y_1), (x_2, y_2) respectively, and (x_m, y_m) the required coordinates of M, the mid-point of PQ. Draw PD and QF and ME perpendiculars to OX. Through M draw NML parallel to OX to meet DP produced at N and QF at L.

Then, from congruent triangles PMN and QML, we get

NM = ML or DE = EF

or $\quad OE - OD = OF - OE$ or $x_m - x_1 = x_2 - x_m$

or $\quad 2x_m = x_1 + x_2 \Rightarrow x_m = \dfrac{x_1 + x_2}{2}$.

Again, from the same congruent triangles, we get

PN = LQ and DN – DP = FQ – FL

or $\quad EM - y_1 = y_2 - EM$ or $y_m - y_1 = y_2 - y_m$

or $\quad 2y_m = y_1 + y_2 \Rightarrow y_m = \dfrac{y_1 + y_2}{2}$

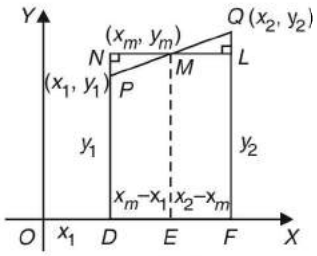


Fig. 15.09

Hence, the coordinates of the mid-point of the join of (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Note. This formula is helpful in (*a*) finding out the middle point of the join of two given points and (*b*) verifying whether two st. lines bisect each other or not.

Ex. 2. Coordinates of the mid-point of the join of points (–1, 5), (7, 3) are $\left(\dfrac{-1+7}{2}, \dfrac{5+3}{2}\right)$, *i.e.*, (3, 4).

15.10. Distance formula

The distance *d* between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points. From P, Q draw PL, QM perpendiculars on the *x*-axis and PR perpendicular on MQ.

Then, $\quad PR = LM = x_2 - x_1$

$\quad RQ = MQ - MR = MQ - LP = y_2 - y_1$

From rt. $\angle d\ \Delta$ PRQ, $PQ^2 = PR^2 + RQ^2$

or $\quad d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\therefore \quad |d| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

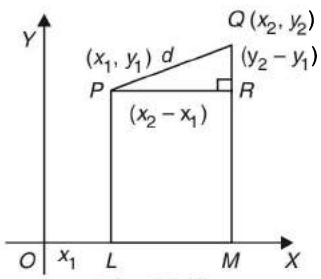


Fig. 15.10

Cor. ***Distance from the origin***

The distance of the point (x_1, y_1) from the origin is

$$|d| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$$

or $\quad |d| = \sqrt{x_1^2 + y_1^2}$

(1)	(2)
(0, 0)	(x_1, y_1)
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	

Note 1. Aid to memory

$$|d| = \sqrt{(\text{diff. of abscissae})^2 + (\text{diff. of ordinates})^2}$$

Note 2. Complete distance formula

$\because \quad d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \therefore d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

We take only the positive square root as we are usually interested only in the magnitude of the segment PQ.

Note 3. Generalisation. We have derived this formula taking P and Q in the first quadrant but the formula is true in whatever quadrants the points may lie. In coordinate geometry all the theorems and formulae proved by supposing points or lines to be in the first quadrant will always remain true in whatever quadrants the points or lines may lie. Only the proper signs will have to be taken with the coordinates.

Ex. 3. Find the distance between points (6, –8), (2, –5).

Sol. $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-4)^2 + (3)^2$
$= 16 + 9 = 25 \therefore |d| = 5$

$x_2 = 2;\ y_2 = -5;\ x_1 = 6;\ y_1 = -8$
$x_2 - x_1 = -4;\quad y_2 - y_1 = 3.$

15.11. Division or section formulae

To find the coordinates of the point which divides internally the line joining two given points in a given ratio.

Let A(x_1, y_1) and B(x_2, y_2) be the two given points and P a point on AB which divides it in the given ratio $m_1 : m_2$. It is required to find the co-ordinates of P. Suppose they are (x, y). Draw the perpendiculars AL, PM, BN on OX, and AK, PT on PM and BN respectively. Then, from similar triangles APK and PBT, we have

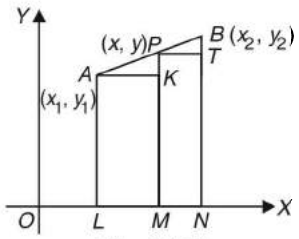


Fig. 15.11

$$\frac{AP}{PB} = \frac{AK}{PT} = \frac{KP}{TB} \qquad \ldots (i)$$

Now AP : PB $= m_1 : m_2$, AK = LM = OM – OL $= x - x_1$ and PT = MN = ON – OM $= x_2 - x$
KP = MP – MK = MP – LA $= y - y_1$, TB = NB – NT = NB – MP $= y_2 - y$.

$\therefore$ From (i), we have $\dfrac{m_1}{m_2} = \dfrac{x - x_1}{x_2 - x} = \dfrac{y - y_1}{y_2 - y}$

The first two relations give $\dfrac{m_1}{m_2} = \dfrac{x - x_1}{x_2 - x}$ or $m_1 x_2 - m_1 x = m_2 x - m_2 x_1$

or $x(m_1 + m_2) = m_1 x_2 + m_2 x_1$ or $x = \dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$

Similarly, from the relation $\dfrac{AP}{PB} = \dfrac{KP}{TB}$, we get $\dfrac{m_1}{m_2} = \dfrac{y - y_1}{y_2 - y}$

which gives on simplification, $y = \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$

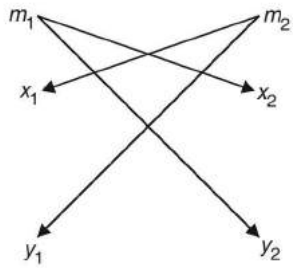


Fig. 15.12

Thus, the coordinates of P are

$$\boldsymbol{x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2},\ y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}}$$

Aid to memory. A convenient schematic arrangement for this formula is as shown in Fig 15.12.

Cor. Putting $m_1 = 1$, and $m_2 = 1$, the co-ordinates of the mid-point are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

15.12. Formula for external division

Construct as shown in Fig. 15.13. As before, from similar triangles APK and PBT, given AP : BP $= m_1 : m_2$, we get

$$\frac{AP}{PB} = \frac{AK}{PT} = \frac{KP}{TB} \text{ or } \frac{m_1}{m_2} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$

Now, $\frac{m_1}{m_2} = \frac{x - x_1}{x - x_2}$ gives $x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$

and $\frac{m_1}{m_2} = \frac{y - y_1}{y - y_2}$ gives $y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$

Hence, the coordinates of P for external division are

$$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}.$$

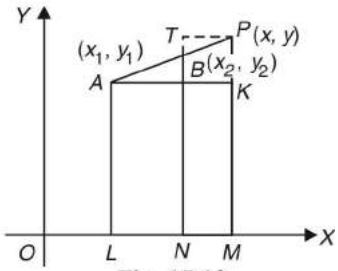


Fig. 15.13

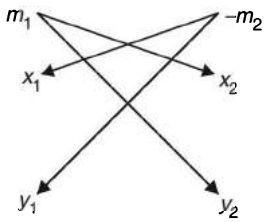


Fig. 15.14

Aid to memory. Put $-m_2$ for m_2 in the formula for internal division. Writing $-m_2$ for m_2 we have the schematic arrangement as shown here. [Fig. 15.14]

Note. Alternative Method for obtaining coordinates for external division. The coordinates of P in this case can be obtained by keeping in mind the fact that here m_1 and m_2 are measured in opposite directions and so if m_1 is reckoned positive, m_2 will be negative [Fig. 15.15]

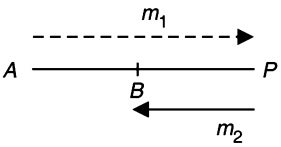


Fig. 15.15

Ex. 4. Find the coordinates of the point which divides the join of the points (8, 9) and (–7, 4), (*i*) internally in the ratio 2 : 3 and (*ii*) externally in the ratio 4 : 3.

Sol. (*i*) The coordinates of the point for internal division are

$$x = \frac{2 \times (-7) + 3 \times 8}{2 + 3} = \frac{-14 + 24}{5} = \frac{10}{5} = 2, y = \frac{2 \times 4 + 3 \times 9}{2 + 3} = \frac{8 + 27}{5} = \frac{35}{5} = 7.$$

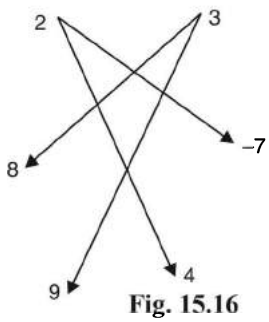


Fig. 15.16

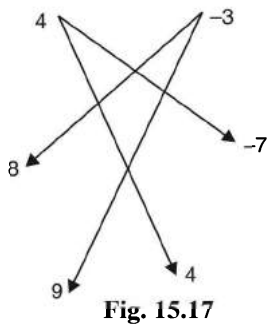


Fig. 15.17

(*ii*) For external division, the coordinates are

$$x = \frac{4 \times (-7) - 3 \times 8}{4 - 3} = \frac{-28 - 24}{1} = -52; \quad y = \frac{4 \times 4 - 3 \times 9}{4 - 3} = \frac{16 - 27}{1} = -11.$$

15.13. Converse problem

To find the ratio in which a given point divides the join of the given points.

Ex. 5. Find the ratio in which the point (2, 1) divides the join of the points (1, –2) and (4, 7).

Sol. Let the required ratio be $\lambda : 1$, then the coordinates of the point of division are

$$x = \frac{4\lambda + 1}{\lambda + 1}, y = \frac{7\lambda - 2}{\lambda + 1}$$

But the coordinates of the point of division are given to be (2, 1).

$x_1 = 1, \ m_1 = \lambda, \ y_1 = -2$

$x_2 = 4, \ m_2 = 1, \ y_2 = 7$

$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$

Equating the abscissae, we get

$$\frac{4\lambda+1}{\lambda+1}=2 \text{ which gives } \lambda=\frac{1}{2}.$$

$\therefore$ The required ratio is $\frac{1}{2}:1$, *i.e.,* 1 : 2.

Note 1. If the value of λ comes out to be negative, it should be inferred that the division is external.

Note 2. Rule to find the ratio when the coordinates of the point of division are given.

1. *Let the required ratio be $\lambda : 1$.*
2. *Find out the coordinates of the point dividing the join of the given points in this ratio.*
3. *Equate the abscissae or ordinates of the point so obtained and the given point.*

Ex. 6. Find the ratios in which the axes divide the line joining the points (2, 5) and (1, 9).

Sol. Division by the *x*-axis.

Let the *x*-axis divide the given line in the ratio $\lambda : 1$. Then, the co-ordinates of the point of division are

$$x=\frac{\lambda+2}{\lambda+1}, y=\frac{5+9\lambda}{\lambda+1}$$

But this is a point on the *x*-axis so

$$\frac{5+9\lambda}{\lambda+1}=0, \text{ which gives } \lambda=\frac{-5}{9}.$$

$x_1=2, \quad m_1=\lambda, \quad y_1=5$

$x_2=1, \quad m_2=1, \quad y_2=9$

On the *x*-axis, the ordinate = 0.

Hence, the *x*-axis divides the join externally in the ratio 5 : 9.

(*b*) Proceed similarly for the *y*-axis. Calculate abscissa and put it equal to 0. **Ans.** 2 : 1 externally

15.14. Centroid of a triangle. To find the co-ordinates of the centroid of a triangle whose vertices are given.

Definition. The centroid of a triangle is the point of intersection of its medians.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC. Let AD be the median bisecting its base. Then

$$D=\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right).$$

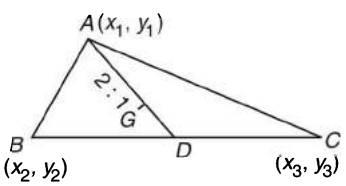


Fig. 15.18

Now, the point G on AD, which divides it internally in the ratio 2 : 1, is the centroid. If (x, y) are the coordinates of G, then (Euclidean Geometry)

$$x=\frac{2\times\frac{x_2+x_3}{2}+1\times x_1}{2+1}=\frac{x_1+x_2+x_3}{3}, y=\frac{2\times\frac{y_2+y_3}{2}+1\times y_1}{2+1}=\frac{y_1+y_2+y_3}{3}.$$

Hence, the co-ordinates of the centroid are given by

$$\boldsymbol{x=\frac{x_1+x_2+x_3}{3}, y=\frac{y_1+y_2+y_3}{3}.}$$

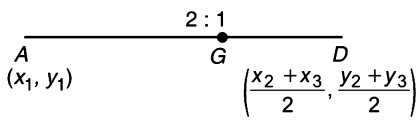


Fig. 15.19

Note. The symmetry of these results shows that the same point G divides the other two medians in the ratio 2 : 1, *i.e.,* the point G lies on the three medians, *i.e.,* **the medians are concurrent.**

15.15. Incentre of a triangle

To find the co-ordinates of the incentre of a triangle whose vertices are given.

Definition and notation. *The point of concurrence of the internal bisectors of the angles of a triangle is called the* ***incentre*** *of the triangle and is generally denoted by I.*

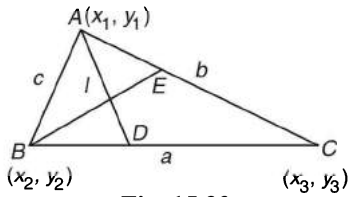


Fig. 15.20

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle. Let AD and BE, the bisectors of angles A and B meet in *I*.

Then, *I* is the in-centre of the triangle.

$\because$ AD bisects $\angle BAC$ of ΔABC.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b},$$

i.e., D divides BC internally in the ratio $c : b$. [Euclidean Geometry]

$\therefore$ The co-ordinates of D are

$$\left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right).$$

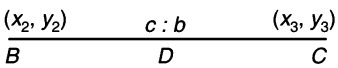


Fig. 15.21

Now, BI bisects $\angle ABD$ of ΔABD $\therefore$ $\dfrac{AI}{ID} = \dfrac{AB}{BD} = \dfrac{c}{BD}$ ...(*i*)

But $\dfrac{BD}{DC} = \dfrac{c}{b}$ (Proved above)

or $\dfrac{BD}{BD + DC} = \dfrac{c}{c+b}$ (Componendo)

or $\dfrac{BD}{BC} = \dfrac{c}{c+b}$ $\therefore$ $BD = \dfrac{ca}{c+b}$ ($\because$ $BC = a$)

$\therefore$ From (*i*), $\dfrac{AI}{ID} = \dfrac{c}{\frac{ca}{c+b}} = \dfrac{c+b}{a}$

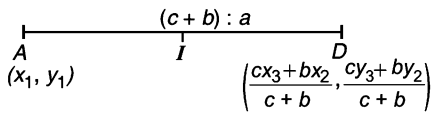


Fig. 15.22

which shows that I divides AD internally in the ratio $c + b : a$.

$\therefore$ The co-ordinates of I are

$$x = \frac{(c+b)\left(\frac{cx_3 + bx_2}{c+b}\right) + ax_1}{(c+b)+a} = \frac{\boldsymbol{ax_1 + bx_2 + cx_3}}{\boldsymbol{a+b+c}},$$

$$y = \frac{(c+b)\left(\frac{cy_3 + by_2}{c+b}\right) + ay_1}{(c+b)+a} = \frac{\boldsymbol{ay_1 + by_2 + cy_3}}{\boldsymbol{a+b+c}}.$$

Ex. 7. Find the centroid of the triangle whose angular points are (3, – 5), (–7, 4) and (10, –2) respectively.

Sol. The co-ordinates of the centroid are

$$x = \frac{3 + (-7) + 10}{3} = \frac{6}{3} = 2, \qquad y = \frac{(-5) + 4 + (-2)}{3} = \frac{-3}{3} = -1.$$

$\therefore$ The centroid is (2, – 1).

Ex. 8. Find the co-ordinates of the incentre of the triangle whose vertices are (4, –2), (–2, 4), (5, 5).

Sol. Let A(4, –2), B(–2, 4), C(5, 5) be the vertices of the triangle. Then,

$$a = \text{BC} = \sqrt{[5-(-2)]^2 + (5-4)^2} = \sqrt{(7)^2 + 1^2} = \sqrt{50} = 5\sqrt{2}.$$

$$b = \text{CA} = \sqrt{(4-5)^2 + (-2-5)^2}$$

$$= \sqrt{(-1)^2 + (-7)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}.$$

$$c = \text{AB} = \sqrt{(-2-4)^2 + [4-(-2)]^2} = \sqrt{(-6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}.$$

$\therefore$ The coordinates of the incentre are

$$x = \frac{ax_1 + bx_2 + cx_3}{a+b+c} = \frac{5\sqrt{2}(4) + 5\sqrt{2}(-2) + 6\sqrt{2}(5)}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{40\sqrt{2}}{16\sqrt{2}} = \frac{5}{2},$$

$$y = \frac{ay_1 + by_2 + cy_3}{a+b+c} = \frac{5\sqrt{2}(-2) + 5\sqrt{2}(4) + 6\sqrt{2}(5)}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{40\sqrt{2}}{16\sqrt{2}} = \frac{5}{2}.$$

$\therefore$ The coordinates of the incentre are $\left(\frac{5}{2}, \frac{5}{2}\right)$.

Ex. 9. Using section formula, prove that the points (–1, –1), (5, 7) and (8, 11) are collinear.

Sol. The points will be collinear, if any one of these points divides the other two in a certain finite ratio.

Let the point B(5, 7) divide the join of the points A (–1, –1) and C(8, 11) in the ratio $\lambda : 1$. Then

$$\frac{8\lambda - 1}{\lambda + 1} = 5, \qquad \frac{11\lambda - 1}{\lambda + 1} = 7$$

$\Rightarrow 8\lambda - 1 = 5\lambda + 5$ | $\Rightarrow 11\lambda - 1 = 7\lambda + 7$

$\Rightarrow 3\lambda = 6$ | $\Rightarrow 4\lambda = 8$

$\Rightarrow \lambda = 2$ | $\Rightarrow \lambda = 2$

A (–1, –1) — $\lambda : 1$ — B — C (8, 11)

Fig. 15.23

Since both the equations yield the same value of λ, so B divides | AC | in the ratio 2 : 1 (internally). Therefore, the given points are collinear.

EXERCISE 15 (*b*)

1. Find the mid-points of the lines joining
   (*i*) (5, 8), (9, 11); (*ii*) (0, 0), (8, –5); (*iii*) (–7, 0), (0, 10); (*iv*) (–4, 3), (6, –7).
2. Find the mid-points of the sides of a triangle whose vertices are A(1, –1), B(4, –1), C(4, 3).
3. Find the centre of a circle if the end points of a diameter are A(–5, 7) and B(3, –11).
4. If M is the mid-point of AB, find the co-ordinates of : (*i*) A if the co-ordinates of M and B are M(2, 8) and B (–4, 19) and (*ii*) B if the co-ordinates of A and M are A(–1, 2), M(–2, 4).
5. Find the distance between each of the following pairs of points:
   (*i*) (7, 9), (4, 5); (*ii*) (15, 11), (3, 6); (*iii*) (4, –5), (0, 0); (*iv*) (2, –11), (–4, –3).
6. Find the radius of the circle that has its centre at (0, – 4) and passes through $(\sqrt{13}, 2)$.
7. Find the lengths of the sides of the triangle whose vertices are A(3, 4), B(2, –1) and C(4, –6).
8. The vertices of ΔABC are A(–1, 3), B(1, 1) and C(5, 1). Find the length of the median to (*i*) AB, (*ii*) AC, (*iii*) BC.
9. A circle has its centre at the origin and a radius of $\sqrt{12}$. State whether each of the following points is on, outside or inside the circle : $(1, -\sqrt{7})$, (3, 5), $(2, 2\sqrt{2})$.

10. A line is of length 10, and one end is at the point (–3, 2). If the ordinate of the other end be 10, prove that the abscissa will be 3 or – 9.

11. Find the coordinates of the point which divides internally the join of the points
(*i*) (8, 9) and (–7, 4) in the ratio 2 : 3; (*ii*) (1, –2) and (4, 7) in the ratio 1 : 2.

12. Find the coordinates of the point which divides externally the join of the points
(*i*) (–4, 4) and (1, 7) in the ratio 2 : 1; (*ii*) (3, 4) and (–6, 2) in the ratio 3 : 2.

13. Find the coordinates of the points of trisection of the line joining the points (2, 3) and (6, 5).

14. The line joining the points (3, 2) and (6, 8) is divided into four equal parts, find the coordinates of the points of section.

15. In what ratio does the point $\left(1, \frac{7}{2}\right)$ divide the join of (–2, –4) and $\left(2, \frac{-10}{3}\right)$?

16. In what ratio is the line joining the points
(*i*) (2, –3) and (5, 6) divided by the *x*-axis; (*ii*) (3, –6) and (–6, 8) divided by the *y*-axis?

17. Find the ratio in which the axes divide the line joining the points (2, 5) and (1, 9).

18. Show by using section formula that the points (3, –2), (5, 2) and (8, 8) are collinear.

19. Find the centroid of the triangle whose angular points are (–4, 6), (2, –2) and (2, 5) respectively.

20. If $(x_1, y_1) = (2, 3)$; $x_2 = 3$ and $y_3 = -2$ and G is (0, 0), find y_2 and x_3.

21. Find the coordinates of the in-centre of the triangle whose vertices are (–36, 7), (20, 7) and (0, –8).

ANSWERS

1. (*i*) $\left(7, \frac{19}{2}\right)$ (*ii*) $\left(4, \frac{-5}{2}\right)$ (*iii*) $\left(\frac{-7}{2}, 5\right)$ (*iv*) (1, – 2) **2.** $\left(\frac{5}{2}, -1\right)$, (4, 1), $\left(\frac{5}{2}, 1\right)$.

3. (–1, –2). **4.** (8, –3), (–3, 6). **5.** (*i*) 5, (*ii*) 13, (*iii*) $\sqrt{41}$, (*iv*) 10.

6. 7. **7.** $\sqrt{26}, \sqrt{29}, \sqrt{101}$. **8.** (*i*) $\sqrt{26}$, (*ii*) $\sqrt{2}$ (*iii*) $2\sqrt{5}$.

9. (*a*) inside, (*b*) outside, (*c*) on. **11.** (*i*) (2, 7) or (–1, 6).

(*ii*) (2, 1) or (3, 4) **12.** (*i*) (6, 10) or (–9, 1). (*ii*) (–24, –2) or (21, 8) **13.** $\left(\frac{10}{3}, \frac{11}{3}\right), \left(\frac{14}{3}, \frac{13}{3}\right)$

14. $\left(\frac{15}{4}, \frac{7}{2}\right), \left(\frac{9}{2}, 5\right), \left(\frac{21}{4}, \frac{13}{2}\right)$. **15.** 3 : 1 **16.** (*i*) 1 : 2, (*ii*) 1 : 2.

17. 5 : 9 and 2 : 1 externally, **19.** (0, 3). **20.** –1, –5. **21.** (–1, 0).

15.16. Area of a triangle

To find the area of a triangle whose vertices are given.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle.

Let Δ be the required area of the triangle. From A, B, C, draw the perpendiculars AL, BM and CN on the *x*-axis.

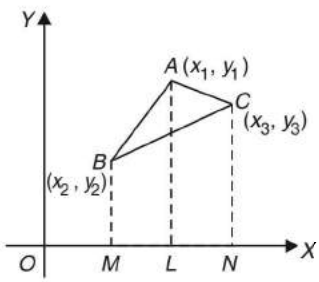


Fig. 15.24

Then from Fig. 15.24,

ΔABC = trap. ABML + trap. ALNC – trap. BMNC

$$= \frac{1}{2}(MB + LA)\,ML + \frac{1}{2}(LA + NC)\,LN - \frac{1}{2}(MB + NC)\,MN$$

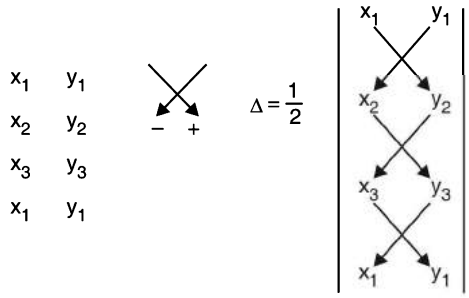
$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] \quad \text{[Form I]}$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \quad \text{[Form II]}$$

Cor. If one vertex of the triangle is the origin and the remaining vertices the points (x_1, y_1) and (x_2, y_2), its area would be $(x_1y_2 - x_2y_1)$.

Note 1. Aid to Memory: Write the abscissae and the ordinates of the vertices in two columns as shown here, the abscissa and the ordinate of the first vertex occurring twice. Multiply diagonally attaching –ve sign to the multiplications from right to left and +ve sign to the multiplications from left to right. Divide the whole result by 2.



Note 2. Complete area formula: The sign of the Δ is positive or negative according as the order of the vertices A, B, C is counter-clockwise or clockwise.

A convenient method of ascertaining the sign of area is as follows:

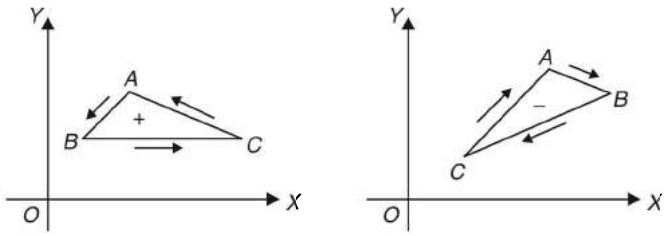


Fig. 15.25

We take the area as positive if the vertices are situated in such a way that in going round the triangle in the given order of the vertices, the area of the triangle lies on the left hand side of the observer. If the area lies on the right, the result will be negative.

Hence, the complete area formula is $\Delta = \pm \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$.

Note 3. When to use the complete area formula: If in employing the area formula, it gives a negative result, we neglect the negative sign to get the area in magnitude. The complete area formula is used when the area of the triangle is given.

15.17. Area of a quadrilateral

The area of any quadrilateral can be found by splitting up the quadrilateral into two triangles and adding up their areas. But if we are not paying any regard to the sign of the areas, care should be taken to ensure that areas of both the triangles are on the left hand side in going along their boundaries.

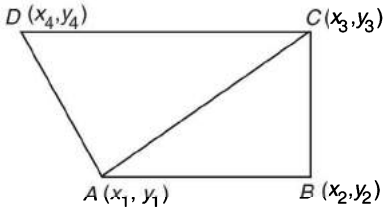


Fig. 15.26

Thus, area of the quadrilateral whose vertices taken in order are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ would be calculated as follows:

quad. ABCD= $\Delta ABC + \Delta ACD$

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

$$+ \frac{1}{2}[(x_1y_3 - x_3y_1) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$$

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)].$$

Note. Aid to Memory: Write down coordinates and multiply as shown here.

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

15.18. Collinear points

Collinear points are points which lie on the same straight line.

15.19. Condition of collinearity of three points

If three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on a straight line, the area of the triangle formed by them is zero,

i.e. $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = 0$

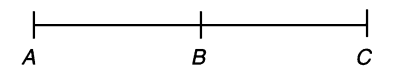


Fig. 15.27

Thus, we have the following **rules** to prove that three given points are collinear.

1. ***Find Δ, the area of the triangle formed by these three points.***

2. ***Show that $\Delta = 0$.***

Ex. 10. Find the area of the triangle whose vertices are (1, 3), (2, 4) and (5, 6).

Sol. The required area

$$= \frac{1}{2}[1 \times 4 - 2 \times 3 + 2 \times 6 - 5 \times 4 + 5 \times 3 - 1 \times 6]$$

$$= \frac{1}{2}(4 - 6 + 12 - 20 + 15 - 6) = -\frac{1}{2}$$

$$\frac{1}{2}\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \\ 1 & 3 \end{bmatrix}$$

$\therefore$ Changing the sign, the area of the triangle in magnitude $= \frac{1}{2}$.

Ex. 11. Find the area of the quadrilateral whose vertices are (3, 4), (0, 5), (2, – 1) and (3, –2).

Sol. The required area

$$= \frac{1}{2}[3 \times 5 - 0 \times 4 + 0 \times (-1) - 2 \times 5 + 2 \times (-2) - 3 \times (-1) + 3 \times 4 - 3 \times (-2)]$$

$$= \frac{1}{2}[15 - 0 - 0 - 10 - 4 + 3 + 12 + 6] = \frac{1}{2} \times 22 = 11.$$

$\frac{1}{2}$ [3, 4; 0, 5; 2, –1; 3, –2; 3, 4]

Ex. 12. Show that the following points are collinear (a, $b + c$), (b, $c + a$) and (c, $a + b$).

Sol. The area of the triangle formed by the three points is

$$\Delta = \frac{1}{2}[a(c + a) - b(b + c) + b(a + b) - c(c + a) + c(b + c) - a(a + b)]$$

$$= \frac{1}{2}(ac + a^2 - b^2 - bc + ab + b^2 - c^2 - ac + bc + c^2 - a^2 - ab)$$

$$= \frac{1}{2} \times 0 = 0$$

$\frac{1}{2}$ [a, $b + c$; b, $c + a$; c, $a + b$; a, $b + c$]

$\therefore$ The points are collinear.

EXERCISE 15 (*c*)

1. Find the area of the triangle whose vertices are

(*i*) (4, 2) (4, 5) and (–2, 2); (*ii*) (0, 0), (–2, 3) and (10, 7); (*iii*) (a, 0), (0, b) and (x, y). **[*SC*]**

2. Find the area of the quadrilateral whose vertices are

(*i*) (1, 1), (7, – 3) (12, 2) and (7, 21); (*ii*) (1, 1), (3, 4), (5, –2) and 4, –7).

3. Show that the following points are collinear:

(*i*) (1, 4), (3, –2) and (–3, 16); (*ii*) (–5, 1), (5, 4) and (10, 7).

4. If (7, a), (–5, 2) and (3, 6) are collinear, find a.

5. If the area of the quadrilateral whose angular points A, B, C, D taken in order are (1, 2), (–5, 6), (7, –4) and (–2, k) be zero, find the value of k.

6. A, B, C are the points (–1, 5), (3, 1) and (5, 7) respectively. D, E, F, are the mid-points of BC, CA, AB respectively. Prove that Δ ABC = 4 Δ DEF.

7. The straight lines $y = m_1x + c_1$, $y = m_2x + c_2$, and $x = 0$ intersect in the three points P, Q, and R. Find the area of the triangle PQR. What is the value of the area if $c_1 = c_2$? **[*ISC*]**

ANSWERS

1. (*i*) 9 sq units (*ii*) 22 sq units (*iii*) $\frac{1}{2}(bx + ay - ab)$ sq. units **2.** (*i*) 132 sq. units (*ii*) $\frac{41}{2}$. sq. units

4. $a = 8$ **5.** $k = 3$ **7.** (*i*) $\frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2}$ (*ii*) 0.

15.20. Shifting of Origin

Let OX and OY be the original axes, O' the new origin and (h, k) its co-ordinates referred to OX and OY.

Let $O'X'$ and $O'Y'$ be drawn parallel to and in the same directions as OX and OY respectively. Let the coordinates of P be (x, y) referred to the original co-ordinate axes OX and OY, and (X, Y) referred to the new pair of axes $O'X'$ and $O'Y'$.

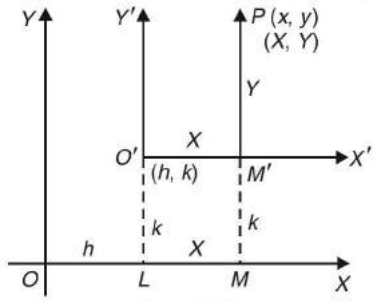


Fig. 15.28

Let $O'L$ and PM be perpendiculars from O' and P respectively to OX. Let PM meet $O'X'$ in M'.

Then $x = OM = OL + LM = h + X = X + h \Rightarrow X = x - h$

$y = MP = MM' + M'P = k + Y = Y + k \Rightarrow Y = y - k$

Thus, we have

(*i*) The point whose coordinates were (x, y) has now the coordinates (X, Y), i.e., $(x - h, y - k)$ w.r.t the new pair of axes.

(*ii*) The curve whose equation was $f(x, y) = 0$ has now the equation $f(X + h, Y + k) = 0$

Note. This transformation is called the **translation of axes.**

Ex. 13. The co-ordinate of a point P are (7, 12). Find its new coordinates with respect to the origin (0, 0) and the axes of coordinates through it. Find the new coordinates of P if the origin is shifted to the point (4, 7).

Sol. Let the new coordinates of P with respect the the new origin (4, 7) be (X, Y). Then $X = 7 - 4$, $Y = 12 - 7$ $[x - h, y - k]$, *i.e.*, $X = 3$, $Y = 5$. So the new coordinates of P with respect to the origin (4, 7) are (3, 5).

Ex. 14. If the coordinates of the new origin O' are (4, 9) and the coordinates of a point P w.r.t. the new origin are (5, 3), find the co-ordinates of P w.r.t. the old origin O (0, 0).

Sol. The new coordinates of P w.r.t. the origin (4, 9) are (5, 3).

If $(h, k) \equiv (4, 9)$ and $(X, Y) \equiv (5, 3)$, then the coordinate of P w.r.t the old origin O(0, 0) are (x, y), then, we have

$$X = x - h,\ Y = y - k$$

$$\Rightarrow \quad 5 = x - 4,\ 3 = y - 9 \Rightarrow x = 9,\ y = 12$$

Hence, the coordinates of P w.r.t. the old origin (0, 0) are (9, 12).

Examples for Practice

1. Find the new coordinates of a point P if its coordinates

(*i*) w.r.t. the old origin are (8, 11) and the origin is shifted to the point (3, 5).

(*ii*) w.r.t. the old origin are (–3, 7) and the origin is shifted to the point (5, 9)

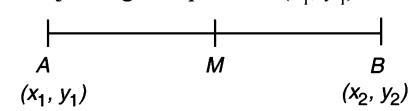
Ans. (*i*) (5, 6) (*ii*) (–8, –2).

2. Find the coordinates of a point P w.r.t. the old origin (0, 0) if its coordinates w.r.t. the new origin (7, 9) are (2, 3). **Ans.** (9, 12)

CHAPTER SUMMARY

1. Mid-point formula

The coordinates of the mid-point M of a line segment AB joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

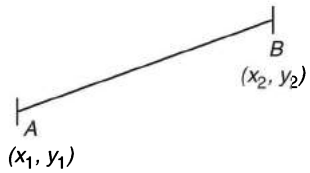


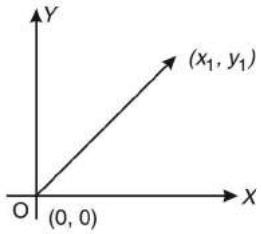
2. Distance formula

The distance between the points $A\ (x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$|\,d\,| \;=\; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance of a point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$.





3. Section formula

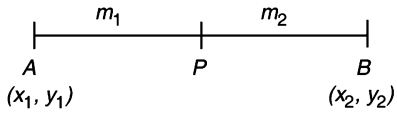
The coordinates of a point P, which divides the line segment AB joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2$ are given by

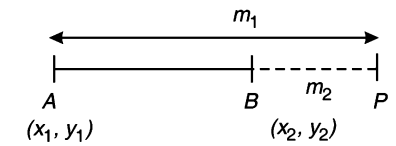
(a) If the division is internal

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

(b) If the division is external

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$





4. Centroid of a triangle

The coordinates of the centroid of a triangle having vertices whose coordinates are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

5. Incentre of a triangle

The coordinates of the incentre of a triangle whose sides are of lengths a, b, c and vertices have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

6. Area of a triangle

The area of a triangle whose vertices have co-ordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the formulae

$$\Delta \;=\; \frac{1}{2}|\,x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\,|$$

7. Condition of collinearity

Three given points are collinear if the area of the triangle formed by them is zero.

8. Shifting the origin

If the coordinates of a point P w.r.t. the origin $(0, 0)$ are (x, y) and the origin is shifted to the point (h, k) then the coordinates of P w.r.t. the new origin are $P(X, Y)$ given by $X = x - h$, $Y = y - k$

CHAPTER TEST

1. Show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right triangle.
2. A quadrilateral has the vertices at the points (–4, 2), (2, 6), (8, 5) and (9, –7). Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.
   [**Hint.** Prove either of the following
   (*i*) opposite side equal
   (*ii*) the mid-point of one diagonal is the same as that of the other diagonal.]
3. Show that the points $(a, b + c)$, $(b, c + a)$, $(c, a + b)$ are collinear.
4. Centroid of a triangle is (1, 4) and two of its vertices are (4, –3) and (–9, 7). Find the area of the triangle. [**Ans.** $91\frac{1}{2}$ sq. units]
5. Find the third vertex of a triangle if two of its vertices are at (–1, 4) and (5, 2) and the medians through these vertices meet at (0, –3). [**Ans.** (–4, –15)]

HISTORICAL NOTE

Rene Descartes (1596–1650). It occasionally happens—perhaps once in a century—that a man is born with such an all round genius and such a mastery of many branches of learning as to make it difficult to classify him. Such a man was Descartes, known also by his Latin name Renatus Cartesius. He was the inventor of Coordinate Geometry or Analytical Geometry.

The great contribution to mathematics made by him was embodied in an Appendix to a treatise on method in science. It bore the simple name La Geometric, but its content was by no means of simple importance, for it was one of the most powerful agencies for the advancement of mathematics in his time. In it is the essence of analytical geometry, the application of algebra to geometry, or more properly the idea of the inter-relation of the two. The most important feature of this inter-relation is that any point in a plane is determined by two coordinates, x and y, the equation $f(x, y) = 0$ expressing a property which is true for every point of the curve. The Appendix proved to be of far more importance than the work itself, constituting what the English philosopher John Stuart Mill described as "the greatest single step ever made in the progress of the exact sciences". This method of representing lines and curves by equations has become of the highest importance for all scientific work, and it has enabled mathematics to assist in all investigations of processes taking place in space.

Jacques Hadamard, an eminent mathematician and fellow countryman of Descartes has remarked: "The mere invention of coordinates was not Descartes' greatest merit, because that had already been done, by the ancients ?... It is quite another thing to recognise (as in the use of co-ordinates) a general method and to follow to the end the idea which it represents. It is exactly this merit, whose importance every real mathematician knows, that was pre-eminently Descartes' in geometry...Descartes indeed understood thoroughly the significance of what he had done and he was right when he boasted that he had as far surpassed all geometry before him as Cicero's rhetoric surpasses the ABC."

CHAPTER 16

THE STRAIGHT LINE

16.01. Slope or Gradient of a straight line—Definition

The slope (or gradient) of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis. The slope of a line is indicated by the letter 'm'.

Thus, if AP be the portion of a line above the x-axis and $\angle XAP = \theta$, then

$$m = \tan\theta.$$

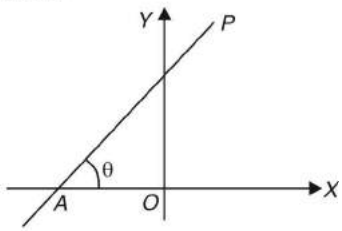


Fig. 16.01

Note 1. θ is measured positively from 0° to 180°.

Note 2. The reader should clearly understand this definition and note that for the slope we take that angle which is measured from the positive direction of the x-axis in the anticlockwise direction to the part of the line above the x-axis. Fig. 16.01 shows different positions of the straight line and the angle measured there on.

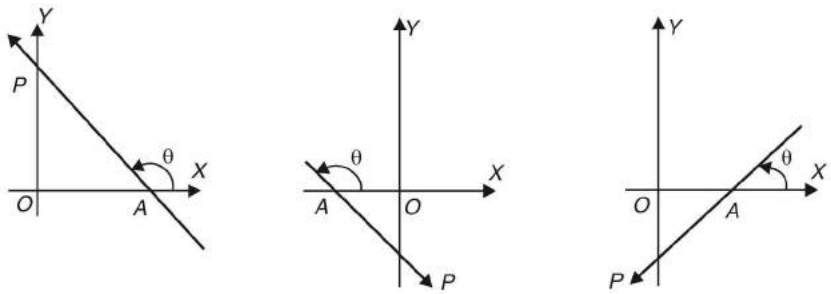


Fig. 16.02

16.02. Slope of a line making equal angles with the axes

In Fig. 16.03 are shown the two positions of a line in which it makes equal angles with the axes.

In (*i*) $\theta = 45°$; $m = \tan 45° = 1$

In (*ii*) $\theta = 135°$; $m = \tan 135°$

$= -1 \therefore \boldsymbol{m = \pm 1.}$

Note. When a line makes equal angles with the axes, it cuts off equal intercepts on the axes.

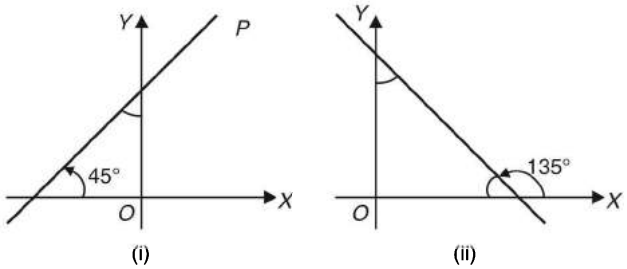


Fig. 16.03

Cor. Lines parallel and perpendicular to the x-axis.

(a) If a line is parallel to the x-axis, its inclination is 0° and so its slope is then tan 0°, *i.e.,* 0

*(b) If a line is perpendicular to the x-axis, its inclination is 90° and so its slope is ∞.

16.03. Slope of a line joining two points

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two points. Let θ be the inclination of the line AB with the x-axis. Construct as shown in Fig. 16.04. Then,

$$\tan\theta = \frac{LB}{AL} = \frac{y_2 - y_1}{x_2 - x_1} \qquad [\theta = \angle XPB = \angle LAB]$$

$$\therefore \quad \boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

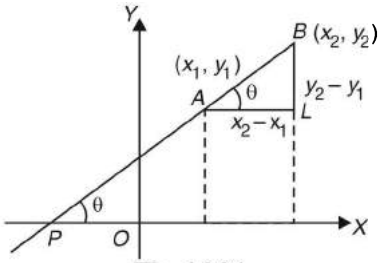


Fig. 16.04

If θ is a obtuse angle as in Fig. 16.05, then

$$m = \tan\theta$$

$$= -\tan(180° - \theta) = -\frac{BR}{RA}$$

$$= -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

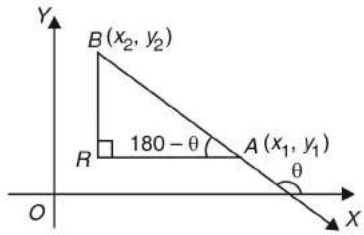


Fig. 16.05

Aid to memory. Slope of a line joining two points is equal to $\dfrac{\text{difference of the ordinates}}{\text{difference of the abscissae}}$.

16.04. Angle between two lines

Suppose we have to find the angle between two lines whose inclinations to the x-axis are θ_1 and θ_2. Let θ and ϕ be the interior and exterior angles respectively between the given lines, then

$$\theta_1 = \theta + \theta_2 \text{ or } \theta = \theta_1 - \theta_2$$

$$\therefore \quad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\left[\text{Using the formula, } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

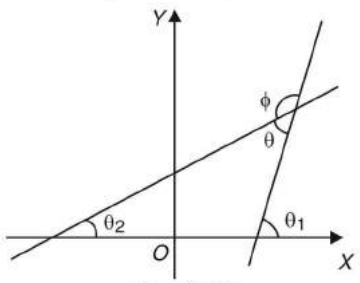


Fig. 16.06

If we put $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, we have

$$\boldsymbol{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

If ϕ be the exterior angle between the lines, then

$$\tan\phi = \tan(\pi - \theta) = -\tan\theta = -\frac{m_1 - m_2}{1 + m_1 m_2}$$

Hence the complete angle formula is $\boldsymbol{\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1 m_2}}$.

In particular, if α is acute, then tan α is +ve. Therefore, the acute angle α between the lines is given by.

* Strictly speaking the slope of a line parallel to the y-axis is not defined as tan 90° does not exist; tan 90° is equal to ∞ in the limiting sense.

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

Note 1. Aid to memory:

$$\tan\theta = \frac{\text{difference of the slopes}}{1 + \text{product of the slopes}}.$$

Note 2. Acute and obtuse angles between the lines. Finding out the angle between two lines generally means finding out the acute angle between them.

Note 3. When to use the complete angle formula. The complete angle formula is used when the angle is given.

Note 4. This formula fails when one of the lines is perpendicular to the x-axis (why?). In such cases, we apply formula given in Art. 16.11. (See note Art. 16.11)

Ex. 1. Find the angle between the lines whose slopes are – 3 and $-\frac{1}{2}$.

Sol. If θ be the required angle, then

$$\tan\theta = \left|\frac{-3-\left(\frac{-1}{2}\right)}{1+(-3)\left(\frac{-1}{2}\right)}\right| = \left|\frac{\frac{-5}{2}}{\frac{5}{2}}\right| = 1 = \tan 45°. \quad \therefore \theta = 45°.$$

16.05. Parallel and perpendicular lines

1. Parallel lines. (*Condition of parallelism.*)

If the lines are parallel, then $\theta = 0°$ $\therefore \tan\theta = \tan 0° = 0$.

i.e., $\frac{m_1 - m_2}{1 + m_1 m_2} = 0$ $\therefore m_1 - m_2 = 0$ or $m_1 = m_2$,

which is the required condition.

Hence, two lines are parallel if their slopes are equal.

2. Perpendicular lines. (*Condition of perpendicularity*)

If the lines are perpendicular, then $\theta = 90°$,

i.e. $\frac{m_1 - m_2}{1 + m_1 m_2} = \infty$ [$\tan 90° = \infty$]

$\therefore$ The denominator $1 + m_1 m_2 = 0$ or $\boldsymbol{m_1 m_2 = -1}$ **or** $\boldsymbol{m_2 = -\frac{1}{m_1}}$

which is the required condition.

Thus, two lines are perpendicular if the product of their slopes = –1.

i.e. if the slope of one line is the negative reciprocal of the slope of the other line.

EXERCISE 16 (*a*)

1. Find the slope of a line whose inclination is

(*i*) 30° (*ii*) 45° (*iii*) 60° (*iv*) 15° (*v*) 135°

2. Find the slope and inclination of the line through each pair of the following points:

(*i*) (1, 2) and (5, 6) (*ii*) (0, 0) and $(-\sqrt{3}, 3)$

(*iii*) (10, 4) and (–2, –2) (*iv*) (–1, –8) and (5, 7).

3. In the hexagon $PQRSTU$, $RS \| PU \| QT$. Which sides or diagonals have

(*i*) positive slope (*ii*) negative slope

(*iii*) zero slope (*iv*) infinite slope?

4. The side BC of an equilateral ΔABC is parallel to the x-axis. What are the slopes of its sides?

5. In a regular hexagon $ABCDEF$, $AB \| ED \|$ the x-axis. What are the slopes of its sides?

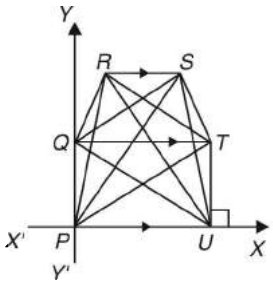


Fig. 16.07

6. Using slopes determine which of the following sets of three points are collinear.

(*i*) (5, –2), (7, 6), (0, –2); (*ii*) (–2, 3), (8, –5), (5, 4);

(*iii*) (6, –1), (5, 0), (2, 3); (*iv*) (–1, 5), (3, 1), (5, 7).

7. Find y if the slope of the line joining (–8, 11), $(2, y)$ is $\frac{-4}{3}$.

8. Find the angle between the lines whose slope are (*i*) 2 and –1 (*ii*) 2 and $\frac{3}{4}$.

9. Find the slope of the line which makes an angle of 45° with a line of slope $\frac{-6}{5}$.

[**Hint.** As the angle is given, so use complete angle formula and get two values of the slope].

10. Find the interior angles of the triangle whose vertices are $A(4, 3)$, $B(-2, 2)$ and $C(2, -8)$.

11. Find the slope of a line parallel to a line whose slope is

(*i*) –3 (*ii*) $\frac{1}{2}$ (*iii*) 2·3 (*iv*) 0.

12. Find the slope of a line parallel to the line which passes through each pair of the following points:

(*i*) (0, 0) and (5, 6), (*ii*) (–1, 3) and (4, 7), (*iii*) (–5, –8) and (3, 0), (*iv*) $(-a, 0)$ and $(0, b)$.

13. Find the slope of a line perpendicular to the line whose slope is

(*i*) $\frac{1}{3}$ (*ii*) $-\frac{5}{6}$ (*iii*) 5 (*iv*) $-5\frac{1}{7}$

(*v*) 0 (*vi*) Infinite

14. Find the slope of a line perpendicular to the line which passes through each pair of the following points:

(*i*) (0, 8) and (–5, 2); (*ii*) (1, –11) and (5, 2); (*iii*) $(-k, h)$ and $(b, -f)$; (*iv*) (x_1, y_1) and (x_2, y_2).

15. In rect. $ABCD$, slope of $AB = \frac{5}{6}$. State the slope of (*i*) BC, (*ii*) CD, (*iii*) DA.

16. In parallelogram $ABCD$, slope of $AB = -2$, slope of $BC = \frac{3}{5}$. State the slope of

(*i*) AD, (*ii*) CD, (*iii*) the altitude to AD, (*iv*) the altitude to CD.

17. The vertices of a ΔABC are $A(1, 1)$, $B(7, 3)$ and $C(3, 6)$. State the slope of the altitude to

(*i*) AB, (*ii*) BC, (*iii*) AC.

18. The line joining $(-5, 7)$ and $(0, -2)$ is perpendicular to the line joining $(1, -3)$ and $(4, x)$. Find x.

ANSWERS

1. (*i*) $\frac{1}{\sqrt{3}}$, (*ii*) 1, (*iii*) $\sqrt{3}$, (*iv*) $2-\sqrt{3}$, (*v*) -1.

2. (*i*) 1, 45° (*ii*) $-\sqrt{3}$, 120° (*iii*) $\frac{1}{2}$, 26° 34′, (*iv*) $\frac{5}{2}$, 68° 12′.

3. (*i*) QR, QS, PS, PT, PR, (*ii*) RT, QU, RU, ST, SU, (*iii*) PU, QT, RS, (*iv*) PQ, UT.

4. $0, -\sqrt{3}, \sqrt{3}$. **5.** $0, \sqrt{3}, -\sqrt{3}$ $0, \sqrt{3}, -\sqrt{3}$. **6.** Only (*iii*) **7.** $\frac{-7}{3}$

8. (*i*) 71° 34′, (*b*) 26° 34′. **9.** 11, $\frac{-1}{11}$. **10.** $A = 70° 14′, B = 77° 40′, C = 32° 6′$.

11. The same for each part. **12.** (*i*) $\frac{6}{5}$ (*ii*) $\frac{4}{5}$ (*iii*) 1 (*iv*) $\frac{b}{a}$.

13. (*i*) -3 (*ii*) $\frac{6}{5}$ (*iii*) $\frac{-1}{5}$ (*iv*) $\frac{7}{36}$. (*v*) Infinite slope (*vi*) 0.

14. (*i*) $\frac{-5}{6}$ (*ii*) $\frac{-4}{13}$ (*iii*) $\frac{b+k}{f+h}$, (*iv*) $-\left(\frac{x_2-x_1}{y_2-y_1}\right)$

15. (*i*) $\frac{-6}{5}$ (*ii*) $\frac{5}{6}$ (*iii*) $\frac{-6}{5}$

16. (*i*) $\frac{3}{5}$ (*ii*) -2 (*iii*) $\frac{-5}{3}$ (*iv*) $\frac{1}{2}$

17. (*i*) -3 (*ii*) $\frac{4}{3}$ (*iii*) $\frac{-2}{5}$ **18.** $x = -\frac{3}{4}$

16.06. Straight lines in simplest forms

The straight line is the simplest geometric curve. Below we give three loci which are the simplest forms of a straight line.

(*i*) A straight line parallel to the *x*-axis at a given distance from it.

Let AB be the straight line parallel to the x-axis at a distance b from it. The straight line can be considered as the locus of a moving point $P(x, y)$ whose distance from the x-axis is equal to b for all positions of P.

Draw PM perpendicular to the x-axis. Then $OM = x$ and $MP = y$. Using the condition $MP = b$, we get

$$y = b$$

which is the required equation of the straight line AB.

Corollary. *The equation of the x-axis is $y = 0$, since $b = 0$ in this case.*

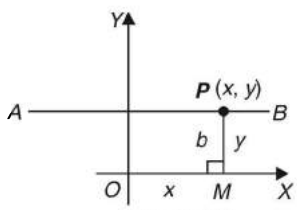


Fig. 16.08

(*ii*) **A straight line parallel to the *y*-axis at a given distance from it.**

Let *AB* be the straight line parallel to the *y*-axis at a distance *a* from it. This straight line can be considered as the locus of a moving point $P(x, y)$ whose distance from the *y*-axis is equal to *a* for all positions of *P*.

Let *PM* be the perpendicular from *P* to the *x*-axis. Then $OM = x$, and $MP = y$. Using the condition $OM = a$, we get

$$\boldsymbol{x = a}$$

which is the required equation of the straight line *AB*.

Corollary. *The equation of the y-axis is $x = 0$, since $a = 0$ in this case.*

Aid to memory. *The equation of a line parallel to the x-axis does not contain x and the equation of a line parallel to the y-axis does not contain y. Thus,*

(*a*) *The equation $3y - 5 = 0$ represents the straight line which is parallel to the x-axis and at a distance of $\frac{5}{3}$ units above it, since $y = \frac{5}{3}$.*

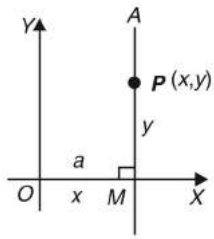


Fig. 16.09

(*b*) *The equation $5y + 7 = 0$ represents the straight line which is parallel to the x-axis and at a distance of $\frac{7}{5}$ units below it, since $y = -\frac{7}{5}$.*

(*c*) *The equation $2x - 11 = 0$ represents the straight line which is parallel to the y-axis and at a distance of $\frac{11}{2}$ units to its right, since $x = \frac{11}{2}$.*

(*d*) *The equation $5x + 9 = 0$ represents the straight line which is parallel to the y-axis and at a distance of $\frac{9}{5}$ units to its left, since $x = -\frac{9}{5}$.*

(*iii*) **A straight line through the origin.**

Let *l* be a straight line through the origin such that it makes an angle θ with the positive direction of the *x*-axis.

This line *l* can be considered as the locus of the moving point *P* (x, y). Let *PM* be the perpendicular to the *x*-axis. Then $OM = x$ and $MP = y$. Then

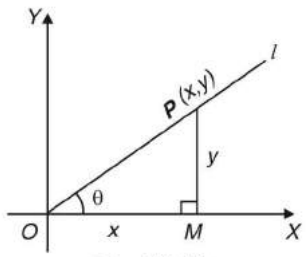


Fig. 16.10

$$\frac{y}{x} = \tan\theta \text{ or } y = \tan\theta \,.\, x$$

or $y = mx$, *if* $m = \tan\theta$ = gradient of the line.

Thus, $\boldsymbol{y = mx}$ is the equation of the straight line passing through the origin and having its gradient equal to *m*.

16.07 Intercepts—Definition

(*a*) If a straight line *l* meets the *x*-axis at *A* and the *y*-axis at *B*, then

(*i*) *OA* (*i.e.*, the distance of *A* from the origin) is called the intercept made by the line on the *x*-axis, or simply *x-intercept,* and

(*ii*) *OB* (*i.e.*, the distance of B from the origin) is called the intercept made by the line on the *y*-axis, or simply *y-intercept.*

(*iii*) the *two together, OA* and *OB* (in this order) are called the *intercepts made by the line on the axes.*

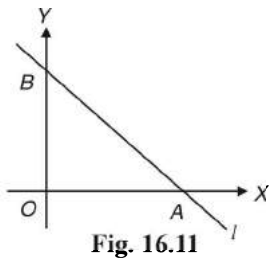


Fig. 16.11

(*b*) **Rule for the signs of the intercepts**

(*i*) *The intercept on the x-axis is positive if measured to the right of the origin and negative if measured to the left of the origin.*

(*ii*) *The intercept on the y-axis is positive if it is measured above the origin and negative if measured below the origin.*

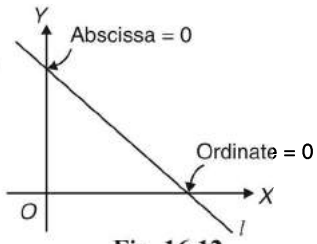


Fig. 16.12

(*c*) **Rule to find the intercepts of a line on the axes**

It is evident that a line cuts the *x*-axis at a point whose ordinate is zero and cuts the *y*-axis at a point whose abscissa is zero. Hence,

(*i*) *to get the x-intercept, we put y = 0 in the equation of the line and find the value of x.*

(*ii*) *to get the y-intercept, we put x = 0 in the equation of the line and find the value of y.*

16.08. Various forms of equations of the straight line

(1) Slope-intercept Form

To find the equation of the straight line whose gradient is m and whose intercept on the y-axis is c.

Let *l* be the straight line which makes an angle θ with *OX* and whose intercept *OB* on the *y* axis is *c*, such that tan θ = *m*.

The straight line can be considered as the locus of a moving point *P* (*x*, *y*).

Take any position of *P* on the straight line. Draw *PM* perpendicular to the *x*-axis. From *B* draw *BL* perpendicular to *PM*.

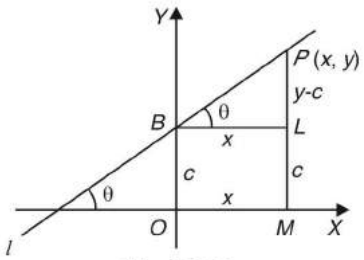


Fig. 16.13

Then, $\frac{y-c}{x} = \tan\theta = m$

Or $\quad y = mx + c$, which is the required equation.

As it involves slope and *y*-intercept of the straight line, it is sometimes referred to as slope-intercept form.

Note 1. If the equation of a straight line is put in the form $y = mx + c$, then *m*, the coefficient of *x*, is the slope, and *c*, the constant term, is the *y*-intercept of the straight line.

Note 2. If the slope and the *y*-intercept of a line are known, we can at once write down its equation by substituting for *m* and *c* their values with proper signs in the equation $y = mx + c$. For example, the equation of the line of slope –3 and *y*-intercept equal to 7 is $y = -3x + 7$.

[$\because$ $m = -3$, $c = 7$]

(2) Point-slope form

To find the equation of the straight line passing through a given point (x_1, y_1) and having a given gradient m.

Let a line *l* make an angle θ with *OX* such that tan θ = *m*.

Let *Q* (x_1, y_1) be a given point on the line.

Let the line be considered as the locus of a moving point *P* (*x*, *y*).

Draw *PM* and *QN* perpendicular to the *x*-axis, and *QL* perpendicular to *PM*.

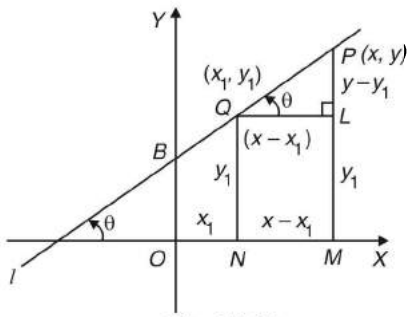


Fig. 16.14

Then $\frac{y - y_1}{x - x_1} = \tan\theta = m$

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

which is the required equation.

Note. The equation is used to write down the equation of a line if its slope and one point are known.

(3) Two-point slope form

To find the equation of the straight line passing through two given points (x_1, y_1) ***and*** (x_2, y_2).

Since the line passes through (x_1, y_1) *and* (x_2, y_2), therefore slope of the line $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Substituting this value of m in the *Point-slope form,* we get

$$\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

which is the required equation.

Note. This formula enables us to find the equation of the line joining two given points. For example, the equation of the line joining the points (5, 7) and (–1,2) is

$$y - 7 = \frac{2 - 7}{-1 - 5}(x - 5), \textit{ i.e., } 5x - 6y + 17 = 0.$$

Ex. 2. Determine the slope and *y*-intercept of the following lines:

(*i*) $y = x + 5$. **(*ii*) $y = \sqrt{3}\,x - 7$.**

Sol. (*i*) Comparing with $y = mx + c$, we get $m = 1$ and $c = 5$

$\therefore$ slope = 1, and y-intercept = 5.

(*ii*) Similarly in $y = \sqrt{3}\,x - 7$, $m = \sqrt{3}$ and $c = |-7|$

$\because$ slope = $\sqrt{3}$, and y-intercept = $|-7| = 7$

Ex. 3. State the equation of the line which has the *y*-intercept equal to $\frac{4}{3}$ and is perpendicular to $3x - 4y + 11 = 0$.

Sol. $3x - 4y + 11 = 0 \Rightarrow 4y = 3x + 11 \Rightarrow y = \frac{3}{4}x + \frac{11}{4}$

Slope of $3x - 4y + 11 = 0$ is $\frac{3}{4}$.

$\therefore$ Slope of a perpendicular line is $-\frac{4}{3}$.

$\therefore$ The equation required is $y = -\frac{4}{3} \times x + \frac{4}{3}$ or $4x + 3y = 4$.

Ex. 4. Find the equation of the straight line through the given point *P*(–1, – 5) and having its slope equal to $\frac{9}{5}$.

Sol. The required equation is $y - (-5) = \frac{9}{5}[x - (-1)]$ [Using $y - y_1 = m(x - x_1)$]

or $5(y + 5) = 9(x + 1)$ or $9x - 5y - 16 = 0$.

Note. The line can be constructed as follows if we want to: As the slope is positive so the line rises 9 units as it moves a horizontal distance of 5 units to the right. Hence to construct the line proceed as given below:

(*i*) Locate the point P (–1, –5).

(*ii*) Move 5 units to the right and then 9 units vertically upward and call this final point as Q.

(*iii*) Join P, Q.

Then PQ is the required line. [Fig. 16.15]

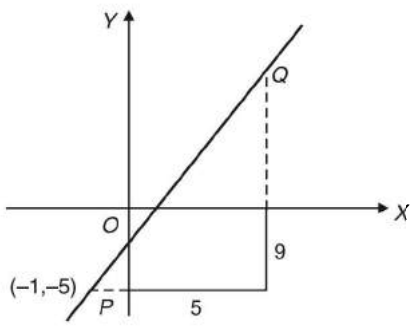


Fig. 16.15

Ex. 5. Find the equation of the line through the point (4, –5) and (*a*) parallel to (*b*) perpendicular to the line $3x + 4y + 5 = 0$.

Sol. Slope of the line $3x + 4y + 5 = 0$ is $\frac{-3}{4}$ Comparing with $y = mx + c$ ...(1)

(*a*) Slope of the line parallel to (1) is also $\frac{-3}{4}$.

$\therefore$ The equation of the line passing through the point (4, – 5) and parallel to (1) is

$$y + 5 = \frac{-3}{4}(x - 4) \qquad [\text{Using } y - y_1 = m(x - x_1)]$$

or $4(y + 5) = -3(x - 4)$ or $3x + 4y + 8 = 0$.

(*b*) Slope of the line perpendicular to (1) is $\frac{4}{3}$.

$\therefore$ The equation of the line perpendicular to (1) and through the point (4, –5) is

$$y + 5 = \frac{4}{3}(x - 4)$$

or $3(y + 5) = 4(x - 4)$ or $4x - 3y - 31 = 0$.

Ex. 6. Find the equations of the two straight lines through the point (2, –1) making an angle of 45° with the line $6x + 5y - 1 = 0$.

Sol. The equation of any line through (2, –1) is $y + 1 = m(x - 2)$. ...(1)

Now we proceed to determine m.

As (1) makes an angle of 45° with $6x + 5y - 1 = 0$ $\left(\text{slope} = \frac{-6}{5}\right)$.

So $$\tan 45° = \pm \frac{m - \left(\frac{-6}{5}\right)}{1 + m\left(\frac{-6}{5}\right)}.$$ (See Note 3 of Art. 16.04)

Taking positive sign, we get $5 - 6m = 5m + 6$ or $m = \frac{-1}{11}$.

Taking negative sign, we get $5 - 6m = -5m - 6$ or $m = 11$

Substituting these values of m in (1), we get $y + 1 = \frac{-1}{11}(x - 2)$ and $y + 1 = 11(x - 2)$

or $x + 11y + 9 = 0$ and $11x - y - 23 = 0$,

which are the required equations of the lines.

EXERCISE 16 (*b*)

1. State the equation of the line which has the y-intercept

(i) 2 and slope 7 ; (ii) – 3 and slope – 4 ; (iii) –1 and is parallel to $y = 5x - 7$;

(iv) 2 and is inclined at 45° to the x-axis; (v) – 5 and is equally inclined to the axes.

2. What will be the value of m and c if the straight line $y = mx + c$ passes through the points (3, –4) and (–1, 2)?

3. Find the equation of the straight line through the given point P and having the given slope m if

(i) $P(-4, 7)$, $m = -\sqrt{3}$; (ii) $P(-1, -5)$, $m = \frac{-6}{11}$.

4. Find the equation of the line through the point (1, –2) making an angle of 135° with the x-axis.

5. Find the equation to the straight line passing through

(i) the origin and perpendicular to $x + 2y = 4$; (ii) the point (4, 3) and parallel to $3x + 4y = 12$;

(iii) the point (4, 5) and (a) parallel to, (b) perpendicular to $3x - 2y + 5 = 0$.

6. Find the equation to the line which is perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ at the point where it meets the x-axis.

7. Find the equation of the two lines through the point (4, 5) which make an acute angle of 45° with the line $2x - y + 7 = 0$. **[*ISC*]**

8. The line through $A(4, 7)$ with gradient m meets the x-axis at P and the y-axis at R. The line through B (8, 3) with gradient $\frac{-1}{m}$ meets the x-axis at Q and the y-axis at S. Find in terms of m, the co-ordinates of P, Q, R and S. Obtain expressions for $OP.OQ$ and $OR.OS$, where O is the point (0, 0). **[*SC*]**

9. Write down the slopes of the lines

(i) joining P(l, 1) and $Q(2, 3)$; (ii) joining $L(-p, q)$ and $M(r, s)$;

(iii) parallel to the line joining $A(-1, 5)$ and $B(-6, -7)$;

(iv) perpendicular to the line joining $B(2, -3)$ and $S(-4, 1)$.

10. Find the equations of the lines joining the points

(i) A(l, 1) and $B(2, 3)$; (ii) $L(a, b)$ and $M(b, a)$; (iii) $P(3, 3)$ and $Q(7, 6)$.

What is the length of the portion of the line in (c) intercepted between the axes of co-ordinates?

11. Given the vertices $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$ of ΔABC, find

(i) the equation of the side AB; (ii) the equation of the median through A;

(iii) the equation of the altitude through B;

(iv) the equation of the perpendicular bisector of the side AB.

12. The points A, B and C are (4, 0), (2, 2) and (0, 6) respectively. AB produced cuts the y-axis at P and CB produced cuts the x-axis at Q. Find the co-ordinates of the points P and Q. Find the equation of the straight line joining the mid-points of AC and OB (where O is the origin), and verify that this line passes through the mid-point of PQ. **[*SC*]**

13. A line through the point (3, 0) meets the variable line $y = tx$ at right angles at the point P. Find, in terms of t, the co-ordinates of P.

Find the value of k for which P lies on the curve $x^2 + y^2 = kx$. **[*ISC*]**

14. The point P is the foot of the perpendicular from $A(0, t)$ to the line whose equation is $y = tx$. Determine

(i) the equation of the line AP, (iii) the co-ordinates of P,

(iii) the area of ΔOAP, where O is the origin. **[*ISC*]**

15. Find the equation of the line joining the origin to the point of intersection of $4x + 3y = 8$ and $x + y = 1$. **[*ISC*]**

16. Find the equation to the straight line which passes through the point of intersection of the lines $3x + 4y - 1 = 0$ and $5x + 8y - 3 = 0$ and is perpendicular to the line $4x - 2y + 3 = 0$. **[*ISC*]**

ANSWERS

1. (*i*) $y = 7x + 2$ (*ii*) $y = -4x - 3$, (*iii*) $y = 5x - 1$, (*iv*) $y = x + 2$ (*v*) $y = x - 5$ or $y = -x - 5$.

2. $m = \frac{-3}{2}, c = \frac{1}{2}$ **3.** (*i*) $\sqrt{3}x + y = 7 - 4\sqrt{3}$ (*ii*) $6x + 11y + 61 = 0$. **4.** $x + y + 1 = 0$.

5. (*i*) $2x - y = 0$, (*ii*) $3x + 4y - 24 = 0$, (*iii*) $3x - 2y - 2 = 0$, $2x + 3y = 23$.

6. $ax + by = a^2$ **7.** $3x + y - 17 = 0$, $x - 3y + 11 = 0$.

8. $P\left(\frac{4m-7}{m}, 0\right)$, $Q(3m + 8, 0)$, $R(0, -4m + 7)$, $S\left(0, \frac{3m+8}{m}\right)$

$\text{OP.OQ} = \frac{1}{m}(4m - 7)(3m + 8)$, and $\text{OR.OS} = -\frac{1}{m}(4m - 7)(3m + 8)$.

9. (*i*) 2 (*ii*) $\frac{(s-q)}{r+p}$ (*iii*) $\frac{12}{5}$ (*iv*) $\frac{3}{2}$.

10. (*i*) $2x - y - 1 = 0$ (*ii*) $x + y = a + b$ (*iii*) $3x - 4y + 3 = 0$, $\frac{5}{4}$.

11. (*i*) $5x + 114y = 106$ (*ii*) $y = 4$, (*iii*) $12x + 5y + 3 = 0$ (*iv*) $28x - 10y = 19$.

12. (0, 4), (3, 0), $2x - y = 1$. **13.** $P\left(\frac{3}{1+r^2}, \frac{3t}{1+t^2}\right)$, $k = 3$.

14. (*i*) $x + ty - t^2 = 0$ (*ii*) $P\left(\frac{t^2}{1+t^2}, \frac{t^3}{1+t^2}\right)$ (*iii*) $\frac{1}{2}\left(\frac{t^3}{1+t^2}\right)$ sq. units.

15. $4x + 5y = 0$. **16.** $x + 2y - 1 = 0$

(4) Intercept form

To find the equation of the straight line which makes intercepts 'a' and 'b' on the x-axis and the y-axis respectively.

Let the straight line l meet the x-axis at A and the y-axis at B such that $OA = a$, and $OB = b$.

The line can be considered as the locus of a moving point $P(x, y)$. Draw PM perpendicular to the x-axis. Then from similar triangles AMP and AOB, we get

$$\frac{MA}{OA} = \frac{MP}{OB} \Rightarrow \frac{a-x}{a} = \frac{y}{b} \Rightarrow 1 - \frac{x}{a} = \frac{y}{b}$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

which is the required equation.

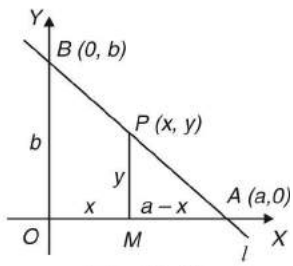


Fig. 16.16

Alternatively, the equation could also be formed by considering the fact that the line passes through two points $A(a, 0)$ and $B(0, b)$. This is left as an exercise to the student.

Aid to memory. $\frac{x}{x\text{-intercept}} + \frac{y}{y\text{-intercept}} = 1$.

Ex. 7. Find the equation of the straight line passing through the point (2, 3) and cutting off (*a*) equal intercepts along the positive directions of both the axes, (*b*) intercepts equal in magnitude and opposite in sign.

Sol. (*a*) Let the intercepts be each equal to '*a*'.

Then, the equation of the line in the intercept form is $\frac{x}{a}+\frac{y}{a}=1$ *i.e.* $x+y=a$. ...(1)

As the point (2, 3) lies on it, so $2+3=a$ or $a=5$

Substituting in (1), we get $x+y=5$.

(*b*) Let the intercepts be '$-a$' and '$+a$'. Then the equation of the line in the intercept form is

$$\frac{x}{-a}+\frac{y}{a}=1, \text{ i.e., } x-y=-a$$

As the point (2, 3) lies on it, so $2-3=-a$ or $a=1$

$\therefore$ The equation of the line is $x-y+1=0$.

Note. If the intercepts are 'a' and '$-a$' on the x-axis and the y-axis respectively, then the line is

$$\frac{x}{a}+\frac{y}{-a}=1, \text{ i.e., } x-y=+a$$

As the point (2, 3) lies on it, so $2-3=+a$ or $a=-1$

$\therefore$ The equation of the line is $-x+y=+1$ or $x-y+1=0$.

which is the same as found above.

(5) Perpendicular form or normal form

To find the equation of a straight line in terms of the length of the perpendicular 'p' from the origin to the line and the angle 'α' which the perpendicular makes with the x-axis.

Method I. Let the given straight line l be such that $OL=p$ and $\angle LOX=\alpha$.

The straight line can be considered as the locus of a moving point $P(x, y)$.

Draw PM perpendicular to the x-axis. Then draw MK perpendicular to OL, and PN perpendicular to MK as shown in Fig. 16.17.

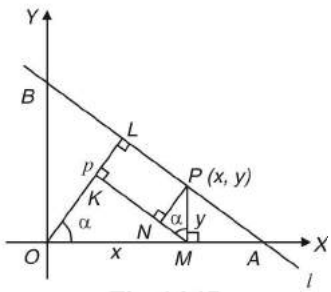


Fig. 16.17

$\angle LOM+\angle KMO=90°$, $\angle KMO+\angle PMK=90°$

$\therefore \quad \angle LOM+\angle KMO=\angle KMO+\angle PMK$

$\therefore \quad \angle LOM=\angle PMK$ or $\alpha=\angle PMK$ or $\angle PMK=\alpha$

Now, $p=OL=OK+KL=OK+PN=x\cos\alpha+y\sin\alpha$ $\left[\because \frac{OK}{OM}=\cos\alpha, \text{ and } \frac{PN}{MP}=\sin\alpha\right]$

Hence, $\boldsymbol{x\cos\alpha+y\sin\alpha=p}$ is the required equation.

Method II. Gradient of the line OL is $\tan\alpha$. Therefore, the gradient of given line is $-\cot\alpha$. The coordinates of L are $(p\cos\alpha, p\sin\alpha)$.

Applying the one-point form of the straight line, the equation required is

$$y-p\sin\alpha=-\cot\alpha(x-p\cos\alpha) \text{ or } y-p\sin\alpha=-\frac{\cos\alpha}{\sin\alpha}(x-p\cos\alpha)$$

or $y\sin\alpha-p\sin^2\alpha=-x\cos\alpha+p\cos^2\alpha$ or $x\cos\alpha+y\sin\alpha=p(\cos^2\alpha+\sin^2\alpha)$

or $\boldsymbol{x\cos\alpha+y\sin\alpha=p}$.

Note. p is always measured away from the origin and is always positive in value, α is a positive angle less than 360° measured from the positive direction OX of the x-axis to the normal from the origin to the line.

Ex. 8. Sketch and write down the equation of the line for which (*i*) $p = 3$, $\alpha = 120°$, (*ii*) $p = 8$, $\alpha = 135°$, (*iii*) $p = 6$, $\alpha = -150°$.

Sol. (*i*) The equation of the line is

$$x \cos 120° + y \sin 120° = 3$$

or $$x\left(\frac{-1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 3$$

or $$x - \sqrt{3}\, y + 6 = 0.$$

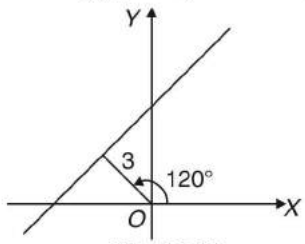


Fig. 16.18

(*ii*) and (*iii*) are left as exercises to the student.

Ex. 9. Sketch roughly the lines satisfying the given conditions and write their equations:

(*i*) inclination $\theta = 150°$, and distance from the origin = 3.

(*ii*) *x* intercept = 7, and distance from the origin = 2.

(*iii*) nearest point to the origin is (3, –4).

Sol. (*i*) The figure shows that $p = 3$ and $\alpha = 60°$ for the line AB, and $p = 3$ and $\alpha = 180° + 60° = 240°$ for the line CD. Therefore their equations are

$x \cos 60° + y \sin 60° = 3$ and $x \cos 240° + y \sin 240° = 3$ [Fig. 16.19 (*i*)]

or $$x \times \frac{1}{2} + y \times \frac{\sqrt{3}}{2} = 3 \text{ and } x \times \left(\frac{-1}{2}\right) + y\left(\frac{-\sqrt{3}}{2}\right) = 3 \text{ or } x + \sqrt{3}\, y = 6 \text{ and } x + \sqrt{3}\, y + 6 = 0.$$

(*ii*) Let $\angle AOL = \alpha$ such that $\cos\alpha = \frac{2}{7}$, and therefore $\sin\alpha = \frac{3\sqrt{5}}{7}$ [Fig. 16.19 (*ii*)]

The figure shows that $p = 2$ for both the lines AB and AC, and if the perpendicular OL makes angle α with OX, the perpendicular OM makes angles $(360° - \alpha)$ with OX.

$\therefore$ Δs OAL and OAM are congruent

$\therefore$ $\angle AOM = \alpha$

$\therefore$ OM makes angle $(360° - \alpha)$ with OX.

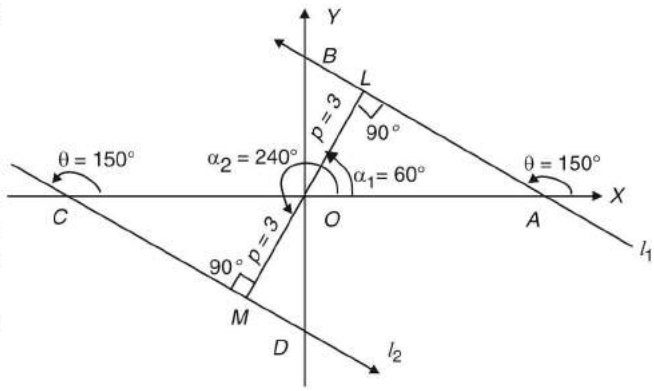


Fig. 16.19 (*i*)

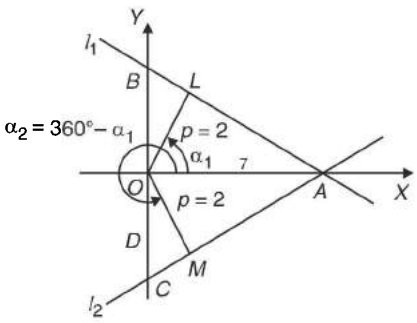


Fig. 16.19 (*ii*)

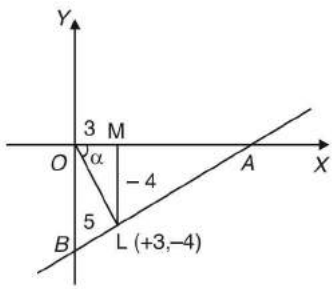


Fig. 16.19 (*iii*)

$\therefore$ The equations of AB and AC are

$x \cos\alpha + y \sin\alpha = 2$ and $x \cos(360° - \alpha) + y \sin(360° - \alpha) = 2$

or $x \times \frac{2}{7} + y \times \frac{3\sqrt{5}}{7} = 2$ and $x \cos \alpha - y \sin \alpha = 2$. or $2x + 3\sqrt{5y} = 14$ and $2x - 3\sqrt{5y} = 14$.

(*iii*) Let $\angle LOM = \alpha$ such that $\cos \alpha = \frac{3}{5}$ and $\sin \alpha = \frac{4}{5}$. [Fig. 16.19 (*iii*)]

Then, OL makes an angle $(360° - \alpha)$ with OX, and $OL = \sqrt{3^2 + 4^2} = 5$.

$\therefore$ The equation of the line AB is $x \cos (360° - \alpha) + y \sin (360° - \alpha) = 5$

or $\cos \alpha - y \sin \alpha = 5$ or $x \times \frac{3}{5} - y \times \frac{4}{5} = 5$ or $3x - 4y = 25$.

EXERCISE 16 (*c*)

1. Write down the equation of the straight line cutting off intercepts a and b from the axes where

(*i*) $a = -2, b = 3$. (*ii*) $a = 5, b = -6$. (*iii*) $a = -\frac{k}{m}, b = k$.

2. Determine the x-intercept *'a'* and the y-intercept *'b'* of the following lines. Sketch each.

(*i*) $3x + 5y - 15 = 0$, (*ii*) $x - y - 7 = 0$.

3. Find the equation of the line which makes equal intercepts on the axes and passes through the point (2, 3).

4. Write down the equation of the line which makes an intercept of $2a$ on the x-axis and $3a$ on the y-axis. Given that the line passes through the point (14, –9), find the numerical value of a. **[*SC*]**

5. Find the equation of the straight line which passes through the point (5, 6) and has intercept on the axes equal in magnitude but opposite in sign.

6. A straight line passes through (2, 3) and the portion of the line intercepted between the axes is bisected at this point. Find its equation.

7. Show that the three points (5, 1), (1, – 1) and (11, 4) lie on a straight line. Further find

(*i*) its intercepts on the axes;

(*ii*) the length of the portion of the line intercepted between the axes;

(*iii*) the slope of the line.

8. Find the equation of the straight line which passes through the point (3, – 2) and cuts off positive intercepts on the x and y-axes which are in the ratio 4 : 3. **[*SC*]**

9. Find the equation of the straight line at a distance of 3 units from the origin such that the perpendicular from the origin to the line makes an angle α, given by the equation $\tan \alpha = \frac{5}{12}$, with the positive direction of the axis of x.

10. Show on a diagram the position of the straight line $x \cos 30° + y \sin 30° = 2$ in relation to the co-ordinate axes, indicating clearly which angle is 30° and which length is 2 units. Find

(*i*) the equation of the straight line parallel to that given and passing through the point (4, 3);

(*ii*) the length of the perpendicular from the origin on to this line;

(*iii*) the distance between the two parallel straight lines. **[*SC*]**

11. A straight line $\frac{x}{a} - \frac{y}{b} = 1$ passes through the point (8, 6) and cuts off a triangle of area 12 units from the axes of co-ordinates. Find the equations of the straight line. **[*ISC*]**

12. A straight line passes through the points $(a, 0)$ and $(0, b)$. The length of the line segment contained between the axes is 13 and the product of the intercepts on the axes is 60. Calculate the values of a and b and find the equation of the straight line. **[*ISC*]**

ANSWERS

1. (*i*) $3x - 2y + 6 = 0$, (*ii*) $6x - 5y - 30 = 0$, (*iii*) $mx - y + k = 0$.
2. (*i*) $a = 5, b = 3$. (*ii*) $a = 7, b = -7$. **3.** $x + y - 5 = 0$ or $x - y + 1 = 0$.
4. $3x + 2y = 6a, a = 4$. **5.** $x - y + 1 = 0$. **6.** $3x + 2y - 12 = 0$.
7. (*i*) $3, \frac{-3}{2}$, (*ii*) $\frac{3}{2}\sqrt{5}$, (*iii*) $\frac{1}{2}$. **8.** $4y + 3x = 1$.
9. $12x + 5y = 39$
10. $x \cos 30° + y \sin 30° = 2\sqrt{3} + \frac{3}{2}$; Length of the perp. from the origin upon the line $= 2\sqrt{3} + \frac{3}{2}$ or 4.96 ; Distance between the two parallel lines = 4.96 – 2 = 2.96.
11. $\frac{x}{4} - \frac{y}{6} = 1$ and $\frac{x}{8} - \frac{y}{3} = -1$ **12.** $a = 12, b = 5; \frac{x}{12} + \frac{y}{5} = 1$; $a = -12, b = -5$; $\frac{x}{12} + \frac{y}{5} = -1$

16.09. General equation of the straight line

If we scrutinise closely the different forms of equations of the straight line found so far we notice that they are *all first* degree equations in both x and y. Now we shall show that the converse is also true. We shall establish the following theorem.

Every equation of the form $ax + by + c = 0$, so long as both a and b are not zero, is the equation of a straight line.

We consider two cases, according as $b = 0$ or $b \neq 0$. If $b = 0$, then we must not have $a = 0$, and the above equation becomes $x = \frac{-c}{a}$, which we know as the equation of a straight line parallel to the y-axis and $\frac{-c}{a}$, units from it.

If $b \neq 0$, we divide by b and solve for y, getting $y = \frac{-a}{b}x - \frac{c}{b}$.

From the slope-intercept form for the equation of a line, we recognize this as the equation of a line with slope $\frac{-a}{b}$ and y-intercept $\frac{-c}{b}$.

16.10. Gradient or slope of ax + by + c = 0

Method I. We put it as $y = \frac{-a}{b}x - \frac{-c}{b}$ and by comparing it with $y = mx + c$, we find that the gradient, which is the coefficient of x, is $\frac{-a}{b}$.

Thus, the gradient of the line $ax + by + c = 0$ is equal to $\frac{-a}{b}$. *i.e.*, $-\frac{\text{coefficient of } x}{\text{coefficient of } y}$.

Method II. The st. line $ax + by + c = 0$ cuts the x-axis at $\left(\frac{-c}{a}, 0\right)$ and the y-axis at $\left(0, \frac{-c}{b}\right)$.

$\therefore$ The gradient $= \frac{\frac{-c}{b} - 0}{0 - \left(\frac{-c}{a}\right)} = -\frac{a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$.

16.11. Angle between two lines

Let the lines be

$$a_1x + b_1y + c_1 = 0 \quad ...(i) \quad \text{and} \quad a_2x + b_2y + c_2 = 0 \quad .... (ii)$$

Let their slopes be m_1 and m_2. Then $m_1 = \frac{-a_1}{b_1}$ and $m_2 = \frac{-a_2}{b_2}$.

If θ is the angle between the given lines, we get

$$\tan\theta = \left|\frac{m_1 \sim m_2}{1 + m_1\, m_2}\right| = \left|\frac{a_1b_2 \sim a_2b_1}{a_1a_2 + b_1b_2}\right| \qquad \text{(A)}$$

Working Rule 1. *Write down the slopes of the given lines by using the relation:*

$$Slope = -\frac{coeff.\ of\ x}{coeff.\ of\ y}.$$

2. *Apply the formula* $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1\, m_2}\right|$

Note. The formula $\tan\theta = \frac{m_1 \sim m_2}{1 + m_1\, m_2}$ fails when one of the lines is parallel to the y-axis. *i.e.,* of the form $x = a$, *whose slope is* ∞. *In such cases we use the form* $\tan\theta = \left|\frac{a_1b_2 \sim a_2b_1}{a_1a_2 + b_1b_2}\right|$.

16.12. To reduce the general equation ax + by + c = 0 to the perpendicular form

The equation $x\cos\alpha + y\sin\alpha = p$ has the following characteristics:

(a) (coeff. of x)^2 + (coeff. of y)^2 = 1.

(b) the constant term occurs on the right side and is positive.

Let the altered form of the equation be $kax + kby + kc = 0$. ...(1)

Then, we should have $(ka)^2 + (kb)^2 = 1$.

Where from $k = \frac{1}{\sqrt{a^2 + b^2}}$ [Taking + ve sign with the square root]

Substituting in (1), $\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y + \frac{-c}{\sqrt{a^2 + b^2}} = 0$

Transposing, $\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{-c}{\sqrt{a^2 + b^2}}$ [Taking constant term to the right side]

If c is positive, the required equation will be $-\frac{a}{\sqrt{a^2 + b^2}}x - \frac{b}{\sqrt{a^2 + b^2}}y = \frac{c}{\sqrt{a^2 + b^2}}$

[Changing sign and making right side positive]

Here, $\cos\alpha = \frac{-a}{\sqrt{a^2 + b^2}}, \sin\alpha = \frac{-b}{\sqrt{a^2 + b^2}}, p = \frac{c}{\sqrt{a^2 + b^2}}$

If c is –ve, the required equation would be, $\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{-c}{\sqrt{a^2 + b^2}}$

$$\left[\because \frac{-c}{\sqrt{a^2 + b^2}} \text{ is + ve}\right]$$

Working Rule 1. *Divide throughout by* $\sqrt{a^2 + b^2}$, *i.e.,* by

$$\sqrt{(coeff.\ of\ x)^2 + (coeff.\ of\ y)^2}.$$

2. *Transpose the constant term to the right hand side and make it positive by changing the sign, if necessary.*

Ex. 10. Reduce the equation $x + y + \sqrt{2} = 0$ to the normal form and find the values of p and α.

Sol. Dividing both sides by $\sqrt{\{(1)^2 + (1)^2\}}$, *i.e.*, $\sqrt{2}$, we get $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 1 = 0$

or $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = -1$ [Taking the constant term to the right side]

or $-\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 1$ ....(1) [*Changing signs to make the constant term +ve*]

This is the required form.

Comparing this equation with the equation $x \cos \alpha + y \sin \alpha = p$

$$\cos \alpha = \frac{-1}{\sqrt{2}}, \qquad \sin \alpha = -\frac{1}{\sqrt{2}}.$$

Considering angles between 0° and 360°,

when $\cos \alpha = \frac{-1}{\sqrt{2}}$, $\alpha = 135°$ or $225°$ [Angles in 2nd and 3rd quadrants]

when $\sin \alpha = -\frac{1}{\sqrt{2}}$, $\alpha = 225°$ or $315°$ [Angles in 3rd and 4th quadrants]

Common angle = 225°

Hence, $\alpha = 225°$ and $p = 1$ and the normal form of the equation is

$$x \cos 225° + y \sin 225° = 1 \qquad ...(2)$$

We can also solve two trigonometric equations as follows:

$$\cos \alpha = \frac{-1}{\sqrt{2}}, \ \sin \alpha = \frac{-1}{\sqrt{2}}.$$

Since both $\sin \alpha$ and $\cos \alpha$ are negative in value, therefore, α lies in the third quadrant. Dividing, we get $\tan \alpha = 1 = \tan 45°$ or $= \tan (180° + 45°)$ $\therefore$ $\alpha = 45°$ or $225°$

Hence, $\alpha = 225°$ as it lies in the third quadrant.

Note. When such values appear in the equation for which angles can be found out with the help of *t*-ratios of 0°, 30°, 45°, 60°, 90°, the required eqn. should be written in the form (2) and not form (1).

16.13. Identical lines (Comparing coefficients)

If two equations

$$ax + by + c = 0 \qquad ...(1)$$

and $$a'x + b'y + c' = 0 \qquad ...(2)$$

represent the same straight line, then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Proof. As the two equations represent the same st. line, so they will have the same slope and the same y-intercepts.

Expressing equations (1) and (2) in the form of $y = mx + c$, we have

$$y = \frac{a}{b}x - \frac{c}{b} \text{ or } y = -\frac{a'}{b'}x - \frac{c'}{b'}$$

$\therefore$ $\quad \frac{-a}{b} = \frac{-a'}{b'} \text{ or } \frac{a}{a'} = \frac{b}{b'}$ [Slopes being equal]

$\quad \frac{-c}{b} = \frac{-c'}{b'} \text{ or } \frac{b}{b'} = \frac{c}{c'}$ [y-intercepts being equal]

$\therefore$ $\quad \boldsymbol{\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}}.$

Aid to Memory. If two equations represent the same st. line,

ratio of coefficients of x = ratio of coefficients of y = ratio of constant terms.

16.14. Position of two points relative to a line

To show *that the points (x_1, y_1) is on one side or the other of the straight line $ax + by + c = 0$ according as the expression $ax_1 + by_1 + c$ is positive or negative.*

Let P be the point (x_1, y_1) and AB the line $ax + by + c = 0$. Draw PM perpendicular to the x-axis to meet AB in Q. Let $MQ = y'$. Take the co-ordinates of Q as (x_1, y').

If P lies above AB, then $PM > QM$

or $\quad y_1 > y' \quad ...(1)$

Now, because Q lies on AB

$\therefore \quad ax_1 + by' + c = 0 \text{ or } y' = -\frac{a}{b}x_1 - \frac{c}{b}$

Substituting in (1), we get $y_1 > -\frac{a}{b}x_1 - \frac{c}{b}$ or $by_1 > -ax_1 - c$

$ax_1 + by_1 + c > 0 \quad ...(2)$

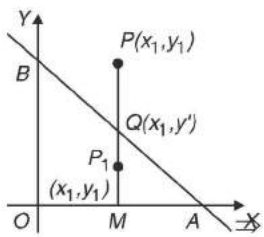


Fig. 16.20

If P lies below AB as in the position of P_1 (see Fig. 16.20), then $PM < QM$ or $y_1 < y'$.

By repeating the above process we shall get $ax_1 + by_1 + c < 0$. ...(3)

Thus two points lie on the same side or opposite sides of the given line $ax + by + c = 0$ according as $ax + by + c$ is of the same sign or opposite signs when the co-ordinates of the given points are substituted successively for x and y in $ax + by + c$.

Ex. 11. Are the points (2, 3) and (1, 3) on the same side or on opposite sides of the line $x - 2y = -3$.

Sol. The equation of the line is

$$x - 2y + 3 = 0 \quad ...(1) \quad \text{(Make right side = 0)}$$

Substituting the co-ordinates (2, 3) in the left side of (1), we get $2 - 2(3) + (3) = -1$ and substituting the co-ordinates (1, 3), we get $1 - 2(3) + 3 = -2$.

As the results are of the same sign, so the points are on the same side of the line.

EXERCISE 16 (*d*)

1. Write down the slopes of the following lines:

(*i*) $2x + 3y + 1 = 0$ (*ii*) $7x - 5y + 8 = 0$ (*iii*) $-6y - 11x = 0$

(*iv*) $xx_1 + yy_1 = a^2$ (*v*) $3x + 4y - 2(x + x_1) - 5(y + y_1) + 2 = 0$.

2. Find the value of k such that the line
$(k-2)x + (k+3)y - 5 = 0$ is (*i*) parallel to the line $2x - y + 7 = 0$ (*ii*) perpendicular to it.

3. Prove that the lines (*i*) $3x + 4y - 7 = 0$ and $28x - 21y + 50 = 0$ are mutually perpendicular;
(*ii*) $px + qy - r = 0$ and $-4px - 4qy + 5s = 0$ are parallel.

4. Find the slope of the line which is perpendicular to the line $7x + 11y - 2 = 0$.

5. Determine the angle between the lines whose equations are
(*i*) $3x + y - 7 = 0$ and $x + 2y + 9 = 0$, (*ii*) $2x - y + 3 = 0$ and $x + y - 2 = 0$.

6. Use tables to find the acute angle between the lines $2y + x = 0$ and $\frac{x}{1} + \frac{y}{2} = 2$. **[*SC*]**

7. Reduce the following equations to the normal form and find the values of p and α.
(*i*) $\sqrt{3}x - y + 2 = 0$, (*ii*) $3x + 4y + 10 = 0$ (Use tables).

8. Put the equation $12y = 5x + 65$ in the form $x\cos\theta + y\sin\theta = p$ and indicate clearly in a rough diagram the position of the straight line and the meaning of the constant θ and p. **[*SC*]**

9. If $Ax + By = C$ and $x\cos\alpha + y\sin\alpha = p$ represent the same line, find p in terms of A, B, C.

10. Show that $(2, -1)$ and $(1, 1)$ are on opposite sides of $3x + 4y = 6$.

11. The sides of a triangle are given by the equations $3x + 4y = 10$, $4x - 3y = 5$, and $7x + y + 10 = 0$; show that the origin lies within the triangle.

12. Find by calculation whether the points $(13, 8)$, $(26, -4)$ lie in the same, adjacent, or opposite angles formed by the straight lines $5x + 6y - 112 = 0$, and $10x + 11y - 217 = 0$. **[*CU*]**

ANSWERS

1. (*i*) $\frac{-2}{3}$ (*ii*) $\frac{7}{5}$ (*iii*) $\frac{-11}{6}$ (*iv*) $\frac{-x_1}{y_1}$ (*v*) 1.

2. (*i*) $\frac{-4}{3}$ (*ii*) 7. **4.** $\frac{11}{7}$ **5.** (*i*) 45° (*ii*) $\tan^{-1}(3) = 71°\ 34'$

6. $\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}(0.75) = 36°\ 52'$.

7. (*i*) $x\cos 150° + y\sin 150° = 1$; $p = 1$, $\alpha = 150°$ (*ii*) $x\cos 233°8' + y\sin 233°8' = 2$

8. $\frac{-5}{13}x + \frac{12}{13}y = 5$; $p = 5, \theta = 112°37'$ nearly. **9.** $p = \frac{|C|}{\sqrt{(A^2 + B^2)}}$ **12.** Opposite.

16.15. Intersection of straight lines

The co-ordinates of the point or points of intersection of two lines or two curves satisfy the equations of both of them and so their co-ordinates are obtained by solving their equations simultaneously as in Algebra. The corresponding values of x and y so obtained give respectively the abscissae and the ordinates of the point or the points of intersection.

Ex. 12. Find the equation of the straight line joining the origin to the point of intersection of $y - x + 7 = 0$ and $y + 2x - 2 = 0$. **[*SC*]**

Sol. $y + 2x = 2$ ...(1)

$y - x = -7$ ...(2)

Subtracting (2) from (1), $3x = 9$ $\therefore\ x = 3$.

Substituting the value of x in (1), $y = -4$.

$\therefore$ The co-ordinates of the point of intersection of (1) and (2) are $(3, -4)$.

The equation of the line joining the origin to the point $(3, -4)$ is

$$y - 0 = \frac{-4 - 0}{3 - 0}(x - 0) \qquad i.e.\ 4x + 3y = 0. \qquad \left[\because (y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$$

16.16. Concurrency of lines

The three given lines are concurrent if they meet in a point, *i.e.,* they have a common point of intersection. Hence, to prove that three given lines are concurrent, we proceed as follows:

Rule. 1. *Solve the equations of two of the lines and find the co-ordinates of their point of intersection.*

2. *Show that these co-ordinates satisfy the third equation.*

Ex. 13. Prove that the three lines

$$2x - y - 5 = 0,\ 3x - y - 6 = 0 \text{ and } 4x - y - 7 = 0 \text{ meet in a point.}$$

Sol. Solving the first two equations, we get

$x = 1$, $y = -3$. Thus, the co-ordinates of the point of intersection of the first two lines are $(1, -3)$

Substituting these co-ordinates in the third equation, we get

$4(1) - (-3) - 7 = 0$, or $0 = 0$, which is true.

Hence, the lines are concurrent.

Ex. 14. For what value of 'a' the three straight lines $3x + y + 2 = 0$, $x - y + 2 = 0$, $x + 2ay + 5 = 0$ are concurrent.

Sol. Solving the first two equations, the point of intersection of the first two lines is obtained as $(-1, 1)$.

If the third line also passes through this point, we must have $-1 + 2a + 5 = 0$, which gives $a = -2$.

Ex. 15. To find the condition that the three straight lines $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ and $A_3x + B_3y + C_3 = 0$ are concurrent.

Sol. The given straight lines are

$$A_1x + B_1y + C_1 = 0 \quad \ldots(i)$$

$$A_2x + B_2y + C_2 = 0 \quad \ldots(ii)$$

and

$$A_3x + B_3y + C_3 = 0 \quad \ldots(iii)$$

Solving (i) and (ii) by the rule of cross-multiplication, we get

$$\frac{x}{B_1C_2 - B_2C_1} = \frac{y}{C_1A_2 - C_2A_1} = \frac{1}{A_1B_2 - A_2B_1}$$

$$\therefore \quad x = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1},\ y = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}$$

$\therefore$ The co-ordinates of the point of intersection of (i) and (ii) are

$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right)$$

For the given lines to meet in a point, this point must lie on (iii).

$$\therefore \quad A_3\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}\right) + B_3\left(\frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right) + C_3 = 0$$

$$\Rightarrow \quad A_3(B_1C_2 - B_2C_1) + B_3(C_1A_2 - C_2A_1) + C_3(A_1B_2 - A_2B_1) = 0.$$

EXERCISE 16 (*e*)

1. Find the co-ordinates of the point of intersection of the straight lines
(*i*) $3x - 5y + 5 = 0, 2x + 3y - 22 = 0$ (*ii*) $2x - 3y - 7 = 0, 3x - 4y - 13 = 0$.

2. Find the area of the triangle formed by the lines $y + x - 6 = 0$, $3y - x + 2 = 0$ and $3y = 5x + 2$.
[Hint. First find the co-ordinates of the vertices of the triangle by solving the equations taken two at a time. Now apply area formula.]

3. Find the orthocentre of the triangle whose angular points are (0, 0), (2, – 1), (– 1, 3).
[**Note.** Orthocentre is the point of intersection of the altitudes]

4. The vertices of a triangle are A (0, 5), B (– 1, – 2) and C (11, 7). Write down the equations of BC and the perpendicular from A to BC and hence find the co-ordinates of the foot of the perpendicular.

5. Find the equation of the straight line passing through the point of intersection of the two lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and parallel to the straight line $y - x = 8$.
[Hint. Find the pt. of intersection and use the eqn. $y - y_1 = m(x - x_1)$].

6. Find the equation of the line through the intersection of $y + x = 9$ and $2x - 3y + 7 = 0$, and perpendicular to the line $2y - 3x - 5 = 0$.

7. Prove that the three lines $5x + 3y - 7 = 0$, $3x - 4y = 10$, and $x + 2y = 0$ meet in a point.

8. For what value of m are the three lines $y = x + 1$, $y = 2(x + 1)$ and $y = mx + 3$ concurrent ?

9. The co-ordinates of A, B and C are (3, 1), (1, 5) and (4, 2) respectively. P is the mid-pt. of BC and (*i*) Q lies on AC and is such that $CQ : QA = 3 : 1$, R lies on AB and is such that $AR : RB = 1 : 3$. Find the equation of the lines AP, BQ and CR and prove that lines are concurrent. **[*SC*]**

10. The sides of a triangle are OA, OB, AB and have equations $2x - y = 0$, $3x + y = 0$, $x - 3y + 10 = 0$, respectively. Find the equation of the three medians of the triangle and verify that they are concurrent. **[*SC & GCE*]**

11. Show that the lines $lx + my + n = 0$, $mx + ny + l = 0$ and $nx + ly + m = 0$ are concurrent if $l + m + n = 0$.

12. Prove that the lines $(b - c)x + (c - a)y + (a - b) = 0$, $(c - a)x + (a - b)y + (b - c) = 0$ and $(a - b)x + (b - c)y + (c - a) = 0$ are concurrent.

13. Prove that the medians of a triangle are concurrent.

ANSWERS

1. (*i*) (5, 4), (*ii*) (11,5), **2.** 12 sq units; **3.** (–4, –3). **4.** $3x - 4y = 5$, $4x + 3y = 15$; (3, 1)
5. $x - y = 0$. **6.** $2x + 3y - 23 = 0$. **8.** $m = 3$.
9. AP is $5x + y = 16$, BQ is $5x + 3y = 20$, CR is $y = 2$. **10.** $7x - y = 0$, $x - y + 2 = 0$, $x + 2y - 5 = 0$.

16.17. Distance of a point from a line

Let the given line be $ax + by + c = 0$ ...(*i*)
and $P(x_1, y_1)$ the given point

Putting $y = 0$ in (*i*), we have $x = \frac{-c}{a}$.

$\therefore$ The line $ax + by + c = 0$ meets the x-axis at $A\left(\frac{-c}{a}, 0\right)$

Similarly, putting $x = 0$ in (*i*), we see that the given line meets the y-axis at $B\left(0, \frac{-c}{b}\right)$.

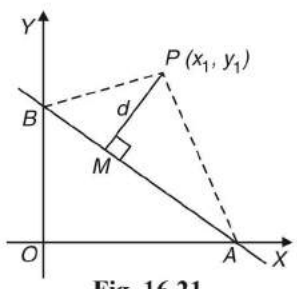


Fig. 16.21

$$\therefore \quad |AB| = \sqrt{\left[0-\left(\frac{-c}{a}\right)\right]^2+\left[\frac{-c}{b}-0\right]^2}$$

$$= \sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}} = \left|\frac{c}{ab}\right|\sqrt{a^2+b^2} \qquad ....(ii)$$

From P, draw $PM \perp AB$, then $d = |MP|$.

Now, Area of $\Delta PAB = \frac{1}{2}|AB|.|MP| = \frac{1}{2}|AB|.d$

$$= \frac{1}{2}\left|\frac{c}{ab}\right|\sqrt{a^2+b^2}\,.d \qquad ....(iii)$$

Also, Area of ΔPAB

$$= \frac{1}{2}\left|\left[0-\left(\frac{-c}{a}\right)y_1+\left(\frac{-c}{a}\right)\left(\frac{-c}{b}\right)-0+0-x_1\left(\frac{-c}{b}\right)\right]\right|$$

$$= \frac{1}{2}\left|\left(\frac{cy_1}{a}+\frac{c^2}{ab}+\frac{cx_1}{b}\right)\right|$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right||ax+by_1+c| \qquad ....(iv)$$

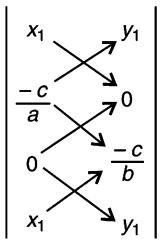


Fig. 16.22

Equating (*iii*) and (*iv*), we get

$$= \frac{1}{2}\left|\frac{c}{ab}\right|\sqrt{a^2+b^2}\,.d = \frac{1}{2}\left|\frac{c}{ab}\right||ax_1+by_1+c|$$

$$\Rightarrow \quad \boxed{d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$$

Working Rule. (*i*) *In the left side of the equation (right side being zero) substitute the co-ordinates of the point, and (ii) divide the result by* $\sqrt{(Coeff.\ of\ x)^2+(Coeff.\ of\ y)^2}$.

Cor. The perpendicular distance of the origin from the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2+b^2}}$.

16.18. Distance between two parallel lines

Let $ax + by + c = 0$ ...(*i*) and $ax + by + c' = 0$ ...(*ii*) be the given parallel lines.

Since, by definition of parallel lines, the distance between them is the same throughout, therefore, to find distance between them, choose any point on one of the lines and find the perpendicular distance from this point to the other line. A convenient point would be $\left(0, \frac{-c}{b}\right)$ [obtained by letting $x = 0 \Rightarrow y = \frac{-c}{b}$]

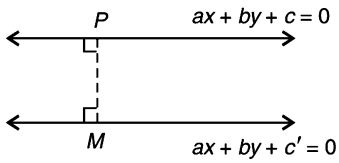


Fig. 16.23

Then, reqd. distance between the lines $= \frac{\left|a.0+b.\left(\frac{-c}{b}\right)+c'\right|}{\sqrt{a^2+b^2}} = \frac{|c'-c|}{\sqrt{a^2+b^2}}$.

Ex. 16. Find the distance of the point $P(-2, 3)$ from the line AB which is $x - y = 5$.

Sol. The equation of the line is

$$x - y - 5 = 0 \qquad \text{[Making right side zero } \textbf{(note this step)}\text{]}$$

$\therefore$ Perp. distance of the pt. $(-2, 3) = \dfrac{|(-2) - 3 - 5|}{\sqrt{(1)^2 + (-1)^2}} = \dfrac{10}{\sqrt{2}} = 5\sqrt{2}.$

[**Rule 1.** *Find the co-ordinates of any point on one of the given lines, preferably putting $x = 0$ or $y = 0$.*

2. *The perp. distance of this point from the other line is the required distance between the lines.*]

Ex. 17. Find the perpendicular distance between the lines.

$$\textbf{3}x + \textbf{4}y - \textbf{5} = \textbf{0} \quad \textbf{...(1) and } \textbf{6}x + \textbf{8}y - \textbf{45} = \textbf{0} \quad \textbf{...(2)}$$

Sol. Putting $x = 0$ in the first eqn., $y = \dfrac{5}{4}$. Hence, $\left(0, \dfrac{5}{4}\right)$ is a point on the first line.

Perp. distance of this pt. from the line $6x + 8y - 45 = 0$

$$= \frac{\left|6 \times 0 + 8 \times \frac{5}{4} - 45\right|}{\sqrt{36 + 64}} = \frac{|-35|}{10} = \frac{35}{10} = \frac{7}{2}$$

which is the required distance.

Ex. 18. Find the equation of the st. line, with a positive gradient, which passes through the point (–5, 0) and is at a perpendicular distance of 3 units from the origin. [*SC & GCE*]

Sol. Let m (> 0) the gradient of the line then, any line through $(-5, 0)$ is $y - 0 = m(x + 5)$

or $$mx - y + 5m = 0 \qquad ...(1)$$

Its perp. distance from the origin is 3, *i.e.*, $\pm\dfrac{|m \cdot 0 - 0 + 5m|}{\sqrt{m^2 + (-1)^2}} = 3 \Rightarrow |5m| = 3\sqrt{m^2 + 1}$

$\Rightarrow$ $$25m^2 = 9(1 + m^2) \text{ } i.e., \text{ } m^2 = \frac{9}{16} \Rightarrow m = \frac{3}{4}. \text{ (since } m \text{ is +ve)}$$

Hence, the required equation of the line with a positive gradient is $y = \dfrac{3}{4}.(x + 5)$, [From (1)]

i.e., $3x - 4y + 15 = 0$.

Ex. 19. Find all points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y = 10$. [*IIT*]

Sol. Let the reqd. point on the line $x + y = 4$ be (h, k), then $h + k = 4$ ...(1)

Its perp. distance from the line $4x + 3y = 10$ is $\dfrac{|4h + 3k - 10|}{\sqrt{4^2 + 3^2}} = 1 \text{ (given)} \Rightarrow |4h + 3k - 10| = 5$

$\Rightarrow$ $4h + 3k - 10 = \pm 5$.

$\Rightarrow$ $4h + 3k - 10 = 5$, *i.e.*, $4h + 3k = 15$ ...(2)

and $4h + 3k - 10 = -5$, *i.e.*, $4h + 3k = 5$ ...(3)

Solving (1) and (2), we get $h = 3$, $k = 1$. Solving (1) and (3), we get $h = -7$, $k = 11$

Hence, the required pts. are (3, 1) and (–7, 11).

EXERCISE 16 (*f*)

1. Find the distance of the point P from the line AB in the following cases :

(*i*) $P(4, 2)$, AB is $5x - 12y - 9 = 0$, **[*SC & GCE*]**

(*ii*) $P(0, 0)$, AB is $h(x + h) + k(y + k) = 0$.

2. Calculate the length of the perpendicular from (7, 0) to the straight line $5x + 12y - 9 = 0$ and show that it is twice the length of the perpendicular from (2, 1). **[*SC & GCE*]**

3. The point $A(0, 0)$, $B(1, 7)$, $C(5, 1)$ are the vertices of a triangle. Find the length of the perp. from A to BC and hence the area of the ΔABC. **[*SC & GCE*]**

4. Find the lengths of altitudes of the triangle whose sides are given by $3x - 4y = 5$, $4x + 3y = 5$ and $x + y = 1$.

5. If p is the perpendicular distance of the origin from the line whose intercepts on the axes are a and b, show that **[*ISC*]**

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

6. Find the perpendicular distance between the lines

(*i*) $3x + 4y + 5 = 0$, $3x + 4y + 17 = 0$. (*ii*) $9x + 40y - 20 = 0$, $9x + 40y + 21 = 0$.

(*iii*) $y = mx + c$, $y = mx + d$.

7. Find the equations of two straight lines which are parallel to the straight line $x + 7y + 2 = 0$, and a unit distance from the point $(2, -1)$.

8. Find the equations of the two straight lines drawn through the point (0, 1) on which the perpendiculars dropped from the point (2, 2) are each of unit length.

9. A straight line is parallel to the lines $3x - y - 3 = 0$ and $3x - y + 5 = 0$, and lies between them. Find its equation if its distances from these lines are in the ratio 3 : 5.

10. Find the equation of the locus of a point P which is equidistant from the st. line $3x - 4y + 2 = 0$ and the origin. **[*SC & GCE*]**

11. A point P is such that the sum of the squares of its distances from the two axes of co-ordinates is equal to the square of its distance from the line $x - y = 1$. Find the equation of the locus of P. **[*SC & GCE*]**

12. Show that the equation to the parallel line mid-way between the parallel lines

$$ax + by + c_1 = 0 \text{ and } ax + by + c_2 = 0 \text{ is } ax + by + \frac{c_1 + c_2}{2} = 0.$$

13. Prove that the line $12x - 5y + 7 = 0$ is mid-parallel to the lines $12x - 5y + 7 = 0$ and $12x - 5y - 13 = 0$.

ANSWERS

1. (*i*) 1. (*ii*) $\sqrt{(h^2 + k^2)}$. **2.** 2 units. **3.** $\frac{17}{\sqrt{13}}$ units; 17 sq. units **4.** $1, \frac{1}{7}, \frac{\sqrt{2}}{10}$

6. (*i*) $\frac{12}{5}$, (*ii*) 1, (*iii*) $\frac{|c \sim d|}{\sqrt{1 + m^2}}$ **7.** $x + 7y + 5(\sqrt{2} + 1) = 0$, $x + 7y - 5(\sqrt{2} - 1) = 0$.

8. $y - 1 = 0$, $4x - 3y + 3 = 0$. **9.** $3x - y = 0$. **10.** $16x^2 + 9y^2 + 24xy - 12x + 16y - 4 = 0$.

11. $x^2 + y^2 + 2xy + 2x - 2y - 1 = 0$.

HINTS TO SELECTED QUESTIONS

9. $3x - y - 3 = 0$ and $3x - y + 5 = 0$ (Given)

Let the required line be $3x - y + c = 0$ ....(*i*)

Take a point, say, $(0, c)$ on it. Then

$$\frac{\frac{|3\times 0 - c - 3|}{\sqrt{10}}}{\frac{|3\times 0 - c + 5|}{\sqrt{10}}} = \frac{3}{5} \Rightarrow 5c + 15 = -3c + 15 \Rightarrow 8c = 0 \Rightarrow c = 0$$

Substituting in (*i*), we get $3x - y = 0$, which is the required equation.

12. $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ (Given)

Let the parallel line mid-way between the given line be $ax + by + c = 0$. ...(1)

Any point, say $\left(0, \frac{-c}{b}\right)$, is equidistant from the given lines but in opposite directions, therefore

$$\frac{\left|(a\times 0) + b\times\left(\frac{-c}{b}\right) + c_1\right|}{\sqrt{a^2+b^2}} = -\frac{\left|(a\times 0) + b\times\left(\frac{-c}{b}\right) + c_2\right|}{\sqrt{a^2+b^2}}$$

$$\Rightarrow -c + c_1 = c - c_2 \Rightarrow c = \frac{c_1 + c_2}{2}$$

Substituting in (1), we get

$ax + by + \frac{c_1 + c_2}{2} = 0$, which is the required equation.

Another form, $ax + by + \frac{c_1 + c_2}{2} = 0 \Rightarrow \frac{1}{2}(2ax + 2by + c_1 + c_2) = 0$

$$\Rightarrow \frac{1}{2}[(ax + by + c_1) + (ax + by + c_2)] = 0$$

16.19. Angular bisectors

An angular bisector of two lines is the locus of a point which is equidistant from the two lines. Then, if the lines be $l_1: a_1 x + b_1 y + c_1 = 0$ and $l_2 : a_2 x + b_2 y + c_2 = 0$ and if (h, k) be any point on the bisector, then we have |PM| = |PN|

i.e., $$\frac{|a_1 h + b_1 k + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2 h + b_2 k + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$\Rightarrow$ Locus of $P(h, k)$ is

$$\frac{|a_1 x + b_1 y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2 x + b_2 y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

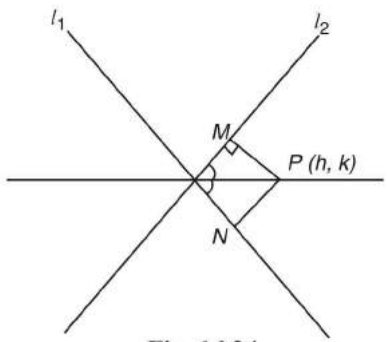


Fig. 16.24

Hence, the equations of the bisectors of the angles between lines l_1 and l_2 are given by

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

Note 1. On taking '+' and '–' signs separately, we get the equations of the bisectors of acute and obtuse angles between the lines. The **bisectors of acute and obtuse angles** can be distinguished by the method explained in Ex. 20.

Note 2. The two bisectors can be **distinguished *with respect to the origin*** by working out as follows:

1. *Make the constant terms positive in the equations of the given lines (right sides being zero).*

2. The equation obtained on taking the +ve sign will be the equation of the bisector b_1 that bisects the angle in which the origin lies. The other equation will represent the bisector b_2 that bisects the angle in which the origin does not lie.

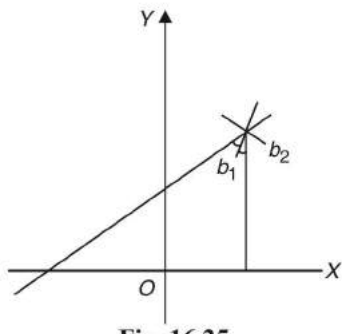


Fig. 16.25

Ex. 20. Find the equations of the bisectors of the angles between the lines $12x + 5y - 4 = 0$ and $3x + 4y + 7 = 0$. Distinguish (*i*) the one which bisects the angle in which the origin lies and (*ii*) the one which bisects the acute angle between the lines.

Sol. *Writing the given equations so that the constant terms are positive, we get

$$-12x - 5y + 4 = 0 \quad ...(i)$$

$$3x + 4y + 7 = 0 \quad ...(ii)$$

The equations of the angular bisectors of (*i*) and (*ii*) are

$$\frac{-12x - 5y + 4}{\sqrt{(-12)^2 + (-5)^2}} = \pm \frac{3x + 4y + 7}{\sqrt{3^2 + 4^2}}$$

The bisector of the angle in which the origin lies is

$$\frac{-12x - 5y + 4}{13} = + \frac{3x + 4y + 7}{5}, \quad i.e.,\ 5\ (-12x - 5y + 4) = 13\ (3x + 4y + 7),$$

$$i.e., \quad 99x + 77y + 71 = 0. \quad ...(iii)$$

The equation of the other bisector is

$$\frac{-12x - 5y + 4}{13} = - \frac{3x + 4y + 7}{5}, \ i.e.,\ 5\ (-12x - 5y + 4) = -13\ (3x + 4y + 7),$$

$$i.e., \quad 21x - 27y - 111 = 0, \quad i.e., \quad 7x - 9y - 37 = 0. \quad ...(iv)$$

Of (*iii*) and (*iv*), that bisector will be the bisector of the acute angle, which makes an angle $\theta < 45°$ with any of the lines (*i*) or (*ii*). To find this, we make use of the fact that if $\theta < 45°$, $\tan\theta$ is numerically < 1.

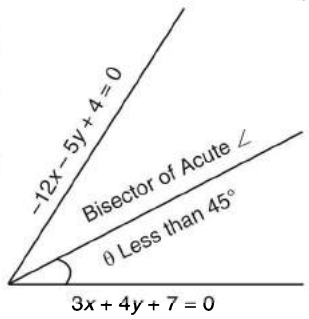


Fig. 16.26

Consider the line (*ii*) and the bisector (*iv*). If θ be the angle between them, then

$$\tan\theta = \frac{28 + 27}{-21 + 36} = \frac{55}{15} > 1 \Rightarrow \theta > 45°$$

$\therefore$ The equation (*iv*), *i.e.*

$7x - 9y - 37 = 0$ is the bisector of the obtuse angle.

Hence, the other bisector, namely, $99x + 77y + 71 = 0$ bisects, the acute angle.

Ex. 21. A triangle is formed by the lines whose equations are $3x + 4y - 6 = 0$, $12x - 5y - 3 = 0$ and $4x - 3y + 12 = 0$. Find the internal bisector of the angle opposite to the side $3x + 4y - 6 = 0$.

Sol. Let the lines be BC, CA, AB.

The bisectors of the angle opposite to $3x + 4y - 6 = 0$ are

$$\frac{12x - 5y - 3}{13} = \pm \frac{4x - 3y + 12}{5},$$

$$i.e., \quad 8x + 14y - 171 = 0 \quad ...(1)$$

$$\text{and} \quad 112x - 64y + 141 = 0 \quad ...(2)$$

Solving the eqns. of AB, BC and BC, CA, we get

$$B = \left(\frac{-6}{5}, \frac{12}{5}\right) \text{ and } C = \left(\frac{2}{3}, 1\right).$$

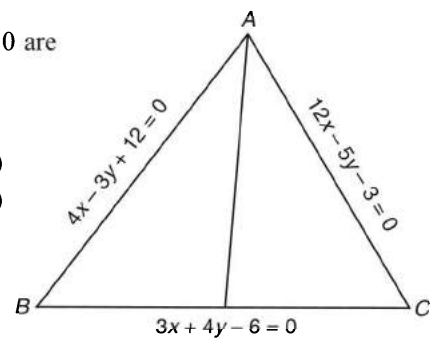


Fig. 16.27

Of (1) and (2), that line will be the internal bisector for which the points B and C are on its opposite sides.

Substituting the co-ordinates of B and C in the L.H.S. of (1), we get

* This ensures, by Art. 16.14, that the origin lies on the positive side of both the lines. Then a point on the bisector of the angle, in which the origin lies, will also lie on the positive side (or the negative side) of both the lines.

For point B, $8\left(\frac{-6}{5}\right)+14\left(\frac{12}{5}\right)-171=-47$. For point C, $8\left(\frac{2}{3}\right)+14(1)-171=\frac{-455}{3}$.

As the results are of the same sign, therefore B and C are on the same side of (1).

$\therefore$ (1) is not the internal bisector, but the other bisector (2), *i.e.,* $112x - 64y + 141 = 0$ is the internal bisector.

EXERCISE 16 (*g*)

1. Find the equations of the lines bisecting the angles between the following pairs of straight lines writing first the bisector of the angle in which the origin lies :

(*i*) $3x - 4y + 10 = 0, 5x - 12y - 10 = 0$; (*ii*) $12x - 5y + 3 = 0, 4x + 3y - 2 = 0$.

2. Find the equations of the bisectors of the angles between $4x + 3y - 4 = 0$ and $12x + 5y - 3 = 0$. Show that these bisectors are at right angles to each other.

3. Find the locus of a point which moves so that the perpendiculars drawn from it to the two straight lines $3x + 4y = 5$, $12x - 5y = 13$ are equal.

4. Find the equations of the lines bisecting the angles between the lines $4x - 3y + 12 = 0$ and $12x + 5y = 20$. Find, without using the tables, the tangent of an angle between one of these bisectors and one of the original lines. **[*CU*]**

5. Find the bisector of the acute angle between the lines :

(*i*) $3x + 4y = 11$ and $12x - 5y = 2$; (*ii*) $5x = 12y + 24$ and $12x = 5y + 10$.

6. Prove that the perpendiculars drawn from any point of the line $2x + 11y = 5$ to the lines $24x + 7y = 20$ and $4x - 3y = 2$ are equal in length.

7. A triangle is formed by the lines whose equations are $AB : x + y - 5 = 0$, $BC : x + 7y - 7 = 0$, and $CA : 7x + y + 14 = 0$. Find (*i*) the bisector of the interior angle at B, and (*ii*) the bisector of the exterior angle at C.

8. Find the centre of the inscribed circle of the triangle the equations of whose sides are $y - 15 = 0$, $12y + 5x = 0$ and $4y - 3x = 0$.

9. The co-ordinates of A, B, C are respectively $(-4, 0)$, $(0, 2)$ and $(-3, 2)$. Find (*i*) the equation of the straight line which bisects the angle CAB internally ; (*ii*) the co-ordinates of the point where this straight line meets the straight line joining C to the middle point of AB. **[*CU*]**

ANSWERS

1. (*i*) $4x - 7y + 5 = 0, 7x + 4y + 90 = 0$. (*ii*) $112x + 14y - 11 = 0, 8x - 64y + 41 = 0$

2. $8x - 14y + 37 = 0, 112x + 64y - 67 = 0$. **3.** $3x - 11y = 0, 99x + 27y - 130 = 0$.

4. $x + 8y - 32 = 0, 8x - y + 4 = 0 ; \frac{4}{7}$ or $\frac{7}{4}$ **5.** (*i*) $11x + 3y - 17 = 0$, (*ii*) $x - y - 2 = 0$.

7. (*i*) $3x + 6y - 16 = 0$, (*ii*) $8x + 8y + 7 = 0$. **8.** $(-1, 8)$. **9.** (*i*) $x - y + 4 = 0$, (*ii*) $(-2.5, 1.5)$.

16.20. Family of lines

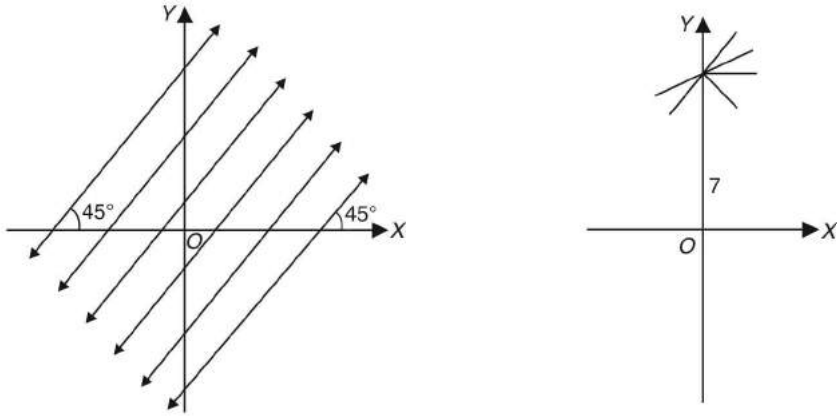
Parameter. We have derived equations of lines in various forms such as

$$y = mx + c, \frac{x}{a} + \frac{y}{b} = 1, x\cos\alpha + y\sin\alpha = p \qquad ...(A)$$

Each of these equations has two constants, which have some geometrical significance. The equation of a line is completely determined when some definite values are assigned to these constants.

Take, for example, the equation $y = mx + c$. The constants m and c are fixed for a particular line, but vary from line to line. To distinguish them from the variables x, y, they are given the name parameters. In (A), m and c ; a and b ; α and p are respectively the parameters for the three equations.

If in these equations one of the parameters is replaced by a fixed value, we obtain lines with one parameter, which represent a set of lines having a 'common' property. Each value given to the parameter yields a definite line. All these lines obtained by assigning different values to this parameter are called **one parameter systems** or **families of lines.**



m = 1 and c, *i.e.,* intercepts different y-intercept = 7 and m different

Fig. 16.28

Thus, the equation $y = x + c$, in which c is the parameter, represents the one parameter family lines with slope equal to 1. As shown in Fig. 16.28 there is an infinite number of lines in the family. Similarly, the equation $y = mx + 7$, where m is a parameter, represents an infinite number of lines whose y-inercept is 7.

Below we give different families of lines involving the use of parameters.

16.21. Parallel lines

The equation $\boldsymbol{ax + by + k = 0}$ represents for different values of k a family of lines parallel to the line $ax + by + c = 0$. The same can be easily seen by considering their slopes.

Working rule. *In the equation of the given line, change only the constant term to a new constant term k, keeping the remaining terms as they are.*

16.22. Perpendicular lines

The equation $\boldsymbol{ay - bx + k = 0}$ represents for different values of k a family of lines perpendicular to the line $ax + by + c = 0$. The same can be easily seen by considering their slopes.

Working rule. *Interchange the coefficients of x and y in the equation of the given line ; change the sign of either x or y and replace the constant term by a new constant k.*

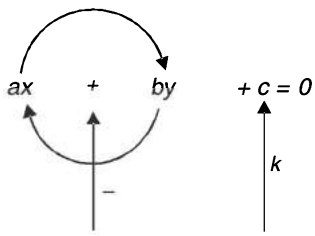


Fig. 16.29

Ex. 22. Find the equation of the line which passes through the point (4, –5) and is (*a*) parallel (*b*) perpendicular to the line $3x + 4y + 5 = 0$.

Method I. This example can be solved by considering the slopes and using the relation

$$y - y_1 = m(x - x_1).$$

Method II. (*a*) Any line parallel to the given line is $3x + 4y + k = 0$ ...(1)

As it passes through the point, (4, –5), so $3 \times 4 + 4 \times (-5) + k = 0 \quad \therefore \quad k = 8.$

Hence, the required equation is $3x + 4y + 8 = 0$. [Substituting the value of k in (1)]

(*b*) Any line perpendicular to the given line is $4x - 3y + k = 0$

As it passes through the point (4, –5), so $4 \times 4 - 3 \times (-5) + k = 0 \quad \therefore \quad k = -31$

Hence, the required equation is $4x - 3y - 31 = 0$.

16.23. Any line through the intersection of two given lines

The equation,

$$a_1 x + b_1 y + c_1 + k (a_2 x + b_2 y + c_2) = 0 \quad \text{... (A)}$$

represents the family of all lines passing through the point of intersection of the lines.

$$l_1 : a_1 x + b_1 y + c_1 = 0$$

and $$l_2 : a_2 x + b_2 y + c_2 = 0$$

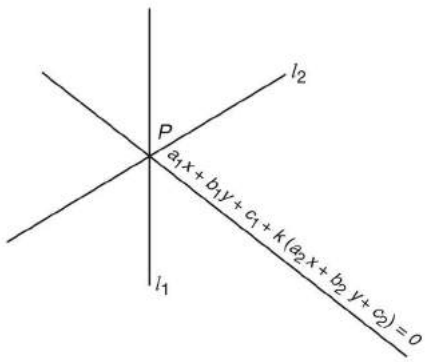


Fig. 16.30

Proof. Let the point of intersection of the lines l_1 and l_2 be $P(x_1, y_1)$. As it lies on these lines, so

$a_1 x_1 + b_1 y_1 + c_1 = 0$ and $a_2 x_1 + b_2 y_1 + c_2 = 0$.

The line (*A*) will pass through *P* if

$a_1 x_1 + b_1 y_1 + c_1 + k (a_2 x + b_2 y + c_2) = 0$,

i.e., $0 + k (0) = 0$ or $0 = 0$,

which is true. Hence proved.

Aid to memory. If $u = 0$ and $v = 0$ be the given lines, the line through their intersection is $u + kv = 0$, *i.e.,* one line + k (other line) = 0.

Ex. 23. Find the equation of the line joining the origin to the point of intersection of $4x + 3y = 8$ and $x + y = 1$.

Sol. The equations of lines are

$4x + 3y - 8 = 0$ ...(1) [Make right sides zero.]

$x + y - 1 = 0$ ...(2) **(Note this step)**

The equation of any line through the point of intersection of (1) and (2) is

$$4x + 3y - 8 + k (x + y - 1) = 0 \quad \text{...(3)}$$

If it passes through (0, 0), we have

$$0 + 0 - 8 + k (0 + 0 - 1) = 0 \text{ or } -8 - k = 0 \quad \therefore \quad k = -8$$

Substituting the value of k in (3), we get

$$4x + 3y - 8 - 8(x + y - 1) = 0$$

or $4x + 5y = 0$, which is the required equation.

EXERCISE 16 (*h*)

Apply the methods of this chapter in the following questions :

1. Find the equation of the straight line which passes through the point (4, 5) and is (*i*) parallel, (*ii*) perpendicular to the straight line $3x - 2y + 5 = 0$.

2. Find the equation of the straight line which is such that

(*i*) it passes through the point (4, 3) and is parallel to the line $3x - 4y + 5 = 0$.

(*ii*) it passes through the point (4, 3) and is perpendicular to the line $3x - 4y + 5 = 0$.

3. Find the equation of the straight line which passes through

(*i*) the origin and the point of intersection of the st. lines $y - x + 7 = 0, y + 2x - 2 = 0$;

(*ii*) the point $(2, -9)$ and the intersection of the lines $2x + 5y - 8 = 0$ and $3x - 4y = 35$;

(*iii*) the origin and the point of intersection of the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$.

4. Show that the equation $n(ax + by + c) = c(lx + my + n)$ represents the line joining the origin to the point of intersection of $ax + by + c = 0$ and $lx + my + n = 0$.

5. Find the equation of the line through the intersection of $x - y = 1$ and $2x - 3y + 1 = 0$ and parallel to $3x + 4y = 12$.

6. Find the equation of the line through the intersection of $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and parallel to $y - x = 8$.

7. Find the equation of the line through the intersection of $y + x = 9$ and $2x - 3y + 7 = 0$, and perpendicular to the line $2y - 3x - 5 = 0$.

ANSWERS

1. $3x - 2y - 2 = 0, 2x + y - 23 = 0$ **2.** (*i*) $3x - 4y = 0$ (*ii*) $4x + 3y = 25$

3. (*i*) $4x + 3y = 0$, (*ii*) $x - y - 11 = 0$, (*iii*) $(ac' - a'c)x - (bc' - b'c)y = 0$

5. $3x + 4y - 24 = 0$ **6.** $x - y = 0$ **7.** $2x + 3y - 23 = 0$

HINT TO SELECTED QUESTION

5. Any line through the intersection of the given lines is $x - y - 1 + k(2x - 3y + 1) = 0$ ...(1)

or $x(1 + 2k) - y(1 + 3k) - 1 + k = 0$

Its slope $= \dfrac{1 + 2k}{1 + 3k}$. [Collect x's and y's at one place. ***Note this step***]

If it is parallel to $3x + 4y - 12 = 0\left(\text{slope} = \dfrac{-3}{4}\right)$. Then $\dfrac{1 + 2k}{1 + 3k} = \dfrac{-3}{4}$, which gives $k = \dfrac{-7}{17}$.

The required eqn. is obtained by substituting the value of k in (1).

LOCUS

16.24. Definition of locus

Locus of a point is the path which it describes when moving under a certain given condition or conditions.

You have already studied *locus* in plane geometry. Now you will study the *equation of the locus*. We give below two illustrations of the equation of the locus.

16.25. Illustrations of equation of the locus

1. Let $A(-1, 2)$ and $B(3, 1)$ be two fixed points. We know from plane geometry that the locus of the moving point P such that $PA = PB$ is the perpendicular bisector LM of the segment AB. $P_1, P_2, P_3, P_4 \ldots$ are different positions of the moving point P.

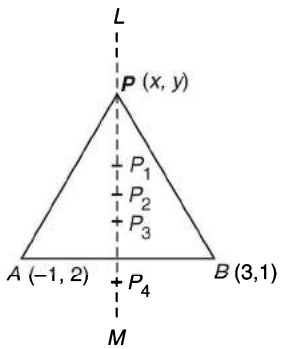


Fig. 16.31

To find the equation of the locus let (x, y) be the co-ordinates of the moving point P. Using the condition that $PA = PB$ or $PA^2 = PB^2$ we get

$(x + 1)^2 + (y - 2)^2 = (x - 3)^2 + (y - 1)^2$

or $x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 - 2y + 1$ or $8x - 2y - 5 = 0$ ...(A)

This is the algebraic form of the given geometrical condition.

It is to be inferred from the above discussion that when any point P satisfies the geometrical condition mentioned in the illustration given above, its co-ordinates satisfy the algebraic relation (A).

This algebraic relation which is satisfied for all positions of the moving point is called the equation of the locus described by the moving point.

The reader can easily convince himself about the following:

(a) All points lying on the perpendicular bisector LM will satisfy the equation (A).

(b) All those points which fulfil the condition of the problem will lie on this perpendicular bisector and satisfy its equation.

(c) All those points which do not satisfy the given condition will not lie on the locus and therefore not satisfy its equation.

Definition. *The* **equation of the locus** *is a relation between x and y which is satisfied by the co-ordinates of all points of the locus and by no others.*

Note 1. The point P when denoted by (x, y) is a general point and covers all its positions on the locus. x and y are variables and (x, y) are sometimes referred to as *current co-ordinates* of the point.

Note 2. For the purpose of this book, the reader can assume that every locus can be represented by some equation and *conversely* every equation represents some locus.

Note 3. The equation will be considered to have been satisfied by a point if by substituting for x and y in the given equation the abscissa and the ordinate respectively of the given point the left-hand side is equal to the right-hand side.

2. Let us take another example.

Let $C(2, 4)$ be a fixed point. We know from plane geometry that the locus of the point P such that the distance PC remains constant and is equal to 5 units is a circle with centre C and radius equal to 5 units.

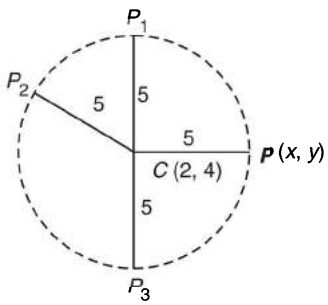


Fig. 16.32

To find the equation of the locus let (x, y) be the co-ordinates of P. Using the condition that $|CP| = 5$, we get

$$\sqrt{(x-2)^2 + (y-4)^2} = 5$$

or $x^2 - 4x + 4 + y^2 - 8y + 16 = 25$ or $x^2 + y^2 - 4x - 8y - 5 = 0$,

which is the equation of the locus (here the circle).

16.26. Method to find the equation of the locus

From the examples given above we conclude that to find the equation of the locus we proceed as follows :

Form I. **Step (*i*)** *Take the moving point as P (x, y) on the locus.*
Step (*ii*) *Form an equation in x and y using the given condition.*
Step (*iii*) *Simplify if need be. Then this is the required equation of the locus.*

Form II. **Step (*i*)** *Take any point (h, k) or* (x_1, y_1) *or* (α, β), *etc. on the locus.*
Step (*ii*) *Use the given condition to write a relation between h and k, etc.*
Step (*iii*) *Simplify the relation if need be.*
Step (*iv*) *Change (h, k) into the current co-ordinates (x, y), i.e. put x for h and y for k. The equation so obtained is the required equation of the locus.*

Ex. 24. Find the locus of a point which moves so that its distance from the x-axis is twice its distance from the y-axis.

Sol. Let (h, k) be any point on the locus. Step (i)

Then, by the given condition $|k| = 2|h|$, $|y| = 2|x|$ Step (ii)

$\therefore$ The locus of (h, k) is $y = \pm 2x$. Step (iii)

Note. Equal distances does not mean equal coordinates.

EXERCISE 16 (i)

Find the locus of a point which moves so that

1. Its distance from the x-axis is c^2.
2. It distance from the origin is 5.
3. The sum of the squares of its distances from the points (2, 4) and (– 3, – 1) is 30.
4. Its distance from the x-axis is half its distance from the y-axis.
5. Its distance from the y-axis is equal to its distance from the point (1, 1).
6. Find the locus of a point which is equidistant from the points (1, 0) and (–1, 0).
7. $A(2, 0)$ and $B(4, 0)$ are two given points. A point P moves so that $PA^2 + PB^2 = 10$. Find the locus of P.
8. Find the locus of a point such that the sum of its distances from the points (0, 2) and (0, –2) is 6.
9. Find the locus of a point, so that the join of points (–5, 1) and (3, 2) subtends a right angle at the moving point.
10. Two points A and B with co-ordinates (5, 3), (3, –2) are given. A point P moves so that the area of ΔPAB is constant and equal to 9 square units. Find the equation to the locus of the point P.
11. Show that (1, 2) lies on the locus $x^2 + y^2 - 4x - 6y + 11 = 0$.
12. Does the point (3, 0) lie on the curve $3x^2 + y^2 - 4x + 7 = 0$?
13. Find the condition that the point (h, k) may lie on the curve $x^2 + y^2 + 5x + 11y - 2 = 0$.
14. If the line $(2 + k)x - (2 - k)y + (4k + 14) = 0$ passes through the point (–1, 21), find k.
15. A is the point (–1, 0) and B is the point (1, 1). Find a point on the line $4x + 5y = 4$, which is equidistant from A and B. **[*SC*]**
16. The co-ordinates of the point S are (4, 0) and a point P has coordinates (x, y). Express PS^2 in terms of x and y. Given that M is the foot of the perpendicular from P to the y-axis and that the point P moves so that the lengths PS and PM are equal, prove that the locus of P is $8x = y^2 + 16$. Find the co-ordinates of one of the two points on the curve whose distance from S is 20 units. **[*SC*]**
17. Find the ratio in which the line joining the points (6, 12) and (4, 9) is divided by the curve $x^2 + y^2 = 4$.
18. AB is a line of fixed length, 6 units, joining the points A $(t, 0)$ and B which lies on the positive y-axis. P is a point on AB distant 2 units from A. Express the co-ordinates of B and P in terms of t. Find the locus of P as t varies. **[*ISC*]**
19. A rod of length l slides with its ends on two perpendicular lines. Find the locus of its mid-point.
20. If O is the origin and Q is a variable, point on $x^2 = 4y$, find the locus of the mid-point of OQ.

ANSWERS

1. $|y| = c^2$. **2.** $x^2 + y^2 = 25$. **3.** $x^2 + y^2 + x - 3y = 0$. **4.** $2y = \pm x$

5. $y^2 - 2x - 2y + 2 = 0$. **6.** $x = 0$. **7.** $x^2 + y^2 - 6x + 5 = 0$.

8. $9x^2 + 5y^2 = 45$ **9.** $x^2 + y^2 + 2x - 3y - 13 = 0$

10. $5x - 2y - 37 = 0$. $5x - 2y - 1 = 0$ **12.** No. **13.** $h^2 + k^2 + 5h + 11k - 2 = 0$.

14. $k = \frac{5}{4}$ **15.** $\left(-\frac{1}{4}, 1\right)$ **16.** $PS^2 = (x-4)^2 + y^2$; $(20, \pm 12)$.

17. Externally in the ratio 4 : 3 and 44 : 31. **18.** $B(0, \sqrt{36-t^2})$, $P\left(\frac{2t}{3}, \frac{\sqrt{36-t^2}}{3}\right)$, $x^2 + 4y^2 = 16$.

19. $4x^2 + 4y^2 = l^2$ **20.** $x^2 = 2y$.

HINTS TO SELECTED QUESTIONS

9. Substitute $x = 1$ and $y = 2$ in the left-hand side of the equation and prove it equal to the right-hand side.

15. If the required ratio be $\lambda : 1$, the co-ordinates of the point of division are

$$\left(\frac{6+4\lambda}{\lambda+1}, \frac{12+9\lambda}{\lambda+1}\right).$$

As this point lies on the curve $x^2 + y^2 = 4$, so $\left(\frac{6+4\lambda}{\lambda+1}\right)^2 + \left(\frac{12+9\lambda}{\lambda+1}\right)^2 = 4$.

Solving this equation, we get, $\lambda = \frac{-4}{3}$ and $\frac{-44}{31}$. Hence the curve cuts the line at two points which divide it externally in the ratio 4 : 3 and 44 : 31.

19. Let AB be a rod of length l. Suppose it intersects the axes at points $(a, 0)$ and $(0, b)$.

Let its mid-point whose locus we have to find be (h, k). Then

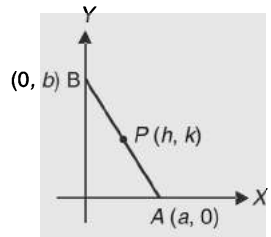
$h = \frac{a+0}{2}, k = \frac{0+b}{2} \Rightarrow h = \frac{a}{2}, k = \frac{b}{2}$

Also, $AB^2 = OA^2 + OB^2 \Rightarrow (a-0)^2 + (0-b)^2 = l^2$

$\Rightarrow a^2 + b^2 = l^2 \Rightarrow (2h)^2 + (2k)^2 = l^2$

$\Rightarrow 4h^2 + 4k^2 = l^2$

$\therefore$ Locus of (h, k) is $4x^2 + 4y^2 = l^2$.



16.27. Locus of the point of intersection of two lines whose equations involve one parameter

Rule. *Eliminate the parameter from the equations of the lines. Eliminant is the required locus.*

Ex. 25. Find the locus of the point of intersection of the lines $x + y = 3 + \lambda$ and $5x - y = 7 + 3\lambda$, where λ is a variable.

Sol. The required equation of the locus will be obtained on eliminating λ between the lines

$$x + y = 3 + \lambda \quad ...(1)$$

and $$5x - y = 7 + 3\lambda \quad ...(2)$$

From (1), $\lambda = x + y - 3$

Substituting the value of λ in (2), we get

$5x - y = 7 + 3(x + y - 3)$ *i.e.,* $2x - 4y + 2 = 0 \Rightarrow x - 2y + 1 = 0$,

which is the required equation of the locus.

Ex. 26. Find the locus of the foot of the perpendicular from the origin to the line which always passes through a fixed point (h, k).

Sol. The equation of any line through (h, k) *is* $y - k = m(x - h)$ ...(1)

The equation of the line perpendicular to it from the origin is $y = -\frac{1}{m}x$. ...(2)

To get the locus of the point of intersection of (1) and (2) we eliminate m from (1) and (2), Multiplying (1) and (2),

$$y(y - k) = -x(x - h) \text{ or } x^2 + y^2 - hx - ky = 0,$$

which is the required locus.

Ex. 27. A is (–2, 0) and *P* is any point on the curve given by $y^2 = 16x$. If *Q* bisects *AP*, find the equation of the locus of *Q*.

Sol. Let the co-ordinates of P be (h, k). Then as P lies on $y^2 = 16x$, we have

$$k^2 = 16h \qquad ...(1)$$

Let the co-ordinates of Q be (α, β). Since Q bisects AP, therefore

$$\alpha = \frac{-2+h}{2} \text{ and } \beta = \frac{0+k}{2} \Rightarrow h = 2\alpha + 2 \text{ and } k = 2\beta \qquad ...(2)$$

Eliminating h and k from (1) and (2), we get $(2\beta)^2 = 16\,(2\alpha + 2) \Rightarrow \beta^2 = 8\alpha + 8$

Changing α to x and β to y, we get the locus of Q (α, β) as $y^2 = 8x + 8$.

Note. We could straightway assume the co-ordinates of Q as (x, y) instead of (α, β).

EXERCISE 16 (*j*)

1. The lines $x - 2y + 6 = 0$ and $2x - y - 10 = 0$ intersect at P. Without finding the co-ordinates of P prove that the equation of the line through P and the origin of co-ordinates is perpendicular to $39x + 33y - 580 = 0$.
2. A point P moves so that its distance from the line given by $x = -3$ is equal to its distance from the point $(3, 0)$. Show that the locus of P is $y^2 = 12x$.
3. $A(2, 5)$, $B(4, -11)$ are two fixed points and C is a point which moves on the line $3x + 4y + 5 = 0$. Find the locus of the centroid of the triangle ABC.
4. Find the cartesian equation of the curve whose parametric equations are:
   (*i*) $x = t, y = 3t + 5$; (*ii*) $x = t, y = t^2$;
   (*iii*) $x = 4\cos\theta, y = 4\sin\theta$; (*iv*) $x = 4\cos\theta, y = 3\sin\theta$.
5. Find the locus of the point of intersection of the lines $x = \frac{a}{m^2}$ and $y = \frac{2a}{m}$, where m is a parameter.
6. Find the intersection S of the lines $x - ty + t^2 = 0$, $tx + y - t^3 - 2t = 0$.
   Show that S lies on the curve whose equation is $y^2 = 4x$. Sketch this curve. **[*ISC*]**
7. Find the locus of the middle point of the portion of the line $x\cos\alpha + y\sin\alpha = p$, where p is a constant, intercepted between the axes. **[*ISC*]**
   [Hint. Find the co-ordinates of the mid-point. Put x equal to its abscissa and y equal to its ordinate, and then eliminate α].

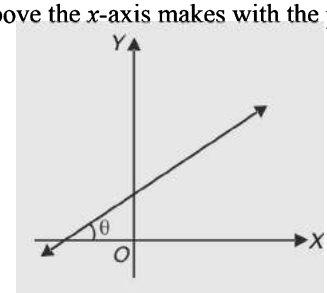
ANSWERS

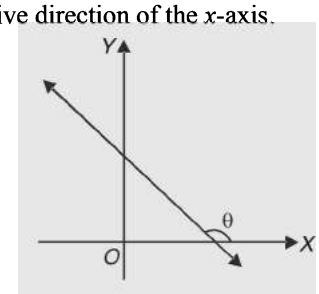
3. $9x + 12y + 11 = 0$. **4.** (*i*) $y = 3x + 5$, (*ii*) $x^2 = y$, (*iii*) $x^2 + y^2 = 16$, (*iv*) $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

5. $y^2 = 4ax$. **6.** $(t^2, 2t)$ **7.** $p^2(x^2 + y^2) - 4x^2y^2 = 0$

CHAPTER SUMMARY

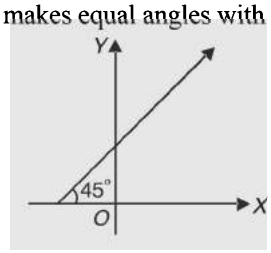
1. **The slope (or gradient)** of a line is given by $\boldsymbol{m = \tan\theta}$, where θ is the angle which the part of the line above the x-axis makes with the positive direction of the x-axis.

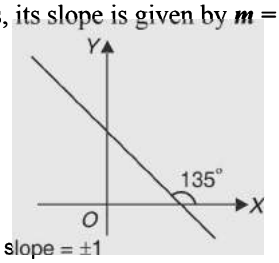




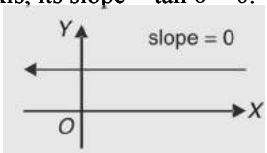
Particular cases of gradient

2. (*a*) If the line makes equal angles with the axes, its slope is given by $m = \pm 1$.

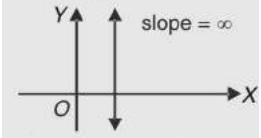




(*b*) If the line is parallel to *x*-axis, its slope = $\tan\theta = 0$.

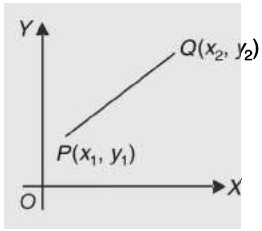


(*c*) If the line is parallel to *y*-axis, its slope is $\tan 90^\circ$ is not defined or we say that $m = \infty$.



(*d*) Slope of a line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \left(= \frac{\text{difference of ordinates}}{\text{difference of abscissas}} \right)$$



(*e*) Slope of a line whose equation is $ax + by + c = 0$ is given by $m = \dfrac{-a}{b} = \dfrac{-\text{coefft. of } x}{\text{coefft. of } y}$

3. Angle θ between two lines whose slopes are m_1 and m_2 is given by

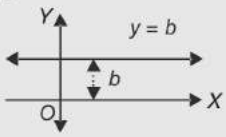
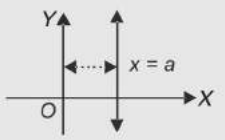
$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{\text{difference of slopes}}{1 + \text{product of slopes}} \right|$$

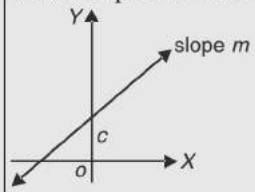
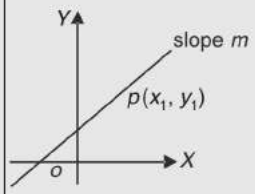
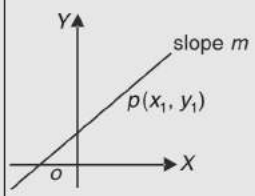
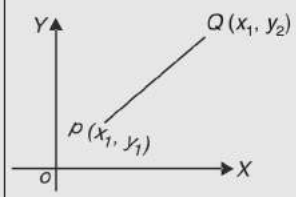
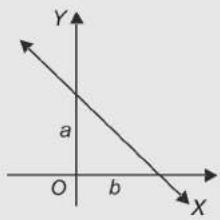
Particulars

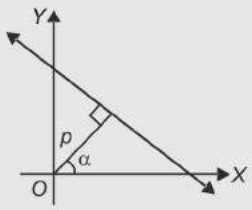
(*i*) Two lines are parallel when $m_1 = m_2$, *i.e.*, their slopes are equal.

(*ii*) Two lines are mutually perpendicular when $m_1 m_2 = -1$, *i.e.*, $m_2 = \dfrac{-1}{m_1}$, *i.e.*, when the product of their slopes = –1 and slope of one line is **negative reciprocal** of the other.

4. Various equations of lines

(*a*) $y = b$	parallel to *x*-axis 
(*b*) $x = a$	parallel to *y*-axis 

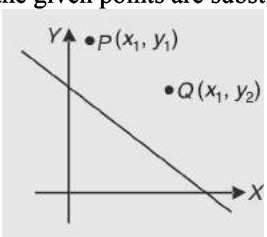
(*c*) $y = mx$	whose slope is m and which passes through the origin. 
(*d*) $y = mx + c$	whose slope is m and which cuts off an intercept 'c' from y-axis 
(*e*) $y - y_1 = m(x - x_1)$	whose slope is m and which passes through the point (x_1, y_1) 
(*f*) $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	line joining two points (x_1, y_1) and (x_2, y_2) 
(*g*) $\frac{x}{a} + \frac{y}{b} = 1$	line cutting off intercepts 'a' and 'b' from x-axis and y-axis respectively. 

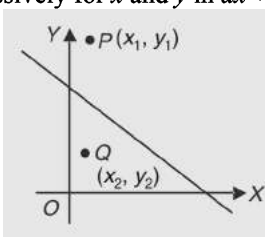
(*h*) $x\cos\alpha + y\sin\alpha = p$	line on which length of the perpendicular from the origin is 'p' and the perpendicular makes angle 'α' with x-axis. 
(*i*) $ax + by + c = 0$	General equation of a line
(*j*) $P + \lambda Q = 0$	Equation of a line through the point of intersection of two lines $P = 0$ and $Q = 0$.

5. The point of intersection of two lines is obtained by solving the equations of the two lines.

6. Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, *i.e.,* ratios of the coeff. of x, coeff. of y and the constant terms are equal.

7. Two points (x_1, y_1) and (x_2, y_2) are on the same side or opposits sides of a given line $ax + by + c = 0$ according as $ax + by + c = 0$ is of the same sign or opposite signs when the coordinates of the given points are substituted successively for x and y in $ax + by + c$.

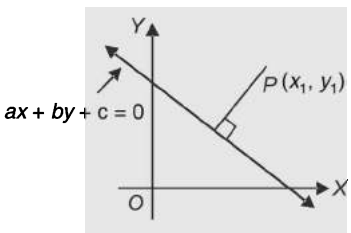




8. If equations of three lines are $P = 0$, $Q = 0$ and $R = 0$ and if $\boldsymbol{P + Q + R = 0}$, then the given lines are concurrent.

9. Perpendicular distance of a point (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$\boldsymbol{d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.}$$



10. Equations of angular bisectors of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\boldsymbol{\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}}$$

One of the signs out of '±' gives the bisector of the acute angle and the other gives the bisector of the obtuse angle. [Refer Solved Ex. 21].

11. **Locus – (Definition).** Locus of a point is the path which it describes when moving under certain given condition or conditions.
12. **Equation of the locus.** Refer to Art. 16.26.

CHAPTER TEST

1. The coordinates of the points A, B, C are (0, 4), (2, 5) and (3, 3) respectively. Prove that $AB = BC$ and the angle ABC is a right angle.
2. Find the coordinates of the point which divides the line joining (5, –2) and (9, 6) in the ratio 3:1.
3. The vertices of a quad. $PMQS$ are $P(0, 0)$, $M(3, 2)$, $Q(7, 7)$ and $S(4, 5)$. Show that $PMQS$ is a parallelogram. **[*SC*]**

   [Hint. Test for a || gm. Prove either of the following:

   (*i*) opposite sides are equal;

   (*ii*) The mid-point of one diagonal in the same as that of the other diagonal.

   (*iii*) opposite sides are parallel.]
4. P, Q and R are three collinear points. P and Q are (3, 4) and (7, 7) respectively, and $PR = 10$ units. Find the coordinates of R.
5. The coordinates of the vertices of a triangle are (4, –3), (–5, 2) and (x, y). If the centroid of the triangle is at the origin, show that $x = y = 1$. **[*WBHS*]**
6. Find the equation of the line passing through the point (–4, –5) and perpendicular to the line joining the points (1, 2) and (5, 6).
7. Find the equation of the straight line which passes through the point (3, 4) and has intercepts on the axes such that their sum is 14.
8. Reduce the equation of the straight line $3x + 4y + 15 = 0$ to normal form and find the perpendicular distance of the line from the origin. **[*WBHS*]**
9. Find the equation of the straight line which passes through the point of intersection of the straight lines $x + y = 8$ and $3x - 2y + 1 = 0$ and is parallel to the straight line joining the points (3, 4) and (5, 6). **[*CBSE*]**
10. Find the equation of the straight line which passes through the point of intersection of the straight lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ and cuts off equal intercepts from the axes.

    [*AICBSE, WBHS AP*]
11. Find the locus of a point such that the line segments having end points (2, 0) and (–2, 0) substend a right angle at that point.
12. Find the coordinates of the orthocentre of the triangle whose vertices are (1, 2), (2, 3) and (4, 3).
13. Find the equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.

ANSWERS

2. (8, 4) **4.** (11, 10) or (–5, –2) **6.** $x + y + 9 = 0$ **7.** $4x + 3y = 24, x + y = 7$

8. $-\frac{3}{5}x - \frac{4}{5}y = 3,\ p = 3$ **9.** $x - y + 2 = 0$ **10.** $23x + 23y - 11 = 0$

11. $x^2 + y^2 = 4$ **12.** (1, 6) **13.** $11x - 3y + 9 = 0$

CHAPTER 17

CIRCLES

17.01. Definition

A circle is the locus of a point which moves in a plane so that it is always at a constant distance from a fixed point in the plane. The fixed point is called the **centre** and the constant distance is called the length of the **radius** of the circle.

17.02. Equations of the circle

Centre—radius form

Let the centre of the circle be at a fixed point C (h, k) and the length of the radius be equal to r. Then if P(x, y) be the moving point on the circle, we have by definition $|\ CP\ | = r$ (Fig. 17.01)

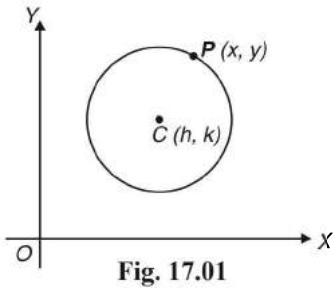


Fig. 17.01

i.e., $\sqrt{(x-h)^2+(y-k)^2} = r$ or $\mathbf{(x-h)^2+(y-k)^2=r^2}$.

This is the *standard form* of the equation of the circle.

[**Aid to memory.** $(x$ – abscissa of the centre$)^2 + (y$ – ordinate of the centre$)^2$ = (radius)^2]

Ex. 1. Equation of the circle whose centre is at (–5, 4) and the radius is 7 is

$$(x+5)^2+(y-4)^2=7^2,$$

i.e., $$x^2+y^2+10x-8y-8=0.$$

17.03. Particular cases

(1) If the *centre be at the origin* (Fig. 17.02) and the radius r, then $h = 0$, $k = 0$ and, so the equation of the circle is $x^2 + y^2 = r^2$.

(2) If the origin lies on the circle (Fig. 17.03), then $h^2 + k^2 = r^2$, and so the equation of the circle in this case is $x^2 + y^2 - 2hx - 2ky = 0$.

Note. The student should note that the equation of a circle passing through the origin does not contain any constant term.

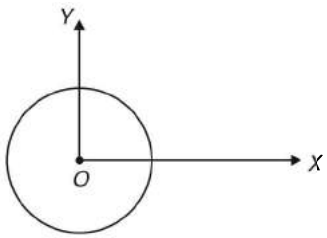


Fig. 17.02

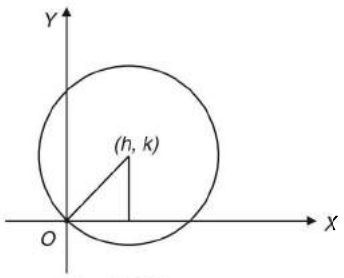


Fig. 17.03

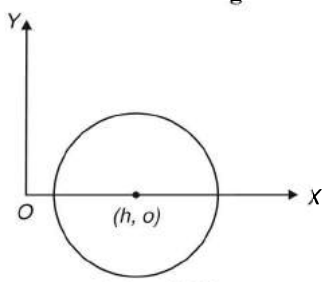


Fig. 17.04

(3) If the centre lies on the x-axis (Fig. 17.04), then the ordinate of the centre is zero, *i.e.* $k = 0$, and so the equation of the circle is $(x - h)^2 + y^2 = r^2$.

If the centre lies on the y-axis, the equation of the circle would be $x^2 + (y - k)^2 = r^2$.

(4) If the centre lies on the x-axis and the origin lies on the circle, then $k = 0$ and $h = r$, and so the equation of the circle is $x^2 + y^2 - 2hx = 0$. [Fig. 17.05]

(5) Similarly, if the centre lies on the y-axis and the origin lies on the circle, the equation of the circle would be $x^2 + y^2 - 2ky = 0$. [Fig. 17.06]

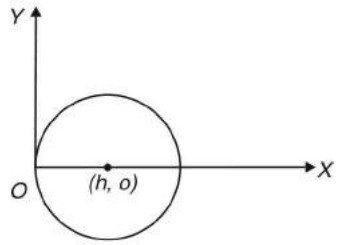


Fig. 17.05

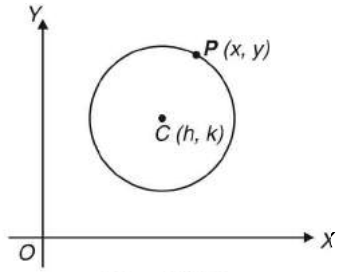
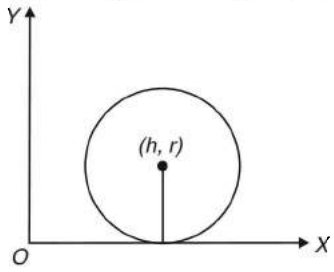


Fig. 17.06

(6) If the circle touches the x-axis or the y-axis [Fig. 17.07], then $|k| = r$ and $|h| = r$ respectively and the equation of the circle in the two cases would be



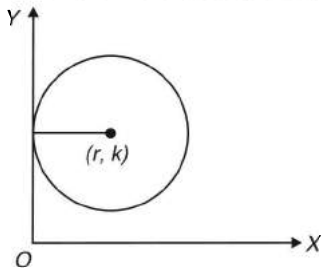


Fig. 17.07

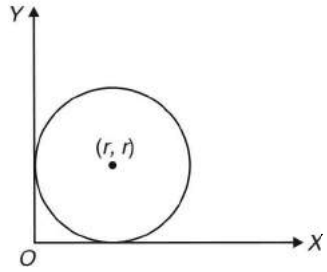


Fig. 17.08

$(x - h)^2 + (y - r)^2 = r^2$ and $(x - r)^2 + (y - k)^2 = r^2$ respectively.

(7) If the circle *touches both the axes* in the positive quadrant, then $h = k = r$ and so the equation of the circle is $(x - r)^2 + (y - r)^2 = r^2$, *i.e.* $x^2 + y^2 - 2rx - 2ry + r^2 = 0$. (Fig. 17.08)

17.04. Diameter form. Equation of the circle, the ends of whose diameter are known

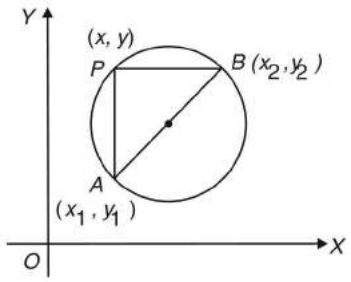


Fig. 17.09

Let the ends of the diameter be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be any point on the circle. Join PA and PB. Then, by geometry, $\angle APB = 90°$, *i.e.* PA $\perp$ PB. (Fig. 17.09)

$$\therefore \quad \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1 \qquad [m_1 m_2 = -1]$$

i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, which is the required equation of the circle.

Ex. 2. The equation of the circle described on the line joining the points (3, 4) and (2, – 7) as diameter is $(x - 3)(x - 2) + (y - 4)(y + 7) = 0$, *i.e.*, $x^2 + y^2 - 5x + 3y - 22 = 0$.

Note. This can also be done by finding the centre and the radius.

EXERCISE 17 (*a*)

1. Find the equation of the circle

(*i*) whose centre is (4, 5), radius is 7 ; (*ii*) whose centre is (0, – 4) and which touches the x-axis ; (*iii*) which passes through the origin and cuts off intercepts of length 'a', each from positive direction of the axes.

2. Find the equation of the circle

(*i*) whose centre is (a, b) and which passes through the origin ;

(*ii*) whose centre is the point (2, 3) and which passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

3. Find the equation of the circle which has A (1, 3) and B (4, 5) as opposite ends of a diameter. Find also the equation of the perpendicular diameter. **[*SC & GCE*]**

4. Find the equation to the circles which pass through the origin and cut off intercepts equal to (*i*) 3 and 4, (*ii*) $2a$ and $2b$ from the x-axis and the y-axis respectively.

5. Find the equation of the circles which touch the axis of x at a distance of 4 from the origin and cut off an intercept of 6 from the axis of y.

6. A circle having its centre in the first quadrant touches the y-axis at the point (0, 2) and passes through the point (1, 0). Find the equation of the circle.

7. Obtain the equation of the circle, radius 2 units, which lies in the positive quadrant and touches both axes of co-ordinates. Find also the equation of the circle with centre (6, 5) which touches the above circle externally.

8. Obtain the equation of the circle, centre (1, 0), which passes through the point P(3, $1\frac{1}{2}$).

Find also the equation of the equal circle which touches the given circle externally at P.

9. Calculate the co-ordinates of the foot of the perpendicular from the point (– 4, 2) to the line $3x + 2y = 5$.

Find the equation of the smallest circle passing through (– 4, 2) and having its centre on the line $3x + 2y = 5$.

10. The point diametrically opposite to the point P(1, 0) on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is

(*a*) (3, – 4) (*b*) (–3, 4) (*c*) (–3, – 4) (*d*) (3, 4). [*AIEEE 2008*]

ANSWERS

1. (*i*) $x^2 + y^2 - 8x - 10y - 8 = 0$, (*ii*) $x^2 + y^2 + 8y = 0$, (*iii*) $x^2 + y^2 - ax - ay = 0$.

2. (*i*) $x^2 + y^2 - 2ax - 2by = 0$, (*ii*) $x^2 + y^2 - 4x - 6y - 12 = 0$.

3. $x^2 + y^2 - 5x - 8y + 19 = 0$; $6x + 4y = 31$.

4. (*i*) $x^2 + y^2 - 3x - 4y = 0$, (*ii*) $x^2 + y^2 - 2ax - 2by = 0$.

5. $x^2 + y^2 \pm 8x \pm 10y + 16 = 0$ **6.** $x^2 + y^2 - 5x - 4y + 4 = 0$.

7. $x^2 + y^2 - 4x - 4y + 4 = 0$, $x^2 + y^2 - 12x - 10y + 52 = 0$.

8. $4x^2 + 4y^2 - 8x - 21 = 0$, $4x^2 + 4y^2 - 40x - 24y + 111 = 0$.

9. (– 1, 4) ; $x^2 + y^2 + 2x - 8y + 4 = 0$ **10.** (*c*)

HINTS TO SELECTED QUESTIONS

4. ∠AOB being equal to 90°, AB is a diameter of the circle. (Fig. 17.10)

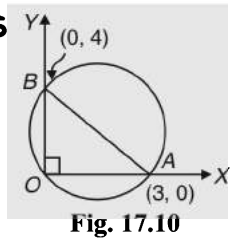


Fig. 17.10

7. For circles which touch externally,
distance between the centres = sum of the radii.

17.05. General equation

The standard equation of the circle can be represented in another form

$$(x - h)^2 + (y - k)^2 = r^2$$

or $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$ or $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$

This is called the general form of the equation of the circle.

Conversely. By completing the squares we may express it in the standard form. Thus, we have

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

or $$[x - (-g)]^2 + [y - (-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2.$$

Comparing it with the standard equation $(x - h)^2 + (y - k)^2 = r^2$, we easily infer that the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle, its centre and radius being as follows:

Centre = $(-g, -f)$, Radius = $\sqrt{g^2 + f^2 - c}$

Centre can also be found from the relations given by (1), namely,

$g = -h$, $f = -k$, and $c = h^2 + k^2 - r^2$, which give $h = -g$, $k = -f$,

$r^2 = h^2 + k^2 - c = (-g)^2 + (-f)^2 - c = g^2 + f^2 - c$ $\quad \therefore r = \sqrt{g^2 + f^2 - c}$

Hence, the centre (h, k) becomes $(-g, -f)$ and the radius = $\sqrt{g^2 + f^2 - c}$.

Note 1. The special features of the equation of the circle, *viz.*

$x^2 + y^2 + 2gx + 2fy + c = 0$, are that

(*i*) It is a second degree equation in both x and y ;

(*ii*) Coeff. of x^2 = coeff. of y^2 ;

(*iii*) It does not contain any term involving the product xy.

Any second degree equation fulfilling these requirements will represent a circle.

Note 2. The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$, also represents a circle. To find its centre and radius *we first make the coefficients of* x^2 and y^2 *each unity* by dividing throughout by a.

Thus, we have, $x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$ $\quad [a \neq 0]$

and so for this circle, centre is at $\left(\frac{-g}{a}, \frac{-f}{a}\right)$ and radius = $\sqrt{\left(\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}\right)}$

Aid to memory. *abscissa of the centre* = $-\frac{1}{2}$ (*coeff. of* x);

ordinate of the centre = $-\frac{1}{2}$ (*coeff. of* y).

Ex. 3. For the circle $x^2 + y^2 - 8x - 10y + 7 = 0$, centre is (4, 5) $[-\frac{1}{2}(-8)$ and $-\frac{1}{2}(-10)]$ and radius $= \sqrt{(4^2 + 5^2 - 7)} = \sqrt{(16 + 25 - 7)} = \sqrt{34}$.

17.06. Intercept on the axes

Knowing that on the x-axis, $y = 0$ and on the y-axis $x = 0$, we may calculate the intercepts as follows (Fig. 17.11.)

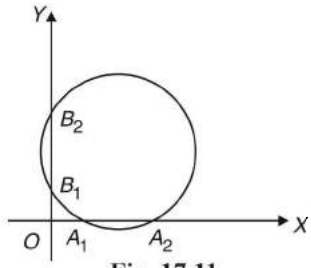


Fig. 17.11

1. *For the x-intercept put $y = 0$ in the equation of the circle and solve the resulting quadratic in x. The algebraic difference of the two values of x gives the length of the intercept.*

2. *For the y-intercept, put $x = 0$ in the equation of the circle and now proceed as in* 1.

Ex. 4. Find the lengths of the intercepts of the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the axes of co-ordinates.

Sol. The equation of the circle is $x^2 + y^2 - 5x - 13y - 14 = 0$

Putting $y = 0$ in it, we get $x^2 - 5x - 14 = 0$ or $(x - 7)(x + 2) = 0 \quad \therefore x = 7, -2$.

Hence, the circle cuts off an intercept of 7 units from the positive direction and 2 units from the negative direction of the x-axis.

$\therefore$ Length of the x intercept = 9.

Now, putting $x = 0$, we get $y^2 - 13y - 14 = 0$ or $(y - 14)(y + 1) = 0 \quad \therefore y = 14, -1$.

$\therefore$ Length of the y-intercept = 15.

17.07. Geometrical conditions required to determine a circle

Both the standard and the general form of the equations of a circle contain three independent arbitrary constants. The circle is completely determined if we know the values of these three constants. To know their values, we require three conditions or informations about the circle. These can be any three geometrical conditions such as

(a) Three given points through which the circle passes;

(b) Two given points on the circle and a line on which its centre lies; and

(c) Central line, point, and radius, etc.

Rule to find the equation of the circle satisfying three given conditions

1. *Let the required equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.*
2. *Use the three given conditions to establish three equations in g, f and c*
3. *Solve these three equations as in algebra for g, f and c and obtain their values.*
4. *Substitute the values of g, f and c in (1). This gives the required equation.*

Ex. 5. Find the equation of the circle which passes through the points (1, 1), (0, –1) and (–2, 0).

Sol. Method I. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{.....(1)}$$

As it passes through the point (1, 1) so $1 + 1 + 2g + 2f + c = 0$

i.e. $$2g + 2f + c = -2 \qquad \text{...(2)}$$

Similarly, for points (0, – 1) and (– 2, 0), we have

$$-2f + c = -1 \qquad \text{...(3)}$$

$$-4g + c = -4 \qquad \text{....(4)}$$

Solving (2), (3) and (4), we get $g = \frac{1}{2}, f = -\frac{1}{2}, c = -2$

$\therefore$ The required equation of the circle is

$$x^2 + y^2 + x - y - 2 = 0$$

[Substituting the values of g, f, and c in (1)]

Method II. The three points are P(1, 1), Q(0, –1), R(– 2, 0).

Let the centre be C(h, k).

$\because$ $|CP| = |CQ|$ or $CP^2 = CQ^2$

$\therefore$ $(h-1)^2 + (k-1)^2 = (h-0)^2 + (k+1)^2$

or $2h + 4k - 1 = 0$ ...(1)

Similarly, $\because$ $|CQ| = |CR|$ or $CQ^2 = CR^2$

$\therefore$ $(h-0)^2 + (k+1)^2 = (h+2)^2 + (k-0)^2$

or $4h - 2k + 3 = 0$ ...(2)

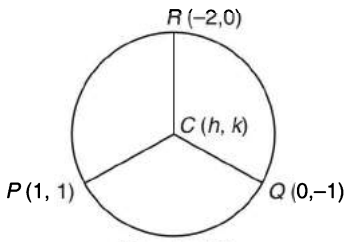


Fig. 17.12

Solving (1) and (2) simultaneously, we get $h = -\frac{1}{2}, k = \frac{1}{2}$

$\therefore$ Centre is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Radius = $|CP| = \sqrt{\left[1-\left(-\frac{1}{2}\right)\right]^2 + \left(1-\frac{1}{2}\right)^2} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$

$\therefore$ The equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$,

i.e. $$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{10}{4} \text{ or } x^2 + y^2 + x - y - 2 = 0.$$

Method III. Find the right bisectors of PQ and QR. Then their point of intersection is the centre of the circle. After finding the centre the radius can be easily found by using the distance formula.

EXERCISE 17 (*b*)

1. Find the centre and radius of the circle

(*i*) $x^2 + y^2 + 4x - 4y - 1 = 0$; (*ii*) $2x^2 + 2y^2 = 3x - 5y + 7$.

2. Find the lengths of the intercepts of the circle $3x^2 + 3y^2 - 5x + 3y = 0$ on the co-ordinate axes.

3. Find the equation of the circle, which passes through the point (5, 4) and is concentric with the circle $x^2 + y^2 - 8x - 12y + 15 = 0$.

4. The radius of the circle $x^2 + y^2 - 2x + 3y + k = 0$ is $2\frac{1}{2}$. Find the value of k. Find also the equation of the diameter of the circle, which passes through the point $(5, 2\frac{1}{2})$. **[*ISC*]**

5. Prove that the circle $x^2 + y^2 - 6x - 2y + 9 = 0$

(*i*) touches the x-axis;

(*ii*) lies entirely inside the circle $x^2 + y^2 = 18$. **[*ISC*]**

6. Find the co-ordinates of the centre of the circle $x^2 + y^2 - 4x + 6y = 3$. Given that the point A, outside the circle, has co-ordinates (a, b) where a and b are both positive, and that the tangents drawn from A to the circle are parallel to the two axes respectively, find the values of a and b. **[*SC*]**

7. Find the equation of the circle whose centre is at the point (4, 5) and which touches the x-axis. Also find the co-ordinates of the points at which the circle cuts the y-axis. **[*SC*]**

8. Prove that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch at the point $(3, -1)$.

Note. Apply the following geometric principles :

(*i*) Circles touch if $d = r_1 \pm r_2$, where d is the distance between the centres and r_1, r_2 the radii of the circles.

(*ii*) The point of contact divides the central line in the ratio of their radii internally for external contact and externally for internal contact.

Here, $C_1 \equiv (2, -3)$ and $r_1 = \sqrt{4 + 9 - 8} = \sqrt{5}$; $C_2 \equiv (5, 3)$ and $r_2 = \sqrt{25 + 9 - 14} = 2\sqrt{5}$

$$C_1C_2 = \sqrt{(2-5)^2 + (-3-3)^2} = \sqrt{9 + 36} = 3\sqrt{5} = r_1 + r_2$$

Hence, the given circles touch externally.

Now, the point of contact (x, y) divides $C_1\ C_2$ in the ratio $1 : 2$.

$$\therefore \quad x = \frac{1 \times 5 + 2 \times 2}{1 + 2} = 3 \qquad \left[x = \frac{m_1\ x_2 + m_2\ x_1}{m_1 + m_2} \right]$$

$$y = \frac{2 \times (-3) + 1 \times 3}{1 + 2} = -1 \qquad \left[y = \frac{m_1\ y_2 + m_2\ y_1}{m_1 + m_2} \right]$$

$\therefore$ The point of contact is $(3, -1)$.

9. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch externally at the point $\left(\frac{1}{5}, \frac{3}{5}\right)$.

10. Show that the circles $x^2 + y^2 + 2x - 6y + 9 = 0$ and $x^2 + y^2 + 8x - 6y + 9 = 0$ touch internally.

11. Find the equation of the circle which passes through the points $(0, 0)$, $(0, 1)$ and $(2, 3)$.

12. Find the centre and radius of the circle which passes through the points $(7, 5)$, $(6, -2)$, $(-1, -1)$.

13. Find the equation of the circle circumscribing the triangle formed by the lines $x + y + 1 = 0$, $3x + y - 5 = 0$, and $2x + y - 4 = 0$.

14. Show that the circle $x^2 + y^2 - 4x + 4y + 4 = 0$ touches the co-ordinate axes. If the points of contact are A and B, find the equation of the circle which passes through A, B and the origin.

15. Find the equation of the circle which passes through the points P(l, 0), Q(3, 0) and R(0, 2). Find also (*i*) the co-ordinates of the other point in which the axis of y cuts the circle, (*ii*) the coordinates of the other end of the diameter through Q. **[*SC*]**

16. Find the equation of the circle which has its centre on the line $y = 2$ and which passes through the points $(2, 0)$ and $(4, 0)$.
[*SC*]

17. Find the equation of the circle which passes through the points $(1, -2)$, $(4, -3)$ and has its centre on the line $3x + 4y + 10 = 0$. **[*LU*]**

18. The vertices A, B, C of a triangle ABC have co-ordinates $(4, 4)$, $(5, 3)$ and $(6, 0)$ respectively. Find the equations of the perpendicular bisectors of AB and BC, the co-ordinates of the circumcentre and the radius of the circumcircle of the triangle ABC. **[*ISC*]**

19. The radius of a circle is 5 units and it touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point $(5, 5)$. Find the equation of the circle. **[*ISC*]**

20. A circle $x^2 + y^2 - 2ax + 2y = 0$ cuts the axis of x at the point A and it circumscribes an equilateral triangle with OA as one of the sides, where O is the origin. Find the value of a and the vertices of the equilateral triangle. **[*ISC 1991*]**

21. Find the equation of the circle which passes through the points $(5, 0)$ and $(1, 4)$ and whose centre lies on the line $x + y - 3 = 0$. **[*ISC 1992*]**

ANSWERS

1. (*i*) (–2, 2); 3. (*ii*) $\left(\frac{3}{4}, -\frac{5}{4}\right); \frac{3}{4}\sqrt{10}$. **2.** $\frac{5}{3}, 1$. **3.** $x^2 + y^2 - 8x - 12y + 47 = 0$.

4. $-3, 2x - 2y = 5$ **6.** $(2, -3)$, $a = 6$, $b = 1$. **7.** $x^2 + y^2 - 8x - 10y + 16 = 0$; (0, 2), (0, 8).

11. $x^2 + y^2 - 5x - y = 0$. **12.** (3, 2); 5.

13. $x^2 + y^2 - 22x - 4y + 25 = 0$. **14.** $x^2 + y^2 - 2x + 2y = 0$.

15. $2x^2 + 2y^2 - 8x - 7y + 6 = 0$; (*i*) $\left(0, \frac{3}{2}\right)$; (*ii*) $\left(1, \frac{7}{2}\right)$.

16. $x^2 + y^2 - 6x - 4y + 8 = 0$. **17.** $x^2 + y^2 - 4x + 8y + 15 = 0$.

18. $x - y = 1$, $x - 3y = 1$; (1, 0); 5. **19.** $x^2 + y^2 - 18x - 16y + 120 = 0$.

21. $x^2 + y^2 - 4x - 2y - 5 = 0$

HINTS TO SELECTED QUESTIONS

5. (*i*) If a circle touches the x-axis, the ordinate of the centre = the radius, and if a circle touches the y-axis, the abscissa of the centre = the radius; (*ii*) Distance between the centres + $r < R$.

20. The circle cuts the axis of x at A $(2a, 0)$ and the axis of y at B $(0, -2)$ so that OA = $2a$, OB = 2.

Let OAC be the equilateral triangle. $\angle OCA = 60°$.

Then $\angle OBA = 60°$ ($\angle$s *in the same segment*)

$\therefore \angle OAB = 30°$

From Δ OAB, $\frac{OA}{OB} = \tan 60°$, or $\frac{2a}{2} = \sqrt{3}$, or $a = \sqrt{3}$

The co-ordinates of C are (OL, –LC), *i.e.* $(a, -a \tan 60°)$, *i.e.* $(\sqrt{3}, -3)$

Hence the co-ordinates of the vertices of the equilateral triangle OAC are O (0, 0),

A$(2\sqrt{3}, 0)$, C$(\sqrt{3}, -3)$.

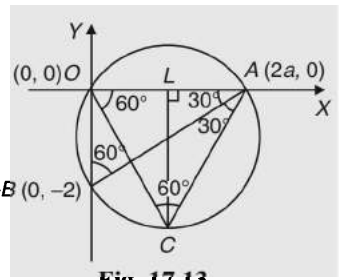


Fig. 17.13

17.08. Intersection of a line and a circle

The points of intersection of a line and a circle can be obtained by solving the equations of the line and the circle, for which we proceed as follows:

1. *Eliminate y or x (whichever is simpler) between the equations of the line and the circle, i.e. find out the value of y or x from the equation of the line and substitute it in the equation of the circle.*

2. *Solve the quadratic equation in x or y so obtained and get two values of the variable.*

3. *Substitute these values of the variable in the equation of the line* and find the two corresponding values of the other variable.*

These pairs of the values of x and y determine the co-ordinates of the points of intersection of the given line and the circle.

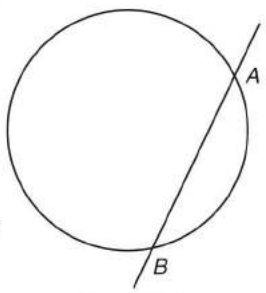


Fig. 17.14

Ex. 6. Find the points in which the line $y = 2x + 1$ cuts the circle $x^2 + y^2 = 2$.

Sol. From the equation of the line, we have $y = 2x + 1$.

Substituting this value of y in the equation of the circle, we get $x^2 + (2x + 1)^2 = 2$,

i.e. $5x^2 + 4x - 1 = 0$,

* **Caution.** Don't substitute in the equation of the circle.

$$i.e.,\ (5x-1)(x+1)-0, \quad \therefore\ x=\frac{1}{5}, -1.$$

Substituting these values in the equation of the line, we get

when $x=\frac{1}{5}$, $y=2\times\frac{1}{5}+1=\frac{7}{5}$; when $x=1$, $y=2\times(-1)+1=-1$.

Hence, the required co-ordinates of the points of intersection are $\left(\frac{1}{5},\frac{7}{5}\right)$ and $(-1,-1)$.

Note. Applying distance formula, we can get the length of the intercept AB. Thus, here

$$|AB|=\sqrt{\left(\frac{1}{5}+1\right)^2+\left(\frac{7}{5}+1\right)^2}$$

$$=\sqrt{\frac{180}{25}}$$

$$=\sqrt{\frac{36}{5}}=\frac{6}{5}\sqrt{5}.$$

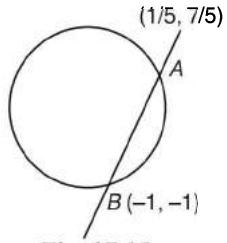


Fig. 17.15

Length of the intercept (simpler method). We know that perp. from the centre meets a chord at its middle point.

Hence, we have

$$|AM|=\sqrt{AC^2-CM^2} \qquad \text{(Pythagoras Theorem)}$$

$$\therefore \quad AB=2\,AM$$

$$=2\sqrt{AC^2-CM^2},$$

i.e. length of the chord or the intercept

$$=2\sqrt{(\text{radius})^2-(\text{central perp.})^2}.$$

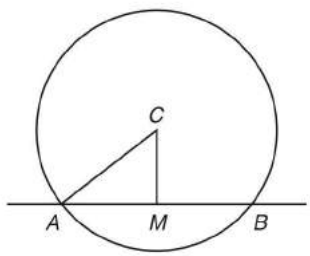


Fig. 17.16

Ex. 7. Find the length of the chord intercepted by the circle $x^2+y^2-8x-6y=0$ on the line $x-7y-8=0$.

Method I. Centre of the circle is C(4, 3). Radius $r=\sqrt{4^2+3^2}=5$

Central perpendicular on the line $=\frac{|4-7\times3-8|}{\sqrt{1+49}}=\frac{|-25|}{\sqrt{50}}=\frac{5}{\sqrt{2}}$

$\therefore$ Length of the chord $=2\sqrt{(\text{radius})^2-(\text{central perp.})^2}.=2\sqrt{5^2-\left(\frac{5}{\sqrt{2}}\right)^2}=5\sqrt{2}.$

Method II. Proceed by eliminating x or y.

17.09. Equation of the tangent at a point on the circle

The equation of a tangent to a circle may be found by making use of the fact that a tangent is perpendicular to the radius through the point of contact.

Ex. 8. Find the equation of the circle, centre C(1, 2) and tangent $x+y-5=0$.

Sol. Since the centre is at C(l, 2), the equation of the circle has the form

$$(x-1)^2+(y-2)^2=r^2.$$

Now r, the radius = Length of the perpendicular from C to the tangent $x+y-5=0$.

$$\therefore \qquad |r|=\frac{|1+2-5|}{\sqrt{1+1}}=\frac{|-2|}{\sqrt{2}}=\sqrt{2}$$

$\therefore$ The required equation of the circle is

$$(x-1)^2+(y-2)^2=(\sqrt{2})^2, \text{ i.e., } x^2+y^2-2x-4y+3=0.$$

Ex. 9. Find the equation of the tangent to the circle $x^2+y^2-2x+8y-23=0$; (*a*) at the point $P(3,-10)$ on it; (*b*) having slope 3 ; (*c*) through the external point (8, – 3).

Sol. (*a*) The centre of the given circle is C(1, – 4). Slope of CP $= \dfrac{-10+4}{3-1}=\dfrac{-6}{2}=-3$. Since the tangent at P is perp. to the radius CP, slope of the tangent is $\dfrac{1}{3}$.

$\therefore$ The equation of the tangent to the given circle at the point P(3, –10) is $y+10=\dfrac{1}{3}(x-3)$ or $x-3y-33=0$.

Note. The student should carefully note the *method* for finding the *equation of the tangent at a given point of a circle.*

1. Find the slope of the radius through the point of contact.
2. Slope of the tangent is its negative reciprocal.
3. Use $y-y_1=m(x-x_1)$ to write down the equation of the tangent.

(*b*) The tangent, having a slope 3, will be of the form $y=3x+b$. As it touches the circle with centre C(1, – 4) and radius $=\sqrt{1+16+23}=2\sqrt{10}$, so

$$\frac{|3(1)-(-4)+b|}{\sqrt{10}}=2\sqrt{10}$$

$$\Rightarrow \quad \frac{7+b}{\sqrt{10}}=\pm 2\sqrt{10} \Rightarrow 7+b=\pm 20$$

$$\Rightarrow \quad b=13,-27 \qquad \text{[Length of perpendicular = radius]}$$

Thus, there are two tangents having equations

$$y=3x+13 \quad \text{and } y=3x-27.$$

(*c*) Any line through the point (8, –3) is

$$y+3=m(x-8) \quad \text{or} \quad mx-y-8m-3=0.$$

If it is to be a tangent to the circle, centre C(l, – 4) and radius $2\sqrt{10}$, we should have

$$\frac{|m-(-4)-8m-3|}{\sqrt{(m^2+1)}}=2\sqrt{10} \quad \Rightarrow \quad (1-7m)^2=40(1+m^2).$$

i.e., $\quad 9m^2-14m-39=0 \Rightarrow (9m+13)(m-3)=0 \quad \therefore \ m=\dfrac{-13}{9}, 3$

Hence, the required equations of the tangents are $y+3=\dfrac{-13}{9}(x-8)$,

and $\quad y+3=3(x-8)$, *i.e.*, $\quad 13x+9y-77=0$ and $3x-y-27=0$.

Note the *method* for finding *the equations of the tangents from an external point.*

Ex. 10. Write the equation of the circle having radius 5 and tangent as the line $3x-4y+5=0$ at (1, 2).

Sol. Let the equation of the circle be $(x-h)^2+(y-k)^2=5^2$.

As the line $3x-4y+5=0$ is a tangent to it, so

(*i*) $\dfrac{|3h-4k+5|}{\sqrt{9+16}}=5 \Rightarrow 3h-4k+5=\pm 25$ [Perp, from centre = radius]

or $3h - 4k = 20$ [Taking +ve sign] ...(1)

and $3h - 4k = -30$ [Taking –ve sign] ...(2)

Also, slope of the radius through $(1, 2) = -\dfrac{1}{\text{slope of the given line}}$,

i.e. $\dfrac{k-2}{h-1} = -\dfrac{4}{3}$ or $4h + 3k = 10$ ....(3)

Solving (1), (3); and (2), (3) simultaneously in pairs, we get $h = 4$, $k = -2$ and $h = -2$, $k = 6$. Hence, the required equations of the circles are $(x - 4)^2 + (y + 2)^2 = 25$, $(x + 2)^2 + (y - 6)^2 = 25$.

Ex. 11. Find the value of p so that the straight line $x \cos\alpha + y \sin\alpha = p$ may touch the circle $x^2 + y^2 - 2ax \cos\alpha - 2ay \sin\alpha = 0$.

Sol. The centre and radius of the given circle are $C(a\cos\alpha, a\sin\alpha)$

and $r = \sqrt{a^2\cos^2\alpha + a^2\sin^2\alpha} = \sqrt{a^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{a^2} = |a|$. *

The given line will touch the given circle if

$$\frac{|a\cos\alpha \times \cos\alpha + a\sin\alpha \times \sin\alpha - p|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = a, \qquad \text{[Perp. from centre = radius]}$$

i.e., if $\dfrac{a(\cos^2\alpha + \sin^2\alpha) - p}{1} = \pm a$, *i.e.* if $(a - p) = \pm a$; $p = 0, 2a$.

***Maths Alert!** As we have done above, $\sqrt{a^2}$ should be taken as $|a|$ and not a unless it is known that $a \geq 0$. Thus, $\sqrt{9} = 3$, but $\sqrt{\cos^2 x} = |\cos x| = \cos x$, if $x \in$ I or IV quad. and $-\cos x$, if $x \in$ II or III quad.

Ex. 12. Find the condition, that the line $lx + my + n = 0$ may be a tangent to the circle $(x - h)^2 + (y - h^2) = r^2$.

Sol. Centre of the given circle is the pt. (h, k) and radius is r. The given line will touch the given circle if $\dfrac{|lh + mk + n|}{\sqrt{l^2 + m^2}} = r$. [Perp. from centre = radius]

i.e. if $(lh + mk + n)^2 = r^2(l^2 + m^2)$ which is the required condition.

Ex. 13. Prove that the straight line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$. Find the point of contact.

Note. We employ another method in solving this example. A tangent, by definition, is a line which cuts a circle in two coincident points. Hence, if we should eliminate y between the two equations and get a quadratic in x, the two values of x obtained from this eqn. will be equal. Hence, we have the following method :

**(*i*) *Eliminate y from the equations of the line and the circle and get a quadratic in x.*

(*ii*) *If the line touches the circle, the quadratic so obtained has equal roots.*

Sol. Eliminating y from the given equations, we have

$$x^2 + (x + a\sqrt{2})^2 = a^2, \text{ i.e., } 2x^2 + 2\sqrt{2}\,.\,ax + a^2 = 0, \text{ i.e. } (x\sqrt{2} + a)^2 = 0,$$

$$\therefore \quad x = \frac{-a}{\sqrt{2}}, \frac{-a}{\sqrt{2}}$$

As this equation has equal roots, so the given line touches the given circle.

** It should be noted that this is a general technique applicable not only to circles but also to other quadratic curves.

Substituting the value of x in the eqn. of the line, we get $y = \frac{-a}{\sqrt{2}} + a\sqrt{2} = \frac{a}{\sqrt{2}}$

Hence, the point of contact is $\left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.

17.10. Tangent in terms of *m*

To find the equation of a tangent to the circle $x^2 + y^2 = a^2$ in terms of its gradient

Let the equation of the circle be $x^2 + y^2 = a^2$ ...(1)

Let the equation of a tangent to (1) be $y = mx + c$ ...(2)

Solving (1) and (2) simultaneously, we have $x^2 + (mx + c)^2 = a^2$

or $(1 + m^2)\, x^2 + 2mcx + (c^2 - a^2) = 0$ ...(3)

(2) is a tangent to (1) if (3) has equal roots.

i.e., $4m^2c^2 - 4\,(1 + m^2)\,(c^2 - a^2) = 0$ [$\because$ Discriminant = 0]

or $m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2 = 0$ or $c^2 = a^2\,(1 + m^2)$ or $c = \pm\, a\sqrt{1 + m^2}$

Substituting in (2) we see that the equation of a tangent to (1) is

$$\boxed{y = mx + a\sqrt{1 + m^2} \text{ or } y = mx - a\sqrt{1 + m^2}}$$

whatever be the value of m.

Cor. It also follows that $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ if either $c = a\sqrt{1 + m^2}$ or $c = -a\sqrt{1 + m^2}$, which is the condition of tangency.

Note. The reason why there are two equations $y = mx \pm a\sqrt{1 + m^2}$ is that there are always two tangents with a given slope to a circle, namely, the tangents at the ends of a diameter.

Tangents from a point outside the circle

Ex. 14. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ drawn from the point (1, 4).

Sol. Any tangent to the given circle is $y = mx + 4\sqrt{1 + m^2}$

This passes through the given point (1, 4) if $4 = m + 4\sqrt{1 + m^2}$

or $4 - m = 4\sqrt{1 + m^2}$ or $(4 - m)^2 = 16(1 + m^2)$ or $15m^2 + 8m = 0 \Rightarrow m = 0, -\frac{8}{15}$

Hence the equations of the tangents are $y = 4$ and $8x + 15y = 68$.

Ex. 15. Find the locus of the point of intersection of perpendicular tangents to the circle $x^2 + y^2 = 4$.

Sol. Any tangent to the circle $x^2 + y^2 = 4$ is $y = mx + 2\sqrt{1 + m^2}$

It passes through (α, β) if

$\beta = m\alpha + 2\sqrt{1 + m^2}$ or $\beta - m\alpha = 2\sqrt{1 + m^2}$

or $(\beta - m\alpha)^2 = 4\,(1 + m^2)$

or $\beta^2 - 2m\,\alpha\,\beta + m^2\alpha^2 = 4 + 4m^2$

or $m^2\,(\alpha^2 - 4) - 2\,\alpha\,\beta m + (\beta^2 - 4) = 0$ ...(1)

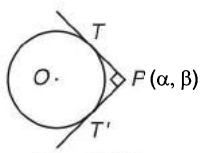


Fig. 17.17

The roots m_1 and m_2 of this quadratic equation are the slopes of PT and PT′. If PT and PT′ are at rt. angles, then $m_1m_2 = -1$.

From (1) $m_1 m_2 = \dfrac{\beta^2 - 4}{\alpha^2 - 4}$ $\therefore$ $\dfrac{\beta^2 - 4}{\alpha^2 - 4} = -1$

or $\beta^2 - 4 = -\alpha^2 + 4$ or $\alpha^2 + \beta^2 = 8$

$\therefore$ The locus of (α, β) is $x^2 + y^2 = 8$.

Ex. 16. Find the equations of the two tangents to the circle $x^2 + y^2 - 8x - 10y - 8 = 0$ which are perpendicular to the line $5x - 12y = 2$.

Sol. Any line perpendicular to $5x - 12y = 2$ is $12x + 5y + k = 0$

This line would be a tangent to the circle if its distance from the centre is equal to the radius.

Centre of the given circle is (4, 5) and radius 7.

$$\Rightarrow \qquad \frac{|12 \times 4 + 5 \times 5 + k|}{\sqrt{12^2 + 5^2}} = 7$$

This gives $k = 18$ or -164

Hence, the required tangents are $12x + 5y + 18 = 0$ and $12x + 5y - 164 = 0$.

EXERCISE 17 (*c*)

Points of intersection and length of a chord

1. The circle $4x^2 + 4y^2 = 25$ cuts the line $3x + 4y - 10 = 0$ at A and B. Calculate the coordinates of A and B. **[*SC*]**
2. Find the length of the chord $x + 2y = 5$ of the circle whose equation is $x^2 + y^2 = 9$. Determine also the equation of the circle described on the chord as diameter. **[*SC*]**
3. Find the intercept made by the circle $4x^2 + 4y^2 - 24x + 5y + 25 = 0$ on the st. line $4x - 2y = 7$.
4. Find the equation of each circle satisfying the given conditions.
   (*i*) Centre C (1, – 3) and tangent to $2x - y - 4 = 0$.
   (*ii*) Tangent to $2x - 3y - 7 = 0$ at (2, – 1) and passes through (4, 1).
   (*iii*) Tangent to $3x + y + 3 = 0$ at (– 3, 6) and tangent to $x + 3y - 7 = 0$.
   (*iv*) Tangent to $2x - 3y + 1 = 0$ at (l, 1), radius $\sqrt{13}$. **[*ISC*]**
   (*v*) Tangent to the *y-axis* at $(0, \sqrt{3})$ and passes through (–1, 0).
5. Find the length of the chord made by the axis of *x,* with the circle whose centre is $(0, 3a)$ and which touches the st. line $3x + 4y = 37a$. **[*LU*]**
6. Find the equation of the circle which has centre C (3, 1) and which touches the line $5x - 12y + 10 = 0$.
7. **Tangents from an external point.** Find the equations of the tangents to the circle $x^2 + y^2 = 10$ through the external point (4, – 2).
8. **Tangent satisfying given condition.** Find the equations of the tangents to the circle.
   (*i*) $x^2 + y^2 = 25$ inclined at an angle of 60° to the *x*-axis.
   (*ii*) $x^2 + y^2 + 2x + 2y = 7$ inclined at an angle of 45° to the *x*-axis.
   (*iii*) $x^2 + y^2 = a^2$ making a triangle of area a^2 with the axes.
   (*iv*) $x^2 + y^2 - 6x + 4y = 12$ and parallel to the line $4x + 3y + 5 = 0$.
   (*v*) $x^2 + y^2 - 22x - 4y + 25 = 0$ and perp. to the line $5x + 12y + 9 = 0$.
9. **Conditions of tangency.** Find the condition that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ may touch
   (*i*) the *x*-axis, (*ii*) the *y*-axis, and (*iii*) the *x*-axis, at the origin.

10. Find the conditions that the line

(*i*) $y = mx + c$ may touch the circle $x^2 + y^2 = a^2$;

(*ii*) $lx + my + n = 0$ may touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

11. For what value of k will the line $4x + 3y + k = 0$ touch the circle $2x^2 + 2y^2 = 5x$?

12. Show that $3x - 4y + 11 = 0$ is a tangent to the circle $x^2 + y^2 - 8y + 15 = 0$ and find the equation of the other tangent which is parallel to the st. line $3x = 4y$. **[*CU*]**

13. Show that $x = 7$ and $y = 8$ touch the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and find the points of contact.

14. Show that the line $3x + 4y + 20 = 0$ touches the circle $x^2 + y^2 = 16$ and find the point of contact.

15. Length of the tangent. *Prove that the length t of the tangent from the point* $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$t = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

Hence, find the length of the tangent

(*i*) to the circle $x^2 + y^2 - 2x - 3y - 1 = 0$ from the point (2, 5);

(*ii*) to the circle $x^2 + y^2 - 6x + 8y + 4 = 0$ from the origin;

(*iii*) to the circle $3x^2 + 3y^2 - 7x - 6y = 12$ from the point (6, –7);

(*iv*) to the circle $x^2 + y^2 - 4y - 5 = 0$ from the point (4, 5). **[*SC*]**

16. If $x = 4 + 5\cos\theta$ and $y = 3 + 5\sin\theta$, show that the locus of the point (x, y) as θ varies, is a circle. Find the centre and radius of the circle. **[*ISC*]**

17. A (1, 0) and B (7, 0) are two points on the axis of x. A point P is taken in the first quadrant such that PAB is an isosceles triangle and PB = 5 units. Find the equation of the circle described on PA as diameter. **[*ISC*]**

18. Find the equation of the circle which touches the line $y = 2$, passes through the origin and the point where the curve $y^2 - 2x + 8 = 0$ meets the x-axis. **[*ISC*]**

19. (*i*) Prove that the line $y = 2x$ touches the circle $x^2 + y^2 + 16x + 12y + 80 = 0$ and find the co-ordinates of the point of contact.

(*ii*) The circle $x^2 + y^2 - 6x - 10y + p = 0$ does not intersect or touch either axis and the point (1, 4) is inside the circle. Calculate the range of possible values of p. **[*ISC*]**

ANSWERS

1. $\left(0, 2\frac{1}{2}\right), \left(2\frac{2}{5}, \frac{7}{10}\right)$. **2.** 4, $x^2 + y^2 - 2x - 4y + 1 = 0$. **3.** $\frac{\sqrt{19}}{2}$.

4. (*i*) $5(x^2 + y^2) - 10x + 30y + 49 = 0$ (*ii*) $x^2 + y^2 + 4x - 10y - 23 = 0$.

(*iii*) $x^2 + y^2 - 6x - 16y + 33 = 0$, $x^2 + y^2 + 9x - 11y + 48 = 0$

(*iv*) $(x - 3)^2 + (y + 2)^2 = 13$; $(x + 1)^2 + (y - 4)^2 = 13$. (*v*) $x^2 + y^2 + 4x - 2\sqrt{3}\,y + 3 = 0$

5. $|8a|$. **6.** $x^2 + y^2 - 6x - 2y + 9 = 0$, $5x - 12y + 10 = 0$.

7. $3x + y - 10 = 0$, $x - 3y - 10 = 0$.

8. (*i*) $y = \sqrt{3}\,x \pm 10$. (*ii*) $x - y \pm 3\sqrt{2} = 0$, (*iii*) $y = \pm x \pm a\sqrt{2}$.

(*iv*) $4x + 3y + 19 = 0$, $4x + 3y - 31 = 0$ (*v*) $12x - 5y + 8 = 0$, $12x - 5y = 252$

9. (*i*) $g^2 = c$, (*ii*) $f^2 = c$, (*iii*) $g = c = 0$

10. (*i*) $c^2 = a^2(1 + m^2)$, (*ii*) $[lg + mf - n]^2 = (l^2 + m^2)(g^2 + f^2 - c)$.

11. $\frac{5}{4}$ or $-\frac{45}{4}$ **12.** $3x - 4y + 21 = 0$. **13.** (7, 3), (2, 8). **14.** $\left(-\frac{12}{5}, -\frac{16}{5}\right)$.

15. (*i*) 3 (*ii*) 2 (*iii*) 9 (*iv*) 4

16. $x^2 + y^2 - 8x - 6y = 0$, centre (4, 3), radius = 5. **17.** $x^2 + y^2 - 5x - 4y + 4 = 0$.

18. $x^2 + y^2 - 4x = 0$. **19.** (*i*) (–4, –8) (*ii*) $25 < p < 29$.

HINTS TO SELECTED QUESTIONS

15. In Fig 17.18.

$C = (-g, -f)$

$CQ^2 = g^2 + f^2 - c$

$CP^2 = (x_1 + g)^2 + (y_1 + f)^2$

In rt. angled Δ PQC, $t^2 = PQ^2 = CP^2 - CQ^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore \quad t = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

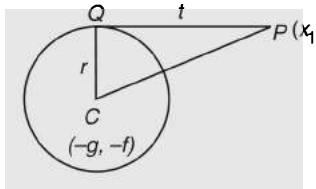


Fig. 17.18

Working Rule. In the left side of the equation of the circle (right side being zero, and coeffs. of x^2, y^2 unity) substitute the co-ordinates of the point, and take the square root of the result. (*a*) The required length of the tangent

$$= \sqrt{(2^2 + 5^2 - 2 \times 2 - 3 \times 5 - 1)} = \sqrt{(4 + 25 - 4 - 15 - 1)} = \sqrt{9} = 3.$$

Note. From the value $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ of t it can be easily deduced that the point (x_1, y_1) will be inside, on, or outside the circle according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \lessgtr 0$. This is a useful test of the position of a point.

19. Use that (*a*) CA < *r*, and (*b*) $r < 3$, where C is the centre and A is the point (1, 4) and *r* the radius.

17.11. Parametric equations of the circle $x^2 + y^2 = a^2$

Let the equation of a circle be $x^2 + y^2 = a^2$.

Let P (x, y) be any point on the circle and PM be the perpendicular from P to the x-axis. Let OP, which is equal to a in length, make an angle θ with OX. Then $\boldsymbol{x = a \cos \theta, y = a \sin \theta}$.

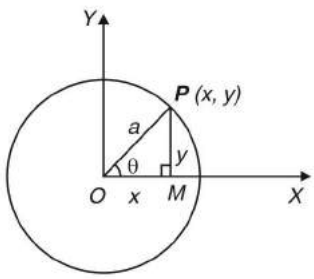


Fig. 17.19

These are the **parametric equations** of the circle $x^2 + y^2 = a^2$, where θ is the parameter.

It also follows that the parametric co-ordinates of any point on the circle are $(\boldsymbol{a \cos \theta, a \sin \theta})$. In future this point will be called 'θ' point. If the equation of the circle be in the general form

$$(x - h)^2 + (y - k)^2 = a^2,$$

then the parametric equations are

$x - h = a \cos \theta$, $y - k = a \sin \theta$, *i.e.* $\boldsymbol{x = h + a \cos \theta, y = k + a \sin \theta}$,where θ is the parameter.

The co-ordinates of any point on the circle are $(\boldsymbol{h + a \cos \theta, k + a \sin \theta})$.

Note 1. If we eliminate the parameter θ from $x = a \cos \theta$, $y = a \sin \theta$, we get $x^2 + y^2 = a^2$. In other words, the co-ordinates $(a \cos \theta, a \sin \theta)$ satisfy the equation of the circle $x^2 + y^2 = a^2$ whatever be the value of θ. This is why θ is called a parameter.

Note 2. The co-ordinates of a point of the circle $x^2 + y^2 = a^2$ can be either taken (x_1, y_1) or $(a \cos \theta, a \sin \theta)$. The former involves 2 unknown quantities x_1, y_1 whereas the latter involves only one quantity θ. This is why it is more convenient to employ parametric co-ordinates than the Cartesian co-ordinates.

EXERCISE 17 (*d*)

1. Find the equations of the tangent to the circle $2x^2 + 2y^2 = 5$ which are perpendicular to $y = 2x$.

2. Find the equations of the tangents to the circle $x^2 + y^2 - 8y - 8 = 0$ which are parallel to the line $5x - 2y = 2$.

3. Find the equation of the circle which has extremities of a diameter the origin and the point $(2, -4)$. Find also the equations of the tangents to the circle which are parallel to this diameter.

4. Show that, whatever be the value of α, the lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = a$ are tangents to the circle $x^2 + y^2 = a^2$.

Hence obtain the locus of the points from which perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$.

5. Find the locus of the feet of the perpendiculars drawn from the point $(b, 0)$ on tangents to the circle $x^2 + y^2 = a^2$.

6. (*i*) Find the equation of that chord of the circle $x^2 + y^2 = 15$, which is bisected at the point $(3, 2)$.

(*ii*) Find the locus of mid-points of all chords of the circle $x^2 + y^2 = 15$ that pass through the point $(3, 4)$, **[*ISC 1990 type*]**

7. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = 4(y + 1)$ drawn through the origin. **[*ISC 1990*]**

ANSWERS

1. $x + 2y \pm \frac{5}{\sqrt{2}} = 0$

2. $5x - 12y + 481 + 26\sqrt{6} = 0$ and $5x - 12y + 48 - 26\sqrt{6} = 0$

3. $x^2 + y^2 - 2x + 4y = 0$; $2x + y \pm 5 = 0$ **4.** $x^2 + y^2 = 2a^2$

5. $(y^2 + x^2 - bx)^2 = a^2\{y^2 + (x - b)^2\}$ **7.** $x^2 + y^2 - 2y = 0$

HINT TO SELECTED QUESTION

6. (*ii*) let P be the point $(3, 4)$, and let (α, β) be the co-ordinates of the mid-point L of a chord RQ that passes through P.

The co-ordinates of the centre C of the circle are $(0, 0)$.

The gradient of CL is $\frac{\beta - 0}{\alpha - 0}$, *i.e.*, $\frac{\beta}{\alpha}$.

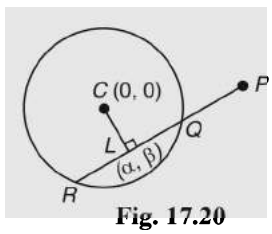


Fig. 17.20

Therefore, the gradient of RQ is $\frac{-\alpha}{\beta}$.

The equation of RQ is $y - \beta = \frac{-\alpha}{\beta}(x - \alpha)$

or $\beta y + \beta^2 = -\alpha x + \alpha^2$ or $\alpha a + \beta y = \alpha^2 + \beta^2$

This passes through $(3, 4)$, therefore $3\alpha + 4\beta = \alpha^2 + \beta^2$

Changing α to x and β to y, the locus of the mid-point is

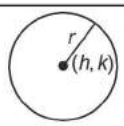
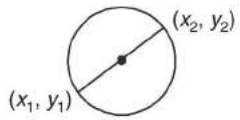
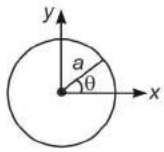
$$x^2 + y^2 - 3x - 4y = 0$$

Method II. $CL \perp RP \Rightarrow \frac{\beta - 0}{\alpha - 0} \times \frac{\beta - 4}{\alpha - 3} = -1 \Rightarrow \alpha^2 + \beta^2 - 3\alpha - 4\beta = 0$

$\therefore$ The locus of (α, β) is $x^2 + y^2 - 3x - 4y = 0$.

CHAPTER SUMMARY

1. Equations of circles

$x^2 + y^2 = r^2$	Centre (0, 0), radius r
$(x - h)^2 + (y - k)^2 = r^2$	Centre (h, k), radius r 
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$	Ends of diameter (x_1, y_1) and (x_2, y_2) 
$x^2 + y^2 + 2gx + 2fy + c = 0$	General equation; centre $\equiv (-g, -f)$; radius $= \sqrt{g^2 + f^2 - c}$ **Special features of the equation :** 1. second degree equation in x and y 2. coefft. of x^2 = coefft. of y^2 3. does not contain the term involving the product 'xy'
$x = a\cos\theta, y = a\sin\theta$ $x = h + a\cos\theta, y = k + a\sin\theta$	**Parametric forms :** radius 'a' and parameter θ radius a, centre (h, k), parameter θ 

2. Equation of a tangent to the circle $x^2 + y^2 = a^2$ in terms of the slope (gradient) of the tangent is given by $\boldsymbol{y = mx \pm a\sqrt{1 + m^2}}$, where m is the slope of the tangent.

3. (*i*) The points of intersection of a line and a circle can be obtained by solving the two equations.

(*ii*) Then, the length of the chord can be obtained by applying distance formula.



(*iii*) Length of the chord can also be obtained by cosidering the fact that perp. from the centre bisects the chord and applying Pythagoras' theorem to the rt. $\angle d$ ΔOPA.

CHAPTER TEST

1. Find the centre and radius of the circle $2x^2 + 2y^2 - x = 0$.

2. Find the equation of the circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

3. Find the equation of the circle drawn on the line joining (–1, 2) and (3, –4) as diameter.

4. Find the equation of the circle passing through the points (4, 1) and (6, 5) and whose centre lies on the line $4x + y = 16$. ***[WBHS, AI CBSE]***

5. Find the equation of a circle of radius 5 whose centre lies on x-axis and passes through the point (2, 3).

6. Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point (–2, –7). **[*CU*]**

7. Find the equation of the circle through the points (0, 0), (2, 0) and (0, 4). Also find the coordinates of its centre and its radius. **[*CBSE*]**

8. Find the parametric representation of the circle $x^2 + y^2 - 2x - 4y - 4 = 0$. **[*AI CBSE, MP*]**

9. Find the length of the chord intercepted by the circle $x^2 + y^2 = 25$ on the line $2x - y + 5 = 0$.

10. Find the equations of the tangents to the circle $x^2 + y^2 = 9$, which are parallel to the line $3x + 4y = 0$.

11. Find the equation of the circle which touches the y-axis at a distance of +4 from the origin and cuts off an intercept 6 from the x-axis. **[*CBSE, J&K*]**

ANSWERS

1. Centre at $\left(\frac{1}{4}, 0\right)$; radius $\frac{1}{4}$.

2. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

3. $x^2 + y^2 - 2x + 2y - 11 = 0$

4. $x^2 + y^2 - 6x - 8y + 15 = 0$

5. $x^2 + y^2 - 12x + 11 = 0$; $x^2 + y^2 + 4x - 21 = 0$

6. $x^2 + y^2 - 8x + 6y - 27 = 0$

7. $x^2 + y^2 - 2x - 4y = 0$; centre (1, 2); radius $= \sqrt{5}$.

8. $x = 1 + 3\cos\theta$, $y = 2 + 3\sin\theta$, where θ is a parameter.

9. $4\sqrt{5}$ units

10. $3x + 4y + 15 = 0$, $3x + 4y - 15 = 0$

11. $x^2 + y^2 - 10x - 8y + 16 = 0$

CHAPTER 18

LIMITS

18.01. Introduction

We first investigate, in an intuitive fashion, the meaning of limit.

(*a*) Suppose we are travelling from Kanpur to Lucknow by a train which will reach Lucknow at 8.00 a.m. As the time gets closer and closer to 8.00 a.m., the distance of the train from Lucknow gets closer and closer to zero (assuming the train is running on time.) If we consider time as our independent variable, denoted by t and distance as a function of time, say $f(t)$, then we see *that* $f(t)$ approaches zero as t approaches 0. We can say that the limit of $f(t)$ is zero as t tends to zero.

(*b*) Let a regular polygon be inscribed in a circle of given radius. We notice the following points from geometry:

(*i*) The area of the polygon cannot be greater than the area of the circle however large the number of sides may be.

(*ii*) As the number of sides of the polygon increases indefinitely, the area of the polygon continually approaches the area of the circle.

(*iii*) Ultimately the difference between the area of the circle and the area of the polygon can be made as small as we please by sufficiently increasing the number of sides of the polygon.

This is expressed by saying that the *limit* of the area of the polygon inscribed in a circle, as the number of sides increases indefinitely (or approaches infinity), is the area of the circle.

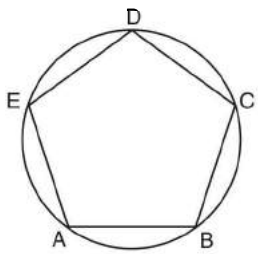


Fig. 18.01

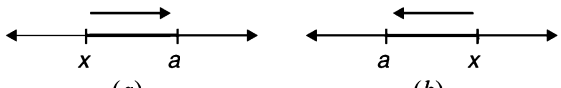
18.02. Meaning of $x \to a$

Let x be a variable and a be a constant. Since x is a variable, we can change its value at pleasure. It can be so changed that its value comes nearer and nearer to a. Then we say that x *approaches* a and express it by the notation $x \to a$.

Further, we know that $|x - a|$ is the distance between x and a on the real number line and $0 < |x - a|$ if $x \neq a$. The phrase *"*x tends to a" implies

(*i*) $x \neq a$, *i.e.,* $0 < |x - a|$ and

(*ii*) x takes up values nearer and nearer to a, *i.e.,* the distance $|x - a|$ between x and 'a' becomes smaller and smaller. One may ask 'how much smaller'? The answer is, as much as we please. It may be less than 0.1, 0.000001, 0.0000001 and so on. In fact, we may choose any positive number δ. Howsoever small it may be $|x - a|$ will always be less than δ. The above discussion leads us to the following definition of $x \to a$.



(*a*) (*b*)

Fig. 18.02

Let x *be a variable and* a *be a constant.*

Definition. ***Given a number $\delta > 0$ howsoever small, if x takes up values such that $0 < |x - a| < \delta$ then x is said to tend to a, and is symbolically written as $x \to a$.***

In simple language we can say that if x can assume values such that the positive difference between x and a remains very small then we say that x tends to a.

Note 1. If x approaches from values less than a, *i.e.,* from the left side of A, we write $x \to a^-$.

If x approaches a from values greater than a, *i.e.,* from the right side of A, then we write $x \to a^+$. $x \to a$ means both $x \to a^+$ and $x \to a^-$. Therefore, x approaches or tends to a means x approaches that value a from both right and left.

2. '*x* tends to zero' is a particular case of the above form, and a is taken as zero. Thus $|x| < \delta$ however small $\Leftrightarrow x \to 0$.

Note. $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$;

$|x| < \delta \Rightarrow |x - 0| < \delta \Rightarrow 0 - \delta < x < 0 + \delta \Rightarrow -\delta < x < \delta$.

18.03. The limit of a function

The following discussion will help you to arrive at a formal definition of limit.

(a) Consider the function $y = f(x) = \dfrac{x^2 - 1}{x - 1}$. In this case y is not defined for $x = 1$.

$\therefore \quad y = \dfrac{x^2 - 1}{x - 1} = x + 1$, where $x \neq 1$.

There are two cases (i) when x approaches 1 through values greater than 1, written as $x \to 1^+$ and (ii) when x approaches 1 through values less than 1 written as $x \to 1^-$.

$x \to 1^+$	1.1	1.01	1.001 ...	1.00000001	$x \to 1^-$	0.9	0.99	0.999	...	0.9999999
$y = x + 1$	2.1	2.01	2.001 ...	2.00000001	$y = x + 1$	1.9	1.99	1.999	...	1.9999999

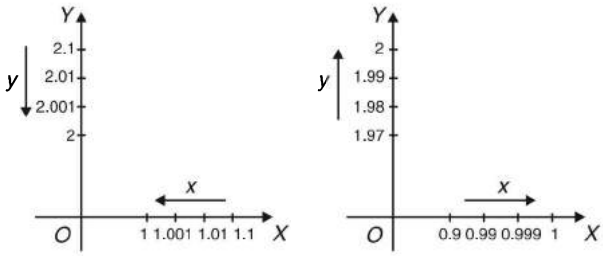


Fig. 18.03

Thus, ***as x approaches 1, y approaches 2.***

Thus, though the function $\dfrac{x^2 - 1}{x - 1}$. has no value for $x = 1$, it has a *limiting value* 2 *as x* approaches 1 ($x \to 1$). Its graph is as shown in Fig. 18.04.

Note that here the difference $|f(x) - 2|$ can be made as small as we please by taking x nearer to 1.

We express it as $\displaystyle\lim_{x \to 1}\left(\frac{x^2 - 1}{x - 1}\right) = 2$

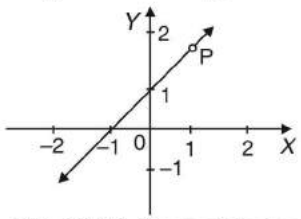


Fig. 18.04. Graph of the line $y = x + 1$ with a gap at P ($x = 1$).

(b) Consider the series

$1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots$. which is a G.P.

Let S_n denote the sum of n terms. Then,

$$S_n = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}$$

We notice that

(*i*) S_n can never be greater than 2.

(*ii*) S_n continually approaches 2 as n increases.

(*iii*) The difference between 2 and S_n can be made as small as we please by sufficiently increasing the value of n.

This is expressed in symbols as

$\lim\limits_{n\to\infty} S_n = 2$ or $S_n \to 2$ as $n \to \infty$

and is read as "the limit of S_n when n approaches infinity is 2."

Definition. ***The limit of a function for*** *a given value of the variable is that constant to which the function continually approaches as the variable approaches the given value such that the difference between the constant and the function may be made as small as we please by making the variable approach sufficiently near its assigned value.*

Or

A function $f(x)$ is said to tend to a limit l when x tends to 'a' if difference between $f(x)$ and l can be made as small as we please by making x sufficiently near 'a' and we write

$$\lim_{x\to a} f(x) = l \quad \text{or} \quad \operatorname{Lt}_{x\to a} f(x) = l.$$

18.04. Arriving at the formal rigorous definition of limit of a function

In Art. 18.02, you have seen that variable x being nearer and nearer to a means $|x-a| < \delta$ where δ is any number greater than 0. Also "$|x-a| < \delta$" and $x \neq a$ can as well be expressed as "$0 < |x-a| < \delta$". Further making $|f(x)-l|$ "as small as we like" means making $|f(x)-l| < \varepsilon$ for any $\varepsilon > 0$.

The function f approaches the limit l, if for every number > 0 we can make $|f(x)-l| < \varepsilon$ by requiring that $|x-a|$ be sufficiently small, and $x \neq a$.

Using the above ideas we can formulate the following definition of limit.

Definition. *Let $f(x)$ be a function of x. If for every positive number ε, however small it may be, there exists a positive number δ such that whenever $0 < |x-a| < \delta$ we have $|f(x)-l| < \varepsilon$, then we say $f(x)$ tends to the limit l as 'x tends to a' and write*

$$\lim_{x\to a} f(x) = l.$$

Illustration. Prove that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0.$

Sol. $\left|x\sin\frac{1}{x}-0\right| = \left|x\sin\frac{1}{x}\right| = |x|\cdot\left|\sin\frac{1}{x}\right| \le |x| \qquad \left(\because \left|\sin\frac{1}{x}\right| \le 1\right)$

$$\left|x\sin\frac{1}{x}-0\right| \le |x| < \varepsilon, \quad \text{whenever } 0 < |x-0| < \varepsilon$$

Hence, $\lim\limits_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0.$

18.05. Right handed and left handed limits

You have seen that there are two ways of arriving at the limiting value 2 of the function $\frac{x^2-1}{x-1}$ as x approaches 1 either from values less than 1 or from values greater than 1. The two limits thus obtained are respectively called the *left hand* and *right hand* limits.

In general, a function $f(x)$ is said to possess a *right hand limit* as x approaches a from values *greater than a* and is expressed as $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a+0} f(x)$.

A function $f(x)$ is said to possess a *left hand* limit as x approaches a from values less than a, and is expressed as $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a-0} f(x)$.

A function will have a limiting value only if its right hand limit equals its left hand limit.

Illustration 1. Let $f(x) = \frac{|x|}{x}, x \neq 0$. (See Solved Ex. 2)

Here, $f(x)$ is undefined at $x = 0$, but its value can be determined at any point as close to the point 0 as we like.

First, let $x \to 0^+$. The following table gives the values of $f(x) - 0$ on the right of the point $x = 0$.

x	0.1	0.01	0.001	0.001
$\lim\limits_{x\to 0^+} \frac{\lvert x\rvert}{x} = 1$	1	1	1	1

Evidently, at all points on the right of the point $x = 0$, the value of $|f(x) - 1|$ is 0,

i.e., $|f(x) - 1| < \varepsilon$ for any $\varepsilon > 0$.

Hence, we can say that the value of $|f(x) - 1|$ can be made smaller than any positive number, however small.

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

Now, let $x \to 0^-$. The following table gives the values of $f(x)$ at points on the left of the point $x = 0$.

x	–0.1	–0.01	–0.001	–0.001
$f(x) = \frac{\lvert x\rvert}{x}$	–1	–1	–1	–1

Thus, the value of $|f(x) - (-1)|$ on the left of the point $x = 0$ is nearly 0, *i.e.,* $|f(x) - (-1)| < \varepsilon$ for any $\varepsilon > 0$.

Hence, we can say that the value of $|f(x) - (-1)|$ can be made smaller than any positive number, however, small.

$\therefore \quad \lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$

$\therefore \quad \lim\limits_{x\to 0^+} \frac{|x|}{x} \neq \lim\limits_{x\to 0^-} \frac{|x|}{x}$, therefore, $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

The diagram given here illustrates this. The graph of the function $\frac{|x|}{x}$ has two branches. One branch is in the first quadrant and, the other in the third quadrant. Both the branches are parallel to the x-axis and are at a distance of 1 unit from the x-axis. In the first branch notice that as x approaches the point 0, (*i.e.*, $x \to 0$) through the points, A_1, A_2..., the value of y always remains 1 and hence it can be inferred that y approaches 1. Similarly, as x approaches 0, from the left through the points A_1', A_2' ..., the values of y remains –1 and hence it can be inferred that in this case y approaches –1.

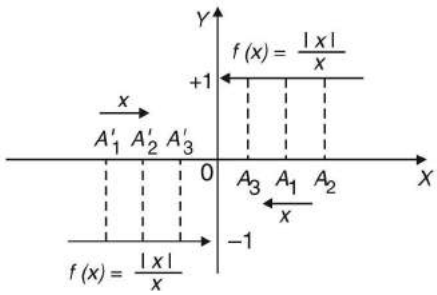


Fig. 18.05

Notice that the graph breaks at the point $x = 0$ and the function has no limiting value at that point. At all other points the graph is continuous (*i.e.*, without any break) and the limit of the function at those points can be determined.

Illustration 2. Let $f(x) = [x]$ and $x \to 2$, where $[x]$ means the greatest integer less than or equal to x.

First, let $x \to 2+$.

The values of $[x]$ for values of x greater than 2 and close to 2 are tabulated below.

x	2.1	2.01	2.001	
$f(x) = [x]$	2	2	2	etc.

$\therefore \quad \lim_{x \to 2^+} [x] = 2$

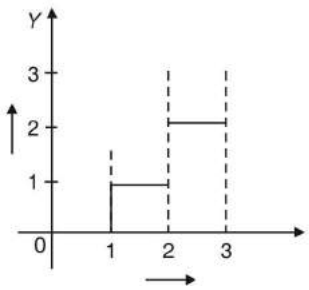


Fig. 18.06

Now let $x \to 2-$. Value of $f(x)$ at points less than 2 and close to 2 are tabulated below.

x	1.9	1.99	1.999	
$f(x) = [x]$	1	1	1	etc.

$\therefore \quad \lim_{x \to 2^-} [x] = 1$

Since the left hand limit and right hand limit are not equal, therefore, $\lim_{x \to 2} [x]$ does not exist.

The graph of the function $[x]$ is broken at the points $x = 1, 2, 3$ etc., and the limit of the function at these points does not exist. [Also see solved Ex. 3]. The graph is continuous at all other points and the limit of the function at these points exists.

Working rule for finding the left and right hand limits

1. *To find the left hand limit as $x \to a$*

(i) Put $x = a - h$ in $f(x)$ (ii) Take the limit of $f(a - h)$ as $h \to 0$ i.e.,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$$

2. *To find the right hand limit*

(i) Put $x = a + h$ in $f(x)$ (ii) Take the limit of $f(a + h)$ as $h \to 0$ i.e.,

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$$

18.06. Theorem

Prove that $\underset{x\to 0^+}{Lim}\ f(-x) = \underset{x\to 0^-}{Lim}\ f(x)$.

Proof. Let $y = -x$. Then as $x \to 0^-, y \to 0^+$

$\therefore \quad \underset{x\to 0^-}{\text{Lim}}\ f(x) = \underset{y\to 0^+}{\text{Lim}}\ f(-y) = \underset{x\to 0^+}{\text{Lim}}\ f(-x)$ [By merely changing the variable throughout]

Hence, $\underset{x\to 0^+}{\text{Lim}}\ f(-x) = \underset{x\to 0^-}{\text{Lim}}\ f(x)$

Ex. 1. If $f(x) = \dfrac{x^2-9}{x-3}$, find if $\underset{x\to 3}{\text{Lim}}\ f(x)$ exists.

Sol. $f(x) = \dfrac{x^2-9}{x-3}$

$$\text{L.H.L.} = \underset{x\to 3^-}{\text{Lim}}\left(\frac{x^2-9}{x-3}\right) = \underset{h\to 0}{\text{Lim}}\frac{(3-h)^2-9}{(3-h)-3} \qquad \text{(Putting } x = 3-h\text{)}$$

$$= \underset{h\to 0}{\text{Lim}}\frac{h^2-6h}{-h} = \underset{h\to 0}{\text{Lim}}\frac{h-6}{-1} = 6 \qquad \text{(We can divide by } h \text{ as } h \neq 0\text{)}$$

$$\text{R.H.L.} = \underset{x\to 3^+}{\text{Lim}}\left(\frac{x^2-9}{x-3}\right) = \underset{h\to 0}{\text{Lim}}\frac{(3+h)^2-9}{(3+h)-3} \qquad \text{(Putting } x = 3+h\text{)}$$

$$= \underset{h\to 0}{\text{Lim}}\left(\frac{h^2+6h}{h}\right) = \underset{h\to 0}{\text{Lim}}\,(h+6) = 6 \qquad \text{(We can divide by } h \text{ as } h \neq 0\text{)}$$

Since L.H.L. = R.H.L. = 6, therefore, $\underset{x\to 3}{\text{Lim}}\ f(x)$ exists and is equal to 6.

Ex. 2. Prove that $\underset{x\to 0}{\text{Lim}}\dfrac{|x|}{x}$, $x \neq 0$ does not exist. **[*Roorkee*]**

Sol. $$\text{R.H.L.} = \underset{h\to 0^+}{\text{Lim}}\ f(x) = \underset{h\to 0}{\text{Lim}}\ f\,(0+h) = \underset{h\to 0}{\text{Lim}}\frac{|0+h|}{0+h} = \underset{h\to 0}{\text{Lim}}\frac{h}{h} = 1 \quad (h > 0)$$

$$\text{L.H.L.} = \underset{x\to 0^-}{\text{Lim}}\ f(x) = \underset{h\to 0}{\text{Lim}}\ f\,(0-h) = \underset{h\to 0}{\text{Lim}}\frac{|0-h|}{0-h} = \underset{h\to 0}{\text{Lim}}\frac{h}{-h} = -1 \quad (h > 0)$$

$\because$ R.H.L. $\neq$ L.H.L. $\quad \therefore \quad \underset{x\to 0}{\text{Lim}}\ f(x)$ does not exist.

Ex. 3. Examine $\underset{x\to 2}{\text{Lim}}\ [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. $$\text{(L.H.L. at } x = 2) = \underset{x\to 2^-}{\text{Lim}}\ f(x) = \underset{h\to 0}{\text{Lim}}\ f\,(2-h) = \underset{h\to 0}{\text{Lim}}\,[2-h] = 1 \qquad [\because\ 1 < 2-h < 2]$$

$$\text{(R.H.L. at } x = 2) = \underset{x\to 2^+}{\text{Lim}}\ f(x) = \underset{h\to 0}{\text{Lim}}\ f\,(2+h) = \underset{h\to 0}{\text{Lim}}\,[2+h] = 2 \qquad [\because\ 2 < 2+h < 3]$$

Since L.H.L $\neq$ R.H.L., therefore, $\underset{x\to 2}{\text{Lim}}\ [x]$ does not exist.

Ex. 4. Let a function f be defined as

$$f(x)=\begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \\ x-1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

Establish the existence of $\lim\limits_{x\to\frac{1}{2}} f(x)$.

Sol. R.H.L. $= \lim\limits_{x\to\frac{1}{2}+} f(x) = \lim\limits_{x\to\frac{1}{2}+} (x-1)$ ($f(x)$ is given to be equal to $x-1$ for $x > \frac{1}{2}$)

Put $x = \frac{1}{2} + h$. Then $h \to 0$ as $x \to \frac{1}{2}$.

$$= \lim_{h\to 0}\left(\frac{1}{2}+h-1\right) = \lim_{h\to 0}\left(h-\frac{1}{2}\right) = -\frac{1}{2}$$

L.H.L. $= \lim\limits_{x\to\frac{1^-}{2}} f(x) = \lim\limits_{x\to\frac{1^-}{2}} (x)$ ($f(x)$ is given to be equal to x for $x < \frac{1}{2}$)

Put $x = \frac{1}{2} - h$. Then $h \to 0$ as $x \to \frac{1}{2}$.

$$= \lim_{h\to 0}\left(\frac{1}{2}-h\right) = \frac{1}{2}$$

Since L.H.L. $\neq$ R.H.L. therefore, $\lim\limits_{x\to\frac{1}{2}} f(x)$ does not exist.

Ex. 5. If f is an odd function and if $\lim\limits_{x\to 0} f(x)$ exists, prove that this limit must be zero.

Sol. $\lim\limits_{x\to 0} f(x)$ exists $\Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) \Rightarrow \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(0+h)$

$\Rightarrow \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0} f(h) \Rightarrow -\lim\limits_{h\to 0} f(h) = \lim\limits_{h\to 0} f(h)$ [$\because f(x)$ is odd, $\therefore f(-h) = -f(h)$]

$\Rightarrow 2\lim\limits_{h\to 0} f(h) = 0 \Rightarrow \lim\limits_{h\to 0} f(h) = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = 0.$

Ex. 6. A function $f(x)$ is defined as $f(x) = \begin{cases} 1, \text{ when } x \neq 0; \\ 2, \text{ when } x = 0. \end{cases}$

Does the Limit of $f(x)$ as $x \to 0$ exist? Explain your answer. **[*Roorkee*]**

Sol. R.H.L. $= \lim\limits_{x\to 0^+} f(x) = 1$ ($\because f(x) = 1$, when $x \neq 0$).

L.H.L. $= \lim\limits_{x\to 0^-} f(x) = 1$ ($\because f(x) = 1$, when $x \neq 0$). $\therefore$ L.H.L. = R.H.L. = 1

$\therefore$ $\lim\limits_{x\to 0} f(x)$ exists and its value is 1.

EXERCISE 18 (*a*)

1. Show that $\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

2. Show that $\lim\limits_{x\to\frac{\pi}{2}}\tan x$ does not exist. **3.** If $f(x)=\begin{cases}x^2+4 & \text{for } x<2\\ x^3 & \text{for } x>2\end{cases}$, find $\lim\limits_{x\to 2} f(x)$

4. If $f(x)=\begin{cases}\dfrac{x}{|x|} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$, find $\lim\limits_{x\to 0} f(x)$. **[*IIT*]**

5. Does $\lim\limits_{x\to 0} f(x)$ exist if $f(x)=\begin{cases}x, & \text{when } x<0\\ 0, & \text{when } x=0\\ x^2, & \text{when } x>0\end{cases}$

6. Show that $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist. **7.** Show that $\lim\limits_{x\to 0} e^{-1/x}$ does not exist.

8. Show that $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ does not exist.

[**Note:** $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}$ exists and is equal to 0. See Ex. 15, Part (7) solved on Page Ch 18-25]

9. If f is an even function, then prove that $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$, whenever they exist.

10. For what values of p does the $\lim\limits_{x\to 1} f(x)$ exist, where f is defined by the rule

$$f(x)=\begin{cases}2px+3 & \text{if } x<1\\ 1-px^2 & \text{if } x>1\end{cases}$$

ANSWERS

1. L.H.L. = – 1, R.H.L. = 1. Since LHL ≠ RHL, therefore, limit does not exist.

2. Limit does not exist as $\lim\limits_{x\to\frac{\pi^+}{2}}\tan x=-\infty$ and $\lim\limits_{x\to\frac{\pi^-}{2}}\tan x=\infty$

3. 8 **4.** Does not exist **5.** Yes, 0 **7.** $\lim\limits_{x\to 0^-} f(x)=\infty$ and $\lim\limits_{x\to 0^+} f(x)=0$ **10.** $-\dfrac{2}{3}$

HINTS TO SELECTED QUESTIONS

3. $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2}(x^2+4)=8,\ \lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2}(x)^3=8$

6. $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\left(\dfrac{1}{x}\right)=\lim\limits_{h\to 0}\left(\dfrac{1}{0-h}\right)=-\infty$ and $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{h\to 0}\dfrac{1}{0+h}=\infty$

8. $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{h\to 0} f(0-h)=\lim\limits_{h\to 0}\sin\left(\dfrac{1}{-h}\right)=-\lim\limits_{h\to 0}\sin\dfrac{1}{h}=-$ (an oscillating number which oscillates between – 1 and + 1. So, $\lim\limits_{x\to 0^-} f(x)$ does not exist. Similarly, $\lim\limits_{x\to 0^+} f(x)$ does not exist. Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.] [See Appendix at the end of the book]

9. f being an even function, $f(-x) = f(x)$ for all x.
$\therefore \quad \underset{x\to 0^+}{\text{Lim}}\ f(x) = \underset{x\to 0^-}{\text{Lim}}\ f(-x) = \underset{x\to 0^-}{\text{Lim}}\ f(x)$

10. $\underset{x\to 1^-}{\text{Lim}}\ f(x) = 2p + 3$ and $\underset{x\to 1^+}{\text{Lim}}\ f(x) = 1 - p$

18.07. Theorems on limits

The following results are of fundamental importance and are accepted here without proof. Let f and g be two functions such that $\underset{x\to a}{\text{Lim}}\ f(x) = l$, $\underset{x\to a}{\text{Lim}}\ g(x) = m$. Then

(*i*) $\underset{x\to a}{\text{Lim}}\,(f+g)x = \underset{x\to a}{\text{Lim}}\ [f(x)+g(x)] = \underset{x\to a}{\text{Lim}}\ f(x) + \underset{x\to a}{\text{Lim}}\ g(x) = l + m$

(*ii*) $\underset{x\to a}{\text{Lim}}\,(f-g)x = \underset{x\to a}{\text{Lim}}\ [f(x)-g(x)] = \underset{x\to a}{\text{Lim}}\ f(x) - \underset{x\to a}{\text{Lim}}\ g(x) = l - m$

(*iii*) $\underset{x\to a}{\text{Lim}}\,(fg)x = \underset{x\to a}{\text{Lim}}\ [f(x).g(x)] = \underset{x\to a}{\text{Lim}}\ f(x).\underset{x\to a}{\text{Lim}}\ g(x) = lm$

(*iv*) $\underset{x\to a}{\text{Lim}}\,(f\div g)x = \underset{x\to a}{\text{Lim}}\ [f(x)\div g(x)] = \underset{x\to a}{\text{Lim}}\ f(x) \div \underset{x\to a}{\text{Lim}}\ g(x) = l \div m,\ (m \neq 0)$

(*v*) $\underset{x\to a}{\text{Lim}}[f(x)]^n = \left[\underset{x\to 0}{\text{Lim}}\ f(x)\right]^n$, where n is a positive integer

(*vi*) $\underset{x\to a}{\text{Lim}}(f(x))^{g(x)} = \left(\underset{x\to 0}{\text{Lim}}\ f(x)\right)^{\underset{x\to a}{\text{Lim}}\ g(x)} = l^m$. In particular, $\underset{x\to a}{\text{Lim}}\ e^{f(x)} = e^{\underset{x\to a}{\text{Lim}}\ f(x)} = e^l$.

(*vii*) If $\underset{x\to a}{\text{Lim}}\ f(x) = +\infty$ or $-\infty$ then $\underset{x\to a}{\text{Lim}}\ \dfrac{1}{f(x)} = 0$

(*viii*) If $\underset{x\to a}{\text{Lim}}\ \log f(x) = \log\left(\underset{x\to a}{\text{Lim}}\ f(x)\right) = \log l$

(*ix*) $\underset{x\to a}{\text{Lim}}\ fog(x) = f\left(\underset{x\to a}{\text{Lim}}\ g(x)\right) = f(m)$

The above results can be stated as under:

(i) The limits of the sum of two functions is equal to the sum of their limits.
(ii) The limits of the difference of two functions is equal to the difference of their limits.
(iii) The limits of the product of two functions is equal to the product of their limits.
(iv) The limit of quotient of two functions is equal to the quotient of their limits provided the limit of the divisor is not zero.

Note. The above results hold good only if l and m both exist.

18.08. A few useful expansions

Some of the algebraic-trigonometric formulae of immense utility in limits are:

(*i*) $(x + a)^n = x^n + {}^nc_1 x^{n-1}\, a + {}^nc_2\, x^{n-2}\, a^2 + \ldots + a^n$ for all x and $n \in N$ (Bionimal theorem for positive integral index).

(*ii*) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$ for $|x| < 1$.

(Binomial theorem for any index other than a positive integer).

(*iii*) $a^x = 1 + x\log_e a + \dfrac{x^2}{2!}(\log_e a)^2 + \ldots$ ad.inf.

(*iv*) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ ad. inf

(*v*) $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$ for $-1 < x \leq 1$

(*vi*) $\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots$ for $-1 < x < 1$

(*vii*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ ad. inf.

(*viii*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ad.inf.

(*ix*) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$ ad. inf .

18.09. Important note

(*a*) **Meaningless Operation of division by zero.** An important property of the aggregate of rational numbers is that the usual operations of addition, subtraction, multiplication, and division can be performed upon any two such numbers and the number obtained is again a rational number. But there is one exception, namely "division by zero" which is a meaningless operation.

A disregard of this rule often leads to absurd results such as shown below:

(*i*) Let $x = 3$.

Then, $x^2 - 9 = x - 3$ (as each side is equal to zero for $x = 3$) or $(x - 3)(x + 3) = x - 3$ or $x + 3 = 1$ (Dividing both sides by $x - 3$), or $3 + 3 = 1$ (as $x = 3$), or $6 = 1$, which is clearly absurd.

Dividing by $(x - 3)$, *i.e.,* by 0 has led to this adsurdity.

(*ii*) $\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3$ only if $x - 3 \neq 0$, *i.e.* $x \neq 3$.

This equality is valid only if $x \neq 3$ because if $x = 3$, we cannot divide by $(x - 3)$, which is zero for $x = 3$.

(*b*) **Indeterminate forms.** When a function involves the independent variable in such a manner that for a certain assigned value of that variable its value *cannot be found by simply substituting* that value of the variable, the function is said to take an indeterminate form.

One of the commonest cases occurring is that of a fraction whose numerator and denominator both vanish for the value of the variable referred to *i.e.,* $\frac{0}{0}$.

It should be noted that $\frac{0}{0}$ is not a meaningless operation like $\frac{a}{0}$. It is only indeterminate in this form. This means that we can find its value by adopting some other method.

Besides $\frac{0}{0}$, there are other indeterminate forms, *e.g.,* $0 \times \infty$, $\infty - \infty$, $\frac{\infty}{\infty}$, 1^∞, ∞° etc. They may be made to depend on the form $\frac{0}{0}$ in certain cases, by some method or the other.

So far, we were examining the existence of limit. Assuming that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and that $\lim_{x \to a} f(x)$ exists, we now proceed to discuss methods of evaluating limits of various types of functions.

18.10. Algebraic limits

Under this category fall those functions which are not trigonometric, inverse trigonometric, exponential and logarithmic functions.

18.11. We generally evaluate algebraic limits by using the following five methods

(*i*) Direct substitution (*ii*) Factorization (*iii*) Form $\frac{x^n - a^n}{x - a}$ (*iv*) Rationalization

(*v*) $x \to \infty$ type.

Method 1. Direct substitution

Directly substitute the value in the given expression and if we get a finite number, then that finite number is the limit of the given expression.

Ex. 7. Evaluate the following limits:

(*i*) $\lim\limits_{x\to 1} [(x-1)^2 + 5]$ **(*ii*)** $\lim\limits_{x\to 0^-} \left\{\frac{|x|}{x} + x^2 + 3\right\}$ **(*iii*)** $\lim\limits_{x\to 3^-} ([x] - x)$

Sol. (*i*) $\lim\limits_{x\to 1} [(x-1)^2 + 5] = \lim\limits_{x\to 1} (x-1)^2 + \lim\limits_{x\to 1} 5 = (1-1)^2 + 5 = 0 + 5 = 5.$

(*ii*) $\lim\limits_{x\to 0^-} \left\{\frac{|x|}{x} + x^2 + 3\right\} = \lim\limits_{x\to 0^-} \left(\frac{-x}{x} + x^2 + 3\right) = \lim\limits_{x\to 0^-} (-1 + x^{2`} + 3) = \lim\limits_{x\to 0^-} (2 + x^2) = 2 + 0 = 2.$

(*iii*) $\lim\limits_{x\to 3^-} ([x] - x) = \lim\limits_{x\to 3^-} [x] - \lim\limits_{x\to 3^-} (x) = 2 - \lim\limits_{h\to 0} (3 - h) = 2 - 3 = -1.$

EXERCISE 18 (*b*)

Evaluate the following limits:

1. $\lim\limits_{x\to 0} (7x^2 - 5x + 1).$ **2.** $\lim\limits_{x\to 1} \frac{x-1}{x+1}.$ **3.** $\lim\limits_{x\to 2} \frac{x^2 + 5x + 6}{2x^2 - 3x}.$ **4.** $\lim\limits_{x\to 0} \frac{ax + b}{cx + d}.$

5. $\lim\limits_{x\to 5^+} \frac{x-5}{|x-5|}.$ **6.** $\lim\limits_{x\to 5^-} \frac{x+5}{|x+5|}.$ **7.** $\lim\limits_{x\to 5^+} (x - [x]).$ **8.** $\lim\limits_{x\to 1^-} \frac{x^2 - 1}{|x-1|}.$

9. Show that $\lim\limits_{x\to 2} \log_{10} \{x^6 + \sqrt{(x^2 + 1292)}\} = 2.$

10. Given that $f(x) = \frac{4 - 7x}{(3x + 4)}$, $l = \lim\limits_{x\to 2} f(x)$ and $m = \lim\limits_{x\to 0} f(x)$, form the equation whose roots, are $\frac{1}{l}, \frac{1}{m}$.

ANSWERS

1. 1 **2.** 0 **3.** 10 **4.** $\frac{b}{d}$ **5.** 1 **6.** –1

7. 0 **8.** –2

9. Hint: Given limit $= \log_{10} \left[\lim\limits_{x\to 2} x^6 + \sqrt{\lim\limits_{x\to 2} (x^2 + 1292)}\right] = \log_{10} [2^6 + \sqrt{1296}] = \log_{10} 100 = 2$

10. $x^2 - 1 = 0$

Method 2. Factorization

Ex. 8. Evaluate the following limits:

(*i*) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$. (*ii*) $\lim\limits_{x\to 1}\dfrac{x-1}{2x^2-7x+5}$. [*IIT*]

(*iii*) $\lim\limits_{x\to 1}\dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$. (*iv*) $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$. [*IIT*]

(*v*) $\lim\limits_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$. (*vi*) $\lim\limits_{x\to 1}\dfrac{x^2-x\log x+\log x-1}{x-1}$.

(*vii*) $\lim\limits_{x\to 3}(x^2-9)\left(\dfrac{1}{x+3}+\dfrac{1}{x-3}\right)$ (*viii*) $\lim\limits_{x\to 1}\dfrac{x^{1/4}-1}{x^{1/3}-1}$.

Sol. (*i*) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{x-1}$

$=\lim\limits_{x\to 1}(x+1)=1+1=2.$ $[\because x\neq 1\Rightarrow x-1\neq 0]$

(*ii*) $\lim\limits_{x\to 1}\dfrac{x-1}{2x^2-7x+5}=\lim\limits_{x\to 1}\dfrac{x-1}{(x-1)(2x-5)}=\lim\limits_{x\to 1}\dfrac{1}{2x-5}=\dfrac{1}{2-5}=\dfrac{-1}{3}.$

(*iii*) $\lim\limits_{x\to 1}\dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^3-2x^2-2x-2)}{(x-1)(x^2-4x-1)}$

[Factorizing by using remainder theorem]

$=\lim\limits_{x\to 1}\dfrac{x^3-2x^2-2x-2}{x^2-4x-1}=\dfrac{(1)^3-2\times 1^2-2\times 1-2}{1^2-4\times 1-1}=\dfrac{1-2-2-2}{-4}=\dfrac{-5}{-4}=\dfrac{5}{4}.$

(*iv*) $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}=\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}=\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$

$=\lim\limits_{x\to 1}\dfrac{2x-3}{(2x+3)(\sqrt{x}+1)}=\dfrac{2\times 1-3}{(2\times 1+3)(\sqrt{1}+1)}=\dfrac{-1}{5\times 2}=-\dfrac{1}{10}.$ $[\because \sqrt{x}-1\neq 0]$

(*v*) Here, $(\sqrt{x}-1)$ does not occur directly in the numerator; therefore, we make substitution.

Put $\sqrt{x}=y$ $\therefore x=y^2$; As $x\to 1$, so $y^2\to 1$ $\therefore y\to 1$

$\therefore \lim\limits_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}=\lim\limits_{y\to 1}\dfrac{y^4-y}{y-1}=\lim\limits_{y\to 1}\dfrac{y(y^3-1)}{y-1}$

$=\lim\limits_{y\to 1}\dfrac{y(y-1)(y^2+y+1)}{y-1}=\lim\limits_{y\to 1}y(y^2+y+1)$ $[\because y-1\neq 0]$

$=1(1+1+1)=3.$

(*vi*) $\lim\limits_{x\to 1}\dfrac{x^2-x\log x+\log x-1}{x-1}=\lim\limits_{x\to 1}\dfrac{(x^2-1)-(\log x)(x-1)}{x-1}$

$=\lim\limits_{x\to 1}\dfrac{(x-1)(x+1-\log x)}{x-1}=\lim\limits_{x\to 1}(x+1-\log x)$

$=1+1-\log 1=2-0=2.$ $[\because x\to 1\ \therefore x\neq 1\ \therefore x-1\neq 0]$

(*vii*) $\underset{x\to 3}{\text{Lim}}\,(x^2-9)\left(\frac{1}{x+3}+\frac{1}{x-3}\right)=\underset{x\to 3}{\text{Lim}}\left(\frac{x^2-9}{x+3}+\frac{x^2-9}{x-3}\right)$ $[\because x+3\neq 0, x-3\neq 0]$

$= \underset{x\to 3}{\text{Lim}}\,\{(x-3)+(x+3)\}=\underset{x\to 3}{\text{Lim}}\,2x=2\times 3=6.$

(*viii*) Since the indices of x are fractions so, it is not possible to factorise.

L.C.M. of the indices 4 and 3 is 12. Let $x=y^{12}$

Then $y=x^{1/12}$. Also, $x^{1/4}=(y^{12})^{1/4}=y^3$ and $x^{1/3}=(y^{12})^{1/3}=y^4$

As $x\to 1$, so $y\to 1$

$\therefore$ Reqd. limit $= \underset{y\to 1}{\text{Lim}}\frac{y^3-1}{y^4-1}=\underset{y\to 1}{\text{Lim}}\frac{(y-1)(y^2+y+1)}{(y-1)(y^3+y^2+y+1)}$

$= \underset{y\to 1}{\text{Lim}}\frac{y^3-1}{y^4-1}=\underset{y\to 1}{\text{Lim}}\frac{(y-1)(y^2+y+1)}{(y-1)(y^3+y^2+y+1)}=\frac{1+1+1}{1+1+1+1}=\frac{3}{4}.$

EXERCISE 18 (*c*)

Evaluate the following limits:

1. $\underset{x\to -1}{\text{Lim}}\frac{x^2-1}{x+1}$ **2.** $\underset{x\to 1}{\text{Lim}}\frac{(2x-3)(x-1)}{2x^2+x-3}$ **[*ISC*]**

3. $\underset{x\to -1}{\text{Lim}}\frac{(x-4)(x+1)}{x^2+3x+2}$ **[*ISC*]** **4.** $\underset{x\to \frac{1}{2}}{\text{Lim}}\frac{4x^2-1}{2x-1}$ **5.** $\underset{x\to 2}{\text{Lim}}\frac{7x^2-11x-6}{3x^2-x-10}$

6. $\underset{x\to 2}{\text{Lim}}\frac{x^2(x^2-4)}{x-2}$ **7.** $\underset{x\to 2}{\text{Lim}}\left(\frac{x^8-16}{x^4-4}+\frac{x^2-9}{x-3}\right)$ **8.** $\underset{x\to 2}{\text{Lim}}\frac{x^3-8}{x-2}$

9. $\underset{x\to \frac{1}{2}}{\text{Lim}}\left(\frac{8x-3}{2x-1}-\frac{4x^2+1}{4x^2-1}\right)$ **10.** $\underset{x\to 3}{\text{Lim}}\frac{x^3-8x^2+45}{2x^2-3x-9}$ **[*ISC*]** **11.** $\underset{x\to 3}{\text{Lim}}\frac{x^3-6x-9}{x^4-81}$

12. $\underset{x\to \sqrt{2}}{\text{Lim}}\frac{x^4-4}{x^2+3x\sqrt{2}-8}$ **13.** $\underset{x\to 1}{\text{Lim}}\frac{1-x^{-1/3}}{1-x^{-2/3}}$ **14.** $\underset{x\to 2}{\text{Lim}}\frac{x-2}{\sqrt{x}-\sqrt{2}}$

15. If $\underset{x\to 1}{\text{Lim}}\frac{x^4-1}{x-1}=\underset{x\to k}{\text{Lim}}\frac{x^3-k^3}{x^2-k^2}$, find the value of k.

ANSWERS

1. –2 **2.** $-\frac{1}{5}$ **3.** –5 **4.** 2 **5.** $\frac{17}{11}$ **6.** 16

7. 25 **8.** 12 **9.** $\frac{7}{2}$ **10.** $-\frac{7}{3}$ **11.** $\frac{7}{36}$ **12.** $\frac{8}{5}$

13. $\frac{1}{2}$ **14.** $2\sqrt{2}$ **15.** $\frac{8}{3}$

HINTS TO SELECTED QUESTIONS

10. Use remainder theorem to factorize the numerator.

12. On factorizing $f(x) = \left[\dfrac{(x-\sqrt{2})\,(x+\sqrt{2})\,(x^2+2)}{(x-\sqrt{2})\,(x+4\sqrt{2})}\right]$

13. Given limit $= \underset{x\to 1}{\text{Lim}}\, x^{1/3}\left(\dfrac{x^{1/3}-1}{x^{2/3}-1}\right)$, put $x^{1/3} = y$.

15. $\underset{x\to 1}{\text{Lim}}\dfrac{x^4-1}{x-1} = \underset{x\to 1}{\text{Lim}}\dfrac{(x^2-1)(x^2+1)}{x-1} = \underset{x\to 1}{\text{Lim}}\ (x+1)\,(x^2+1) = 4$

$$\underset{x\to k}{\text{Lim}}\frac{x^3-k^3}{x^2-k^2} = \underset{x\to k}{\text{Lim}}\frac{(x-k)(x^2+kx+k^2)}{(x-k)(x+k)}$$

$$= \underset{x\to k}{\text{Lim}}\frac{x^2+xk+k^2}{x+k} = \frac{3}{2}k$$

Method 3. Rationalization

In functions which involve square roots, rationalisation of either numerator or denominator will facilitate the work.

Ex. 9. Evaluate the following limits:

(*i*) $\underset{h\to 0}{\text{Lim}}\left[\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\right]$ **[*Roorkee*]**

(*ii*) $\underset{x\to a}{\text{Lim}}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}, a\neq 0$ **[*IIT*]**

(*iii*) $\underset{x\to 0}{\text{Lim}}\dfrac{\sqrt{(1+x)}-\sqrt{(1+x^2)}}{x}$ **[*SC*]**

(*iv*) $\underset{h\to 0}{\text{Lim}}\dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}\right)$

(*v*) $\underset{x\to 0}{\text{Lim}}\dfrac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1-x^2}-\sqrt{1-x}}$

Sol. (*i*) $\underset{h\to 0}{\text{Lim}}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \underset{h\to 0}{\text{Lim}}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\times\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$$= \underset{h\to 0}{\text{Lim}}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \underset{h\to 0}{\text{Lim}}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \underset{h\to 0}{\text{Lim}}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(*ii*) $\sqrt{a+2x}-\sqrt{3x} = (\sqrt{a+2x}-\sqrt{3x})\times\dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} = \dfrac{(a+2x)-3x}{\sqrt{a+2x}+\sqrt{3x}} = \dfrac{a-x}{\sqrt{a+2x}+\sqrt{3x}}$

Similarly, $\sqrt{3a+x}-2\sqrt{x} = \dfrac{3(a-x)}{\sqrt{3a+x}+2\sqrt{x}}$

$$\therefore\quad \underset{x\to a}{\text{Lim}}\frac{\sqrt{a+2x}-\sqrt{3x}}{(\sqrt{3a+x}-2\sqrt{x})} = \underset{x\to a}{\text{Lim}}\frac{a-x}{\sqrt{a+2x}+\sqrt{3x}}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{3(a-x)}$$

$$= \underset{x\to a}{\text{Lim}}\frac{\sqrt{3a+x}+2\sqrt{x}}{3\left(\sqrt{a+2x}+\sqrt{3x}\right)} = \frac{\sqrt{4a}+2\sqrt{a}}{3(\sqrt{3a}+\sqrt{3a})} = \frac{4\sqrt{a}}{6\sqrt{3a}} = \frac{2}{3\sqrt{3}}.$$

(*iii*) $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x)}-\sqrt{1+x^2}}{x}=\lim\limits_{x\to 0}\dfrac{[\sqrt{1+x}-\sqrt{1+x^2}][\sqrt{1+x}+\sqrt{1+x^2}]}{x[\sqrt{1+x}+\sqrt{1+x^2}]}$

$=\lim\limits_{x\to 0}\dfrac{(1+x)-(1+x^2)}{x[\sqrt{1+x}+\sqrt{1+x^2}]}=\lim\limits_{x\to 0}\dfrac{x\,(1-x)}{x\,[\sqrt{1+x}+\sqrt{1+x^2}]}$

$=\lim\limits_{x\to 0}\dfrac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}}=\dfrac{1-0}{\sqrt{1+0}+\sqrt{1+0}}=\dfrac{1}{1+1}=\dfrac{1}{2}.$

(*iv*) $\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}\right)=\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)=\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\times\dfrac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right]$

$=\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right]=\lim\limits_{h\to 0}\dfrac{-1}{\sqrt{x}\sqrt{x+h}[\sqrt{x}+\sqrt{x+h}]}$

$=\dfrac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})}=\dfrac{-1}{2x\sqrt{x}}.$

(*v*) In this case, we rationalize both the numerator and the denominator.

Given limits

$=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x}-\sqrt{1+x^2})\,(\sqrt{1+x}+\sqrt{1+x^2})}{(\sqrt{1+x}+\sqrt{1+x^2})}\cdot\dfrac{(\sqrt{1-x^2}+\sqrt{1-x})}{(\sqrt{1-x^2}-\sqrt{1-x})\,(\sqrt{1-x^2}+\sqrt{1-x})}$

$=\lim\limits_{x\to 0}\dfrac{(1+x)-(1+x^2)}{(1-x^2)-(1-x)}\cdot\dfrac{(\sqrt{1-x^2}+\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1+x^2})}=\lim\limits_{x\to 0}\left[\dfrac{x-x^2}{x-x^2}\cdot\dfrac{\sqrt{1-x^2}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1+x^2}}\right]$

$=\lim\limits_{x\to 0}\dfrac{\sqrt{1-x^2}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1+x^2}}=\dfrac{1+1}{1+1}=1.$

EXERCISE 18 (*d*)

Evaluate the following limits:

1. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ **2.** $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$ **3.** $\lim\limits_{x\to 4}\dfrac{3-\sqrt{(5+x)}}{x-4}$

4. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$ **5.** $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{(1+x)}-1}$ **6.** $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ ***[NMOC]***

7. $\lim\limits_{x\to 1}\dfrac{\sqrt{(x^2-1)}+\sqrt{(x-1)}}{\sqrt{(x^2-1)}}$ $\left[\textbf{Hint. Given limit} = \lim\limits_{x\to 1}\left(1+\dfrac{1}{\sqrt{(x+1)}}\right)\right]$

8. $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x+x^2)}-1}{x}$ **9.** $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x^3)}-\sqrt{(1-x^3)}}{x^2}$ ***[AP]***

ANSWERS

1. $\frac{1}{2}$ **2.** $\frac{1}{2\sqrt{a}}$ **3.** $\frac{-1}{6}$ **4.** $\frac{1}{2\sqrt{2}}$ **5.** 2

6. $\frac{1}{2}$ **7.** $1+\frac{1}{\sqrt{2}}$ **8.** $\frac{1}{2}$ **9.** 0

Method 4. Using expansions

Ex. 10. Evaluate the following limits.

(i) $\lim\limits_{x\to 0}\dfrac{(1+x)^6-1}{(1+x)^2-1}$ ***(ii)*** $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$ ***[ISC]***

Sol. (*i*) $\lim\limits_{x\to 0}\dfrac{(1+x)^6-1}{(1+x)^2-1}=\lim\limits_{x\to 0}\dfrac{\left(1+6x+\frac{6.5}{1.2}x^2+\ldots+x^6\right)-1}{(1+2x+x^2)-1}$

$=\lim\limits_{x\to 0}\dfrac{6x+15x^2+\ldots+x^6}{2x+x^2}=\lim\limits_{x\to 0}\dfrac{6+15x+\ldots+x^5}{2+x}=\dfrac{6}{2}=3.\quad [x\to 0\Rightarrow x\neq 0]$

(*ii*) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}=\lim\limits_{x\to 0}\left[\dfrac{(1+nx+\frac{1}{2}n(n-1)x^2+\ldots)-1}{x}\right.$

(Expanding by Binomial Theorem)

$=\lim\limits_{x\to 0}[n+\frac{1}{2}n(n-1)x+$ terms containing higher powers of $x]=n.$

Method 5. Using the standard result $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, where $a>0$.

Proof. Put $x=a+h$ so that when $x\to a$, $h\to 0$.

Case 1. ***When n is a positive integer***

$\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=\lim\limits_{h\to 0}\dfrac{(a+h)^n-a^n}{(a+h)-a}=\lim\limits_{h\to 0}\dfrac{1}{h}\{(a+h)^n-a^n\}$

$=\lim\limits_{h\to 0}\dfrac{1}{h}[a^n+{}^nc_1a^{n-1}h+{}^nc_2a^{n-2}h^2+\ldots+h^n-a^n]$ (Using Binomial theorem)

$=\lim\limits_{h\to 0}\dfrac{1}{h}[{}^nc_1a^{n-1}h+{}^nc_2a^{n-2}h^2+\ldots+h^n]=\lim\limits_{h\to 0}\dfrac{h}{h}[{}^nc_1a^{n-1}+{}^nc_2a^{n-2}h+\ldots+h^{n-1}]$

$=\lim\limits_{h\to 0}[{}^nc_1a^{n-1}+{}^nc_2a^{n-2}h+{}^nc_3a^{n-3}h^2+\ldots h^{n-1}]$ (We can divide by h as $h\neq 0$)

$={}^nc_1a^{n-1}=na^{n-1}.$

Case 2. ***When n is a negative integer or a fraction***

$\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=\lim\limits_{h\to 0}\dfrac{(a+h)^n-a^n}{(a+h)-a}=\lim\limits_{h\to 0}\dfrac{a^n\left(1+\frac{h}{a}\right)^n-a^n}{h}$

$$= \lim_{h\to 0} \frac{a^n\left[\left\{1+n\cdot\frac{h}{a}+\frac{n(n-1)}{1.2}\cdot\frac{h^2}{a^2}+....\right\}-1\right]}{h}$$

$$= \lim_{h\to 0} \frac{a^n . n}{h}\cdot\frac{h}{a} \quad [1+\frac{(n-1)}{1.2}\cdot\frac{h}{a} + \text{terms containing higher powers of } h]$$

$$= a^n\cdot\frac{n}{a} = na^{n-1}.$$

Cor. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}, (m, n\in Q) = \dfrac{m}{n}a^{m-n}$.

$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n} = \lim_{x\to a}\frac{\frac{(x^m-a^m)}{(x-a)}}{\frac{(x^n-a^n)}{(x-a)}} = \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}a^{m-n}.$$

Ex. 11. Evaluate: (*i*) $\lim\limits_{x\to a}\dfrac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x^{\frac{1}{3}}-a^{\frac{1}{3}}}$ (*ii*) $\lim\limits_{x\to 2}\dfrac{x^5-32}{x^2-4}$

(*iii*) $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a}$ [*ISC*] (*iv*) $\lim\limits_{x\to 0}\dfrac{(1+x)^6-1}{(1+x)^2-1}$.

Sol. We use the result $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$

(*i*) $$\lim_{x\to a}\frac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x^{\frac{1}{3}}-a^{\frac{1}{3}}} = \lim_{x\to a}\frac{\frac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x-a}}{\frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{x-a}} = \frac{\frac{3}{5}a^{\frac{3}{5}-1}}{\frac{1}{3}a^{\frac{1}{3}-1}} = \frac{9}{5}\frac{a^{\frac{-2}{5}}}{a^{\frac{-2}{3}}} = \frac{9}{5}a^{\frac{4}{15}}.$$

(*ii*) $$\lim_{x\to 2}\frac{x^5-32}{x^2-4} = \lim_{x\to 2}\frac{1}{(x+2)}\cdot\frac{x^5-2^5}{x-2} = \lim_{x\to 2}\frac{1}{x+2}\cdot\lim_{x\to 2}\frac{x^5-2^5}{x-2}$$

$$= \frac{1}{2+2}\cdot 5\times 2^{5-1} = \frac{1}{4}\times 5\times 2^4 = 20 \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

(*iii*) $$\lim_{x\to a}\frac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a} = \lim_{x\to a}\frac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{(x+2)-(a+2)}$$

$$= \frac{5}{3}(a+2)^{\frac{5}{3}-1} = \frac{5}{3}(a+2)^{\frac{2}{3}}. \qquad \left[\text{Using } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

(*iv*) Put $1+x=y$. Then, $x\to 0 \Rightarrow y\to 1$

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^2-1}=\lim_{y\to 1}\frac{y^6-1}{y^2-1}=\frac{\lim\limits_{y\to 1}\dfrac{y^6-1}{y-1}}{\lim\limits_{y\to 1}\dfrac{y^2-1}{y-1}}=\frac{6(1)^{6-1}}{2(1)^{2-1}}=\frac{6}{2}=3.$$

Note. You can also find the limit by expanding the numerator by using binomial expansion.

Ex. 12. If $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$ and if n is a positive integer, find n.

Sol. $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=n.2^{n-1}$ $\qquad$ $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

$\therefore \quad n.2^{n-1}=80 \quad \Rightarrow n\,.\,2^{n-1}=5\times 2^4=5\times 2^{5-1}\Rightarrow n=5.$

EXERCISE 18 (*e*)

Evaluate the following limits:

1. $\lim\limits_{x\to 1}\dfrac{x^n-1}{x-1}$. **2.** $\lim\limits_{x\to 3}\dfrac{x^5-243}{x^2-9}$. **3.** $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$.

4. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$. **5.** $\lim\limits_{x\to 5}\dfrac{x^4-625}{x^3-125}$. **6.** $\lim\limits_{x\to 2}\dfrac{x^{10}-1024}{x^5-32}$.

7. $\lim\limits_{x\to 9}\dfrac{x^{3/2}-27}{x-9}$. **8.** $\lim\limits_{x\to a}\dfrac{x\sqrt{x}-a\sqrt{a}}{x-a}$. **9.** $\lim\limits_{x\to a}\dfrac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$.

[*ISC*]

10. If $\lim\limits_{x\to -a}\dfrac{x^9+a^9}{x+a}=9$, find all possible values of a.

11. If $\lim\limits_{x\to a}\dfrac{x^5-a^5}{x-a}=405$, find all possible values of a.

12. If $\lim\limits_{x\to a}\dfrac{x^9-a^9}{x-a}=9$, find all possible values of a. **[*HPSB*]**

13. Prove that $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{(1+x)}-\sqrt[3]{(1-x)}}{x}=\dfrac{2}{3}$. **[*AP, EAMCET*]**

ANSWERS

1. n **2.** $\dfrac{135}{2}$ **3.** n **4.** $\dfrac{m}{n}$ **5.** $\dfrac{20}{3}$

6. 64 **7.** $\dfrac{9}{2}$ **8.** $\dfrac{3}{2}\sqrt{a}$ **9.** $\dfrac{3}{2}(a+2)^{1/2}$

10. ± 1 **11.** ± 3 **12.** $1, -1$

HINTS TO SELECTED QUESTIONS

2. Given $\lim\limits_{x\to 3}\dfrac{x^5-3^5}{x-3}\div\lim\limits_{x\to 3}\dfrac{x^2-3^2}{x-3}$ **7.** $27=9^{\frac{3}{2}}$ $\therefore \lim\limits_{x\to 9}\dfrac{x^{\frac{3}{2}}-27}{x-9}=\dfrac{3}{2}(9)^{\frac{3}{2}-1}$

13. Given limit $=\lim\limits_{x\to 0}\dfrac{(1+x)^{1/3}}{x}-\lim\limits_{x\to 0}\dfrac{(1-x)^{1/3}}{x}=\lim\limits_{y\to 1}\dfrac{y^{1/3}-1}{y-1}+\lim\limits_{z\to 1}\dfrac{z^{1/3}-1}{z-1}$

$$=\frac{1}{3}.1^{-2/3}+\frac{1}{3}.1^{-2/3}=\frac{2}{3}$$

18.12. Meaning of the symbol ∞

Let $y = \frac{1}{x}$ and $x \to 0^+$. Let us tabulate the values of $\frac{1}{x}$ for values of x close to 0.

x	1	0.1	0.01	0.00001	10^{-9}
y	1	10	100	100000	10^9

etc.

We notice that as the value of x approaches from the right the value 0, the value of $\frac{1}{x}$ gradually increases. Taking the value of x sufficiently close to 0, we can make the value of $y = \frac{1}{x}$ greater than any positive number, however large. In such cases we say that y tends to infinity and indicate it by the symbol $y \to \infty$.

Hence, $x \to \infty$ means that the value of x so changes that it can be made greater than any pre-assigned positive number however large.

Now, if $f(x)$ be a function of x, and if $f(x) \to l$, as $x \to \infty$, then we say that the limiting value of $f(x)$ is l as x tends to infinity. In symbols, it is written as $\lim_{x\to\infty} f(x) = l$.

In less formal language, $\lim_{x\to\infty} f(x) = l$

means that $f(x)$ can be made as close as desired to l by making x large enough and positive.
Similarly,

$$\lim_{x\to-\infty} f(x) = l$$

Example. To show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

The function $y = \frac{1}{x}$ graphed in Fig. 18.07 is defined for all real numbers except $x = 0$. Study the following statements.

(a) When x is small and positive, $1/x$ is large and positive. For example, $\left(\frac{1}{0.001}\right) = 1000$.

(b) When x is small and negative, $1/x$ is large and negative. For example, $\left(\frac{1}{-0.001}\right) = -1000$.

(c) When x is large and positive, $1/x$ is small and positive.

For example, $\left(\frac{1}{10000}\right) = 0.0001$.

(d) When x is large and negative, $1/x$ is small and negative.

For example, $\left(\frac{1}{-10000}\right) = -0.0001$.

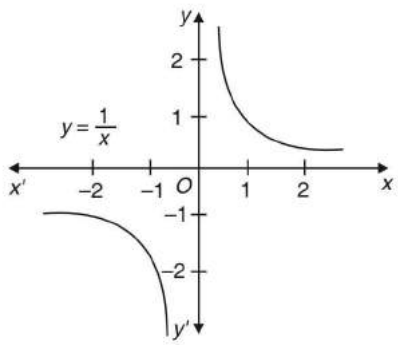


Fig. 18.07

These facts are sometimes abbreviated by saying:

(a) As x approaches 0 from the right, $1/x$ tends to ∞.

(b) As x approaches 0 from left, $1/x$ tends to $-\infty$.

(c) As x tends to ∞, $1/x$ approaches 0.

(d) As x tends to $-\infty$, $1/x$ approaches 0.

The symbol ∞, infinity, does not represent any real number. We cannot use ∞ in arithmetic in the usual way, but it is convenient to be able to say things like "the limit $1/x$ as x approaches infinity is 0".

Note. From the above we find that as $x \to \infty$, then $y \to 0^+$. Hence, for any function $f(x)$

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0^+} f\left(\frac{1}{y}\right) \qquad [\text{ Putting } x = \frac{1}{y}]$$

18.13. Infinite limit of a function

Let $f(x)$ be a function of x. If the value of $f(x)$ can be made greater than any pre-assigned number by taking x close to a, then we say that the function $f(x)$ becomes positively infinite as x approaches a. We write this $\lim_{x\to a} f(x) = +\infty$. Similarly, if the value of $f(x)$ can be made less than any pre-assigned number by taking x close to a, then we say that the function $f(x)$ becomes negatively infinite as x approaches a. We write this as

$$\lim_{x\to a} f(x) = -\infty$$

18.14. To find $\lim_{x\to\infty} f(x)$

Method 1. ***(i) Divide the numerator and denominator of the fraction by the highest power of x present in the fraction.***

(ii) Use the idea of $\frac{1}{x}, \frac{1}{x^2}, \ldots \to 0$ ***as*** $x \to \infty$.

Method 2. Put $x = \frac{1}{y}$. As $x \to \infty$, $y \to 0$

Ex. 13. Evaluate the following limits:

(i) $\lim_{x\to\infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$ [*AP*]

(ii) $\lim_{x\to\infty} [x - \sqrt{(x^2+x)}]$ [*IIT*]

(iii) $\lim_{x\to\infty} (\sqrt{x^2+5x+4} - \sqrt{x^2-3x+4})$

(iv) $\lim_{x\to\infty} 2x(\sqrt{x^2+1} - x)$

(v) $\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt[3]{x^2-1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$

Sol. (*i*) $\lim_{x\to\infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \lim_{x\to\infty} \frac{2x^2+5x+3}{3x^2+10x+8}$

$= \lim_{x\to\infty} \frac{2+\frac{5}{x}+\frac{3}{x^2}}{3+\frac{10}{x}+\frac{8}{x^2}}$ (Dividing the num. and den. by x^2) $= \frac{2}{3}$. $\left[\because \frac{5}{x}, \frac{3}{x^2}, \frac{10}{x}, \frac{8}{x^2} \text{ all} \to 0 \text{ as } x \to \infty\right]$

Alternatively, Put $x = \frac{1}{y}$. As $x \to \infty$, $y \to 0$. Then

$$= \lim_{x\to\infty} \frac{2x^2+5x+3}{3x^2+10x+8} = \lim_{y\to 0} \frac{\frac{2}{y^2}+\frac{5}{y}+3}{\frac{3}{y^2}+\frac{10}{y}+8} = \lim_{y\to 0} \frac{2+5y+3y^2}{3+10y+8y^2} = \frac{2+0+0}{3+0+0} = \frac{2}{3}.$$

(*ii*) Put $x = \dfrac{1}{y}$ so that $y \to 0$ when $x \to \infty$. Then

$$\lim_{x\to\infty}[x-\sqrt{(x^2+x)}] = \lim_{y\to 0}\left[\frac{1}{y}-\sqrt{\left(\frac{1}{y^2}+\frac{1}{y}\right)}\right]$$

$$= \lim_{y\to 0}\frac{1-\sqrt{1+y}}{y} = \lim_{y\to 0}\frac{1-\sqrt{1+y}}{y}\times\frac{1+\sqrt{1+y}}{1+\sqrt{1+y}} \qquad \text{(Rationalizing the num.)}$$

$$= \lim_{y\to 0}\frac{1-(1+y)}{y(1+\sqrt{1+y})} = \lim_{y\to 0}\frac{-1}{1+\sqrt{1+y}} = \frac{-1}{1+\sqrt{1}} = -\frac{1}{2}.$$

(*iii*) $\lim\limits_{x\to\infty}(\sqrt{x^2+5x+4}-\sqrt{x^2-3x+4})$

$$= \lim_{x\to\infty}\frac{\sqrt{x^2+5x+4}-\sqrt{x^2-3x+4}}{1}\cdot\frac{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}} \quad \text{(Rationalizing the num.)}$$

$$= \lim_{x\to\infty}\frac{(x^2+5x+4)-(x^2-3x+4)}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}} = \lim_{x\to\infty}\frac{8x}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}}$$

$$= \lim_{x\to\infty}\frac{8}{\sqrt{1+\frac{5}{x}+\frac{4}{x^2}}+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}}$$

$$= \frac{8}{\sqrt{1+0+0}+\sqrt{1-0+0}} = \frac{8}{\sqrt{1}+\sqrt{1}}$$

$$= \frac{8}{2} = 4. \quad \left[\text{As } x\to\infty, \frac{1}{x}, \frac{1}{x^2} \text{ all} \to 0\right]$$

Divide each term by the highest power of x.
Now $\sqrt{x^2} = x$
$\therefore$ the highest power is x.
$\therefore$ divide each term by x.

(*iv*) First, we rationalize the numerator, but keep $2x$ outside the bracket as it is.

$$\lim_{x\to\infty} 2x(\sqrt{x^2+1}-x) = \lim_{x\to\infty} 2x\,(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{2x[(x^2+1)-x^2]}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}+\frac{x}{x}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{2}{\sqrt{1+0}+1} = \frac{2}{2} = 1. \quad \left[\text{As } x\to\infty, \frac{1}{x^2}\to 0\right]$$

(Divide each term by the highest power of x. This is x here.)

(*v*) Dividing each term by the highest power of x. Here $\sqrt{x^2} = x$; $\sqrt[3]{x^2} = x^{2/3}$; $\sqrt[4]{x^4} = x$

$\sqrt[5]{x^4} = x^{4/5}$ $\therefore$ The highest power is x $\therefore$ Divide each term by x.

$$\therefore \quad \lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}} = \lim_{x\to\infty}\frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}-\sqrt[3]{\frac{x^2}{x^3}+\frac{1}{x^3}}}{\sqrt[4]{\frac{x^4}{x^4}+\frac{1}{x^4}}-\sqrt[5]{\frac{x^4}{x^5}+\frac{1}{x^5}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}} - \sqrt[3]{\frac{1}{x}+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}} - \sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}} = \frac{\sqrt{1+0}-\sqrt[3]{0+0}}{\sqrt[4]{1+0}-\sqrt[5]{0+0}} = \frac{1-0}{1-0} = 1. \quad \left[\text{As } x\to\infty, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \text{etc.} \to 0\right]$$

Ex. 14. It is given that $f(x) = \frac{ax+b}{x+1}$, $\lim_{x\to 0} f(x) = 2$ and $\lim_{x\to\infty} f(x) = 1$. Prove that $f(-2) = 0$.

Sol. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{ax+b}{x+1} = \frac{a(0)+b}{0+1} = b.$ It is given that $\lim_{x\to 0} f(x) = 2 \therefore b = 2$

Again, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{ax+b}{x+1} = \lim_{x\to\infty} \frac{a+\frac{b}{x}}{1+\frac{1}{x}} = \frac{a+0}{1+0} = a$

It is given that $\lim_{x\to\infty} f(x) = 1 \quad \therefore \quad a = 1$

$\therefore \quad f(x) = \frac{ax+b}{x+1} = \frac{1\times x+2}{x+1} = \frac{x+2}{x+1}$ and $\therefore \quad f(-2) = \frac{-2+2}{-2+1} = \frac{0}{-1} = 0.$

EXERCISE 18 (*f*)

Evaluate the following limits:

1. $\lim_{x\to\infty} \frac{4x-3}{2x+7}$ **2.** $\lim_{x\to\infty} \frac{3x^2+2x-5}{x^2+5x+1}$ **3.** $\lim_{x\to\infty} \frac{x^3+6x^2+1}{x^4+3}$ **[*AP*]**

4. $\lim_{x\to\infty} \frac{3x^3+x^2-1}{x^2-x+7}$ **[*ISC*]** **5.** $\lim_{x\to\infty} \frac{5x-6}{\sqrt{4x^2+9}}$ **6.** $\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

7. $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+c}-\sqrt{x})$ **8.** $\lim_{x\to\infty} \sqrt{x^2+x+1}-\sqrt{x^2+1}$

9. $\lim_{n\to\infty} \frac{1+2+3+....+n}{n^2}$ $\left(\text{or } \lim_{n\to\infty} \frac{\Sigma n}{n^2}\right)$ **10.** $\lim_{n\to\infty} \frac{\Sigma n^3}{n^4}$

ANSWERS

1. 2 **2.** 3 **3.** 0 **4.** ∞ **5.** $\frac{5}{2}$ **6.** $\frac{\sqrt{3}-\sqrt{2}}{4}$

7. $\frac{c}{2}$ **[Hint.** First rationalize the numerator] **8.** $\frac{1}{2}$ **[Hint.** First rationalize the numerator]

9. $\frac{1}{2}$ **[Hint.** Given limit $= \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2}$ **]** **10.** $\frac{1}{4}$ **[Hint.** $\Sigma n^3 = \frac{n^2(n+1)^2}{4}$ **]**

18.15. Trigonometric limits

Standard results

1. $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} \cos x = 1$

Let x be an angle in the standard position. Draw PM perp. from any point P. Then $\sin x = \frac{MP}{OP}$ and $\cos x = \frac{OM}{OP}$.

Imagine x becoming smaller and smaller. This brings P closer and closer to M which implies that as

$$x \to 0,\ MP \to 0 \text{ and } OP \to OM,\ i.e.,\ \frac{MP}{OP} \to 0 \text{ and } \frac{OM}{OP} \to 1.$$

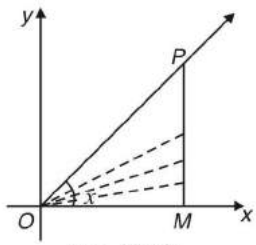


Fig. 18.08

Therefore, as $x \to 0$, $\sin x \to 0$ and $\cos x \to 1$. Hence ,

$\lim\limits_{x\to 0} \sin x = 0$ and $\lim\limits_{x\to 0} \cos x = 1$.

2. $\mathbf{\lim\limits_{x\to 0} \frac{\sin x}{x} = 1}$ **3.** $\mathbf{\lim\limits_{x\to 0} \frac{\tan x}{x} = 1}$ $\left[\frac{0}{0}\text{forms}\right]$

Let there be a unit circle with centre O. Let $\angle AOB = x$ radians, Join AB. Draw $AC \perp OA$ at A. Produce OB to meet AC at C.

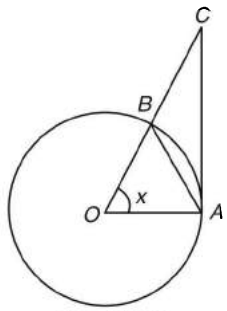


Fig. 18.09

From the figure it is clear that

$$area\ (\Delta\ OAB) < area\ (\text{sector}\ OAB) < area\ (\Delta\ OAC)$$

$$\Rightarrow \quad \frac{1}{2}(OA)\,(OB)\ \sin x < \frac{1}{2}(OA)^2 \cdot x < \frac{1}{2}(OA)\,(AC)$$

$$\Rightarrow \quad \frac{1}{2}(1)\,(1)\ \sin x < \frac{1}{2}(1)^2 \cdot x < \frac{1}{2}(1)\,(1 \tan x) \left[\because \frac{AC}{OA} = \tan x\right]$$

$$\Rightarrow \quad \sin x < x < \tan x \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad \left[\because 0 < x < \frac{\pi}{2}\ \therefore \sin x > 0\right]$$

$$\Rightarrow \quad 1 > \frac{\sin x}{x} > \cos x$$

Thus, we see that $\frac{\sin x}{x}$ always lies between 1 and $\cos x$. Now $\lim\limits_{x\to 0} \cos x = 1$

$\therefore$ When $x \to 0$, $\frac{\sin x}{x}$ lies between 1 and 1, *i.e.,* $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, or $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Caution. In $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, x is expressed in radians. Whenever degree measure of an angle is given, it should be converted into radian measure by the conversion formula $x^\circ = \frac{\pi x}{180}$ radians.

Moreover, x should tend to zero, and the angle x in the numerator, x in the denominator should be the same.

3. $\lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right) = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \lim\limits_{x\to 0} \frac{1}{\cos x} = 1 \times \frac{1}{\cos 0} = 1 \times \frac{1}{1} = 1.$

SOLVED EXAMPLES

Type 1. Evaluation of trigonometric limits, when $x \to 0$.

Ex. 15. Evaluate the following limits:

(*i*) $\lim\limits_{x\to 0} \frac{\sin 3x}{5x}$ **(*ii*)** $\lim\limits_{x\to 0} \frac{\sin x \cos x}{3x}$ **(*iii*)** $\lim\limits_{x\to 0} \frac{x^3}{\sin x^2}$ **(*iv*)** $\lim\limits_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x}$

(*v*) $\lim\limits_{x\to 0} \left(\frac{\sin ax}{\sin bx}\right)^k$ **(*vi*) (*a*)** $\lim\limits_{x\to 0} \frac{\sin x^\circ}{x}$ **(*CEE*)** **(*b*)** $\lim\limits_{x\to 0} \frac{3 \sin x^\circ - \sin 3x^\circ}{x^3}$

(*vii*) $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right)$ **(*viii*)** $\lim\limits_{x\to 0} \frac{x \sin 5x}{\sin^2 4x}$ **(*ix*)** $\lim\limits_{x\to \infty} 5^x \sin\left(\frac{a}{5^x}\right)$.

Sol. (*i*) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{3x}\right)\times\dfrac{3}{5}=\dfrac{3}{5}\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{3x}\right)=\dfrac{3}{5}\times 1=\dfrac{3}{5}.$

(*ii*) $\lim\limits_{x\to 0}\dfrac{\sin x\cos x}{3x}=\dfrac{1}{3}\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\cdot\cos x\right)=\dfrac{1}{3}\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)\times\lim\limits_{x\to 0}\cos x=\dfrac{1}{3}\times 1\times 1=\dfrac{1}{3}.$

(*iii*) $\lim\limits_{x\to 0}\dfrac{x^3}{\sin x^2}=\lim\limits_{x\to 0}\left(\dfrac{x^2}{\sin x^2}\times x\right)=\lim\limits_{x\to 0}\left(\dfrac{x^2}{\sin x^2}\right)\times\lim\limits_{x\to 0}x=1\times 0=0.$

(*iv*) $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin x}=\lim\limits_{x\to 0}\left(\dfrac{2\cos 4x\sin x}{\sin x}\right)$ [Numerator is of the form sin C – sin D].

$=\lim\limits_{x\to 0}\ (2\cos 4x)=2\times\lim\limits_{x\to 0}\ (\cos 4x)=2\times 1=2$

(*v*) $\lim\limits_{x\to 0}\left(\dfrac{\sin ax}{\sin bx}\right)^k=\lim\limits_{x\to 0}\left[\dfrac{\sin ax}{ax}.ax.\dfrac{bx}{\sin bx}.\dfrac{1}{bx}\right]^k=\left(\dfrac{a}{b}\right)^k\lim\limits_{x\to 0}\left(\dfrac{\sin ax}{ax}\right)^k\lim\limits_{x\to 0}\left(\dfrac{bx}{\sin bx}\right)^k$

$=\left(\dfrac{a}{b}\right)^k\times 1\times 1=\left(\dfrac{a}{b}\right)^k.$

(*vi*) (*a*) Remember that $\lim\limits_{x\to 0}\dfrac{\sin x^\circ}{x}=1$ only where x is measured in radians. **[Note]**

$\because\ 180^\circ=\pi$ radians $\therefore\ \lim\limits_{x\to 0}\dfrac{\sin x^\circ}{x}=\lim\limits_{x\to 0}\dfrac{\sin\frac{\pi x}{180}}{x}$

$=\lim\limits_{x\to 0}\dfrac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}.\dfrac{\pi}{180}=\dfrac{\pi}{180}\lim\limits_{x\to 0}\dfrac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}=\dfrac{\pi}{180}\times 1=\dfrac{\pi}{180}.$

(*b*) $\lim\limits_{x\to 0}\dfrac{3\sin x^\circ-\sin 3x^\circ}{x^3}=\lim\limits_{x\to 0}\dfrac{3\sin x^\circ-3\sin x^\circ+4\sin^3 x^\circ}{x^3}$ [$\because\sin 3x=3\sin x-4\sin^3 x$]

$=\lim\limits_{x\to 0}4\left(\dfrac{\sin\frac{\pi x}{180}}{x}\right)^3=\lim\limits_{\frac{\pi x}{180}\to 0}4\left[\dfrac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\right]^3\times\left(\dfrac{\pi}{180}\right)^3=4\left(\dfrac{\pi}{180}\right)^3\quad\left(x\to 0\Rightarrow\dfrac{\pi x}{180}\to 0\right)$

(*vii*) As $x\to 0$, the value of $\sin\left(\dfrac{1}{x}\right)$ oscillates between –1 and + 1, passing through zero and intermediate values, an infinite number of times. Thus, there is no definite value to which $\sin\left(\dfrac{1}{x}\right)$ tends as x tends to zero. Hence, $\lim\limits_{x\to 0}\ \sin\left(\dfrac{1}{x}\right)$ does not exist.

Therefore, it is incorrect to write $\lim\limits_{x\to 0}\ \left(x\sin\dfrac{1}{x}\right)$ as $\lim\limits_{x\to 0}(x).\ \lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$.

Thus, the product theorem on limits cannot be used here.

Since $\lim\limits_{x\to 0} x\sin\frac{1}{x} = \lim\limits_{x\to 0}(x) \times$ a finite quantity between –1 and +1.

$= 0 \times$ a finite quantity $= 0$

Therefore, $\boldsymbol{\lim\limits_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0}$

(viii) $\lim\limits_{x\to 0}\dfrac{x\sin 5x}{\sin^2 4x} = \lim\limits_{x\to 0}\left[\dfrac{\sin 5x}{5x}\times 5.\dfrac{1}{\left(\dfrac{\sin 4x}{4x}\right)^2}\times\dfrac{1}{16}\right]$

$= \dfrac{5}{16}.\lim\limits_{5x\to 0}\dfrac{\sin 5x}{5x}\times\dfrac{1}{\left(\lim\limits_{4x\to 0}\dfrac{\sin 4x}{4x}\right)^2} = \dfrac{5}{16}\times 1\times 1 = \dfrac{5}{16}.$

(ix) $\lim\limits_{x\to\infty} 5^x\sin\left(\dfrac{a}{5^x}\right) = \lim\limits_{\frac{a}{5^x}\to 0}\left[\sin\left(\dfrac{a}{5^x}\right)/\dfrac{a}{5^x}\right]\times a = 1\times a = a.$

Ex. 16. Evaluate the following limits:

(i) $\boldsymbol{\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{x}}$ ***(ii)*** $\boldsymbol{\lim\limits_{x\to 0}\dfrac{\sin 2x(1-\cos 2x)}{x^3}}$ ***(iii)*** $\boldsymbol{\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}}$

(iv) $\boldsymbol{\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}}$ ***(v)*** $\boldsymbol{\lim\limits_{x\to 0}\dfrac{\tan(x+a)-\tan a}{3x}}$ ***(vi)*** $\boldsymbol{\lim\limits_{x\to 0}\left(\dfrac{\tan x-\sin x}{x^3}\right)}$ ***[IIT]***

(vii) $\boldsymbol{\lim\limits_{x\to 0}\left(\dfrac{\tan x-\sin x}{\sin^3 x}\right)}$ ***(viii)*** $\boldsymbol{\lim\limits_{x\to 0}\dfrac{\operatorname{cosec} x-\cot x}{x}}$ ***(ix)*** $\boldsymbol{\lim\limits_{x\to 0}\dfrac{1-\cos ax}{x\sin 3x}}$

Sol. ***(i)*** $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{x} = \lim\limits_{x\to 0}\left(\dfrac{2\sin^2 3x}{x}\right) = \lim\limits_{x\to 0}\left(6.\dfrac{\sin 3x}{3x}\cdot\sin 3x\right)$

$= 6\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\times\lim\limits_{x\to 0}\sin 3x = 6\times 1\times 0 = 0.$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin 2x\,(1-\cos 2x)}{x^3} = \lim\limits_{x\to 0}\dfrac{2\sin x\cos x\cdot 2\sin^2 x}{x^3} = 4\times\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^3\times\lim\limits_{x\to 0}\cos x$

$= 4\times 1\times 1 = 4.$

(iii) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x/2}{x^2} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x/2}{x^2/4}\cdot\dfrac{1}{4} = \dfrac{1}{2}\lim\limits_{x\to 0}\left(\dfrac{\sin x/2}{x/2}\right)^2 = \dfrac{1}{2}\times 1 = \dfrac{1}{2}.$

(iv) $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h} = \lim\limits_{h\to 0}\dfrac{2\cos\left(x+\dfrac{h}{2}\right)\sin\dfrac{h}{2}}{h} = \lim\limits_{h\to 0}\cos\left(x+\dfrac{h}{2}\right)\dfrac{\sin\dfrac{h}{2}}{\dfrac{h}{2}} = \cos x.1 = \cos x.$

(v) $\lim\limits_{x\to 0}\dfrac{\tan(x+a)-\tan a}{3x} = \lim\limits_{x\to 0}\dfrac{\sin[(x+a)-a]}{(3x)\cos(x+a)\cos a}$

$= \lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1}{3\cos(x+a)\cos a} = 1\times\dfrac{1}{3\cos a\cos a} = \dfrac{1}{3}\sec^2 a.$

(*vi*) $\underset{x\to0}{\text{Lim}}\left(\frac{\tan x-\sin x}{x^3}\right)=\underset{x\to0}{\text{Lim}}\left(\frac{\sin x-\sin x\cos x}{x^3\cos x}\right)$.

$=\underset{x\to0}{\text{Lim}}\frac{\sin x(1-\cos x)}{x^3\cos x}=\underset{x\to0}{\text{Lim}}\frac{\sin x\,(2\sin^2 x/2)}{x^3\cos x}=2\underset{x\to0}{\text{Lim}}\left\{\frac{\sin x}{x}.\left(\frac{\sin x/2}{x/2}\right)^2.\frac{1}{4\cos x}\right\}$

$=\frac{2}{4}\left[\underset{x\to0}{\text{Lim}}\frac{\sin x}{x}\cdot\underset{x\to0}{\text{Lim}}\left(\frac{\sin x/2}{x/2}\right)^2.\underset{x\to0}{\text{Lim}}\frac{1}{\cos x}\right]=\frac{1}{2}\times1\times1^2\times1=\frac{1}{2}$.

(*vii*) $\underset{x\to0}{\text{Lim}}\left(\frac{\tan x-\sin x}{\sin^3 x}\right)=\underset{x\to0}{\text{Lim}}\frac{\sin x-\sin x\cos x}{\sin^3 x\cos x}=\underset{x\to0}{\text{Lim}}\frac{\sin x\,(1-\cos x)}{\sin^3 x.\cos x}$

$=\underset{x\to0}{\text{Lim}}\frac{1-\cos x}{\sin^2 x.\cos x}=\underset{x\to0}{\text{Lim}}\frac{1-\cos x}{(1-\cos^2 x)\cos x}=\underset{x\to0}{\text{Lim}}\frac{1}{(1+\cos x)\cos x}=\frac{1}{(1+1)\times1}=\frac{1}{2}$.

(*viii*) $\underset{x\to0}{\text{Lim}}\frac{\text{cosec}\,x-\cot x}{x}=\underset{x\to0}{\text{Lim}}\frac{1-\cos x}{x\sin x}$

$=\underset{x\to0}{\text{Lim}}\frac{2\sin^2 x/2}{x.2\sin x/2\cos x/2}=\underset{x\to0}{\text{Lim}}\frac{\sin x/2}{x\cos x/2}=\underset{x\to0}{\text{Lim}}\frac{\tan x/2}{x/2}\cdot\frac{1}{2}=1\times\frac{1}{2}=\frac{1}{2}$.

(*ix*) $\underset{x\to0}{\text{Lim}}\frac{1-\cos a\,x}{x\sin 3x}=\underset{x\to0}{\text{Lim}}\frac{2\sin^2 ax/2}{x^2}\cdot\underset{3x\to0}{\text{Lim}}\frac{3x}{\sin 3x}\times\frac{1}{3}$

$=2\underset{\frac{ax}{2}\to0}{\text{Lim}}\left(\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right)^2\cdot\frac{a^2}{4}\times1\times\frac{1}{3}=2\times1\times\frac{a^2}{4}\times\frac{1}{3}=\frac{a^2}{6}$.

Ex. 17. Evaluate the following limits:

(*i*) $\underset{x\to0}{\textbf{Lim}}\ \frac{\tan3x-2x}{3x-\sin^2 x}$ (*ii*) $\underset{x\to0}{\textbf{Lim}}\ \frac{x^3\cot x}{1-\cos x}$

Sol. (*i*) $\underset{x\to0}{\text{Lim}}\frac{\tan3x-2x}{3x-\sin^2 x}=\underset{x\to0}{\text{Lim}}\left[\frac{\frac{\tan3x}{3x}-\frac{2}{3}}{1-\left(\frac{\sin x}{3x}\right).\sin x}\right]$ (Dividing numerator and denominator by $3x$)

$=\left\{\frac{\underset{3x\to0}{\text{Lim}}\frac{\tan3x}{3x}-\frac{2}{3}}{1-\frac{1}{3}\underset{x\to0}{\text{Lim}}\frac{\sin x}{x}.\underset{x\to0}{\text{Lim}}\sin x}\right\}=\frac{1-\frac{2}{3}}{1-\frac{1}{3}\times1\times0}=\frac{1}{3}$.

(*ii*) $\underset{x\to0}{\text{Lim}}\left(\frac{x^3\cot x}{1-\cos x}\right)=\underset{x\to0}{\text{Lim}}\left\{\frac{x^3\cot x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right\}$ **(Note this step)**

$=\underset{x\to0}{\text{Lim}}\frac{x^3\cot x(1+\cos x)}{1-\cos^2 x}=\underset{x\to0}{\text{Lim}}\frac{x^3\cot x\,(1+\cos x)}{\sin^2 x}=\underset{x\to0}{\text{Lim}}\frac{x^3\cos x(1+\cos x)}{\sin^3 x}$

$=\frac{1}{\underset{x\to0}{\text{Lim}}\left(\frac{\sin x}{x}\right)^3}\cdot\underset{x\to0}{\text{Lim}}\cos x.\underset{x\to0}{\text{Lim}}(1+\cos x)=\frac{1}{1^3}\times1\times(1+1)=2$.

EXERCISE 18 (*g*)

Evaluate the following limits:

1. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

2. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$

3. $\lim\limits_{x\to 0}\dfrac{\tan \frac{1}{2}x}{3x}$

4. $\lim\limits_{x\to 0}\dfrac{\sin^2 5x}{\sin 15x}$ [*ISC*]

5. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$

6. $\lim\limits_{x\to 0}\dfrac{\sin^2 ax}{\sin^2 bx}$ [*ISC*]

7. $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x^2}$

8. $\lim\limits_{x\to 0}\dfrac{\tan ax}{\tan bx}$

9. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{2x}$

10. $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x}$ [*SC*]

11. $\lim\limits_{x\to 0}\dfrac{\sin^3 a\theta}{\sin^2 b\theta}$ [*AP*]

12. $\lim\limits_{x\to 0}\dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$ [*ISC*]

13. $\lim\limits_{x\to 0}\dfrac{\cos mx-\cos nx}{x^2}$ [*ISC*]

14. $\lim\limits_{x\to 0}\dfrac{2\sin^2 3x}{x^2}$ $\left[\textbf{Hint. } \lim\limits_{x\to 0} 18\left(\dfrac{\sin 3x}{3x}\right)^2\right]$

15. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x}$ [*ISC*]

16. $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{x^2}$

17. $\lim\limits_{x\to 0}\dfrac{1-\cos mx}{1-\cos nx}$

18. $\lim\limits_{x\to 0}\dfrac{\cos Ax-\cos Bx}{x^2}$

19. $\lim\limits_{x\to 0}\dfrac{3\sin x-\sin 3x}{x^3}$

20. $\lim\limits_{x\to 0}\dfrac{\sin 3x\cos 2x}{\sin 2x}$

21. $\lim\limits_{x\to 0}\dfrac{x^2}{1-\cos x}$

22. $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{\sin x}$ [*ISC*]

ANSWERS

1. 2 **2.** 2 **3.** $\frac{1}{6}$ **4.** 0 **5.** $\frac{a}{b}$ **6.** $\frac{a^2}{b^2}$

7. 9 **8.** $\frac{a}{b}$ **9.** 0 **10.** 0 **11.** 0 **12.** 4

13. $\frac{n^2-m^2}{2}$ **14.** 18 **15.** 0 **16.** 8 **17.** $\frac{m^2}{n^2}$ **18.** $\frac{B^2-A^2}{2}$

19. 4 **20.** $\frac{3}{2}$ **21.** 2 **22.** 2 [**Hint.** $\sin 3x = 3\sin x - 4\sin^3 x$]

Type 2. Evaluation of trigonometric limits when $x \to a$ ($a \neq 0$)

Since we have assumed that $\lim\limits_{x\to a} f(x)$ exists, so

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \qquad [\because \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x)] = \lim_{h\to 0} f(a+h).]$$

Thus, if $x \to a$ ($a \neq 0$), we proceed by replacing x by $a + h$ and $x \to a$ by $h \to 0$.

Ex. 18. Evaluate the following limits:

(*i*) $\lim\limits_{x\to\pi}\dfrac{\sin 2x}{\sin x}$ **(*ii*)** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos^2 x}{1-\sin x}$ **(*iii*)** $\lim\limits_{\theta\to\frac{\pi}{2}}(\sec\theta-\tan\theta)$ **[*IIT*]**

(*iv*) $\lim\limits_{\theta\to\frac{\pi}{2}}\left(\dfrac{2\theta-\pi}{\cos\theta}\right)$ **(*v*)** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2}$ **[*HPSB*]**

(*vi*) $\lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sin\theta-\cos\theta}{\theta-\frac{1}{4}\pi}$ **[*IIT*]** **(*vii*)** $\lim\limits_{x\to\pi}\dfrac{\sin x}{x-\pi}$ **(*viii*)** $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{1-\tan x}{x-\frac{\pi}{4}}$

(*ix*) $\lim\limits_{x\to 1}(1-x)\left(\tan\dfrac{\pi x}{2}\right)$ **[*IIT*]** **(*x*)** $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{\cot\theta}{\frac{\pi}{2}-\theta}$

(*xi*) (*a*) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$ **[*IIT*]** **(*b*)** $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}$ **(*c*)** $\lim\limits_{x\to\infty} x\tan\left(\dfrac{1}{x}\right)$

(*xii*) $\lim\limits_{x\to\infty}\sqrt{\left(\dfrac{x+\sin x}{x-\cos x}\right)}$ **is equal to**

(*i*) 0 **(*ii*) 1** **(*iii*) –1** **(*iv*) None of these** **[*Roorkee 1994*]**

Sol. (*i*) Put $x=\pi+\theta$ so that as $x\to\pi$, $\theta\to 0$

$$\therefore\quad \lim_{x\to\pi}\frac{\sin 2x}{\sin x}=\lim_{\theta\to 0}\frac{\sin 2(\pi+\theta)}{\sin(\pi+\theta)}=\lim_{\theta\to 0}\frac{\sin(2\pi+2\theta)}{\sin(\pi+\theta)}$$

$$=-\lim_{\theta\to 0}\left(\frac{\sin 2\theta}{\sin\theta}\right)=-\lim_{\theta\to 0}(2\cos\theta)=-2\lim_{\theta\to 0}(\cos\theta)=-2\times 1=-2.$$

(*ii*) Put $x=\dfrac{\pi}{2}+\theta$ so that $\theta\to 0$ as $x\to\dfrac{\pi}{2}$ $\quad\therefore\quad \lim\limits_{x\to\pi/2}\left(\dfrac{\cos^2 x}{1-\sin x}\right)=\lim\limits_{\theta\to 0}\dfrac{\cos^2\left(\frac{\pi}{2}+\theta\right)}{1-\sin\left(\frac{\pi}{2}+\theta\right)}$

$$=\lim_{\theta\to 0}\frac{\sin^2\theta}{1-\cos\theta}=\lim_{\theta\to 0}\frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=2\lim_{\theta\to 0}\cos^2\frac{\theta}{2}=2\times 1=2.$$

(*iii*) $\sec\theta-\tan\theta=\dfrac{1-\sin\theta}{\cos\theta}=\dfrac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$

$$=\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}=\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore\quad \lim_{\theta\to\pi/2}(\sec\theta-\tan\theta)=\lim_{\theta\to\pi/2}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]=\tan\left(\frac{\pi}{4}-\frac{\pi}{4}\right)=\tan 0=0.$$

(*iv*) Put $= \frac{\pi}{2} - \theta = t$; so that $2\theta - \pi = -2t$

Also when $\theta \to \pi/2$, $t \to 0$

$$\therefore \quad \lim_{\theta \to \frac{\pi}{2}} \left(\frac{2\theta - \pi}{\cos \theta} \right) = \lim_{t \to 0} \left[\frac{-2t}{\cos\left(\frac{\pi}{2} - t \right)} \right] = -2 \lim_{t \to 0} \left(\frac{t}{\sin t} \right) = -2 \times 1 = -2.$$

(*v*) Put $\frac{\pi}{2} - x = t$; so that $\pi - 2x = 2t$. Also when $x \to \frac{\pi}{2}$, $t \to 0$

$$\therefore \quad \lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{t \to 0} \frac{1 + \cos(\pi - 2t)}{4t^2} = \lim_{t \to 0} \frac{1 - \cos 2t}{4t^2}$$

$$= \lim_{t \to 0} \frac{2 \sin^2 t}{4t^2} = \frac{1}{2} \lim_{t \to 0} \left(\frac{\sin t}{t} \right)^2 = \frac{1}{2} \times 1 = \frac{1}{2}.$$

(*vi*) $$\lim_{\theta \to \frac{\pi}{4}} \frac{\sin \theta - \cos \theta}{\theta - \frac{1}{4}\pi} = \lim_{\theta \to \frac{\pi}{4}} \frac{\sqrt{2}\left(\sin \theta . \cos \frac{\pi}{4} - \cos \theta . \sin \frac{\pi}{4} \right)}{\theta - \frac{1}{4}\pi}$$

$$= \sqrt{2} \lim_{\theta \to \frac{\pi}{4}} \frac{\sin\left(\theta - \frac{\pi}{4} \right)}{\theta - \frac{\pi}{4}} = \sqrt{2} \lim_{h \to 0} \frac{\sin h}{h} = \sqrt{2}.1 = \sqrt{2} \quad \left[\text{Let } \theta - \frac{\pi}{4} = h . \text{As } \theta \to \frac{\pi}{4}, h \to 0 \right]$$

(*vii*) Put $x = (\pi + t)$ so that when $x \to \pi$, $t \to 0$. Then

$$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = \lim_{t \to 0} \frac{\sin(\pi + t)}{t} = \lim_{t \to 0} \frac{-\sin t}{t} = -1.$$

(*viii*) Put $x = \frac{\pi}{4} + t$; so that when $x \to \frac{\pi}{4}$, $t \to 0$

$$\lim_{x \to \frac{\pi}{4}} \left(\frac{1 - \tan x}{x - \frac{\pi}{4}} \right) = \lim_{t \to 0} \frac{1 - \tan\left(t + \frac{\pi}{4} \right)}{\left(t + \frac{\pi}{4} \right) - \frac{\pi}{4}} = \lim_{t \to 0} \frac{1}{t} \left[1 - \frac{1 + \tan t}{1 - \tan t} \right]$$

$$= \lim_{t \to 0} \frac{-2 \tan t}{t(1 - \tan t)} = \lim_{t \to 0} \left(\frac{-2}{1 - \tan t} . \frac{\tan t}{t} \right) = \frac{-2}{1} \times 1 = -2.$$

(*ix*) Put $x = 1 - h$; so that when $x \to 1$, $h \to 0$

$$\therefore \lim_{x \to 1} (1 - x) \tan\left(\frac{1}{2}\pi x \right) = \lim_{h \to 0} [1 - (1 - h)] \tan\left[\frac{1}{2}\pi(1 - h) \right] = \lim_{h \to 0} h \cot\left(\frac{1}{2}\pi h \right)$$

$$= \lim_{h \to 0} \frac{1}{\left(\frac{\pi}{2} \right)} . \left(\frac{\frac{1}{2}\pi h}{\sin \frac{1}{2}\pi h} \right) \cos \frac{1}{2}\pi h. \quad [\textbf{Note}] = \frac{1}{\left(\frac{\pi}{2} \right)} \times 1 \times 1 = \frac{2}{\pi}.$$

Remember

$\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$

$\lim\limits_{x\to\infty}\dfrac{\cos x}{x}=0$

(*x*) Put $\dfrac{\pi}{2}-\theta = y$ so that as $\theta\to\dfrac{\pi}{2}$, , $y\to 0$

$$\lim_{\theta\to\frac{\pi}{2}}\frac{\cot\theta}{\frac{\pi}{2}-\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{2}-\theta\right)}{\frac{\pi}{2}-\theta}=\lim_{y\to 0}\frac{\tan y}{y}=1.$$

(*xi*) (*a*) Put $x=\dfrac{1}{y}$. Then $y=\dfrac{1}{x}$ $\therefore\ y\to 0$ when $x\to\infty$

$$\therefore\quad \lim_{x\to\infty}\frac{\sin x}{x}=\lim_{y\to 0}\left(y\sin\frac{1}{y}\right)=0\ \text{[solved Ex. 15, No. 7]}$$

(*b*) Similar to (*a*). Do as in Solved Ex. 15, No. 7. **Ans. 0**

(*c*) $\lim\limits_{x\to\infty} x\tan\left(\dfrac{1}{x}\right)=\lim\limits_{y\to 0}\dfrac{1}{y}\tan y$ where $y=\dfrac{1}{x}$. As $x\to\infty, y\to 0$

$$=\lim_{y\to 0}\frac{\tan y}{y}=1$$

(*xii*) $$\lim_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)}=\lim_{x\to\infty}\sqrt{\left(\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}\right)}$$

$= \lim\limits_{x\to\infty}\sqrt{1}=1$, since $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$ and $\dfrac{\cos x}{x}$ both are equal to 0.

Type 3. Evaluation of *t*-limits by factorization

Ex. 19. Evaluate the limits:

$$\lim_{\theta\to\frac{\pi}{6}}\frac{\cot^2\theta-3}{\operatorname{cosec}\theta-2}$$

Sol. $$\lim_{\theta\to\frac{\pi}{6}}\frac{\cot^2\theta-3}{\operatorname{cosec}\theta-2}=\lim_{\theta\to\frac{\pi}{6}}\frac{(\operatorname{cosec}^2\theta-1)-3}{\operatorname{cosec}\theta-2}=\lim_{\theta\to\frac{\pi}{6}}\frac{(\operatorname{cosec}^2\theta-4)}{\operatorname{cosec}\theta-2}$$

$$=\lim_{\theta\to\frac{\pi}{6}}\frac{(\operatorname{cosec}\theta-2)(\operatorname{cosec}\theta+2)}{(\operatorname{cosec}\theta-2)}=\lim_{\theta\to\frac{\pi}{6}}(\operatorname{cosec}\theta+2)=\operatorname{cosec}\left(\frac{\pi}{6}\right)+2=2+2=4.$$

Ex. 20. If $\lim\limits_{\theta\to 0} k\theta\operatorname{cosec}\theta=\lim\limits_{\theta\to 0}\theta\operatorname{cosec} k\theta$, prove that k must be ± 1.

Sol. $$\lim_{\theta\to 0} k\theta\operatorname{cosec}\theta=k\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=k.1=k$$

$$\lim_{\theta\to 0}\theta\operatorname{cosec} k\theta=\frac{1}{k}\lim_{\theta\to 0}\frac{k\theta}{\sin k\theta}=\frac{1}{k}.1=\frac{1}{k}$$

It is given that $\lim\limits_{\theta\to 0} k\theta\operatorname{cosec}\theta=\lim\limits_{\theta\to 0}\theta\operatorname{cosec} k\theta$ $\quad\therefore\ k=\dfrac{1}{k}$ or $k^2=1\Rightarrow k=\pm 1$.

Ex. 21. Show that $\lim\limits_{x\to 0}\dfrac{|\sin x|}{x}$ does not exist. [*NDA April 2006*]

Sol. L.H.L. $= \lim\limits_{h\to 0^-}\dfrac{|\sin(0-h)|}{0-h} = \lim\limits_{h\to 0}\dfrac{|\sin h|}{-h} = \lim\limits_{h\to 0}\dfrac{\sin h}{-h} = -1$

R.H.L. $= \lim\limits_{h\to 0^+}\dfrac{|\sin(0+h)|}{0+h} = \lim\limits_{h\to 0}\dfrac{|\sin h|}{h} = \lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$

Since L.H.L. ≠ R.H.L., therefore $\lim\limits_{x\to 0}\dfrac{|\sin x|}{x}$ does not exist.

EXERCISE 18 (*h*)

1. $\lim\limits_{x\to\pi}\dfrac{1+\cos x}{x-\pi}$ **2.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2}$ [*ISC*] **3.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$

4. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\frac{\pi}{2}-x}$ [*AP*] **5.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{(\pi-2x)^2}$ **6.** $\lim\limits_{x\to\pi}\dfrac{1-\sin\frac{x}{2}}{(\pi-x)^2}$

7. $\lim\limits_{x\to 1}\dfrac{1-\frac{1}{x}}{\sin\pi(x-1)}$ **8.** $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}$ **9.** $\lim\limits_{x\to\pi}\dfrac{1+\sec^3 x}{\tan^2 x}$

ANSWERS

1. 0 **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$ **4.** 1 **5.** $\frac{1}{8}$ **6.** $\frac{1}{8}$ **7.** $\frac{1}{\pi}$

8. $2\sqrt{a}\cos a$ [**Hint.** Rationalise the denominator, apply formula for sin C – sin D and then put $x = a + h$]

9. $-\frac{3}{2}$

HINTS TO SELECTED QUESTIONS

1. Put $x = \pi + h$, when $x \to \pi$, $h \to 0$ $\therefore$ $\lim\limits_{x\to\pi}\dfrac{1+\cos x}{x-\pi} = \lim\limits_{h\to 0}\dfrac{1+\cos(\pi+h)}{h}$

2. Put $x = \dfrac{\pi}{2}+h$. Then, $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2} = \lim\limits_{h\to 0}\dfrac{1-\cos 2h}{4h^2} = \lim\limits_{h\to 0}\dfrac{2\sin^2 h}{4h^2}$

3. Put $x = \dfrac{\pi}{2}+h$, when $x \to \dfrac{\pi}{2}$, $h \to 0$ **6.** Put $x = \pi + h$. Then, $\lim\limits_{x\to\pi}\dfrac{1-\sin\frac{x}{2}}{(\pi-x)^2} = \lim\limits_{h\to 0}\dfrac{1-\sin\left(\frac{\pi}{2}+\frac{h}{2}\right)}{(-h)^2}$

7. Given limit $= \lim\limits_{x\to 1}\dfrac{x-1}{x\sin\pi(x-1)}$. Put $x = 1 + h$, when $x \to 1$, $h \to 0$

$= \dfrac{1}{\pi}\lim\limits_{h\to 0}\dfrac{\pi h}{\sin\pi h}.\lim\limits_{h\to 0}\dfrac{1}{1+h}$

18.16. Evaluation of exponential and logarithmic limits

To evaluate the exponential and logarithmic limits, we will use the following results, which we state without proof.

1. $\lim\limits_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$ **2.** $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e^x$

Note that the 2nd term and the power (index) are reciprocal of each other.

3. For all values of x, $\lim\limits_{x\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

4. $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \log_e a$ **5.** $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$ **6.** $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$

7. $\lim\limits_{x\to 1}\left(\frac{\log x}{x-1}\right) = 1$

Ex. 22. Evaluate:

(*i*) $\lim\limits_{x\to 0}(1+ax)^{1/x}$ **(*ii*)** $\lim\limits_{x\to 0}\frac{a^x-1}{b^x-1}$ **(*iii*)** $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{x}$

(*iv*) $\lim\limits_{x\to 0}\frac{a^x-b^x}{x}$ **(*v*)** $\lim\limits_{x\to 0}\frac{e^{\sin x}-1}{x}$ **[*CBSE 1992*]** **(*vi*)** $\lim\limits_{x\to\infty} 5^x \sin\left(\frac{a}{5^x}\right)$

(*vii*) $\lim\limits_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$ **[*MNREC 1994*]** **(*viii*)** $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+3}\right)^{x+3}$

(*ix*) $\lim\limits_{x\to 0}\left(\frac{1+6x^2}{1+4x^2}\right)^{\frac{1}{x^2}}$ **(*x*)** $\lim\limits_{x\to 0}\frac{3^x-2^x}{\tan x}$ **(*xi*)** $\lim\limits_{x\to e}\frac{\log x-1}{x-e}$

(*xii*) $\lim\limits_{x\to\frac{\pi}{2}}\frac{e^{\cos x}-1}{\cos x}$ **(*xiii*)** $\lim\limits_{x\to 0}\frac{10^x-2^x-5^x+1}{\sin^2 x}$

Sol. (*i*) $\operatorname{Lim}\limits_{x\to 0}(1+ax)^{1/x} = \operatorname{Lim}\limits_{x\to 0}(1+ax)^{1/x} = \operatorname{Lim}\limits_{ax\to 0}[(1+ax)^{1/ax}]^a = e^a$

(As $x\to 0$, $\operatorname{Lim}\limits_{ax\to 0}(1+ax)^{1/ax} = e$)

(*ii*) $\operatorname{Lim}\limits_{x\to 0}\frac{a^x-1}{b^x-1} = \operatorname{Lim}\limits_{x\to 0}\left[\frac{a^x-1}{x}\times\frac{x}{b^x-1}\right]$

$= \operatorname{Lim}\limits_{x\to 0}\frac{a^x-1}{x} \div \operatorname{Lim}\limits_{x\to 0}\frac{b^x-1}{x} = \log_e a \div \log_e b = \frac{\log_e a}{\log_e b} = \log_b a$

(*iii*) $\operatorname{Lim}\limits_{x\to 0}\frac{e^x-e^{-x}}{x} = \operatorname{Lim}\limits_{x\to 0}\frac{e^{2x}-1}{xe^x}$ **[Note]** $= \operatorname{Lim}\limits_{x\to 0}\frac{e^{2x}-1}{2x}\cdot\frac{2}{e^x}$

$= \operatorname{Lim}\limits_{x\to 0}\frac{e^{2x}-1}{2x}\cdot\operatorname{Lim}\limits_{x\to 0}\frac{2}{e^x} = 1\times\frac{2}{1} = 2$ Using $\operatorname{Lim}\limits_{x\to 0}\frac{e^x-1}{x} = 1.$

Alternatively. You may write expansions of e^x and e^{-x}.

(*iv*) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}=\lim\limits_{x\to 0}\dfrac{(a^x-1)-(b^x-1)}{x}=\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}-\dfrac{b^x-1}{x}\right)$

$=\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}\right)-\lim\limits_{x\to 0}\left(\dfrac{b^x-1}{x}\right)=\log_e a-\log_e b=\log_e\left(\dfrac{a}{b}\right)$

(*v*) $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{x}=\lim\limits_{x\to 0}\left(\dfrac{e^{\sin x}-1}{\sin x}\cdot\dfrac{\sin x}{x}\right)=\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\sin x}\cdot\lim\limits_{\sin x\to 0}\dfrac{\sin x}{x}=1\times 1=1$

(*vi*) Put $y=\dfrac{a}{5^x}$ so that as $x\to\infty$, $y\to 0$

$\lim\limits_{x\to\infty}5^x\sin\left(\dfrac{a}{5^x}\right)=\lim\limits_{y\to 0}\left(\dfrac{a}{y}\sin y\right)=\lim\limits_{y\to 0}\left(a\cdot\dfrac{\sin y}{y}\right)=a\times 1=a$

(*vii*) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x+3}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x+1}\right)^{x+3}=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{x+1}\right)^{x+1}\right]^{\frac{x+3}{x+1}}$

$=e^{\lim\limits_{x\to\infty}\frac{x+3}{x+1}}=e^1=e$ $\qquad\left[\because e^{\lim\limits_{x\to\infty}\frac{1+\frac{3}{x}}{1+\frac{1}{x}}}=e^1=e.\right]$

(*viii*) $\lim\limits_{x\to\infty}\left(\dfrac{x-3}{x+3}\right)^{x+3}=\lim\limits_{x\to\infty}\left[1+\dfrac{x-3}{x+3}-1\right]^{x+3}=\lim\limits_{x\to\infty}\left[1+\dfrac{x-3-x-3}{x+3}\right]^{x+3}$

$=\lim\limits_{(x+3)\to\infty}\left[1-\dfrac{6}{x+3}\right]^{x+3}=\lim\limits_{y\to\infty}\left[1+\dfrac{(-6)}{y}\right]^{y}$ on putting $x+3=y$

$=e^{-6}\left(\text{using}\lim\limits_{x\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x\right).$

(*ix*) $\lim\limits_{x\to 0}\left[\dfrac{1+6x^2}{1+4x^2}\right]^{\frac{1}{x^2}}=\lim\limits_{x\to 0}\left[1+\dfrac{1+6x^2}{1+4x^2}-1\right]^{\frac{1}{x^2}}=\lim\limits_{x\to 0}\left[1+\dfrac{1+6x^2-1-4x^2}{1+4x^2}\right]^{\frac{1}{x^2}}$

$=\lim\limits_{x\to 0}\left[1+\dfrac{2x^2}{1+4x^2}\right]^{\frac{1}{x^2}}=\lim\limits_{x\to 0}\left\{\left[1+\dfrac{2x^2}{1+4x^2}\right]^{\frac{1+4x^2}{2x^2}}\right\}^{\frac{2}{1+4x^2}}$

$=e^{\lim\limits_{x\to 0}\frac{2}{1+4x^2}}=e^2.$

(*x*) $\lim\limits_{x\to 0}\dfrac{3^x-2^x}{\tan x}=\lim\limits_{x\to 0}\dfrac{(3^x-1)-(2^x-1)}{x}\cdot\dfrac{x}{\tan x}$

$=\left[\lim\limits_{x\to 0}\dfrac{3^x-1}{x}-\lim\limits_{x\to 0}\dfrac{2^x-1}{x}\right]\cdot\lim\limits_{x\to 0}\dfrac{x}{\tan x}=(\log_e 3-\log_e 2)\cdot 1=\log_e\dfrac{3}{2}$

(*xi*) $\underset{x\to e}{\text{Lim}}\dfrac{\log x-1}{x-e}=\underset{h\to 0}{\text{Lim}}\dfrac{\log(e+h)-1}{e+h-e}=\underset{h\to 0}{\text{Lim}}\dfrac{\log(e+h)-\log_e e}{h}$

$=\underset{h\to 0}{\text{Lim}}\dfrac{\log\left(1+\dfrac{h}{e}\right)}{h}=\underset{h\to 0}{\text{Lim}}\dfrac{\log\left(1+\dfrac{h}{e}\right)}{h/e}\cdot\dfrac{1}{e}=\dfrac{1}{e}=\dfrac{1}{e}\left(\because\underset{x\to 0}{\text{Lim}}\dfrac{\log(1+x)}{x}=1\right)$

(*xii*) $\underset{x\to\frac{\pi}{2}}{\text{Lim}}\dfrac{e^{\cos x}-1}{\cos x}$. Put $\dfrac{\pi}{2}-x=y$, then, $x=\dfrac{\pi}{2}-y$ and as $x\to\dfrac{\pi}{2}$, $y\to 0$

$\therefore\ \underset{x\to\frac{\pi}{2}}{\text{Lim}}\dfrac{e^{\cos x}-1}{\cos x}=\underset{y\to 0}{\text{Lim}}\dfrac{e^{\cos\left(\frac{\pi}{2}-y\right)}-1}{\cos\left(\dfrac{\pi}{2}-y\right)}$

$=\underset{y\to 0}{\text{Lim}}\dfrac{e^{\sin y}-1}{\sin y}=\underset{\sin y\to 0}{\text{Lim}}\cdot\dfrac{e^{\sin y}-1}{\sin y}=1\quad\left[\text{Using }\underset{x\to 0}{\text{Lim}}\dfrac{e^x-1}{x}=1\right]$

(*xiii*) $\underset{x\to 0}{\text{Lim}}\dfrac{10^x-2^x-5^x+1}{\sin^2 x}=\underset{x\to 0}{\text{Lim}}\dfrac{2^x(5^x-1)-(5^x-1)}{\sin^2 x}=\underset{x\to 0}{\text{Lim}}\dfrac{(2^x-1)(5^x-1)}{\sin^2 x}$

$=\underset{x\to 0}{\text{Lim}}\dfrac{(2^x-1)(5^x-1)}{x^2}\cdot\dfrac{x^2}{\sin^2 x}=\underset{x\to 0}{\text{Lim}}\dfrac{2^x-1}{x}.\underset{x\to 0}{\text{Lim}}\dfrac{5^x-1}{x}\div\underset{x\to 0}{\text{Lim}}\left(\dfrac{\sin x}{x}\right)^2$

$=(\log 2)(\log 5)\div 1=(\log_e 2)(\log_e 5)\quad\left[\text{Using }\underset{x\to 0}{\text{Lim}}\dfrac{a^x-1}{x}=\log_e a\right]$

Ex. 23. Evaluate the following limits:

(*i*) $\underset{x\to 0}{\text{Lim}}\dfrac{\log\left(1-\dfrac{x}{2}\right)}{x}$ (*ii*) $\underset{x\to 0}{\text{Lim}}\dfrac{e^x+e^{-x}-2}{x}$

(*iii*) $\underset{x\to 2}{\text{Lim}}\dfrac{\sin(e^{x-2}-1)}{\log(x-1)}$

Sol. (*i*) $\underset{x\to 0}{\text{Lim}}\dfrac{\log_e\left(1-\dfrac{x}{2}\right)}{x}=\underset{x\to 0}{\text{Lim}}\dfrac{-\left(\dfrac{1}{2}x\right)-\dfrac{1}{2}\left(\dfrac{1}{2}x\right)^2-\dfrac{1}{3}\left(\dfrac{1}{2}x\right)^3\ldots}{x}$

$=\underset{x\to 0}{\text{Lim}}\left[-\dfrac{1}{2}-\dfrac{1}{2}\left(\dfrac{x}{4}\right)-\dfrac{1}{3}\left(\dfrac{x^2}{8}\right)\ldots\right]=-\dfrac{1}{2}-\dfrac{1}{2}(0)-\dfrac{1}{3}(0)=\ldots=-\dfrac{1}{2}.$

(*ii*) Expanding e^x and e^{-x}

$e^x+e^{-x}=\left[1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\right]+\left[1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots\right]$

$\therefore\quad\dfrac{e^x+e^{-x}-2}{x}=\underset{x\to 0}{\text{Lim}}\dfrac{2\left(\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots\right)}{x}$

$$= 2\lim_{x\to\infty}\frac{x\left[\frac{x}{2!}+\frac{x^3}{4!}+\ldots\right]}{x}$$

$$= 2\times 0 = 0$$

$$(iii)\ \lim_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(x-1)} = \lim_{x\to 2}\left[\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}\cdot\frac{e^{x-2}-1}{x-2}\cdot\frac{x-2}{\log(1+(x-2))}\right]$$

$$= \left[\lim_{x-2\to 0}\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}\cdot\lim_{x-2\to 0}\frac{e^{x-2}-1}{x-2}\cdot\lim_{x-2\to 0}\frac{x-2}{\log\{1+(x-2)\}}\right]$$

$$= 1\times 1\times 1 = 1.\left[\text{Using }\lim_{x\to 0}\frac{\sin x}{x}=1,\lim_{x\to 0}\frac{e^x-1}{x}-1,\lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$

Ex. 24. Use the formula $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$ to find $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$.

[*NMOC 1999, CBSE 1996, I.I.T.*]

Sol. $\lim_{x\to 0}\frac{2^x-1}{\sqrt{(1+x)}-1} = \lim_{x\to 0}\frac{(2^x-1)[\sqrt{(1+x)}+1]}{\sqrt{[(1+x)-1][\sqrt{(1+x)}+1]}}$, rationalizing the denominator.

$$= \lim_{x\to 0}\frac{(2^x-1)[\sqrt{(1+x)}+1]}{(1+x)-1} = \lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\left[\sqrt{(1+x)}+1\right]$$

$$= \lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\times\lim_{x\to 0}[\sqrt{(1+x)}+1] = (\log_e 2)\times 2 = 2\log_e 2.$$

Ex. 25. Find the limit of the function $f(x)$, if it exists, defined as follows:

$f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}, x\neq 0$ as x tends to zero.

Sol. R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$

$$= \lim_{h\to 0}\left[\frac{e^{1/0+h}-1}{e^{1/0+h}-1}\right] = \lim_{h\to 0}\left[\frac{1-e^{-1/h}}{1+e^{-1/h}}\right] = \frac{1-0}{1+0} = 1 \qquad (\because e^{-\infty}=0)$$

$$\text{L.H.L} = \lim_{x\to 0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{h\to 0}\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right) = \frac{0-1}{0+1} = -1$$

$\because$ R.H.L $\neq$ L.H.L $\therefore \lim_{x\to 0} f(x)$ does not exist.

EXERCISE 18 (*i*)

Evaluate the following limits:

1. (*i*) $\lim_{x\to 0}\frac{e^{4x}-1}{x}$ **(*ii*)** $\lim_{x\to 0}\frac{3^x-1}{x}$ **(*iii*)** $\lim_{x\to 0}\frac{e^x-x-1}{x}$

2. (*i*) $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos 2x}$ **(*ii*)** $\lim_{x\to 0}\frac{x(2^x-1)}{1-\cos x}$ **(*iii*)** $\lim_{x\to 0}\frac{e^x-\sin x-1}{x}$ **(*iv*)** $\lim_{x\to 0} e^x$

3. (*i*) $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{x}$ (*ii*) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{e^{\sin x}-1}{\sin x}$ (*iii*) $\lim\limits_{x\to 2}\dfrac{e^{x}-e^{2}}{x-2}$ (*iv*) $\lim\limits_{x\to 1}\dfrac{e^{x}-e}{x-1}$

4. $\lim\limits_{x\to 1}\dfrac{e^{-x}-e^{-1}}{x-1}$ **5.** $\lim\limits_{x\to 0}\dfrac{3^{x}-1}{\sqrt{1+x}-1}$ **6.** $\lim\limits_{x\to 1}x^{\frac{1}{x-1}}$

7. $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}$ **8.** $\lim\limits_{x\to 0}\dfrac{8^{x}-2^{x}}{x}$ **9.** $\lim\limits_{x\to 0}\dfrac{a^{x}-b^{x}}{\sin x}$

10. $\lim\limits_{x\to 0}\dfrac{a^{\sin x}-1}{\sin x}$ **11.** $\lim\limits_{x\to 0}\dfrac{3^{2x}-2^{3x}}{x}$ **12.** $\lim\limits_{x\to 1}\dfrac{x-1}{\log_e x}$

13. $\lim\limits_{x\to 0}\dfrac{e^{x}+e^{-x}-2}{x^{2}}$ **14.** $\lim\limits_{x\to 5}\dfrac{\log x-\log 5}{x-5}$ **15.** $\lim\limits_{x\to 1}\dfrac{e^{x}-1}{\sqrt{1-\cos x}}$

[*AICBSE 2002, 2003*]

16. (*i*) $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{2n}$ **[*CET (Karnataka) 1992*]** $\left[\textbf{Hint. } \lim\limits_{x\to\infty}\left[\left(1+\dfrac{2}{n}\right)^{\frac{n}{2}}\right]^{4}=e^{4}\right]$

(*ii*) $\lim\limits_{x\to\infty}\left[1+\dfrac{1}{a+bx}\right]^{c+dx}$ **[*EAMCET 1992*]** (*iii*) $\lim\limits_{x\to 0}(1+ax)^{\frac{b}{x}}$ **[*CET (Karnataka) 1993*]**

17. (*i*) $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$ **[*I.I.T. 1990*]** (*ii*) $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+1}\right)^{x}$ **[*MNREC 1994*]**

(*iii*) $\lim\limits_{x\to 0}\left(\dfrac{1+5x^{2}}{1+3x^{2}}\right)^{\frac{1}{x^{2}}}$ **[*NMOC 1997, I.I.T. 1996*]**

18. (*i*) $\lim\limits_{x\to 0}\dfrac{e^{ax}-1}{\sin x}$ **[*WBHS 1991*]** (*ii*) $\lim\limits_{x\to 0}\dfrac{e^{x^{2}}-1}{\sin x^{2}}$ **[*WBHS*]**

(*iii*) $\lim\limits_{x\to 0}\dfrac{3^{2x}-1}{2^{3x}-1}$ **(T.S. Rajendra)** (*iv*) $\lim\limits_{x\to 1}\dfrac{\sin(e^{x}-1)}{\log x}$

[Hint. *Type solved Ex. 23 (3)*]

19. Evaluate: $\lim\limits_{x\to 0}\dfrac{\log(a+x)-\log(a-x)}{x}, a>0$ **[*CBSE (S.P.) 2003*]**

ANSWERS

1. (*i*) 4 (*ii*) log 3 (*iii*) 0 **2.** (*i*) $\frac{1}{2}$ (*ii*) 2 log 2 (*iii*) 0 (*iv*) 1 **3.** (*i*) a – b (*ii*) $e-1$ (*iii*) e^2 (*iv*) e

4. $-\frac{1}{e}$ **5.** $2\log_e 3$ **6.** e **7.** e **8.** log 4 **9.** $\log\left(\frac{a}{b}\right)$

10. log a **11.** $\log\left(\frac{9}{8}\right)$ **12.** 1 **13.** 1 **14.** $\frac{1}{5}$ **15.** $\sqrt{2}$

16. (*i*) e^4 (*ii*) $e^{a/b}$ (*iii*) e^{ab} **17.** (*i*) e^{-2} (*ii*) e^{-2} (*iii*) e^2

18. (*i*) a (*ii*) 1 (*iii*) $\dfrac{\log_e 9}{\log_e 8}$ (*iv*) 1 **19.** $\dfrac{2}{a}$

HINTS TO SELECTED QUESTIONS

1. (*iii*) $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{x}=\lim\limits_{x\to 0}\dfrac{e^x-1}{x}-\lim\limits_{x\to 0}\dfrac{x}{x}$

2. (*i*) $\lim\limits_{x\to 0}\dfrac{x(e^x-1)}{1-\cos 2x}=\lim\limits_{x\to 0}\dfrac{x(e^x-1)}{2\sin^2 x}=\lim\limits_{x\to 0}\dfrac{x(e^x-1)}{x^2}\Big/\lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}$

(*iv*) $\lim\limits_{x\to 0}e^x=\lim\limits_{x\to 0}(e^x-1)+1=\lim\limits_{x\to 0}\dfrac{e^x-1}{x}\times\left(\lim\limits_{x\to 0}x\right)+1$

3. (*i*) $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{x}=\lim\limits_{x\to 0}\dfrac{(e^{ax}-1)-(e^{bx}-1)}{x}=a\lim\limits_{x\to 0}\dfrac{e^{ax-1}}{ax}-b\lim\limits_{x\to 0}\dfrac{e^{bx}-1}{bx}$

(*ii*) Put $\dfrac{\pi}{2}-x=y\Rightarrow x=\dfrac{\pi}{2}-y$ and as $x\to\dfrac{\pi}{2}, y\to 0$

(*iii*) Put $2-x=y\Rightarrow x=2-y$, As $x\to 2, y\to 0$

4. Put $x=1+h$ so that, as $x\to 1$, $h\to 0$

6. $\lim\limits_{x\to 1}x^{\frac{1}{x-1}}=\lim\limits_{x\to 1}[1+x-1]^{\frac{1}{x-1}}=\lim\limits_{z\to 0}(1+z)^{\frac{1}{z}}$. Putting $x-1=z$

7. $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}=\left[\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}}\right]^{\cos x}$

13. $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{x^2}=\lim\limits_{x\to 0}\dfrac{e^{2x}+1-2e^x}{x^2e^x}=\lim\limits_{x\to 0}\left(\dfrac{e^x-1}{x}\right)^2\cdot e^x$

14. $\lim\limits_{x\to 5}\dfrac{\log x-\log 5}{x-5}=\lim\limits_{h\to 0}\dfrac{\log(5+h)-\log 5}{h}=\lim\limits_{h\to 0}\dfrac{\log\left(1+\frac{h}{5}\right)}{h/5}\cdot\dfrac{1}{5}$

15. $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sqrt{1-\cos x}}=\dfrac{1}{\sqrt{2}}\cdot\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin x/2}=\dfrac{1}{\sqrt{2}}\cdot\dfrac{e^x-1}{x}\cdot\dfrac{2x/2}{\sin x/2}$

16. $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{2n}=\lim\limits_{n\to\infty}\left[\left(1+\dfrac{2}{n}\right)^{\frac{n}{2}}\right]^4$

17. Type Solved Example 22(*xiii*)

18. Type Solved Example 23(*iii*)

*18.17. Indeterminate limits

If $f(x)$ takes any of the forms $\frac{0}{0}, \frac{\infty}{\infty}, (\infty - \infty), (0 \times \infty), 0^\circ, 1^\infty$ and ∞° at $x = a$, then we say that $f(x)$ is indeterminate at $x = a$.

18.18. *L-Hospital's rule

If $\frac{f(a)}{g(a)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\underset{x\to a}{\text{Lim}} \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Lim}} \frac{f'(x)}{g'(x)}$.

Note. May do this article after studying differentiation in Chapter 19.

*Also, called Hospital's Rule.

Ex. 26. Evaluate the following limits:

(*i*) $\underset{x\to 0}{\text{Lim}} \frac{\sin x}{x}$ (*ii*) $\underset{x\to 0}{\text{Lim}} \frac{\log x}{\cot x}$ (*iii*) $\underset{x\to 0}{\text{Lim}} \frac{1-\cos mx}{1-\cos nx}$

Sol. (*i*) $\underset{x\to 0}{\text{Lim}} \frac{\sin x}{x}\left[\frac{\mathbf{0}}{\mathbf{0}}\textbf{form}\right] = \underset{x\to 0}{\text{Lim}} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)}$. [Hospital's rule]

$$= \underset{x\to 0}{\text{Lim}} \frac{\cos x}{1} = \mathbf{1}.$$

(*ii*) Since $\log 0 = \infty$ and $\cot 0 = \infty$, we have

$$\underset{x\to 0}{\text{Lim}} \frac{\log x}{\cot x}\left[\textbf{Form}\frac{\infty}{\infty}\right] = \underset{x\to 0}{\text{Lim}} \frac{\frac{d}{dx}(\log x)}{\frac{d}{dx}(\cot x)}$$ [Hospital's rule]

$$= \underset{x\to 0}{\text{Lim}} \frac{\frac{1}{x}}{-\text{cosec}^2 x} = \underset{x\to 0}{\text{Lim}} \frac{-\sin^2 x}{x}\left[\text{Form}\frac{0}{0}\right]$$

$$= \underset{x\to 0}{\text{Lim}} \frac{-2\sin x \cos x}{1} = -\underset{x\to 0}{\text{Lim}} \sin 2x = 0.$$

(*iii*) $\underset{x\to 0}{\text{Lim}} \frac{1-\cos mx}{1-\cos nx}\left(\frac{\mathbf{0}}{\mathbf{0}}\textbf{form}\right) = \underset{x\to 0}{\text{Lim}} \frac{m\sin mx}{n\sin nx} = \frac{m}{n}.\underset{x\to 0}{\text{Lim}} \frac{\sin mx}{\sin nx}$ [Hospital's Rule]

$$= \frac{m}{n}.\underset{x\to 0}{\text{Lim}}\left(\frac{\sin mx}{mx}\right)\underset{x\to 0}{\text{Lim}}\left(\frac{nx}{\sin nx}\right).\frac{mx}{nx} = \frac{m}{n}.1.1.\frac{m}{n} = \frac{\boldsymbol{m^2}}{\boldsymbol{n^2}}.$$

EXERCISE 18 (*j*)

Evaluate the following limits:

1. $\underset{x\to 0}{\text{Lim}} \frac{(1+x)^m - 1}{(1+x)^n - 1}$ **2.** $\underset{x\to 0}{\text{Lim}} \frac{\log x - \log a}{x - a}$ **3.** $\underset{x\to a}{\text{Lim}} \frac{x^a - a^a}{a^x - a^a}$

4. $\underset{x\to \pi/6}{\text{Lim}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\frac{\sqrt{3}}{2} - \cos x}$ **5.** $\underset{x\to a}{\text{Lim}} \frac{\tan x - \tan a}{\sin a - \sin x}$ **6.** $\underset{x\to 4}{\text{Lim}} \frac{3 - \sqrt{5+x}}{x - 4}$

7. $\lim_{x\to a}\frac{a^x-1}{b^x-1}$ **8.** $\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{\cos\theta}$ **9.** $\lim_{x\to\infty}\frac{x}{2^x}$

10. $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{x\cos x}$

ANSWERS

1. $\frac{m}{n}$ **2.** $\frac{1}{a}$ **3.** $\frac{1}{\log a}$ **4.** 2 **5.** $-\sec^3 a$ **6.** $-\frac{1}{6}$

7. $\log_b a$ **8.** 0 **9.** 0 **10.** 0

CHAPTER SUMMARY

1. Limit-Definition. Let $f(x)$ be a function of x. If for every positive number ε, however small it may be, there exists a positive number δ such that whenever $0 < |x - a| < \delta$ we have $|f(x) - R| < \varepsilon$ then we say $f(x)$ tends to the limit l as 'x tends to a, and write $\lim_{x\to a} f(x) = l$.

Note. The above statement means that smaller is the difference between x and a, smaller will be the difference between $f(x)$ and l.

2. Standard limits.

(1) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$, $a \neq 0$ and $n \in Q$. (2) $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n} = \frac{m}{n}a^{m-n}$, where $m, n \in Q$.

(3) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (4) $\lim_{x\to 0}\frac{\tan x}{x} = 1$ (where x is in radians)

3. L-Hospital's rule. If $\frac{f(x)}{g(x)}$ is of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ where $x = a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

CHAPTER TEST

Evaluate the following limits:

1. $\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}$ **2.** $\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}$ ***[WBHS]*** **3.** $\lim_{x\to 0}\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$ ***[AI CBSE]***

4. $\lim_{x\to 0}\frac{\tan 3x-2x}{3x-\sin^2 x}$ ***[AP, AI CBSE, WB]*** **5.** $\lim_{x\to\infty}\frac{2x^2+7x+5}{4x^2+3x-1}$ ***[CU]***

6. $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$ ***[WB JEE]*** **7.** $\lim_{x\to a}\frac{\sqrt{1+ax}-\sqrt{1-ax}}{x}$ ***[WBHS]***

8. $\lim_{x\to\pi}\frac{\sin x}{\pi-x}$ ***[AI CBSE]*** **9.** $\lim_{x\to 0}\frac{3^{2x}-1}{2^{3x}-1}$

10. $\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}$

ANSWERS

1. 2 **2.** $\frac{3}{2}$ **3.** 4 **4.** $\frac{1}{3}$ **5.** $\frac{1}{2}$ **6.** 1

7. $\frac{1}{a}(\sqrt{1+a^2}-\sqrt{1-a^2})$ **8.** 1 **9.** $\frac{\log 9}{\log 8}$ **10.** $\log\left(\frac{9}{8}\right)$

APPENDIX

(For Tetra Brilliant)

Sandwich theorem (or the pinching or squeeze theorem)

We state this important theorem without proof.

If $f(x)$, $g(x)$ and $h(x)$ are three functions such that $f(x) \le g(x) \le h(x)$ for all x in an open interval containing a except possibly at a itself, and if $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} g(x) = l$. The theorem tells us that if a function can be sandwiched between two other functions, each of which approaches the same limit, say l, as x approaches a, then the sandwiched function also approaches the same limit l as x approaches a.

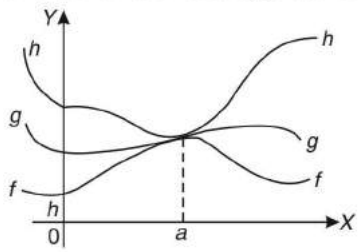


Illustration: Show that $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x^2} = 0$.

For $x \neq 0$, we have $-1 \le \sin(1/x^2) \le 1$. Since $x^2 > 0$ for all $x \neq 0$, we get $-x^2 \le x^2 \sin(1/x^2) \le x^2$ for all $x \neq 0$.

Now $\lim\limits_{x\to 0}(-x^2) = 0$ and $\lim\limits_{x\to 0} x^2 = 0$

Therefore, by the sandwich theorem, $\lim\limits_{x\to 0}\left(x^2 \sin\frac{1}{x^2}\right) = 0$.

(Not in syllabus. It is being given here as it is very useful for competitions.)

CHAPTER 19

DIFFERENTIATION

19.01. Increment

In this chapter, we will study the changes in the values of y corresponding to small changes in the values of x where y and x are related to each other by the equation $y = f(x)$.

Let $y = x^3$. When $x = 2$ we have $y = 8$.

If x changes from 2 to 2.1 then increment in $x = 2.1 - 2 = 0.1$. The corresponding value of y will be $(2.1)^3 = 9.261$ and increment in y in this case $9.261 - 8 = 1.261$.

If x changes from 2 to 0.5, then increment in $x = 0.5 - 2 = -1.5$ and the corresponding value of y is 0.125. Increment in y in this case $0.125 - 8 = -7.875$.

A small change in the value of x as it increases or decreases from one value $x = x_1$ to another value $x = x_2$ in its range is denoted by δx or Δx (read as 'delta x') and is called increment in x. Here, $\delta x = x_2 - x_1$ and we may write $x_2 = x_1 + \delta x$.

It should be noted that δx is not the product of δ and x but is merely a symbol for a small change in x. Similarly, a small change in y or increment in y is δy or Δy.

19.02. Incremental ratio

Consider the function $y = f(x)$ whose domain is R. As x changes to $(x + \delta x)$, there will be a corresponding change in y of amount, say δy.

$\therefore$ y will change from y to $(y + \delta y)$ *i.e.*, $(y + \delta y) = f(x + \delta x) \Rightarrow \delta y = f(x + \delta x) - y$

$= f(x + \delta x) - f(x)$ ...(*i*)

Thus, the increment δy in the dependent variable y depends upon the increment δx in the independent variable x.

Now dividing both sides of equation (*i*) by δx, we get $\dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$, $[\therefore \delta x \neq 0]$

The above ratio of the increment in y corresponding to the increment in x is called the *incremental ratio or difference quotient*.

Now let us study the pattern of change in the ratio $\dfrac{\delta y}{\delta x}$ corresponding to changes in x.

Let $y = x^2$. Let us take initially $x = 3$. We take $\delta x = 4, 3, 2, 1, 0.1, 0.01, \ldots$ etc. Below Table gives the corresponding values of δy and that of $\dfrac{\delta y}{\delta x}$.

Table

δx	$(x + \delta x)$	$(y + \delta y)$	δy	$\dfrac{\delta y}{\delta x}$
4	7	49	40 (= 49 – 9)	10
3	6	36	27 (= 36 – 9)	9
2	5	25	16 (= 25 – 9)	8
1	4	16	7 (= 16 – 9), etc.	7

δx	$(x + \delta x)$	$(y + \delta 9)$	δy	$\frac{\delta y}{\delta x}$
0.1	3.1	9.61	0.61	6.1
0.01	3.01	9.0601	0.0601	6.01
0.001	3.001	9.006001	0.006001	6.001
0.0001	3.0001	9.0006001	9.0006001	6.0001
h	$3 + h$	$9 + 6h + h^2$	$6h + h^2$	$6 + h$

We observe from the above table that if δx decreases and approaches 0, δy also decreases and approaches 0. But the incremental ratio doesn't approach 0. Instead, it approaches a definite value 6. In other words, we can say that $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = 6$.

19.03. Derivative of a function

The limit of the ratio $\frac{\delta y}{\delta x}$ as δx tends to zero is called the derivative or differential coefficient of y with respect to x. This limit may or may not exist.

Definition. *Let $y = f(x)$ be a continuous function of x, then*

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}, \text{ i.e., } \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x},$$

if it exists, is called the ***differential coefficient or the derivative of f (x) w.r.t. x*** *and may be denoted by any one of the following symbols:*

$$\frac{dy}{dx}, \frac{d}{dx} y, Dy, y', f'(x), \text{ or } \frac{d}{dx} f(x)$$

Caution. $\frac{dy}{dx}$ is not the quotient of two quantities dy and dx. It is just a symbol for derivative and is not read as $dy \div dx$ or "*dee y by dee x*" when expressed in the form $\frac{d}{dx}(y)$ it reads $d \div dx$ of y. The symbol $\frac{d}{dx}$ is an operator, which means rate of change w.r.t. x.

Remark. If the small increment in x is taken as h instead of δx (as is done by many authors), then the increment ratio would be $\frac{f(x + h) - f(x)}{h}$ and $f'(x)$ is written as

$f'(x)$ or $\frac{dy}{dx} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

19.04. Derivative of f (x) at the point x = a

The derivative of a function $y = f(x)$ with respect to x at the point $x = a$ is defined as $\lim\limits_{h \to 0} \frac{f(a + h) - f(a)}{h}$ provided the limit exists and is denoted by $\boldsymbol{f'(a)}$.

A function is said to be derivable or differentiable at $x = a$, if it has a derivative there. A function is said to be differentiable on an interval if it is differentiable at every point of the interval.

For the existence of derivative, the limit must be the same whether h approaches zero from the right or from the left. If these limits are different, the derivative will not exist and the function will not be derivable.

The function is not derivable, i.e., $f'(x)$ does not exist if either of the left handed or right handed limits

(i) does not exist, or

(ii) both exist but have different values.

Note. If $f(x)$ possesses derivative at every point of the interval (a, b), then it is said to be derivable in the interval (a, b).

Ex. 1. If $f(x) = \dfrac{3+x}{3-x}$ and $x \neq 3$, then find from definition $f'(2)$. [*Aligarh Engg.*]

Sol. By definition,

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\dfrac{3+(2+h)}{3-(2+h)} - \dfrac{3+2}{3-2}}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{5+h}{1-h} - \dfrac{5}{1}}{h} = \lim_{h\to 0}\frac{6h}{h(1-h)} = \lim_{h\to 0}\frac{6}{1-h} = \frac{6}{1-0} = 6$$

Ex. 2. Find $f'(5)$, for every value of x and y, $f(x+y) = f(x)\,f(y)$ and $f(5) = 2, f'(0) = 3$. [*IIT*]

Sol. It is given that $f(x+y) = f(x)\,f(y)$.

Putting $x = 5$ and $y = 0, f(5+0) = f(5)\,f(0) \quad \Rightarrow \quad f(5) = f(5)\,.\,f(0)$

$\Rightarrow \quad f(0) = 1 \quad (\because f(5) = 2 \neq 0)$

By definition of the derivative,

$$f'(5) = \lim_{h\to 0}\frac{f(5+h) - f(5)}{h} = \lim_{h\to 0}\frac{f(5)\,f(h) - f(5)}{h} \quad [\because f(x+y) = f(x)\,f(y)$$

$$\therefore f(5+h) = f(5)\,f(h)]$$

$$= f(5)\,.\left[\lim_{h\to 0}\frac{f(h)-1}{h}\right] = f(5)\,.\left[\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}\right]$$

$= 2.f'(0) = 2 \times 3 = 6$ (Putting $f(5) = 2$, given)

Ex. 3. Find the derivative of the following at $x = 1$.

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2-7x+5} & \text{when } x \neq 1 \\ -\dfrac{1}{3} & \text{when } x = 1 \end{cases}$$ [*IIT*]

Sol. By definition, $f'(1) = \lim_{h\to 0}\dfrac{f(1+h) - f(1)}{h}$ ...(*i*)

When $x \neq 1, f(1+h) = \dfrac{(1+h)-1}{2(1+h)^2 - 7(1+h) + 5} = \dfrac{h}{2h^2 - 3h}$. When $x = 1$, $f(1) = -\dfrac{1}{3}$

$\therefore$ From (*i*), $f'(1) = \lim_{h\to 0}\dfrac{\left(\dfrac{h}{2h^2-3h}\right) - \left(-\dfrac{1}{3}\right)}{h}$

$$= \lim_{h\to 0}\frac{3h + (2h^2 - 3h)}{3h(2h^2 - 3h)} = \lim_{h\to 0}\frac{2}{3(2h-3)} = \frac{2}{3(0-3)} = -\frac{2}{9}.$$

19.05. Physical interpretation of the derivative

Let a particle move along a line and let us measure its displacement x from a point O on the line. Since the displacement x depends on time t, we may write $x = f(t)$.

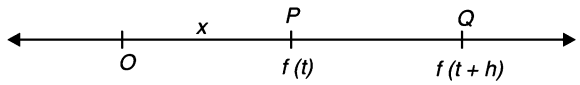


Fig. 19.01

Displacement $OP = x$ at time $t = f(t)$, Displacement PQ at time $(t + h) = f(t + h)$

$\therefore$ Displacement in time $(t + h) - t = h = f(t + h) - f(t)$

$\therefore$ Average velocity of the particle in the interval $h = \dfrac{f(t + h) - f(t)}{h}$

Now as we make h smaller and smaller, we obtain the average velocity in smaller and smaller intervals near P. The limit of the average velocity as $h \to 0$ represents the instantaneous velocity of the particle at time t.

$\therefore$ Velocity (v) at time $t = \underset{h \to 0}{\text{Lim}} \dfrac{f(t + h) - f(t)}{h}$. It is denoted by $\dfrac{dx}{dt}$.

Thus, the derivative is related to a function in the same way as velocity is related to the distance travelled by a moving particle.

19.06. Geometrical interpretation of $\dfrac{dy}{dx}$

Let $y = f(x)$ be a continuous function and let its graph be as shown in the figure.

Let P and Q be two points on the curve whose coordinates are (x, y) and $(x + \delta x, y + \delta y)$.

From P and Q draw PM and QN perpendiculars on the x-axis and PR perp. on NQ.

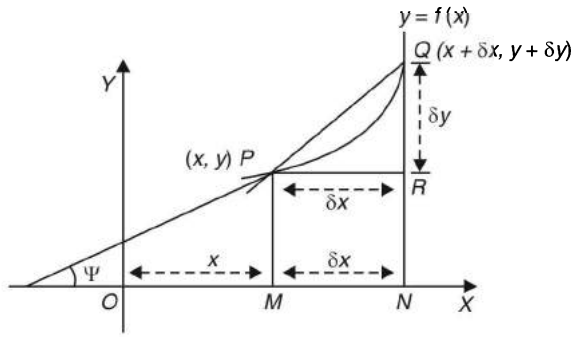


Fig. 19.02

From the figure, $\tan\angle QPR = \dfrac{\delta y}{\delta x}$ ...(*i*)

It is obvious that as $\delta x \to 0$, $Q \to P$. In the limiting position the secant PQ becomes the tangent at P and $\angle QPR$ will tend to angle ψ which the tangent at P makes with the positive direction of the x-axis.

Thus in the limit as $Q \to P$, $\delta x \to 0$ and (*i*) becomes $\underset{8x \to 0}{\text{Lim}} \dfrac{\delta y}{\delta x} = \underset{Q \to P}{\text{Lim}} \tan\angle QPR$

or $\dfrac{dy}{dx} = \tan\psi = \text{slope of the tangent at } P(x, y).$

Hence, the derivative at any point P is the tangent of the angle that the tangent to the curve at P makes with the positive direction of the x-axis.

Cor. 1. If the tangent is parallel to the x-axis, then $\psi = 0 \quad \therefore \quad \frac{dy}{dx} = \tan\psi = \tan 0 = 0.$

Cor. 2. If the tangent is perpendicular to the x-axis, then $\psi = 90°$ and $\cot\psi = 0$ and, therefore, $\frac{dx}{dy} = 0.$

DIFFERENTIATION

19.07. Differentiation from first principles

The process of finding the differential coefficient of a function $f(x)$ w.r.t. the independent variable x is called differentiation.

The process consists of the following steps:

Step 1. *Put the function to be differentiated equal to y, i.e., let* $y = f(x)$ ... (i)

Step 2. *Let* δx *be an increment in the value of x and* δy*, the corresponding increment in the value of y so that* $y + \delta y = f(x + \delta x)$. ...$(ii)$

Step 3. *Find* δy *by subtracting (i) from (ii), i.e.,* $\delta y = f(x + \delta x) - f(x)$.

Step 4. *Find the ratio* $\frac{\delta y}{\delta x}$ *by dividing both sides by* δx, *i.e.,* $\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$

***Step 5.** *Take the limits of both sides as* $\delta x \to 0$. *This gives* $\frac{dy}{dx}$ *or* $f'(x)$.

$$\frac{dy}{dx} \text{ or } f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

The above process of differentiation (by applying all the five steps) is called **differentiation from definition or from first principles or abinitio, or delta method.**

Stated in words, the various **steps** of working are:

1. *Give the independent variable an increment.*
2. *Find the corresponding increment of the function.*
3. *Write the ratio of the two increments.*
4. *Find the limit of this ratio as the increment of the independent variable approaches zero.*

Ex. 4. Differentiate the following functions with respect to x from definition.

(*i*) x^2 **(*ii*)** $\sqrt{x}$ **(*iii*)** $\frac{1}{x^3}$ **(*iv*)** $\sqrt{ax + b}$ **(*v*)** $\frac{ax + b}{ax + d}$

Sol. (*i*) Let $y = x^2$ ...(i)

Let δx be an increment in the value of x and δy, the corresponding increment in the value of y so that $y + \delta y = (x + \delta x)^2$ ...(ii)

Subtracting (i) from (ii), $\delta y = (x + \delta x)^2 - x^2 = 2x\,\delta x + (\delta x)^2$

Dividing both sides by δx, $\frac{\delta y}{\delta x} = 2x + \delta x$

Taking limits as $\delta x \to 0$, $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x + 0 = 2x$ $\boxed{\therefore \frac{d}{dx}(x^2) = 2x}$

---

* It should be noted that when $\delta x \to 0$, then $\delta y \to 0$ also. Therefore $\frac{\delta y}{\delta x}$ takes the form $\frac{0}{0}$, which is indeterminate but not meaningless.

(*ii*) Let $y = \sqrt{x}$ ...(*i*)

Let δx be an increment in the value of x and δy, the corresponding increment in the value of y so that $y + \delta y = \sqrt{x + \delta x}$ ...(*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \sqrt{x + \delta x} - \sqrt{x} = \dfrac{\sqrt{x + \delta x} - \sqrt{x}}{\sqrt{x + \delta x} + \sqrt{x}} \times (\sqrt{x + \delta x} + \sqrt{x})$

[Note this step]

$$= \frac{(x + \delta x) - x}{\sqrt{x + \delta x} + \sqrt{x}} = \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}} \qquad \left[\because (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + 0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad \boxed{\therefore \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}}.$$

(*iii*) Let $y = \dfrac{1}{x^3}$ ...(*i*)

Let δx be the increment in the value of x and δy the corresponding increment in the value of y, so that $y + \delta y = \dfrac{1}{(x + \delta x)^3}$. ...(*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \dfrac{1}{(x + \delta x)^3} - \dfrac{1}{x^3} = \dfrac{x^3 - (x + \delta x)^3}{(x + \delta x)^3 \; x^3}$

$$= \frac{x^3 - [x^3 + 3x^2\,\delta x + 3x\,.\,(\delta x)^2 + (\delta x)^3]}{(x + \delta x)^3\,.\,x^3} = \frac{-[3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3]}{(x + \delta x)^3\,.\,x^3}$$

Dividing both sides by δx, $\quad \therefore \quad \dfrac{\delta y}{\delta x} = \dfrac{-3x^2 - 3x\,(\delta x) - (\delta x)^2}{(x + \delta x)^3\; x^3}$

Taking limits as $\delta x \to 0$, $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{-3x^2}{x^3\,.\,x^3} = \dfrac{-3}{x^4}$ $\qquad \boxed{\therefore \dfrac{d}{dx}\left(\dfrac{1}{x^3}\right) = \dfrac{-3}{x^4}}.$

(*iv*) Let $y = \sqrt{ax + b}$ ...(*i*)

Let δx be an increment in the value of x and δy the corresponding increment in the value of y, so that $y + \delta y = \sqrt{a\,(x + \delta x) + b}$ ...(*ii*)

Subtracting (*i*) from (*ii*), $\quad \delta y = \sqrt{(ax + a\delta x + b)} - \sqrt{ax + b}$

$$= \frac{\sqrt{ax + a\delta x + b} - \sqrt{ax + b}}{\sqrt{ax + a\delta x + b} + \sqrt{ax + b}} \times (\sqrt{ax + a\delta x + b} + \sqrt{ax + b})$$

$$= \frac{(ax + a\delta x + b) - (ax + b)}{\sqrt{ax + a\delta x + b} + \sqrt{ax + b}} = \frac{a\delta x}{\sqrt{ax + a\delta x + b} + \sqrt{ax + b}}$$

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{a}{\sqrt{ax + a\delta x + b} + \sqrt{ax + b}}$

Taking limits as $\delta x \to 0$, $\dfrac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}} \dfrac{\delta y}{\delta x} = \dfrac{a}{\sqrt{ax + b} + \sqrt{ax + b}} = \dfrac{a}{2\sqrt{ax + b}}$

$\therefore$ $\boxed{\dfrac{d}{dx}(\sqrt{ax + b}) = \dfrac{a}{2\sqrt{ax + b}}.}$

Remember: $\dfrac{d}{dx}\sqrt{\text{A linear function of } x} = \dfrac{1}{2\sqrt{\text{that linear function}}} \times \text{coefft. of } x$

(v) Let $y = \dfrac{ax + b}{ax + d}$ ... (*i*)

Let δx be an increment in the value of x and δy, the corresponding increment in the value of y so that $y + \delta y = \dfrac{a(x + \delta x) + b}{c(x + \delta x) + d}$ ...(*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \dfrac{a(x + \delta x) + b}{c(x + \delta x) + d} - \dfrac{ax + b}{cx + d}$

$= \dfrac{(ax + a\delta x + b)(cx + d) - (ax + b)(cx + c\delta x + d)}{(cx + c\delta x + d)(cx + d)} = \dfrac{(ad - bc)\,\delta x}{(cx + c\delta x + d)(cx + d)}$

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{(ad - bc)}{(cx + c\delta x + d)(cx + d)}$

Taking limits as $\delta x \to 0$, $\dfrac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}} \dfrac{\delta y}{\delta x} = \dfrac{ad - bc}{(cx + d)(cx + d)} = \dfrac{ad - bc}{(cx + d)^2}$

$\therefore$ $\dfrac{d}{dx}\left(\dfrac{ax + b}{cx + d}\right) = \dfrac{ad - bc}{(cx + d)^2}.$

Ex. 5. Using first principles, prove that $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{-f'(x)}{\{f(x)\}^2}.$ **[*Pb*]**

Sol. Let $\phi(x) = \dfrac{1}{f(x)}$

Then, $\dfrac{d}{dx}(\phi(x)) = \underset{h \to 0}{\text{Lim}} \dfrac{\phi(x + h) - \phi(x)}{h} = \underset{h \to 0}{\text{Lim}} \dfrac{\dfrac{1}{f(x + h)} - \dfrac{1}{f(x)}}{h}$

$= \underset{h \to 0}{\text{Lim}} \dfrac{f(x) - f(x + h)}{hf(x)\,f(x + h)} = \underset{h \to 0}{\text{Lim}} \dfrac{f(x) - f(x + h)}{h} \cdot \underset{h \to 0}{\text{Lim}} \dfrac{1}{f(x)\,f(x + h)}$

$= -\underset{h \to 0}{\text{Lim}} \dfrac{f(x + h) - f(x)}{h} \cdot \underset{h \to 0}{\text{Lim}} \dfrac{1}{f(x)\,f(x + h)}$

$= -f'(x) \cdot \dfrac{1}{f(x)\,f(x)}$ [$f(x)$ is differentiable $\Rightarrow f(x)$ is continuous $\Rightarrow \underset{h \to 0}{\text{Lim}} f(x + h) = f(x)$]

$= \dfrac{-f'(x)}{\{f(x)\}^2}.$

EXERCISE 19 (*a*)

Differentiate from first principles:

1. $2x$ **2.** $(x-1)^2$ **3.** x^3 **4.** $\frac{1}{\sqrt{x}}$ **5.** $\sqrt{x+1};\ x>-1$ **[*HSB*]**

6. $\frac{2x+3}{3x+2}$ **[*HSB*]** **7.** $\frac{1}{\sqrt{x+a}}$ **8.** $x+\frac{1}{x}$ **9.** $\sqrt{2x+3}$ **10.** $\frac{1}{x^{\frac{3}{2}}}$

11. $(x+1)(2x-3)$ **12.** $\frac{x^2+1}{x}$

ANSWERS

1. 2 **2.** $2(x-1)$ **3.** $3x^{2}$ **4.** $\frac{-1}{2x^{3/2}}$ **5.** $\frac{1}{2\sqrt{x+1}}$

6. $\frac{-5}{(3x+2)^2}$ **7.** $\frac{-1}{2(x+a)^{3/2}}$ **8.** $1-\frac{1}{x^2}$ **9.** $\frac{1}{\sqrt{2x+3}}$ **10.** $-\frac{3}{2}x^{-5/2}$

11. $4x-1$ **12.** $1-\frac{1}{x^2}$

19.8. Differential coefficient of x^n and $(ax+b)^n$, $x>0$ and $x\in R$, n is an integer or a rational number

First Method. Let $y=x^n$, then $y+\delta y=(x+\delta x)^n \Rightarrow \delta y=(x+\delta x)^n-x^n$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{(x+\delta x)^n-x^n}{\delta x} \quad \Rightarrow \quad \frac{\delta y}{\delta x}=\frac{(x+\delta x)^n-x^n}{(x+\delta x)-x}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(x+\delta x)^n-x^n}{(x+\delta x)-x}=\lim_{\delta x\to 0}\frac{z^n-x^n}{z-x} \text{ where } z=x+\delta x \text{ and } z\to x \text{ as } \delta x\to 0$$

$$=nx^{n-1}.$$

Second Method

(*i*) Let $y=x^n$ ....(*i*)

Let δx be increment in x and δy, the corresponding increment in y so that

$$y+\delta y=(x+\delta x)^n \qquad ....(ii)$$

Subtracting (*i*) from (*ii*), $\delta y=(x+\delta x)^n-x^n=x^n\left(1+\frac{\delta x}{x}\right)^n-x^n=x^n\left[\left(1+\frac{\delta x}{x}\right)^n-1\right]$

As δx is small, $\frac{\delta x}{x}$ may be considered less than 1 numerically, therefore, applying Binomial theorem for any index, we get

$$\delta y=x^n\left[\left\{1+n.\frac{\delta x}{x}+\frac{n(n-1)}{1.2}.\left(\frac{\delta x}{x}\right)^2+\text{terms containing higher powers of } \delta x\right\}-1\right]$$

$$=x^n.\,\delta x\left[\frac{n}{x}+\frac{n(n-1)}{1.2}.\frac{\delta x}{x^2}+\text{terms containing } (\delta x)^2 \text{ higher powers of } \delta x\right]$$

Dividing both sides by δx,

$$\frac{\delta y}{\delta x} = x^n\left[\frac{n}{x} + \frac{n(n-1)}{1.2}\cdot\frac{\delta x}{x^2} + \text{terms containing } (\delta x)^2 \text{ and higher powers of } \delta x\right]$$

Taking limits as $\delta x \to 0$, $\dfrac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}}\ \dfrac{\delta y}{\delta x} = x^n \cdot \dfrac{n}{x} = n x^{n-1}$

Hence, $$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}\ .$$

Remark. In the above proof, we have considered the exponent as an integer or a rational number. But the formula can be extended to the case when n is irrational, *i.e.*, when n is a real number. (For proof see appendix)

Thus, we have $\dfrac{d}{dx}(x^n) = nx^{n-1}$, $x \in R, n \in R$ and $x > 0$

Aid to memory:

Diff. Coeff. of a (variable)$^{\text{constant}}$ = constant . (variable)$^{\text{constant}-1}$

For example, $\dfrac{d}{dx}(x^7) = 7x^6, \dfrac{d}{dx}(x^{5/2}) = \dfrac{5}{2}\cdot x^{\frac{5}{2}-1} = \dfrac{5}{2}x^{3/2}$, $\dfrac{d}{dx}(x^{-6}) = -6x^{-6-1} = -6x^{-7}$

$$\boldsymbol{\frac{d}{dx}(x) = 1.x^{1-1} = x^0 = 1} \qquad \textbf{(Note)}$$

$$\boldsymbol{\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1.(x^{-1-1}) = -1.x^{-2} = -\frac{1}{x^2}}. \qquad \textbf{(Note)}$$

$$\boldsymbol{\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}.x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}}. \qquad \textbf{(Note)}$$

Remember:

$$\boldsymbol{\frac{d(x)}{dx} = 1,\ \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2},\ \frac{1}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}}$$

(*ii*) Let $y = (ax + b)^n$ ...(*i*)

Let δx be the increment in x and δy, the corresponding increment in y so that

$$y + \delta y = [a\ (x + \delta x) + b]^n \qquad ...(ii)$$

Subtracting (*i*) from (*ii*), $\delta y = (ax + a\delta x + b)^n - (ax + b)^n$

$$= (ax+b)^n\left(1+\frac{a\delta x}{ax+b}\right)^n - (ax+b)^n = (ax+b)^n\left[\left(1+\frac{a\delta x}{ax+b}\right)^n - 1\right]$$

$$= (ax+b)^n\left[\left\{1+n.\frac{a\delta x}{ax+b}+\frac{n(n-1)}{1.2}\left(\frac{a\delta x}{ax+b}\right)^2\right.\right.$$

$$\left.\left. + \text{terms containing higher powers of } \delta x\right\} - 1\right]$$

$$= (ax+b)^n\,\delta x\left[\frac{na}{ax+b}+\frac{n(n-1)}{1.2}.\frac{a^2\delta x}{(ax+b)^2} + \text{terms containing higher powers of } \delta x\right]$$

Dividing both sides by δx,

$$\frac{\delta y}{\delta x} = (ax+b)^n \left[\frac{na}{ax+b} + \frac{n(n-1)}{1.2} . \frac{a^2 \delta x}{(ax+b)^2} + \text{terms containing higher powers of } \delta x \right]$$

Taking limits as $\delta x \to 0$, $\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = (ax+b)^n \frac{na}{ax+b} = na(ax+b)^{n-1}$

$$\boxed{\frac{d}{dx}(ax+b)^n = n.(ax+b)^{n-1}.a}$$

Aid to memory : $\frac{d}{dx}$ (a linear function of $x)^n = n$ (linear function)$^{n-1}$ .Coefft. of x.

Caution : Do not forget to multiply by the coefft. of *x*.

For example, $\frac{d}{dx}(5x+7)^{10} = 10(5x+7)^9 \times 5 = 50(5x+7)^9$.

$$\frac{d}{dx}(\sqrt{1-8x}) = \frac{d}{dx}[(1-8x)^{1/2}] = \frac{1}{2}(1-8x)^{1/2-1}(-8) = -4(1-8x)^{-1/2} = -\frac{4}{\sqrt{1-8x}}.$$

19.09. General theorems on differentiation

Theorem 1. ***The differential coefficient of a constant is zero, i.e.,*** $\frac{dc}{dx} = 0$ ***,where c is a constant.***

Let $y = c$, where c is a constant. Let there be a small increment δx in x. Corresponding to this increment in x, let there be an increment δy in y. Then

$$y+\delta y = c \Rightarrow \delta y = 0 \Rightarrow \frac{\delta y}{\delta x} = 0 \Rightarrow \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lim}}\ 0 = 0$$

Hence $$\boxed{\frac{d}{dx}(c) = 0}$$

For example, $\frac{d}{dx}(7) = 0$, $\frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$.

Remark. The above result shows that the rate of change of a constant quantity is zero and this is in agreement with our intuitive definition of a constant quantity.

Theorem 2. ***An additive constant disappears in differentiation, i.e.,***

$$\frac{d}{dx}[f(x)+c] = \frac{d}{dx}[f(x)], \quad \textit{where c is a constant.}$$

Let $y = f(x) + c$

Let there be a small increment δx in value of x. Corresponding to this increment, let there be the increment δy in the value of y. Then $y + \delta y = f(x + \delta x) + c$

$$\Rightarrow \quad \delta y = f(x+\delta x)+c-[f(x)+c] = f(x+\delta x)-f(x) \therefore \frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore \quad \underset{\delta x \to 0}{\text{Lim}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$$

Hence, $$\boxed{\frac{dy}{dx} \text{ or } \frac{d}{dx}[f(x)+c] = \frac{d}{dx}(f(x))}$$

For example, $\dfrac{d}{dx}(x^7+10) = \dfrac{d}{dx}(x^7) = 7x^6$.

Cor. Functions whose differential coefficients are equal may differ by a constant.

Theorem 3. ***The differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function i.e.,*** $\dfrac{d}{dx}[kf(x)] = k\dfrac{d}{dx}[f(x)]$.

Let $y = [kf(x)]$ be a function of x.

Let there be a small increment δx in the value of x. Corresponding to this increment, let there be the increment δy in the value of y. Then $y + \delta y = k[f(x + \delta x)]$

$$\therefore \qquad \delta y = k[f(x+\delta x) - f(x)] \Rightarrow \frac{\delta y}{\delta x} = k.\frac{f(x+\delta x)-f(x)}{\delta x}$$

Taking limits as $\delta x \to 0$, $\underset{\delta x \to 0}{\text{Lim}} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx} = \underset{\delta x \to 0}{\text{Lim}}\, k.\dfrac{f(x+\delta x)-f(x)}{\delta x}$

Now we know that $\text{Lim}\, kf(x) = k.\,\text{Lim}\, f(x)$, therefore

$$\frac{dy}{dx} = k.\underset{\delta x \to 0}{\text{Lim}}\frac{f(x+\delta x)-f(x)}{\delta x} = k.\frac{d}{dx}f(x)$$

Hence $$\boxed{\frac{dy}{dx} = \frac{d}{dx}[kf(x)] = k.\frac{d}{dx}f(x)}$$

For example, if $y = 6x^3$, then $\dfrac{dy}{dx} = 6\dfrac{d}{dx}(x^3) = 6 \times 3x^2 = 18x^2$.

Theorem 4. ***The differential coeffficient of the sum of two functions is equal to the sum of the differential coefficients of these functions,***

i.e., $\dfrac{d}{dx}(u+v) = \dfrac{d}{dx}(u) + \dfrac{d}{dx}(v)$

where u and v are functions of x whose differential coefficients exist.

Let $y = u + v$

Let there be a small increment δx in the value of x.

Let the corresponding increments in u, v and y be δu, δv and δy respectively. As $\delta x \to 0$, so δu, δv and δy all approach zero.

Now $y + \delta y = u + \delta u + v + \delta v \;\therefore\; \delta y = u + \delta u + v + \delta v - (u+v)$

$$\Rightarrow \qquad \delta y = \delta u + \delta v \qquad \Rightarrow \qquad \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

Taking limits as $\delta x \to 0$, $\underset{\delta x \to 0}{\text{Lim}}\dfrac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lim}}\left(\dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta x}\right)$

or $$\underset{\delta x \to 0}{\text{Lim}}\frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lim}}\frac{\delta u}{\delta x} + \underset{\delta x \to 0}{\text{Lim}}\frac{\delta v}{\delta x} \qquad \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Hence, $$\boxed{\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)}$$

Cor. $\dfrac{d}{dx}(u-v) = \dfrac{du}{dx} - \dfrac{dv}{dx}$

Generalized form: $\frac{d}{dx}(u+v+w+....) = \frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+....$

Ex. 6. (*i*) Let $y = ax^3 + bx^2 + cx + d$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(ax^3)+\frac{d}{dx}(bx^2)+\frac{d}{dx}(cx)+\frac{d}{dx}(d) = a\frac{d}{dx}(x^3)+b\frac{d}{dx}(x^2)+c\frac{d}{dx}(x)+0$$

$$= a.(3x^2)+b(2x)+c.1 = 3ax^2+2bx+c.$$

(*ii*) Let $y = x+\frac{1}{x}$. Then $\frac{dy}{dx} = \frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{1}{x}\right) = 1-\frac{1}{x^2}$.

(*iii*) Let $y = (x-1)(x-2) = x^2-3x+2$. Then $\frac{dy}{dx} = \frac{d}{dx}(x^2-3x+2) = 2x-3$

(*iv*) Let $y = x^{\frac{2}{3}}+2x^2$. Then $\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{2}{3}})+\frac{d}{dx}(2x^2) = \frac{2}{3}x^{\frac{2}{3}-1}+2(2x) = \frac{2}{3}x^{\frac{-1}{3}}+4x.$

(*v*) Let $y = \frac{(x^2+1)(x+3)}{x} = \frac{x^3+3x^2+x+3}{x} = x^2+3x+1+\frac{3}{x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+3x+1+\frac{3}{x}\right) = 2x+3+0-\frac{3}{x^2} = 2x+3-\frac{3}{x^2}.$$

(*vi*) Let $y = \left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2x.\frac{1}{x} = x^2+\frac{1}{x^2}+2$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)+\frac{d}{dx}(x^{-2})+\frac{d}{dx}(2) = 2x-2x^{-3}+0 = 2x-\frac{2}{x^3}.$$

(*vii*) Let $y = \frac{(2x-1)(5x^{\frac{1}{2}}+7)}{x^{\frac{1}{2}}}$

$$= \frac{10x^{\frac{3}{2}}+14x-5x^{\frac{1}{2}}-7}{x^{\frac{1}{2}}} = 10x+14x^{\frac{1}{2}}-5-7x^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = \frac{d}{dx}(10x+14x^{\frac{1}{2}}-5-7x^{\frac{-1}{2}})$$

$$= 10\frac{d}{dx}(x)+14\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\frac{d}{dx}(5)-7\frac{d}{dx}\left(x^{\frac{-1}{2}}\right)$$

$$= 10\times 1+14\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)-0-7\left(-\frac{1}{2}x^{\frac{-3}{2}}\right) = 10+7x^{\frac{-1}{2}}+\frac{7}{2}x^{\frac{-3}{2}}.$$

EXERCISE 19 (*b*)

Differentiate the following functions:

1. $(ax)^m + b^m$ **2.** $x^3 + 4x^2 + 7x + 2$ **3.** $7x^6 + 8x^5 - 3x^4 + 11x^2 + 6x + 7$

4. $3 + 4x - 7x^2 - \sqrt{2}\,x^3 + \pi x^4 - \frac{2}{5}x^5 + \frac{4}{3}$ **5.** $\frac{3}{x^5}$ **6.** $x^{\frac{5}{3}}$

7. $\frac{7}{x^{\frac{2}{3}}}$ **8.** $(\sqrt{x} + \frac{1}{\sqrt{x}})^2, x \neq 0$ **9.** $\sqrt{x} - \frac{1}{\sqrt{x}}, x \neq 0$ **10.** $\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3}$

11. $2x^{\frac{1}{2}} + 6x^{\frac{1}{3}} - 2x^{\frac{3}{2}}$ **12.** $8x^3 - x^2 + 5 - \frac{2}{x} + \frac{4}{x^3}$ **13.** $\frac{3x^7 + x^5 - 2x^4 + x - 3}{x^4}$

14. (*i*) $(2x-3)^2$ (*ii*) $(2x-3)^{100}$ **15.** $\sqrt{3x+2.}$ **16.** Given $f(x) = \frac{7}{4}x^2$, find $f'\left(\frac{1}{7}\right)$

17. Find the derivative with respect to *x* of the following:

(*i*) $x - \frac{1}{x}$ **[*SC*]** (*ii*) $\sqrt{x} + \frac{1}{\sqrt{x}}$ **[*SC*]** (*iii*) $3x^2 + \frac{3}{x^2}$ **[*SC*]**

(*iv*) $\frac{x^2+1}{x}$ **[*SC*]** (*v*) $\frac{2x+x^4}{x^2}$ **[*SC*]** (*vi*) $\frac{1+x^2}{x^3}$ **[*SC*]**

18. If $y = x + \frac{1}{x}$, prove that $x^2\frac{dy}{dx} - xy + 2 = 0$

19. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, prove that $2x\frac{dy}{dx} + y = 2\sqrt{x}.$

20. If $y = \frac{1}{a-z}$, show that $\frac{dz}{dy} = (z-a)^2$

21. If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\text{to}\,\infty$, show that $\frac{dy}{dx} = y.$

ANSWERS

1. $ma^m x^{m-1}$ **2.** $3x^2 + 8x + 7$ **3.** $42x^5 + 40x^4 - 12x^3 + 22x + 6$

4. $4 - 14x - 3\sqrt{2}x^2 + 4\pi x^3 - 2x^4$ **5.** $\frac{-15}{x^6}$ **6.** $\frac{5}{3}x^{\frac{2}{3}}$

7. $\frac{-14}{3x^{\frac{5}{3}}}$ **8.** $1 - \frac{1}{x^2}$ **9.** $\frac{1}{2\sqrt{x}} + \frac{1}{2x^{\frac{3}{2}}}$ **10.** $\frac{-1}{x^2} - \frac{6}{x^3} - \frac{6}{x^4}$

11. $\frac{1}{x^{1/2}} + \frac{2}{x^{2/3}} - 3x^{1/2}$ **12.** $24x^2 - 2x + \frac{2}{x^2} - \frac{12}{x^4}$ **13.** $9x^2 + 1 - \frac{3}{x^4} + \frac{12}{x^5}$

14. (*i*) $8x - 12$, (*ii*) $200\,(2x-3)^{99}$ **15.** $\frac{3}{2\sqrt{3x+2}}$ **16.** $\frac{1}{2}$

17. (*i*) $1 + \frac{1}{x^2}$ (*ii*) $\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$ (*iii*) $6x - \frac{6}{x^3}$ (*iv*) $1 - \frac{1}{x^2}$ (*v*) $\frac{-2}{x^2} + 2x$ (*vi*) $\frac{-3}{x^4} - \frac{1}{x^2}$

HINTS TO SELECTED QUESTIONS

18. $y=x+\frac{1}{x};\ \Rightarrow \frac{dy}{dx}=1-\frac{1}{x^2}=\frac{x^2-1}{x^2}=\frac{x^2+1-2}{x^2}=\frac{x^2+1}{x^2}-\frac{2}{x^2}$

$$=\frac{1}{x}\left(\frac{x^2+1}{x}\right)-\frac{2}{x^2}=\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{2}{x^2}=\frac{1}{x}y-\frac{2}{x^2}\Rightarrow x^2\frac{dy}{dx}=xy-2$$

$$\Rightarrow x^2\frac{dy}{dx}-xy+2=0$$

Method II. Substitute the values of $\frac{dy}{dx}$ and y in terms of x in L.H.S. of the result to be proved.

21. $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots\ldots\text{to}\infty.$

$$\therefore \frac{dy}{dx}=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\ldots..\text{to}\infty=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.\text{to}\infty=y.$$

19.10. Rule for differentiating product of functions

Theorem 5. ***The differential coefficient of the product of two functions is the sum of the products of each function with the derivatives of other, i.e.,***

$$\boldsymbol{\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx},}$$

where u and v are differentiable functions of x.

Proof. Let $y = uv$ ... (i)

Let δx be a small increment in the value of x. Corresponding to the increment δx in x, let there be the increments δu, δv, δy, in u, v and y respectively.

Then $y + \delta y = (u + \delta u)(v + \delta v) = uv + u\delta v + \delta u.v + \delta u.\delta v$ ... (ii)

Subtracting (i) from (ii), $\delta y=u\delta v+v\delta u+\delta u.\delta v$

Dividing both sides by δx, $\frac{\delta y}{\delta x}=u\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}+\delta u.\frac{\delta v}{\delta x}$

Taking limits as $\delta x \to 0$ and hence $\delta u, \delta v, \delta y$ all $\to 0$.

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=u\frac{dv}{dx}+v\frac{du}{dx}+0.\frac{dv}{dx}$$

Hence, $\boxed{\boldsymbol{\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}}}$

Aid to Memory. Derivative of the product of two functions

= First function × Derivative of the second + Second function × Derivative of the first.

Extension of the rule: If *u*, *v* and *w* are all functions of *x*, then

$$\boldsymbol{\frac{d}{dx}(uvw)=uv\frac{dw}{dx}+vw\frac{du}{dx}+wu\frac{dv}{dx}}$$

Let $y = uvw = (uv)w$. Then $\frac{dv}{dx}=uv\frac{dw}{dx}+w\frac{d}{dx}(uv)$ *Product rule*

$$=uv\frac{dw}{dx}+w\left[u\frac{dv}{dx}+v\frac{du}{dx}\right]=uv\frac{dw}{dx}+wu\frac{dv}{dx}+wv\frac{du}{dx}$$

$$\therefore \quad \frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}$$

Aid to Memory: To differentiate the product of three functions multiply the differential coefficient of each function by the product of the remaining two functions and add up the results.

Note. The rule can be extended to the product of any finite number of functions, the derivatives being the sum of the products as under:

(Derivative of a function × Product of the remaining functions); when all the functions have been considered.

Ex. 7. (*i*) Let $y = (x^2 - 4x + 5)(x^3 - 2)$

Differentiating w.r.t. x, we get $\frac{dy}{dx} = (x^2-4x+5)\frac{d}{dx}(x^3-2)+(x^3-2)\frac{d}{dx}(x^2-4x+5)$

$$= (x^2-4x+5)(3x^2)+(x^3-2)(2x-4) = 5x^4-16x^3+15x^2-4x+8.$$

(*ii*) Let $y = x^3.\sqrt{3x-4}$

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = x^3\frac{d}{dx}(\sqrt{3x-4}) + \sqrt{3x-4}.\frac{d}{dx}(x^3)$$

$$= x^3.\frac{1}{2\sqrt{3x-4}}.3+\sqrt{3x-4}(3x^2)$$

$$= \frac{3x^2}{2\sqrt{3x-4}}[x+2(3x-4)] = \frac{3x^2(7x-8)}{2\sqrt{3x-4}}.$$

(*iii*) Let $y = (x+1)(2x+3)^3(5x+7)^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=(2x+3)^3(5x+7)^2\frac{d}{dx}(x+1) + (x+1)(5x+7)^2\frac{d}{dx}(2x+3)^3 + (x+1)(2x+3)^3\frac{d}{dx}(5x+7)^2$$

$$= (2x+3)^3(5x+7)^2.1+(x+1)(5x+7)^2.3(2x+3)^2.2+(x+1)(2x+3)^3.2(5x+7).5$$

$$= (2x+3)^2.(5x+7)[(2x+3)(5x+7)+6(x+1)(5x+7)+10(x+1)(2x+3)]$$

$$= (2x+3)^2(5x+7)(60x^2+151x+93).$$

19.11. Rule for differentiating quotient of two functions

Theorem 6. ***If u and v are functions of x and v ≠ 0, then***

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}, v \neq 0.$$

Proof. Let $y = \frac{u}{v}$, where u and v are functions of x. ... (*i*)

Let x receive an increment δx and due to this, let u and v receive increment δu and δv and consequently for y the increment is δy. Then $y + \delta y = \frac{u+\delta u}{v+\delta v}$ ... (*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{uv+v\delta u-uv-u\delta v}{v(v+\delta v)} = \frac{v\delta u-u\delta v}{v(v+\delta v)}$

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{v.\dfrac{\delta u}{\delta x} - u.\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$

Taking limits as $\delta x \to 0$, so that $\delta u, \delta v$ and δy all $\to 0$.

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{v.\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v(v+0)} = \frac{v.\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v^2}$$

Hence, $$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}}$$

Aid to Memory:

Derivative of the quotient of two functions

$$= \frac{\textbf{Denom.(Derivative of the Num.) – Num. (Derivative of Denom.)}}{\textbf{Denom}^2.} = \frac{\textbf{D.N′ – N.D′}}{\textbf{D}^2}$$

Another Form: *If f and g are differentiable functions of x and h is such that*

$$h(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0,\ \textit{show that}\ \ h'(x) = \frac{g(x).f'(x) - .f(x)g'(x)}{[g(x)]^2}.$$

Cor. $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-\dfrac{d}{dx}f(x)}{(f(x))^2}.$$

Proof. $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{f(x).\dfrac{d}{dx}(1) - 1.\dfrac{d}{dx}f(x)}{(f(x))^2} = \frac{f(x).0 - \dfrac{d}{dx}f(x)}{(f(x))^2} = \frac{-\dfrac{d}{dx}f(x)}{(f(x))^2}.$$

Ex. 8. (*i*) Let $y = \dfrac{(2+5x)^2}{x^3 - 1}$ $\left[\textbf{Form } y = \dfrac{u}{v}\right]$

$$\frac{dy}{dx} = \frac{(x^3-1)\dfrac{d}{dx}(2+5x)^2 - (2+5x)^2.\dfrac{d}{dx}(x^3-1)}{(x^3-1)^2}$$

$$= \frac{(x^3-1).2(2+5x)\times 5 - (2+5x)^2.3x^2}{(x^3-1)^2}$$

$$= \frac{(2+5x)[10(x^3-1) - 3x^2(2+5x)]}{(x^3-1)^2} = \frac{-(2+5x)(5x^3+6x^2+10)}{(x^3-1)^2}.$$

(*ii*) Let $y = \dfrac{x}{a^2 + x^2}$ $\left[\textbf{Form } \dfrac{u}{v}\right]$

$$\frac{dy}{dx} = \frac{(a^2+x^2).\dfrac{d}{dx}(x) - x.\dfrac{d}{dx}(a^2+x^2)}{(a^2+x^2)^2}$$

$$= \frac{(a^2+x^2).1 - x.2x}{(a^2+x^2)^2} = \frac{a^2+x^2-2x^2}{(a^2+x^2)^2} = \frac{a^2-x^2}{(a^2+x^2)^2}.$$

(*iii*) Let $y = \dfrac{ax^2+bx+c}{px^2+qx+f}$

$$\frac{dy}{dx} = \frac{(px^2+qx+f).\frac{d}{dx}(ax^2+bx+c)-(ax^2+bx+c).\frac{d}{dx}(px^2+qx+f)}{(px^2+qx+f)^2}$$

$$= \frac{(px^2+qx+f)(2ax+b)-(ax^2+bx+c).(2px+q)}{(px^2+qx+f)^2}$$

$$= \frac{2apx^3+2aqx^2+2afx+bpx^2+bqx+bf-2apx^3-2bpx^2-2pcx-aqx^2-bqx-cq}{(px^2+qx+f)^2}$$

$$= \frac{aqx^2-bpx^2+2afx-2pcx+bf-cq}{(px^2+qx+f)^2} = \frac{(aq-bp)x^2+2(af-pc)x+bf-cq}{(px^2+qx+f)^2}$$

(*iv*) Let $y = \dfrac{1}{ax^2+bx+c}$

$$\frac{dy}{dx} = \frac{(ax^2+bx+c).\frac{d}{dx}(1)-1.\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$= \frac{(ax^2+bx+c).0-(2ax+b)}{(ax^2+bx+c)^2} = \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Aliter: By chain rule

$$\frac{dy}{dx} = -\frac{1}{(ax^2+bx+c)^2}.\frac{d}{dx}(ax^2+bx+c)$$

$$= -\frac{1}{(ax^2+bx+c)^2}.(2ax+b)$$

(*v*) Let $y = \sqrt{\dfrac{1+x}{1-x}} = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$ $\left[\text{Form } y = \dfrac{u}{v}\right]$

$$\frac{dy}{dx} = \frac{\sqrt{1-x}\frac{d}{dx}(\sqrt{1+x})-\sqrt{1+x}\frac{d}{dx}(\sqrt{1-x})}{\sqrt{(1-x)^2}}$$

$$= \frac{\sqrt{1-x}.\frac{1}{2\sqrt{1+x}}-\sqrt{1+x}.\frac{1}{2\sqrt{1-x}}(-1)}{(1-x)}$$

Caution. Do not forget to multiply by –1, the coefft. of x.

$$= \frac{\frac{(1-x)+(1+x)}{2\sqrt{1+x}\sqrt{1-x}}}{1-x} = \frac{2}{2\sqrt{1+x}\sqrt{1-x}(1-x)} = \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$$

(*vi*) Let $y = \dfrac{1}{(x-a)(x-b)(x-c)}$

$$\frac{dy}{dx} = -\frac{1}{\{(x-a)(x-b)(x-c)\}^2}\cdot\frac{d}{dx}\left[(x-a)(x-b)(x-c)\right]$$

$$= -\frac{1}{\{(x-a)(x-b)(x-c)\}^2}\cdot\left[(x-a)(x-b).1+(x-b)(x-c).1+(x-a)(x-c).1\right]$$

$$= -\frac{1}{\{(x-a)(x-b)(x-c)\}^2}\cdot\left[(x-a)(x-b)+(x-b)(x-c)+(x-a)(x-c)\right]$$

$$= -\frac{1}{(x-a)(x-b)(x-c)}\left[\frac{1}{x-c}+\frac{1}{x-a}+\frac{1}{x-b}\right].$$

EXERCISE 19 (*c*)

Differentiate the following w.r.t. *x* or *t* or *u* as the case may be:

1. $(ax + b)\,(cx + d)$

2. $(x^{100} + 2x^{50} - 3)\,(7x^8 + 20x + 5)$

3. $x\,(2x - 1)\,(x + 2)$

4. $(x - 2)\,(x + 3)\,(2x + 5)$

5. $y = \dfrac{2x+5}{3x-2}$

6. $y = \dfrac{x^2-3}{x+4}$

7. $y = \dfrac{2x-3}{3x+4}$

8. $y = \dfrac{x^5-x+2}{x^3+7}$

9. $s = t^2\,(t + 1)^{-1}$

10. $z = \dfrac{u}{u^2+1}$

11. $y = \dfrac{x^2+2x+5}{x^2+2x+4}$

12. $f(x) = \dfrac{x^3+2x}{x^2+4}$

13. If $f(x) = \dfrac{x+2}{x-2}$ for all $x \neq 2$, find $f'(-2)$.

14. Differentiate $\dfrac{x+2}{x^2-3}$ and find the value of the derivative at $x = 0$.

15. If $y = \dfrac{x}{x+a}$, prove that $x\dfrac{dy}{dx} = y(1-y)$.

16. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$.

17. Given that $y = \sqrt{\dfrac{1-x}{1+x}}$ show that $(1 - x^2)\,\dfrac{dy}{dx} + y = 0$ **[*ISC*]**

18. Given that $y = (3x - 1)^2 + (2x - 1)^3$, find $\dfrac{dy}{dx}$ and the points on the curve for which $\dfrac{dy}{dx} = 0$.

19. (*i*) If $y=\dfrac{x-1}{2x^2-7x+5}$, find $\dfrac{dy}{dx}$ at $x=2$. **[*ISC*]** (*ii*) If $y=\dfrac{x^2+3}{x^3+2x}$, find $\dfrac{dy}{dx}$ at $x=1$. **[*ISC*]**

20. Find the coordinates of the points on the curve

$y=\dfrac{x}{1-x^2}$ for which $\dfrac{dy}{dx}=1$. **[*ISC*]**

ANSWERS

1. $c(ax+b)+a(cx+d)$ **2.** $(x^{100}+2x^{50}-3)(56x^7+20)+(7x^8+20x+5)(100x^{99}+100x^{49})$

3. $x(2x-1)+2x(x+2)+(2x-1)(x+2)$ **4.** $2(x-2)(x+3)+(x-2)(2x+5)+(x+3)(2x+5)$

5. $\dfrac{dy}{dx}=\dfrac{-19}{(3x-2)^2}$ **6.** $\dfrac{dy}{dx}=\dfrac{x^2+8x+3}{(x+4)^2}$ **7.** $\dfrac{dy}{dx}=\dfrac{17}{(3x+4)^2}$

8. $\dfrac{dy}{dx}=\dfrac{2x^7-35x^4+2x^3-6x^2-7}{(x^3+7)^2}$ **9.** $\dfrac{ds}{dt}=\dfrac{t^2+2t}{(t+1)^2}$ **10.** $\dfrac{dz}{du}=\dfrac{1-u^2}{(u^2+1)^2}$

11. $\dfrac{dy}{dx}=\dfrac{-x^4-4x^3-13x^2+8x-2}{(x^3+2x+4)^2}$ **12.** $f'(x)=\dfrac{x^4+10x^2+8}{(x^2+4)^2}$ **13.** $-\dfrac{1}{4}$

14. $\dfrac{-x^2-4x-3}{(x^2-3)^2}$, $-\dfrac{1}{3}$ **18.** $\dfrac{dy}{dx}=6x(4x-1)$; $(0,0)$, $\left(\dfrac{1}{4},\dfrac{-1}{16}\right)$ **19.** (*i*) -2 (*ii*) $\dfrac{-14}{9}$

20. $(0,0)\left(\sqrt{3},\dfrac{-\sqrt{3}}{2}\right),\left(-\sqrt{3},\dfrac{\sqrt{3}}{2}\right)$

HINTS TO SELECTED QUESTIONS

16. $x\sqrt{1+y}+y\sqrt{1+x}=0\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$

$\Rightarrow \quad x^2(1+y)=y^2(1+x)\Rightarrow x^2-y^2+x^2y-xy^2=0 \Rightarrow y=-\dfrac{x}{1+x}$

18. $y=8x^3-3x^2$

19. $y=\dfrac{x-1}{2x^2-7x+5}=\dfrac{x-1}{(x-1)(2x-5)}=\dfrac{1}{2x-5}$

TRIGONOMETRIC FUNCTIONS

19.12. Derivatives of trigonometric functions

We will make use of the following limits in finding out the derivative of the *t*-functions from first principles.

1. $\lim\limits_{\theta\to 0}\sin\theta=0$ **2.** $\lim\limits_{\theta\to 0}\cos\theta=1$ **3.** $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ **4.** $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1$

where θ is measured in radians.

Caution. Since $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ is valid only when θ is expressed in radians, therefore while differentiating *t*-functions, angles must be expressed in radians. If it is not so, it should first be converted into radians by using the formula $x^\circ=\dfrac{\pi x}{180}$ radians.

Recall the following formulae from trigonometry. You will be using these for the differentiation of t-functions.

(*i*) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ (*ii*) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(*iii*) $\tan(A \pm B) = \dfrac{\tan A \pm B}{1 \mp \tan A \tan B}$ (*iv*) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(*v*) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}; \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(*vi*) $\sin 3x = 3\sin x - 4\sin^3 x; \cos 3x = 4\cos^3 x - 3\cos x$

(*vii*) $1 - \cos x = 2\sin^2 \dfrac{x}{2}, 1 + \cos x = 2\cos^2 \dfrac{x}{2}$

(*viii*) $\sin C + \sin D = 2\sin \dfrac{C+D}{2}\cos \dfrac{C-D}{2}$; $\sin C - \sin D = 2\cos \dfrac{C+D}{2}\sin \dfrac{C-D}{2}$;

$\cos C + \cos D = 2\cos \dfrac{C+D}{2}\cos \dfrac{C-D}{2}$; $\cos C - \cos D = 2\sin \dfrac{C+D}{2}\sin \dfrac{D-C}{2}$

1. Differential coefficient of sin x, where x is in radians

Let $y = \sin x$ ...(*i*)

Let x be changed to $x + \delta x$ and consequently y to $y + \delta y$

$\therefore \quad y + \delta y = \sin(x + \delta x)$ ...(*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \sin(x + \delta x) - \sin x = 2\cos \dfrac{(x+\delta x)+x}{2}\sin \dfrac{x+\delta x-x}{2}$

$$\left(\because \sin C - \sin D = 2\cos \frac{C+D}{2}\sin \frac{C-D}{2}\right)$$

$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin \frac{\delta x}{2}$$

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{2\cos\left(x+\frac{\delta x}{2}\right)\sin \frac{\delta x}{2}}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right).\dfrac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$

[**Note this step.** We have put it in the form $\dfrac{\sin\theta}{\theta}$]

Proceeding to their limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right).\lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

i.e., $\dfrac{dy}{dx} = \cos x \times 1 = \cos x$ $\left[\because \lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1\right]$

Hence, $\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$

2. Differential coefficient of cos x, where x is in radians

Let $y = \cos x$ ... (*i*)

Let x be changed to $x + \delta x$ and consequently y to $(y + \delta y)$

$\therefore$ $y + \delta y = \cos(x + \delta x)$ ... (*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \cos(x + \delta x) - \cos x$

or $\delta y = 2\sin\dfrac{x+\delta x+x}{2}\sin\dfrac{x-x-\delta x}{2}$ $\left(\because \cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}\right)$

$= 2\sin\left(x+\dfrac{\delta x}{2}\right)\sin\left(\dfrac{-\delta x}{2}\right) = -2\sin\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}$ $[\because \sin(-\theta) = -\sin\theta]$

Dividing both sides by δx,

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x} = -\sin\left(x+\frac{\delta x}{2}\right).\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

Proceeding to the limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x \to 0}\sin\left(x+\frac{\delta x}{2}\right).\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

i.e., $\dfrac{dy}{dx} = -\sin x \times 1 = -\sin x$ $\left(\because \lim_{\theta \to 0}\dfrac{\sin\theta}{\theta} = 1\right)$

Hence, $\boxed{\dfrac{d}{dx}(\cos x) = -\sin x}$

3. Differential coefficient of tan x, where x is in radians

Let $y = \tan x$. ... (*i*)

Let x be changed to $x + \delta x$ and consequently y to $(y + \delta y)$. Then

$\therefore$ $y + \delta y = \tan(x + \delta x)$ ... (*ii*)

Subtracting (*i*) from (*ii*),

$$\delta y = \tan(x+\delta x) - \tan x = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin[(x+\delta x)-x]}{\cos(x+\delta x)\cos x} = \frac{\sin\delta x}{\cos(x+\delta x)\cos x}$$

[$\because$ sin A cos B – cos A sin B = sin (A – B)]

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{\sin\delta x}{\delta x\cos(x+\delta x)\cos x} = \dfrac{1}{\cos(x+\delta x)\cos x}.\dfrac{\sin\delta x}{\delta x}$

Proceeding to the limit as $\delta x \to 0$, we have

Hence, $\boxed{\dfrac{d}{dx}(\tan x) = \sec^2 x}$

Alternatively. *Differentiate tan x by writing it in the form* $\frac{\sin x}{\cos x}$ *and applying the formula for differentiating a quotient of functions, we get*

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x . \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x . \cos x - \sin x . (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4. Differential coefficient of cot *x*, where *x* is in radians

Let $y = \cot x$ ... (*i*)

Let x be changed to $x + \delta x$ and consequently y to $y + \delta y$. Then

$\therefore$ $y + \delta y = \cot (x + \delta x)$ ... (*ii*)

Subtracting (*i*) from (*ii*), $\delta y = \cot (x+\delta x) - \cot x = \frac{\cos (x+\delta x)}{\sin (x+\delta x)} - \frac{\cos x}{\sin x}$

$$= \frac{\cos (x+\delta x)\sin x - \sin (x+\delta x)\cos x}{\sin (x+\delta x)\sin x}$$

$$= \frac{\sin [x-(x+\delta x)]}{\sin (x+\delta x)\sin x} = \frac{\sin (-\delta x)}{\sin (x+\delta x)\sin x}$$

$$= \frac{-\sin \delta x}{\sin (x+\delta x)\sin x}$$

[$\because$ sin A cos B – cos A sin B = sin (A – B)]

Dividing both sides by δx, $\frac{\delta y}{\delta x} = -\frac{\sin \delta x}{\delta x . \sin (x+\delta x)\sin x} = -\frac{1}{\sin (x+\delta x)\sin x} . \frac{\sin \delta x}{\delta x}$

Proceeding to the limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{1}{\sin (x+\delta x)\sin x} . \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}.$$

i.e., $\frac{dy}{dx} = \frac{-1}{\sin x . \sin x} . 1 = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$

Hence, $\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$

Try yourself. *Differentiate cot x; regarding it as a fraction* $\frac{\cos x}{\sin x}$.

5. Differential coefficient of sec *x*, where *x* is in radians

Let $y = \sec x = \frac{1}{\cos x}$ ... (*i*)

Let x be changed to $x + \delta x$ and consequently y to $(y + \delta y)$. Then

$\therefore$ $y + \delta y = \sec (x+\delta x) = \frac{1}{\cos (x+\delta x)}$ ... (*ii*)

Subtracting (i) from (ii), $\delta y = \dfrac{1}{\cos(x+\delta x)} - \dfrac{1}{\cos x} = \dfrac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x}$

$$= \frac{2\sin\dfrac{x+x+\delta x}{2}\sin\dfrac{x+\delta x-x}{2}}{\cos(x+\delta x)\cos x}$$

$$= \frac{2\sin\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

Dividing both sides by δx, $\dfrac{\delta y}{\delta x} = \dfrac{2\sin\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\delta x\cos(x+\delta x)\cos x} = \dfrac{\sin\left(x+\dfrac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\cdot\dfrac{\sin\dfrac{\delta x}{2}}{\dfrac{\delta x}{2}}$

Proceeding to the limit as $\underset{\delta x\to 0}{\text{Lim}}\dfrac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Lim}}\dfrac{\sin\left(x+\dfrac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\cdot\underset{\delta x\to 0}{\text{Lim}}\dfrac{\sin\dfrac{\delta x}{2}}{\dfrac{\delta x}{2}}$,

i.e., $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos x.\cos x}.1 = \dfrac{1}{\cos x}.\dfrac{\sin x}{\cos x} = \sec x\tan x$

Hence, $\boxed{\dfrac{d}{dx}(\sec x) = \sec x\tan x}$

6. Differential coefficient of cosec x, where x is in radians

Let $y = \text{cosec } x = \dfrac{1}{\sin x}$ ... (i)

Let x be changed to $(x + \delta x)$ and consequently y to $(y + \delta y)$. Then

$\therefore$ $y+\delta y = \text{cosec}(x+\delta x) = \dfrac{1}{\sin(x+\delta x)}$ ... (ii)

Subtracting (i) from (ii), $\delta y = \dfrac{1}{\sin(x+\delta x)} - \dfrac{1}{\sin x} = \dfrac{\sin x - \sin(x+\delta x)}{\sin(x+\delta x)\sin x}$

$$= \frac{2\cos\dfrac{x+x+\delta x}{2}\sin\dfrac{x-x-\delta x}{2}}{\sin(x+\delta x)\sin x}$$

$$= \frac{2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\left(-\dfrac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}$$

$$= -\frac{2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\sin(x+\delta x)\sin x}$$

$\therefore \sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\sin(-\theta) = -\sin\theta$

Dividing both sides by δx,

$$\frac{\delta y}{\delta x} = -\frac{2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\delta x\sin(x+\delta x)\sin x} = -\frac{\cos\left(x+\dfrac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}\cdot\frac{\sin\dfrac{\delta x}{2}}{\dfrac{\delta x}{2}}$$

Proceeding to the limit as $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\cos\left(x+\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x} \cdot \lim_{\delta x \to 0} \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\cos x}{\sin x . \sin x} . 1 = -\frac{1}{\sin x} . \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

Hence, $\boxed{\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$

Aid to Memory : The derivatives of all the *t*-functions which begin with 'co' are negative. Thus,

$$\frac{d}{dx}(\cos x) = -\sin x; \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x; \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

7. Differential coefficient of sin (*ax* + *b*)

Let $y = \sin(ax + b)$

Let x be changed to $x + \delta x$ and consequently y to $(y + \delta y)$. Then

$$y + \delta y = \sin[a(x+\delta x)+b] = \sin(ax + a\delta x + b)$$

$$\delta y = \sin(ax + a\delta x + b) - \sin(ax + b)$$

$$= 2\cos\frac{ax + a\delta x + b + ax + b}{2} \sin\frac{ax + a\delta x + b - ax - b}{2}$$

$$= 2\cos\left(ax + b + \frac{a\delta x}{2}\right)\sin\frac{a\delta x}{2} \quad \therefore \frac{\delta y}{\delta x} = \cos\left(ax + b + \frac{a\delta x}{2}\right) \frac{\sin\frac{a\delta x}{2}}{\frac{a\delta x}{2}} . a$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \cos(ax+b).1.a = \boldsymbol{a\cos(ax+b)}.$$

Ex. $\dfrac{d}{dx}(\sin(5x+7)) = 5\cos(5x+7)$, $\dfrac{d}{dx}(\sin mx) = m(\cos mx)$.

8. Differential coefficient of cos (*ax* + *b*)

Proceeding as above, you can find

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$$

Ex. $\dfrac{d}{dx}(\cos(3x-2)) = -3\sin(3x-2)$, $\dfrac{d}{dx}(\cos px) = -p\sin px$.

Likewise you can find from first principles the differential coefficients of tan (*ax* + *b*), cot (*ax* + *b*), sec (*ax* + *b*), cosec (*ax* + *b*).

For example,

Let $y = \operatorname{cosec}(ax+b)$

$$y + \delta y = \operatorname{cosec}[a(x+\delta x)+b] = \operatorname{cosec}(ax + a\delta x + b)$$

$$\delta y = \operatorname{cosec}(ax + a\delta x + b) - \operatorname{cosec}(ax + b)$$

$$= \frac{1}{\sin(ax+a\delta x+b)} - \frac{1}{\sin(ax+b)} = \frac{\sin(ax+b)-\sin(ax+a\delta x+b)}{\sin(ax+a\delta x+b)\sin(ax+b)}$$

$$= \frac{2\cos\left(ax+b+\frac{a\delta x}{2}\right).\sin\left(\frac{-a\delta x}{2}\right)}{\sin(ax+a\delta x+b)\sin(ax+b)}$$

$$\frac{\delta y}{\delta x} = -\frac{\cos\left(ax+b+\frac{a\delta x}{2}\right)}{\sin(ax+a\delta x+b)\sin(ax+b)}.\frac{\sin\frac{a\delta x}{2}}{\frac{a\delta x}{2}} \times a$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -\frac{\cos(ax+b)}{\sin(ax+b)\sin(ax+b)} \times 1 \times a$$

$$= -\frac{a}{\sin(ax+b)}.\frac{\cos(ax+b)}{\sin(ax+b)} = -\boldsymbol{a\,\mathrm{cosec}\,(ax+b)\cot(ax+b).}$$

Note. $\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$

$\frac{d}{dx}\cot(ax+b) = -a\,\mathrm{cosec}^2(ax+b)$, $\frac{d}{dx}\sec(ax+b) = a\sec(ax+b)\tan(ax+b)$.

EXERCISE 19 (*d*)

Differentiate the following functions w.r.t. *x*:

1. $\sin 5x$ **2.** $\cos 8x$ **3.** $\sin(5x+9)$ **4.** $\cos(2x-3)$

5. $\tan 7x$ **6.** $\cot nx$ **7.** $\tan(6x+11)$ **8.** $\sin\frac{x}{3}$

9. $\sec mx$ **10.** $\sec\left(\frac{x}{2}-1\right)$ **11.** $\mathrm{cosec}\frac{2}{3}x$ **12.** $x\sin x$

13. $x^2\cos 5x$ **14.** $\sqrt{x}\,\mathrm{cosec}(5x+7)$ **15.** $\frac{\sin 3x}{x-6}$ **16.** $\frac{\cos x}{5x}$

17. $\frac{\tan x}{2x+3}$ **18.** $\frac{\sec(ax-b)}{x^2-2}$ **19.** $\sin 2x$ **20.** $\cos 3x$

21. $\tan 2x$ **22.** $\sin\frac{x}{2}$ **23.** $\sec ax$ **24.** $\sec(px+q)$

25. $\tan(4x-7)$

ANSWERS

1. $5\cos 5x$ **2.** $-8\sin 8x$ **3.** $5\cos(5x+9)$ **4.** $-2\sin(2x-3)$

5. $7\sec^2 7x$ **6.** $-n\,\mathrm{cosec}^2 nx$ **7.** $6\sec^2(6x+11)$ **8.** $\frac{1}{3}\cos\frac{x}{3}$

9. $m\sec mx\tan mx$ **10.** $\frac{1}{2}\sec\left(\frac{x}{2}-1\right)\tan\left(\frac{x}{2}-1\right)$ **11.** $\frac{-2}{3}\mathrm{cosec}\frac{2}{3}x\cot\frac{2}{3}x$

12. $x\cos x+\sin x$ **13.** $-5x^2\sin 5x+2x\cos 5x$

14. $-5\sqrt{x}\operatorname{cosec}(5x+7)\cot(5x+7)+\dfrac{1}{2\sqrt{x}}\operatorname{cosec}(5x+7)$ **15.** $\dfrac{3(x-6)\cos 3x-\sin 3x}{(x-6)^2}$

16. $\dfrac{-x\sin x-\cos x}{5x^2}$ **17.** $\dfrac{(2x+3)(\sec^2 x)-2\tan x}{(2x+3)^2}$

18. $\dfrac{a(x^2-2)\sec(ax-b)\tan(ax-b)-2x\sec(ax-b)}{(x^2-2)^2}$ **19.** $2\cos 2x$

20. $-3\sin 3x$ **21.** $2\sec^2 2x$ **22.** $\dfrac{1}{2}\cos\dfrac{x}{2}$

23. $a\sec ax\tan ax$ **24.** $p\sec(px+q)\tan(px+q)$ **25.** $4\sec^2(4x-7)$

CHAPTER SUMMARY

1. Let x be a point in the domain of the real function f such that $y=f(x)$. Let δx (or h) be a small increment of x and δy the corresponding increment of $y=f(x)$ when the value of f changes from $f(x)$ to $f(x+\delta x)$ or $f(x+h)$. Then the increment ratio $\dfrac{\delta y}{\delta x}$ tends to a definite finite limit as $\delta x \to 0$ and is called the derivative (or differential coefficient) of f at the point x and is denoted by

$$f'(x)=\frac{dy}{dx}=\frac{d}{dx}f(x)=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \text{ or } \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists. We say that $f(x)$ is differentiable at x.

2. A few results

1. $\dfrac{d}{dx}(x^n)=nx^{n-1}$ **2.** $\dfrac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}\cdot a$

3. $\dfrac{d}{dx}(\text{a constant})=0$ **4.** $\dfrac{d}{dx}[f(x)+c]=\dfrac{d}{dx}[f(x)]$

5. $\dfrac{d}{dx}[k(f(x))]=k\dfrac{d}{dx}[f(x)]$ **6.** $\dfrac{d}{dx}(u\pm v)=\dfrac{du}{dx}\pm\dfrac{dv}{dx}$

7. $\dfrac{d}{dx}(u+v+w+\ldots)=\dfrac{du}{dx}+\dfrac{dv}{dx}+\dfrac{dw}{dx}+\ldots$ **8.** $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

9. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$ **10.** $\dfrac{d}{dx}(\sin x)=\cos x$

11. $\dfrac{d}{dx}(\cos x)=-\sin x$ **12.** $\dfrac{d}{dx}(\tan x)=\sec^2 x$

13. $\dfrac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$ **14.** $\dfrac{d}{dx}(\sec x)=\sec x\tan x$

15. $\dfrac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$

5. To differentiate a given function $f(x)$ from first principles

Step 1. Put the function to be differentiated equal to y, *i.e.*, let $y=f(x)$

Step 2. Let δx be an increment in the value of x and δy, the corresponding increment in the value of y so that $y+\delta y=f(x+\delta x)$.

Step 3. Find δy. We have $\delta y = f(x + \delta x) - f(x)$

Step 4. Divide both sides by δx to find the increment ratio $\frac{\delta y}{\delta x}$. Thus, $\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$

Step 5. Take limits of both sides as $\delta x \to 0$. This gives $\frac{dy}{dx}$ or $f'(x) = \underset{\delta x \to 0}{\text{Lim}} \frac{f(x+\delta x) - f(x)}{\delta x}$.

CHAPTER TEST

1. Find from first principles the differential coefficient of $2x^2 + 3x$.
2. Find from first principles the differential coefficient of $\sin 2x$.

Differentiate the following functions w.r.t. x.

3. $f(x) = \sqrt{3x+4}$, $x > -1$
4. $f(x) = \sqrt{4-x}$, $x < 4$
5. $f(x) = \frac{3x+4}{4x+3}$ $\left(x \neq \frac{-3}{4}\right)$
6. $f(x) = \sqrt{x^2+1}$
7. $(2x + 3)(x^3 - x + 2)$
8. $\tan(5x + 7)$
9. $\sin^2(3x - 2)$
10. $(x^3 + \sin x)^5$

ANSWERS

1. $4x + 3$
2. $2 \cos 2x$
3. $\frac{3}{2} \cdot \frac{1}{\sqrt{3x+4}}$
4. $-\frac{1}{2\sqrt{4-x}}$
5. $-\frac{7}{(4x+3)^2}$
6. $\frac{x}{\sqrt{x^2+1}}$
7. $6x^2 + 2x + 1$
8. $5 \sec^2(5x + 7)$
9. $6 \sin(3x - 2) \cos(3x - 2) = 3 \sin(6x - 4)$
10. $5(x^3 + \sin x)^4 . (3x^2 + \cos x)$

CHAPTER 20

MEASURES OF CENTRAL TENDENCY

20.01. Introduction

In classes IX and X you studied how to collect and tabulate the raw data in frequency distributions - **simple distribution, ungrouped** and **grouped frequency distributions, continuous and discrete distributions,** in **inclusive** and **exclusive** forms, **cumulative frequency distribution.** You also studied a few technical terms regarding distributions like **class limits, class boundaries, class interval, class mark** or **class mid-mark, frequency density, variables and variates** etc.

The students are advised to revise quickly what they learnt about frequency distributions in classes IX and X before doing this chapter.

20.02. Averages (Measures of Central Tendency)

Owing to difficulty which the mind experiences in grasping a large mass of figures, the necessity for an average arises to sum up shortly the character of the mass, and various kinds of averages are proposed. An average then, in general, we may regard as one of a class of statistical constants which concisely label a set of observations or measurements pertaining to a common family. It is designed to describe the family type more nearly than is possible by observing any chance member, and *in value it should therefore come somewhere near the middle of the family group, so that if the individual members of the family chance to be equal, each to each, in respect to the organ or character observed it should have the same value as they have.* This constitutes a test for the validity of any formula giving the average of a set of observations.

There are three common measures of central tendency:

1. **Arithmetic mean** or **simply mean**
2. **Median** 3. **Mode**

In this chapter, we will study only mean. Median and mode will be taken up in Chapter 28 in Section C.

20.03. The arithmetic mean

We have the following techniques to evaluate the arithmetic mean of a given data.

Case 1. Simple Distribution or Individual Observation

The arithmetic mean or the mean of a set of numbers $x_1, x_2, x_3, ..., x_n$ is denoted by $\overline{x}$ (real "$\overline{x}$") and is defined as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \quad \textbf{(Direct Method)}$$

The above formula can be stated in simple form as $\bar{x} = \frac{\sum x_i}{n}$.

If to each score x is added a number k, then the new mean = $\bar{x} + k$. If each score x is multiplied by a number k, then the new mean = $k\bar{x}$. (See solved Ex. 4 and Ex. 5).

If the numbers are large, then we assume some suitable number (A) as the mean and use the following formula :

$$\bar{x} = A + \frac{\sum d_i}{n} \quad \textbf{(Short-cut Method)}$$

where A is the assumed mean and $d_i = x_i - A$.

Ex. 1. Calculate the arithmetic mean of the following scores by both direct and short-cut methods : 17, 39, 21, 50, 27, 19, 24, 23, 10, 35.

Sol. Direct Method.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1}{10}(17 + 39 + 21 + 50 + 27 + 19 + 24 + 23 + 10 + 35) = \frac{265}{10} = \mathbf{26.5}$$

Short-cut Method. Let the assumed mean be 23. Then, we have

x_i	$d_i = x_i - 23$
10	– 13
17	– 6
19	– 4
21	– 2
23	0
24	1
27	4
35	12
39	16
50	27
	$\sum d_i = 35$

$$\bar{x} = A + \frac{\sum d_i}{n}$$
$$= 23 + \frac{35}{10}$$
$$= 23 + 3.5$$
$$= \mathbf{26.5.}$$

Ex. 2. The arithmetic mean of the marks obtained by 10 students of class X in Mathematics in a certain examination is 30. The marks obtained are 25, 30, 21, 55, 47, 10, 15, x, 45, 35. Find the value of x.

Sol. Let $\bar{x}$ be the average marks. Then

$$\bar{x} = \frac{1}{10}(25 + 30 + 21 + 55 + 47 + 10 + 15 + x + 45 + 35) = \frac{283 + x}{10}$$

But $\bar{x} = 30$ (given) $\therefore \frac{283 + x}{10} = 30 \Rightarrow 283 + x = 300 \Rightarrow x = 300 - 283 = 17$

Ex. 3. Using the formula $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, prove that $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.

Sol. $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \Rightarrow \sum_{i=1}^{n} x_i = n\bar{x}$ ....(1)

Also, $\sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + ... + (x_n - \bar{x})$

$= (x_1 + x_2 + x_3 + ... + x_n) - n\bar{x} = \sum_{i=1}^{n} x_i - n\bar{x} = n\bar{x} - n\bar{x} = 0.$ [Using (1)]

Ex. 4. If $\bar{x}$ is the mean of $x_1, x_2, ..., x_n$ show that the mean of $x_1 + a, x_2 + a, ..., x_n + a$ is $\bar{x} + a$ where a is any number positive or negative.

Sol. $\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}$ ...(1)

Let $\bar{X}$ be the mean of the numbers $x_1 + a, x_2 + a, . . ., x_n + a,$ then

$$\bar{X} = \frac{(x_1 + a) + (x_2 + a) + (x_3 + a) + ... + (x_n + a)}{n} = \frac{(x_1 + x_2 + ... + x_n) + na}{n}$$

$$= \frac{x_1 + x_2 + ... + x_n}{n} + \frac{na}{n} = \bar{x} + a.$$

Ex. 5. If $\bar{x}$ is the mean of $x_1, x_2, ..., x_n$, show that mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\bar{x}$.

Sol. $\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}$

Let $\bar{X}$ be the mean of $ax_1, ax_2, ax_3, ..., ax_n,$ then

$$\bar{X} = \frac{ax_1 + ax_2 + ax_3 + ... + ax_n}{n} = \frac{a(x_1 + x_2 + ... + x_n)}{n}$$

$$= a.\frac{x_1 + x_2 + ... + x_n}{n} = a.\bar{x}$$

Note. Averages can be very misleading. Suppose a doctor with lucrative practice earns ₹ 300000 a month. He has a wife and three children none of whom are earning members. In his household he has a cook whom he pays ₹ 6000 a month and a gardener who receives ₹ 4000 a month. Thus, seven people between them earn ₹ 310000 a month. The average earning of this group is ₹ $\frac{310000}{7}$ = ₹ 44285.71 approximately. Clearly, this figure does not represent a single person in the group. It gives the reader a totally meaningless figure, because he cannot make one single reliable deduction from it. Such averages create very false impressions.

Keeping such cases in view, several kinds of averages have been invented so that as wide a field as possible may be covered with the minimum of representation. We have a choice of averages and we pick out the one which is appropriate both to our data and our purpose.

Case 2. Arithmetic Mean of a Discrete, *i.e.*, Ungrouped Frequency Distribution

Formula 1. If $x_1, x_2, ..., x_n$ are the values of a variable with frequencies $f_1, f_2, ..., f_n$, then

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

The above formula can be stated in simpler form as

$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$ or more simply as $\boxed{\bar{x} = \frac{\Sigma fx}{\Sigma f}}$*.

Formula 2. (Short-cut Method). In general, it is found that the frequencies and the values of the variable are quite large numbers. The product $f_i x_i$ will also be large. Hence, a short-cut method has been devised as

$$\bar{x} = A + \frac{\sum_{i=1}^{n} f_id_i}{\sum_{i=1}^{n} f_i}$$

or simply $\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$ or more simply as $\boxed{\bar{x} = A + \frac{\Sigma fd}{\Sigma f}}$

where A is any assumed mean and $d_i = x_i - A$ are the deviations of x_i from A.

Note. Any number can be taken as assumed mean A but for the sake of simplicity, it is generally taken as that value of the variable which has the greatest frequency in the frequency distribution or which is near about the middle of the frequency distribution.

Proof. $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + ... + f_nx_n}{\Sigma f_i}$

$$= \frac{f_1 (x_1 - A) + f_2 (x_2 - A) + f_3 (x_3 - A) + ... + f_n (x_n - A)}{\Sigma f_i} + \frac{Af_1 + Af_2 + Af_3 + ... + Af_n}{\Sigma f_i}$$

$$= \frac{\Sigma f_i d_i}{\Sigma f_i} + \frac{A (f_1 + f_2 + f_3 + ... + f_n)}{\Sigma f_i} = \frac{\Sigma f_i d_i}{\Sigma f_i} + \frac{A \,.\, \Sigma f_i}{\Sigma f_i} = \frac{\Sigma f_i d_i}{\Sigma f_i} + A$$

$$\therefore \quad \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} \qquad \text{... (1)}$$

For a simple distribution the formula becomes

$$\bar{x} = A + \frac{\Sigma d_i}{\Sigma f_i} = A + \frac{\Sigma d_i}{n}$$

[In a simple distribution, we can say that the frequency of each item is 1, therefore, $\Sigma f_i = 1 + 1 + ...$ n times $= n$]

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* If each frequency is multiplied by a number k, then the mean will not change in value.

It should be noted that the expression $\frac{\Sigma fd}{\Sigma f}$ is the mean of the deviations. Hence, (1) states,

that the real mean $\bar{x}$ is equal to the sum of the assumed mean and the mean of the deviations.

If $A = 0$, then, from (1), $\bar{x} = \frac{\Sigma fx}{\Sigma f}$. In other words, the expression $\frac{\Sigma fx}{\Sigma f}$ for the real mean is obtained by taking zero as the assumed mean. Another way of saying the same thing is that (1) gives the mean when A is taken as the origin (or the origin is shifted to A) and $\frac{\Sigma fx}{\Sigma f}$ gives the mean when zero is taken as the origin. In other words, we use the transformation $x = A + d$, where d is the new variable and x is the old variable.

Formula 3. (Step-Deviation Method)

Sometimes the deviations d_i's are found to be divisible by a common number h (say), then we can make the calculation easier by putting

$$u_i = \frac{x_i - A}{h}, i = 1, 2, ..., n \Rightarrow x_i = hu_i + A, i = 1, 2, 3, ..., n$$

$$\Rightarrow \quad f_i x_i = f_i (hu_i + A) = h (f_i u_i) + Af_i, i = 1, 2, 3, ..., n$$

$$\Rightarrow \quad \frac{\sum_{i=1}^{n} f_i x_i}{N} = h\left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right) + A \cdot \frac{1}{N}\left(\sum_{i=1}^{n} f_i\right), \text{ where } N = \sum_{i=1}^{n} f_i$$

$$\Rightarrow \quad \boldsymbol{\bar{x} = A + h \cdot \frac{\sum_{i=1}^{n} f_i u_i}{\Sigma f_i}} \textbf{ or simply, } \boldsymbol{\bar{x} = A + h \cdot \frac{\Sigma f_i u_i}{\Sigma f_i}}, \textbf{ where } \boldsymbol{u_i = \frac{d_i}{h}}$$

In a still simpler form, the above formula can be stated as

$$\boxed{\bar{x} = A + i \cdot \frac{\Sigma ft}{\Sigma f}}$$

where A is the assumed mean, i is the length of the class interval and $t = \frac{x_i - A}{i}$.

Note. Finding mean by using the above formula is known as the step deviation method.

It is a very short method and should always be used for grouped data where ***class interval sizes are equal.***

Note. While writing the mean, **don't forget to mention the unit of measurement** in which the quantitative variable has been measured as well as the units of observation from whom the observations have come. Thus, if the average marks of a class of students equal 58.6, we will say that the mean is 58.6 marks per student.

Ex. 6. Find by the direct as well as by the short-cut method the arithmetic mean for the following:

Variable value	5	6	7	8
Frequency	15	25	20	10

Sol. The calculations are shown in the following table:

x	f	fx	$d = x - 6$	fd
5	15	75	–1	–15
6	25	150	0	0
7	20	140	1	20
8	10	80	2	20
Total	70	445		25

By direct method, $\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{445}{70} = \mathbf{6.36.}$

By short-cut method, taking $A = 6$, we have $\bar{x} = A + \frac{\Sigma fd}{\Sigma f} = 6 + \frac{25}{70} = \mathbf{6.36.}$

Ex. 7. Calculate the mean for the distribution:

x	73	72	71	70	69	68	67	66	65
f	2	4	6	10	11	7	5	4	1

Sol. Let the assumed mean be $A = 69$.

x	f	$d = x - A = x - 69$	fd	fx
73	2	4	8	146
72	4	3	12	288
71	6	2	12	426
70	10	1	10	700
69	11	0	0 (= + 42)	759
68	7	–1	– 7	476
67	5	–2	–10	335
66	4	–3	–12	264
65	1	–4	– 4 (= –33)	65
	$\Sigma f = 50$		$\Sigma f.d = 9$	$\Sigma fx = 3459$

By direct method, $\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{3459}{50} = \frac{6918}{100} = 69.18.$

By short-cut method, $\bar{x} = A + \frac{\Sigma fd}{\Sigma f} = 69 + \frac{9}{50} = 69 + \frac{18}{100} = 69 + 0.18 = \mathbf{69.18.}$

Ex. 8. Find the mean wage from the following data:

Wage (in ₹)	800	820	860	900	920	980	1000
No. of workers	7	14	19	25	20	10	5

Sol. Let the assumed mean be $A = 900$.

Wage (in ₹)	*No. of workers*	$d = x - A$ $= x - 900$	$u = \frac{x - 900}{20}$	*fu*
800	7	–100	–5	–35
820	14	–80	–4	–56
860	19	–40	–2	–38
900	25	0	0	0
920	20	20	1	20
980	10	80	4	40
1000	5	100	5	25
	$N = \Sigma f = 100$			$\Sigma fu = -44$

Mean $\bar{x} = A + h.\frac{\Sigma fu}{\Sigma f} = 900 + 20\left(\frac{-44}{100}\right) = 891.2$. Hence, mean wage = **₹ 891.2.**

Case 3. Arithmetic Mean of Continuous or Grouped Frequency Distribution

In this case also, the arithmetic mean is computed by applying any of the formulas given for discrete frequency distribution. The values $x_1, x_2, ..., x_n$ are taken as the mid-points or class marks of the various class intervals. If the given frequency distribution is inclusive, then it should first be converted to exclusive distribution.

Ex. 9. Following is the distribution of earnings of 200 workers in a flour mill.

Monthly wages (in rupees)	**80 – 100**	**100 – 120**	**120 – 140**	**140 – 160**	**160 – 180**
No. of workers	**20**	**30**	**20**	**40**	**90**

Find the average earning of the workers.

Sol. Method 1.

Monthly wages (in rupees)	*Number of workers (f)*	*Class-marks (x)*	*fx*
80 – 100	20	90	1800
100 – 120	30	110	3300
120 – 140	20	130	2600
140 – 160	40	150	6000
160 – 180	90	170	15300
Total	200		29000

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{29000}{200} = 145$$

Hence, average earning of the workers = **₹ 145 per worker.**

Method 2. Since the values of x are large, it will be more convenient to use the short-cut method. Let the assumed mean $A = 130$.

Monthly wages (in rupees)	*Number of workers (f)*	*Class-marks (x)*	$d = x - 130$	fd
80 – 100	20	90	–40	–800
100 – 120	30	110	–20	–600
120 – 140	20	130	0	0
140 – 160	40	150	20	800
160 – 180	90	170	40	3600
Total	200			3000

$$\bar{x} = A + \frac{\Sigma fd}{\Sigma f} = 130 + \frac{3000}{200} = 130 + 15 = 145.$$

Hence, average earning of the workers = **₹ 145** per worker.

Method 3. Since the class intervals are of equal size, it will be most convenient to use the step deviation method. Let the assumed mean be $A = 130$. Here, $i = 20$.

Monthly wages (in rupees)	*Number of workers (f)*	*Class-marks (x)*	$t = \frac{x - 130}{20}$	ft
80 – 100	20	90	– 2	– 40
100 – 120	30	110	–1	–30
120 – 140	20	130	0	0
140 – 160	40	150	1	40
160 – 180	90	170	2	180
Total	$N = \Sigma f = 200$			150

$$\bar{x} = A + i.\frac{\Sigma ft}{\Sigma f} = 130 + 20 \times \frac{150}{200} = \textbf{₹ 145 per worker.}$$

Ex. 10. Estimate the arithmetic mean for the following frequency distribution given by the first two columns, if the wages are taken to the nearest rupee.

Variable Value (wages)	**Actual Class Limits**	**Mid-mark x**	f	$d = x - A = x - 8$	$t = \frac{d}{i} = \frac{x-8}{3}$	ft
₹ 1 – 4	**0.5 – 3.5**	**2**	**8**	**– 6**	**– 2**	**– 16**
4 – 7	**3.5 – 6.5**	**5**	**12**	**– 3**	**– 1**	**– 12**
7 – 10	**6.5 – 9.5**	**8**	**14**	**0**	**0**	**0**
10 – 13	**9.5 – 12.5**	**11**	**7**	**3**	**1**	**7**
Total			**41**			**– 21**

Sol. The mid-marks of the classes are taken as the values of x. Length of the uniform class interval (i) = 3.

Here, we take $A = 8$, since 8 is the value of x corresponding to the maximum frequency 14.

Now, $\bar{x} = A + i.\frac{\Sigma ft}{\Sigma f} = 8 + 3.\frac{-21}{41} = 8 - \frac{63}{41} = ₹\, 6\frac{19}{41} = \textbf{₹ 6.46.}$

Note. It should be noted that from a grouped frequency table it is possible only to *estimate* and not find exactly the arithmetic mean because of the assumption that the mid mark represents all values in a class interval.

Ex. 11. The table below gives the frequency distribution of a number of teaching posts sanctioned in Higher Secondary Schools in 1997.

Teaching posts sanctioned	No. of schools
6 – 10	899
11 – 15	1149
16 – 20	1553
21 – 25	1464
26 – 30	1307
31 – 35	1003
36 – 40	833
41 – 45	642
46 – 50	440
Total	9290

Find the average number of teaching posts sanctioned per Higher Secondary School.

Sol.

Teaching posts sanctioned	*No. of schools* f	*Mid-point* x	$A = 28,\ i = 5$ $t = \frac{x - 28}{5}$	ft
6 – 10	899	8	–4	–3596
11 – 15	1149	13	–3	–3447
16 – 20	1553	18	–2	–3106
21 – 25	1464	23	–1	–1464
26 – 30	1307	28	0	0
31 – 35	1003	33	1	1003
36 – 40	833	38	2	1666
41 – 45	642	43	3	1926
46 – 50	440	48	4	1760
Total	$\Sigma f = 9290$			$\Sigma ft = -5258$

$$\therefore \quad \bar{x} = A + i \cdot \frac{\Sigma ft}{\Sigma f} = 28 + 5 \times \frac{-5258}{9290} = 28 + 5\ (-0.566) = 28 - 2.830 = \mathbf{25.17.}$$

Hence, the average number of teaching posts sanctioned per Higher Secondary School is **25,** as it cannot be fractional.

Ex. 12. Estimate the average wage of a worker from the following table:

Wages (in rupees)	No. of Workers	Wages (in rupees)	No. of Workers
Below 10	15	Below 50	96
" 20	35	" 60	127
" 30	60	" 70	198
" 40	84	" 80	250

Sol. The given distribution is a cumulative frequency distribution and hence it must be changed to an absolute frequency table as shown below.

Wages (in rupees)	*Actual class limits*	*Mid-mark* x	f	$A = 44.5,\ i = 10$ $t = \frac{x - 44.5}{10}$	ft
0 – 9	–0.5 – 9.5	4.5	15	– 4	– 60
10 – 19	9.5 – 19.5	14.5	20	– 3	–60
20 – 29	19.5 – 29.5	24.5	25	– 2	–50
30 – 39	29.5 – 39.5	34.5	24	– 1	–24
40 – 49	39.5 – 49.5	44.5	12	0	0
50 – 59	49.5 – 59.5	54.5	31	1	31
60 – 69	59.5 – 69.5	64.5	71	2	142
70 – 79	69.5 – 79.5	74.5	52	3	156
Total			$\Sigma f = 250$		135

$$\bar{x} = A + i \cdot \frac{\Sigma ft}{\Sigma f}$$

$$= 44.5 + 10 \times \frac{135}{250} = 44.5 + 5.4 = 49.9.$$

Hence, the average wage of a worker is **₹ 49.90.**

Ex. 13. Following is the distribution of marks obtained by 60 students in Economics test.

Marks	No. of students	Marks	No. of students
More than 0	60	More than 30	20
More than 10	56	More than 40	10
More than 20	40	More than 50	3

Calculate the arithmetic mean.

Sol. We will have first to change the given cumulative frequency series to an absolute frequency series. All the 60 students have secured more than 0 marks; but 56 students have secured more than 10 marks, and so on. It means 4 students have secured marks between 0 and 10.

Marks	*No. of students* f	*Mid-mark* x	$A = 35,\ i = 10$ $t = \frac{x - 35}{10}$	ft
0 – 10	4	5	–3	–12
10 – 20	16	15	–2	–32
20 – 30	20	25	–1	–20
30 – 40	10	35	0	0
40 – 50	7	45	1	7
50 – 60	3	55	2	6
Total	$\Sigma f = 60$			$\Sigma ft = -51$

$$\bar{x} = A + i. \frac{\Sigma ft}{\Sigma f} = 35 + 10 \cdot \frac{(-51)}{60} = 35 - 8.5 = \textbf{26.5 marks}$$

Ex. 14. The following table gives the weekly wages in rupees of workers in a certain commercial organisation. The frequency of the class 49 – 52 is missing. It is known that the mean of the frequency distribution is 47.2. Find the missing frequency.

Weekly wages (₹)	40 – 43	43 – 46	46 – 49	49 – 52	52 – 55
Number of workers	31	58	60	?	27

Sol. Let the missing frequency be f_1.

Classes	*Mid-mark* x	f	$A = 47.5, i = 3$ $t = \frac{x - A}{3}$	ft
40 – 43	41.5	31	–2	–62
43 – 46	44.5	58	–1	–58
46 – 49	47.5	60	0	0
49 – 52	50.5	f_1	1	f_1
52 – 55	53.5	27	2	54
Total		$\Sigma f = 176 + f_1$		$\Sigma ft = f_1 - 66$

$$\bar{x} = A + i \cdot \frac{\Sigma ft}{N} = 47.5 + 3 \cdot \frac{f_1 - 66}{176 + f_1}$$

But $\bar{x} = 47.2$ (given) $\therefore \quad 47.2 = 47.5 + 3 \cdot \frac{f_1 - 66}{176 + f_1} \Rightarrow -0.3 = \frac{3f_1 - 198}{176 + f_1}$

$$\Rightarrow \quad -52.8 - 0.3f_1 = 3f_1 - 198 \quad \Rightarrow \quad 3.3f_1 = 145.2 \quad \Rightarrow f_1 = \frac{145.2}{3.3} = 44$$

Hence, the missing frequency is **44.**

Corrected mean

Ex. 15. Mean of 25 observations was found to be 78.4. But later it was found that 96 was misread as 69. Find the correct mean.

Sol. Here, $n = 25$, $\bar{x} = 78.4 \quad \therefore \quad \bar{x} = \frac{\Sigma x}{n} \Rightarrow \Sigma x = n\bar{x} = 25 \times 78.4 = 1960.$

But this value of Σx is incorrect as 96 was misread as 69.

$\therefore$ Correct Σx = (Incorrect Σx) – (sum of incorrect values) + sum of correct values

$= 1960 - 69 + 96 = 1987.$

$\therefore$ Correct mean $= \frac{1987}{25} =$ **79.48.**

20.05. Properties of arithmetic mean

Property 1. ***Algebraic sum of the deviations of a set of values from their arithmetic mean is zero.***

Proof. Let x, $i = 1, 2, \ldots, n$ be the set of values and f the corresponding frequencies of a frequency distribution, then the algebraic sum of the deviations from the arithmetic mean is given by

$$\sum_{i=1}^{n} f(x_i - \bar{x}) = f_1 (x_1 - \bar{x}) + f_2 (x_2 - \bar{x}) + f_3 (x_3 - \bar{x}) + \ldots + f (x_n - \bar{x})$$

$$= (f_1 x_1 + f_2 x_2 + f_3 x_3 + \ldots + f_n x_n) - \bar{x} (f_1 + f_2 + \ldots + f_n)$$

$$= \sum_{i=1}^{n} f_i x_i - \bar{x} \cdot \sum_{i=1}^{n} f_i = N\bar{x} - N\bar{x} = 0 \quad \left(\because \bar{x} = \frac{\Sigma f_i x_i}{N} \Rightarrow \Sigma f_i x_i = N\bar{x} \text{ and } N = \Sigma f_i \right)$$

* **Property 2.** ***The sum of the squares of the deviations of a set of values is minimum when taken about the mean.***

EXERCISE 20

1. Find the mean of :

(*i*) the first 6 natural numbers. (*ii*) the first ten odd natural numbers.
(*iii*) the first eight even natural numbers. (*iv*) $x, x+2, x+3, x+6$ and $x+8$.
(*v*) Squares of the first n natural numbers. (*vi*) Squares of the first 10 natural numbers.
(*vii*) Cubes of first n even natural numbers.

2. (*i*) Find the mean of the following sets of numbers:

(*a*) 2.5, 2.4, 3.5, 2.8, 2.9, 3.3, 3.6 (*b*) $-6, -2, -1, 0, 1, 2, 5, 9$.

(*ii*) The mean of the numbers $6, y, 7, x, 14$ is 8. Express y in terms of x.

3. Duration of sunshine (in hours) in Delhi for first ten days of month as reported by Meteorological department are as under :

5.1, 4.7, 6.1, 1.7, 5.4, 5.0, 11.7, 11.6, 11.9, 3.2

(*i*) Calculate $\bar{x}$; (*ii*) Check $\sum_{i=1}^{10} (x_i - \bar{x}) = 0$.

4. M being the mean of x_1, x_2, x_3, x_4, x_5 and x_6. Find the value of $\sum_{i=1}^{6} (x_i - M)$.

5. Find the mean of the following frequency distributions:

(*i*)

x	19	21	23	25	27	29	31
f	13	15	16	18	16	15	13

(*ii*)

x	2.5	7.5	12.5	17.5	22.5	27.5
f	4	5	7	12	7	5

6. Calculate the mean of the following data by short-cut method:

x	2.5	7.5	12.5	17.5	22.5	27.5
f	4	5	7	12	7	5

7. Calculate by step-deviation method, the arithmetic mean of the following marks obtained by students in English.

Marks	5	10	15	20	25	30	35	40	45	50
No. of students	20	43	75	67	72	45	39	9	8	6

8. The frequency distribution of marks obtained by 40 students of a class is as under. Calculate the Arithmetic Mean.

Marks	0 – 8	8 – 16	16 – 24	24 – 32	32 – 40	40 – 48
Students	5	3	10	16	4	2

* The proof is not given as it depends on "maxima and minima" which you will learn in class XII.

9. Compute the mean of the following frequency table by:

(*i*) direct method and (*ii*) short-cut method

Class	*Frequency*	*Class*	*Frequency*
5 – 10	10	30 – 35	4
10 – 15	6	35 – 40	2
15 – 20	4	40 – 45	1
20 – 25	12	45 – 50	3
25 – 30	8		

10. In a city, the following weekly observations were made in a study of cost of living index for year 1970 – 71.

Cost of living index	140 – 150	150 – 160	160 – 170	170 – 180	180 – 190	190 – 200
Number of weeks	5	10	18	9	6	4

(*i*) Calculate the average weekly cost of living index.

(*ii*) Verify $\Sigma f_i (x_i - \bar{x}) = 0$, where x is the mid value.

11. Calculate the mean by step-deviation method for the following data: **[*ISC*]**

Height (in cm)	*No. of boys*	*Height (in cm)*	*No. of boys*
135 – 140	4	155 – 160	24
140 – 145	9	160 – 165	10
145 – 150	18	165 – 170	5
150 – 155	28	170 – 175	2

12. Given :

Variable	20	19	18	17	16	15	14	13	12	11
Frequency	1	2	4	8	11	10	7	4	2	1

Find the mean variable by taking 11 as the assumed mean, and verify by the direct method.

13. The following table shows the distribution of orders of a firm, according to their value :

Value	Under ₹ 10	₹ 10 and Under ₹ 20	₹ 20 and Under ₹ 30	₹ 30 and Under ₹ 40	₹ 40 and Under ₹ 50	₹ 50 and Under ₹ 60	₹ 60 and Under ₹ 70
No. of buyers	245	383	205	89	47	21	10

Estimate (*i*) the total turnover, (*ii*) the mean value of a single order. **[*ISC*]**

14. The mean of the following data is 20.5. Find the missing frequency.

x	10	15	20	25	30
f	5	7	...	12	6

15. The mean of 40 observations was 160. It was detected on re-checking that the value 125 was wrongly copied as 165 for the computation of the mean. Find the correct mean.

16. The mean of the following frequency table is 50. But the frequencies f_1 and f_2 in classes 20 – 40 and 60 – 80 are missing. Find the missing frequencies.

Class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100	
Frequency	19	f_1	32	f_2	19	(Total 120)

17. The mean of 30 values was 150. It was detected on re-checking that the value 165 was wrongly copied as 135 for the computation of the mean. Find the correct mean. **[*ISC*]**

18. The sum of deviations of set of values $x_1, x_2, ..., x_n$ measured from 50 is –10 and the sum of deviations of values from 46 is 70. Find the value of n and the mean.

ANSWERS

1. (*i*) $\frac{7}{2}$ (*ii*) 10 (*iii*) 9 (*iv*) $x + 4$ (*v*) $\frac{(n+1)(2n+1)}{6}$
(*vi*) 38.5 (*vii*) $2n(n+1)^2$ **2.** (*i*) (*a*) 3 (*b*) 1 (*ii*) $y = 13 - x$ **3.** (*i*) 6.6
4. 0 **5.** (*i*) 25 (*ii*) 16 **6.** 16 **7.** 22.2
8. 23.4 marks **9.** 22 **10.** (*i*) 167.5 **11.** 153.45 **12.** 15.54
13. ₹ 19130, ₹ 19.13, ₹ 16.67 **14.** 40 **15.** 159
16. $f_1 = 25, f_2 = 25$ **17.** 151 **18.** 49.5

HINTS TO SELECTED QUESTIONS

13. The total turn over = Σfx.

18. $\sum_{i=1}^{n}(x_i - 50) = -10$ (given) $\Rightarrow \sum_{i=1}^{n} x_i - 50n = -10$ ...(1)

$\sum_{i=1}^{n}(x_i - 46) = 70$ (given) $\Rightarrow \sum_{i=1}^{n} x_i - 46n = 70$ ...(2)

(2) – (1) gives $4n = 80 \Rightarrow n = 20$. Substitute in (1).

CHAPTER SUMMARY

Formulas for finding arithmetic mean

1. Direct method. Using the individual observations of a simple distribution

(1) $\bar{x} = \frac{\Sigma x_i}{n}$ (2) $\bar{x} = A + \frac{\Sigma d_i}{n}$ (Short cut method)

2. (1) $\bar{x} = \frac{\Sigma fx}{\Sigma f}$ (2) $\bar{x} = A + \frac{\Sigma fd}{\Sigma f}$ (Using assumed mean)

(3) $\bar{x} = A + i\frac{\Sigma ft}{\Sigma f}$ (Step deviation method)

CHAPTER TEST

1. The weights of 50 apples were recorded as given below. Calculate the mean weight to the nearest gram, by step deviation method.

Weight in grams	80–85	85–90	90–95	95–100	100–105	105 –110	110 –115
Number of apples	5	8	10	12	8	4	3

2. Find the value of p if the mean of the following distribution is 7.5:

x	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12	12 – 14
f	6	8	15	p	8	4

3. The mean of the following frequency distribution is 57.6 and the number of observations is 50. Find the missing frequencies f_1 and f_2.

Class	0–20	20–40	40–60	60–80	80–100	100–120
Frequency	7	f_1	12	f_2	8	5

[AI CBSE]

ANSWERS

1. 95.9 grams **2.** $p = 3$ **3.** $f_1 = 8, f_2 = 10$

CHAPTER 21

MEASURES OF DISPERSION

21.01. Dispersion

Suppose, in the case of two frequency distributions, the mean comes out to be the same, what is to be our conclusion from this? Are the two distributions identical, as is indicated by the equality of the two means? Or have they any difference? In case there is a difference, how to express it and what is their difference? These are the questions we have to answer at this stage.

The measures of central tendency reveal only one characteristic of the data, *i.e.*, the point of central tendency. They fail to give us an idea about the extent to which the items of a distribution deviate from a central value. We get this information by the study of dispersion. Dispersion, thus, measures the variability of a series.

Suppose the scores of two teams A and B in 5 matches of football are as follows:

Total Mean score

Team A	5	3	7	2	8
Team B	5	6	5	4	5

Total score of team $A = 25$

$$\text{Mean score} = \frac{25}{5} = 5$$

Total score of team $B = 25$

$$\text{Mean score} = \frac{25}{5} = 5.$$

From the above we find that although the total score or the mean score of both the teams is the same, the score of team A is more spread or dispersed than that of B, that is, there is greater variation in team A which implies that it is less reliable than team B.

Definition. ***The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data.***

Various measures of dispersion or variation are available, the most common being the **range, mean deviation, semi-interquartile range** and the **standard deviation.** In this chapter, we will confine our study to mean deviation about mean and median and standard deviation.

21.02. Introduction

You are familiar with the concept of deviation. Recall that while finding out mean by step deviation method, you found out deviations from the assumed mean.

The term mean deviation literally means the arithmetic mean of the deviations of a series computed from any measure of central tendency, i.e., the mean, median or mode. Since a measure of central tendency lies between the minimum and maximum values of the set of observations, therefore, some of the deviations will be negative and some positive. Also, you know that the sum

of deviations from the mean $\bar{X}$, *i.e.*, $\sum_{i=1}^{n}(x_1 - \bar{X})$ of a set of observations $x_1, x_2, x_3, ..., x_n$ is zero, so finding the mean of the deviations from the mean is meaningless. Therefore, we ignore the + and – signs and take all deviations as positive. Thus, we take the sum of the absolute deviations as

$$\sum_{i=1}^{n}|x_i - \bar{X}| = |x_1 - \bar{x}| + |x_2 - \bar{x}| + |x_3 - \bar{x}| + ... + |x_n - \bar{x}|$$

and divide it by n to get the average or means of these deviations as $\dfrac{\sum_{i=1}^{n}|x_i - \bar{X}|}{n}$. This value is the **mean deviation**.

Mean deviation is denoted by the greek letter, small delta (δ) or M.D.

We usually compute M.D. about any of the three averages, mean, median or mode. Sometimes mode may be ill-defined and so, mean deviation is computed from mean and median. But due to wide applications of mean, the mean deviation is generally computed from mean.

21.03. Computation of mean deviation

The mean deviation of a set of n number $x_1, x_2, ..., x_n$ is defined by

Mean deviation = M.D. = $\dfrac{\sum_{i=1}^{n}|x_1 - \bar{X}|}{n}$, which in simpler form may be expressed as

M.D = $\dfrac{\Sigma|x_i - \bar{x}|}{n} = \dfrac{\Sigma|d_i|}{n}$, where $d_i = x_i - \bar{X}$

If $x_1, x_2, ..., x_n$ occur with frequencies $f_1, f_2, ..., f_n$ resprectively, then the mean deviation can be written as

Mean deviation = M.D. = $\dfrac{\sum_{i=1}^{n} f_i|x_i - \bar{X}|}{N}$ where N = Σf_i. In similar form this may be expressed as

M.D. = $\dfrac{\Sigma f_i|x_i - \bar{X}|}{N} = \dfrac{\Sigma f_i|d_i|}{N}$ or more simply as M.D. = $\dfrac{\Sigma f|d|}{N}$

where $N = \sum_{i=1}^{n} f_i = \Sigma f_i$ = Total frequency.

An interesting property of the sum $\sum_{i=1}^{n}|x_1 - a|$ is that it is a minimum when a is the median, *i.e.*, the mean deviation about the median is a minimum.

Procedure

Step 1. Find the average (mean, median or mode) of the series.

Step 2. Find the deviations of items from the average.

Step 3. Ignore the '+' and '–' signs, *i.e.,* find the absolute values of these deviations.

Step 4. Compute the sum of these deviations, *i.e.*, find $\sum_{i=1}^{n}|(x_i - \bar{X})|$

Step 5. Divide the sum obtained in step 4 by the number of items to get the average of these absolute deviations, *i.e.,* the M.D.

21.04. Coefficient of mean deviation

Mean deviation calculated by using any measure of central tendency is an absolute measure. For the purpose of comparing variation among different series which are expressed in the same or different units, a relative mean deviation is required. The relative mean deviation is obtained by dividing the mean deviation by the average used for calculating the mean deviation and is called coefficient of mean deviation. Thus,

$$\text{Coefficient of mean deviation} = \frac{\text{Mean deviation}}{\text{Average used (Mean or Median or Mode)}}$$

If the result is desired in percentage, then the coefficient of mean deviation is computed as under:

$$\text{Coefficient of mean deviation} = \frac{\text{Mean deviation}}{\text{Average speed}} \times 100$$

Ex. 1. Find the mean deviation from the mean for the following marks:
37, 48, 50, 23, 47, 58, 29, 27, 31, 40.

Sol. Mean $\bar{X} = \frac{\Sigma x_i}{n} = \frac{390}{10} = 39$

Marks	*Deviation from the mean* $d_i = (x_i - \bar{X})$	*(Absolute values)* $\lvert d_i \rvert = \lvert x_i - \bar{X} \rvert$
37	– 2	2
48	9	9
50	11	11
23	– 16	16
47	8	8
58	19	19
29	– 10	10
27	– 12	12
31	– 8	8
40	1	1
390		$\Sigma \lvert d_i \rvert = 96$

$$\text{Mean deviation} = \frac{\Sigma \lvert d_i \rvert}{n} = \frac{96}{10} = 9.6$$

Ex. 2. The scores of a batsman in ten innings are:
48, 80, 58, 44, 52, 65, 73, 56, 64, 54. Find the mean deviation from the median.

Sol. Arranging the data in ascending order, we have

44, 48, 52, 54, 56, 58, 64, 65, 73, 80

Here, $n = 10$. So, median is the mean of 5th and 6th terms.

$$\therefore \quad \text{Median } M_d = \left(\frac{56+58}{2}\right) = 57$$

Scores	*Deviations from the median* $d_i = (x_i - M_d)$	*Absolute values of deviations* $\lvert d_i \rvert = \lvert x_i - M_d \rvert$
44	– 13	13
48	– 9	9
52	– 5	5
54	– 3	3
56	–1	1
58	1	1
64	7	7
65	8	8
73	16	16
80	23	23
		$\Sigma \lvert d_i \rvert = 86$

$\therefore$ Mean deviation $= \dfrac{\Sigma \mid d_i \mid}{n} = \dfrac{86}{10} = 8.6.$

Mean deviation of discrete series

In such series, we multiply the absolute deviations by the respective frequencies and find out their total, *i.e.*, $\sum_{i=1}^{n} f_i \mid d_i \mid$. On dividing this sum by the total frequency, *i.e.*, $N = \Sigma f_i$, we obtain the mean deviation.

Ex.3. Find the mean deviation from the mean for the following data:

x	5	10	15	20	25
f	7	4	6	3	5

[*NCERT*]

Sol.

x_i	f_i	$f_i x_i$	$x_i - \bar{X}$	$\lvert d_i \rvert = \lvert x_i - \bar{X} \rvert$	$f_i \lvert d_i \rvert$
5	7	35	5 – 14 = – 9	9	63
10	4	40	10 – 14 = –4	4	16
15	6	90	15 – 14 = 1	1	6
20	3	60	20 – 14 = 6	6	18
25	5	125	25 – 14 = 11	11	55
Total	**25**	**350**			158

$$\text{Mean } \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{350}{25} = 14$$

$$\text{Mean deviation for the mean} = \frac{\Sigma f_i \mid x_i - \bar{X} \mid}{\Sigma f_i} = \frac{158}{25} = 6.32$$

Ex.4. Find the mean deviation from the median for the folloing data:

34, 66, 30, 38, 44, 50, 40, 60, 42, 51.

Sol. Arranging the given data in ascending order, we have

30, 34, 38, 40, 42, 44, 50, 51, 60, 66

Here, the data contains even number of terms (n = 10). So, the median is the mean of 5th and 6th terms, *i.e.*,

$$M_d = \frac{42+44}{2} = 43$$

Now, we take deviations from the median.

x_i	$d_i = x_i - 43$	$\lvert d_i\rvert = \lvert x_i - 43\rvert$
30	30 – 43 = –13	13
34	34 – 43 = – 9	9
38	38 – 43 = – 5	5
40	40 – 43 = – 3	3
42	42 – 43 = – 1	1
44	44 – 43 = 1	1
50	50 – 43 = 7	7
51	51 – 43 = 8	8
60	60 – 43 = 17	17
66	66 – 43 = 23	23
	Total	87

$\therefore$ Mean deviation from the median = $\frac{\Sigma \lvert d_i \rvert}{x} = \frac{87}{10} = 8.7$.

Ex.5. Calculate the mean deviation and coefft. of mean deviation for the following frequency distribution from both mean and median.

Variable (x)	8	10	15	20	25	32	35
Frequency (f)	3	2	4	7	4	3	7

Sol. (*i*) Mean deviation from mean

Variable (x_i)	*Frequency* f_i	fx_i	*Deivation from the mean* (d_i)	$\lvert d_i\rvert$	$f_i\lvert d_i\rvert$
8	3	24	– 14.8	14.8	44.4
10	2	20	– 12.8	12.8	25.6
15	4	60	– 7.8	7.8	31.2
20	7	140	– 2.8	2.8	19.6
25	4	100	2.2	2.2	8.8
32	3	96	9.2	9.2	27.6
35	7	245	12.2	12.2	85.4
	30	685			242.6

Mean = $\frac{\Sigma f_i x_i}{N} = \frac{685}{30} = 22.8$ approx.

Mean deviation $= \dfrac{\Sigma f_i \mid d_i \mid}{N} = \dfrac{242.6}{30} = 8.09$

Coefft. of mean deviation $= \dfrac{\text{M.D.}}{\text{Mean}} = \dfrac{8.09}{22.8} = 0.355$

(*ii*) **Mean deviation from the median**

First, we compute the median.

Variable (x_i)	*Frequency* (f_i)	*Comulative frequency (c.f.)*	$d_i = x_i - M_d$	$\lvert d_i \rvert = \lvert x_i - M_d \rvert$	$f_i \lvert d_i \rvert$
8	3	3	– 12	12	36
10	2	5	– 10	10	20
15	4	9	– 5	5	20
20	7	16	0	0	0
25	4	20	5	5	20
32	3	23	12	12	36
35	7	30	15	15	105
					237

Here, $N = \Sigma f_i = 30$, $\dfrac{N+1}{2} = \dfrac{30+1}{2} = \dfrac{31}{2} = 15.5$

This lies under the c.f., column in 16. Size of the variable corresponding to this is 20.

$\therefore$ The median is 20.

$\therefore$ Median deviation $= \dfrac{\Sigma f_i \mid d_i \mid}{N} = \dfrac{237}{30} = 7.9$

Coefft. of mean deviation $= \dfrac{\text{M.D.}}{\text{Median}} = \dfrac{7.9}{20} = \mathbf{0.395}$.

Ex.6. Find the mean deviation from the median for the following data:

x_i	**15**	**21**	**27**	**30**	**35**
f_i	**3**	**5**	**6**	**7**	**8**

(NCERT)

Sol.

x_i	f_i	*c.f.*	$d_i = x_i - M_d$	d_i	$f_i \lvert d_i \rvert$
15	3	3	15 – 30 = – 15	15	45
21	5	8	21 – 30 = – 9	9	45
27	6	14	27 – 30 = – 3	3	18
30	7	21	30 – 30 = 0	0	0
35	8	29	35 – 30 = 5	5	40
Total			32		148

Since $\Sigma f_i = 29$ is odd, the median is the $\left(\dfrac{n+1}{2}\right)$ th observation, *i.e.*, $\dfrac{29+1}{2}$ th = 15th observation which corresponds to the variable 30. So, $M_d = 30$.

Hence, mean deviation from the median $= \dfrac{\Sigma f_i \mid d_i \mid}{\Sigma f_i} = \dfrac{148}{29} = 5.1$

Note. The methods of determining median of a given distribution are given in Chapter 28.

Mean deviation of a grouped or continuous frequency distribution

The method is the same as for a discrete frequency distribution. The only difference is that in this case we obtain the mid-points of various classes and take deviations of these mid-points from the given central value (Mean or Median, here we are considering only the mean).

Ex. 7. Calculate the mean deviation and coefficient of mean deviation for the following distribution:

Class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Frequency	7	3	5	2	8

Sol.

Class	*Frequency* (f_i)	*Class-mark* (x_i)	fx_i	$d_i = x_i - \bar{x}$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$	$f_i \lvert d_i \rvert$
0 – 10	7	5	35	– 20.4	20.4	142.8
10 – 20	3	15	45	– 10.4	10.4	31.2
20 – 30	5	25	125	– 0.4	0.4	2.0
30 – 40	2	35	70	9.6	9.6	19.2
40 – 50	8	45	360	19.6	19.6	156.8
			635			352.0

Arithmetic mean ($\bar{x}$) $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{635}{25} = 25.4$

$\therefore$ M.D. $= \dfrac{\Sigma f_i \mid d_i \mid}{N} = \dfrac{352}{25} = 14.08$

Coefft. of mean deviation $= \dfrac{\text{M.D.}}{\text{Mean}} = \dfrac{14.08}{25.4} = 0.554$

EXERCISE 21 (*a*)

From the following data, using mean, calculate mean deviation and the coefficient of mean deviation.

[**Hint:** Coeft. of M.D. $= \dfrac{\text{M.D.}}{\text{mean}}$.]

1. 15, 17, 19, 25, 30, 35, 48

2. 21, 23, 25, 28, 30, 32, 38, 39, 46, 48

3. 10, 70, 50, 53, 20, 95, 55, 42, 60, 48, 80

Calculate the mean deviation from the mean for the following frequency distributions.

4.

x_i	3	9	17	23	27
f_i	8	10	12	9	5

5.

x_i	10	11	12	13	14
f_i	3	12	18	12	3

6.

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of students	5	8	15	16	6

7.

Scores	140 – 150	150 – 160	160 – 170	170 – 180	180 – 190	190 – 200
No. of students	4	6	10	18	9	3

8.

Class Interval	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100	100 – 120
Frequency	3	50	84	32	10	3

For the following, using median, calculate mean deviation and coefficient of mean deviation.

9. 3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21

10. 100, 150, 200, 250, 360, 490, 500, 600, 671

11.

x	10	11	12	13	14
f	3	12	18	12	3

12.

x	3	6	9	12	13	15	21	22
f	3	4	5	2	4	5	4	3

ANSWERS

1. M.D. = 9.14, coefft. of M.D. = 0.338
2. M.D. = 7.8, coefft. of M.D. = 0.24
3. M.D. = 17.27, coefft. of M.D. = 0.326
4. 7.09
5. 0.75
6. 9.44 marks,
7. 10.56
8. M.D. = 13.15, coefft. of M.D. = 0.26
9. M.D. = 5.27, coefft. of M.D. = 0.59
10. M.D. = 173.44, coefft. of M.D. = 0.48
11. M.D. = 0.75, coefft. of M.D. = 0.063
12. M.D. = 4.99, coefft. of M.D. = 0.384

21.05. The standard deviation (S.D)

Introduction. It is the most important measure of dispersion and is widely used in many statistical formulae. In this type of dispersion, the drawback of ignoring the algebraic sign in mean deviation is overcome by taking the square of deviations thereby making all the deviations positive.

Defintion. *It is defined as the positive square-root of the arithmetic mean of the squares of the deviations of the given observations from their arithmetic mean and is denoted by the Greek letter σ(small sigma).*

21.06. Computation of S.D.

The following methods are used to calculate standard deviation.

(*a*) Deviation taken from actual mean.

(*b*) By making direct use of the variable without calculating deviations.

(*c*) Deviation taken from assumed mean.

(*d*) Step deviation method.

A. By taking deviations from the actual mean

If $x_1, x_2, ..., x_n$ be a set of numbers, then,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{N}}$$ which can be written in simpler form as

$$\sigma = \sqrt{\frac{\sum(x_i - \bar{X})^2}{N}} = \sqrt{\frac{\sum d_{x_i}^2}{N}} \text{ or } \sqrt{\frac{\sum d_i^2}{N}} \text{ or simply } \sqrt{\frac{\sum d^2}{N}}$$

where d represents the deviations of each number x_i from the mean $\bar{X}$. If $x_1, x_2, ..., x_n$ occur with frequencies $f_1, f_2, ..., f_n$ respectively, the standard deviation can be

written as $\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i(x_i - \bar{X})^2}{N}}$ ... (2)

which in simpler form can be written as

$$\sigma = \sqrt{\frac{\sum f_i(x_i - \bar{X})^2}{N}} = \sqrt{\frac{f_i d_x^2}{N}} = \sqrt{\frac{\sum f d_x^2}{\sum f}} \text{ or simply } \sigma = \sqrt{\frac{\sum f d^2}{\sum f}}$$

where σ (small sigma) is the notation for standard deviation, x is the variable, X is the arithmetic mean, n or Σf is the total number of frequencies.

Note 1. The standard deviation is used in preference to other deviations as it is a neat method of removing the negatives.

Note 2. In certain books ***s*** is used instead of σ as a notation for the standard deviation. In recent years, the practice has been to distinguish between a *sample* and the population from which it is drawn, by using Latin letters for quantities characterising the sample and Greek letters for corresponding quantities characterising the population. But we will use σ for standard deviation throughout this book, as it is a more familiar notation with the majority of the mathematicians.

Note 3. If x is multiplied by k, then $\bar{X}$ and σ respectively change to $k\bar{X}$ and $k\sigma$.

If x is increased by k, then the mean is increased by k, but σ remains unchanged.

21.07. Variance

The square of the standard deviation σ² is called the variance.

In certain respects variance is even more important than the standard deviation. The importance of the variance will be seen later.

On the other hand, variance requires square units for its measurement. Thus, if the population is of heights of persons measured in centimetres, the variance is in square centimetres.

21.08. Short-cut method to find the standard deviation

Following are the two short-cut methods for finding the standard deviation:

(*a*) $\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{\sum x^2}{n} - \bar{X}^2}$ (For a simple distribution)

$$\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \bar{X}^2} \quad \text{(For a frequency distribution)} \qquad ...(1)$$

(*b*) If any arbitrary number A is chosen as the assumed mean, instead of the true mean $\bar{X}$ and if the deviation of x from A is represented by d, then

$$\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \quad \text{(For a simple distribution)}$$

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} \quad \text{(For a frequency distribution)}$$

where $d = x - A$ or $x = A + d$. ...(2)

If A = 0, then (2) reduces to (1).

Proof. (*a*) For a frequency distribution,

$$\sigma^2 = \frac{\Sigma f(x - \bar{X})^2}{\Sigma f}, \text{ where } \bar{X} = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{\Sigma fx^2}{\Sigma f} - 2\bar{X}\frac{\Sigma fx}{\Sigma f} + \frac{\bar{X}^2 \Sigma f}{\Sigma f}$$

$$= \frac{\Sigma fx^2}{\Sigma f} - 2\bar{X}.\bar{X} + \bar{X}^2 = \frac{\Sigma fx^2}{\Sigma f} - 2\bar{X}^2 + \bar{X}^2 = \frac{\Sigma fx^2}{\Sigma f} - \bar{X}^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2$$

$$\Rightarrow \sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \bar{X}^2}.$$

For a simple distribution the formula becomes,

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{\Sigma x^2}{n} - \bar{X}^2}.$$

(*b*) Let $x - A = d$ then $x = A + d$

or $fx = fA + fd$ or $\Sigma fx = \Sigma fA + \Sigma fd$

or $\frac{\Sigma fx}{\Sigma f} = \frac{A\Sigma f}{\Sigma f} + \frac{\Sigma fd}{\Sigma f}$ or $\bar{X} = A + \frac{\Sigma fd}{\Sigma f}$ or $\bar{X} - A = \frac{\Sigma fd}{\Sigma f}$ ...(*i*)

Now, $x - A = (x - \bar{X}) + (\bar{X} - A)$ [Subtracting and adding $\bar{X}$]

$\therefore \quad (x - A)^2 = (x - \bar{X})^2 + (\bar{X} - A)^2 + 2(x - \bar{X})(\bar{X} - A)$

or $\Sigma(x - A)^2 = \Sigma(x - \bar{X})^2 + \Sigma(\bar{X} - A)^2 + 2(\bar{X} - A).\Sigma(x - \bar{X})$

$\Rightarrow \quad \Sigma(x - A)^2 = \Sigma(x - \bar{X})^2 + \Sigma(\bar{X} - A)^2 \qquad \left[\because \Sigma(x - \bar{X}) = \Sigma x - n\bar{X} = \Sigma x - n.\frac{\Sigma x}{n} = 0\right]$

$\Rightarrow \quad \Sigma f(x - A)^2 = \Sigma f(x - \bar{X})^2 + (\bar{X} - A)^2.\ \Sigma f$

$\Rightarrow \quad \Sigma f(x - \bar{X})^2 = \Sigma f(x - A)^2 - (\bar{X} - A)^2.\ \Sigma f$

$$\Rightarrow \quad \frac{\Sigma f(x-\overline{X})^2}{\Sigma f} = \frac{\Sigma f(x-\mathrm{A})^2}{\Sigma f} - (\overline{X}-\mathrm{A})^2 \qquad \Rightarrow \quad \sigma^2 = \frac{\Sigma fd^2}{\Sigma f} - (\overline{X}-\mathrm{A})^2 \qquad ...(ii)$$

$$\Rightarrow \qquad \sigma^2 = \frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2 \qquad \text{[Using } (i)\text{]}$$

or $$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} \qquad ...(iii)$$

For a simple distribution the formula becomes

$$\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \qquad ...(iv)$$

Corollary 1. If s denotes the root-mean-square deviation from the assumed mean A, then, by definition,

$$s^2 = \frac{\Sigma f(x-A)^2}{\Sigma f} = \frac{\Sigma fd^2}{\Sigma f}$$

From (*ii*), we get $\sigma^2 = s^2 - (\overline{\mathrm{X}} - A)^2$ or $\sigma^2 < s^2$.

as $(\overline{\mathrm{X}} - A)^2$, being a perfect square, is positive. $\therefore\ \sigma < s$.

This proves that the root-mean-square deviation is least when the arithmetic mean is taken as the origin. (Art. 21.06, Property 2)

Corollary 2. (*i*) From the formula $\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$,

it is clear that if each score x is multiplied by a number k, the net effect of this operation is to multiply the σ by k.

(*ii*) By comparing the two results, namely,

$$\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2} \quad \text{and} \quad \sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$$

it is clear that if a constant A is subtracted from each score x, the σ remains unchanged.

21.09. Step-deviation method

The above formula can be modified as under:

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} = i\sqrt{\frac{\Sigma f\left(\frac{d}{i}\right)^2}{\Sigma f} - \left(\frac{\Sigma f\frac{d}{i}}{\Sigma f}\right)^2} \qquad \text{[Multiplying and dividing by } i\text{]}$$

or $$\sigma = i\sqrt{\frac{\Sigma fu^2}{\Sigma f} - \left(\frac{\Sigma fu}{\Sigma f}\right)^2}$$

where i is the uniform class interval, and $u = \frac{d}{i}$.

For a simple distribution, $\sigma = i\sqrt{\frac{\Sigma u^2}{n} - \left(\frac{\Sigma u}{n}\right)^2}$.

This formula simplifies calculations to some extent and is applicable only to those distributions which have *uniform* class interval.

Note. In some books instead of *u, t* is used.

21.10. Variance and coefficient of variation

The square of the standard deviation is called **variance** and is denoted by σ^2.

The coefficient of variation denoted by C.V. is given by the formula $\text{C.V.} = \frac{\sigma}{\bar{X}} \times 100$

where σ is the s.d. and $\bar{X}$ is the mean of the given series.

It may be noted that the coefficient of variation is always a percentage. The expression $\frac{\sigma}{\bar{X}}$ is called the **coefficient of standard deviation.**

Note. The coefficient of variation has a great practical significance and is the best measure of comparing the variability of the two given series. The series for which the coefficient of variation is greater is said to be more variable (less consistent). On the other hand, the series for which the variation is less is said to be less variable or (more consistent).

Ex. 8. Find the standard deviation of the first five even natural numbers.

Sol. The first five even natural numbers are 2, 4, 6, 8, 10.

Method 1.

x	x^2
2	4
4	16
6	36
8	64
10	100
$\Sigma x = 30$	$\Sigma x^2 = 220$

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\frac{220}{5} - \left(\frac{30}{5}\right)^2} = \sqrt{44 - 36}$$

$$= \sqrt{8} = 2\sqrt{2} = 2 \times 1.414$$

$= 2 \times 8.28 =$ **2.83** to 2dp.

Method 2.

x	$d = x - \bar{X}$	d^2
2	−4	16
4	−2	4
6	0	0
8	2	4
10	4	16
30	0	40

$$\bar{X} = \frac{\Sigma x}{n} = \frac{30}{5} = 6$$

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{40}{5}} = \sqrt{8}$$

$= 2\sqrt{2} = 2 \times 1.414 =$ **2.83** to 2dp.

Ex. 9. Calculate the standard deviation from the following set of observations:
8, 9, 15, 23, 5, 11, 19, 8, 10, 12

Sol. Method 1. ***(By taking deviations from actual mean)***

Values x	*Deviation* $d = x - \bar{X}$	d^2
8	–4	16
9	–3	9
15	3	9
23	11	121
5	–7	49
11	–1	1
19	7	49
8	–4	16
10	–2	4
12	0	0
120	0	274

$$\bar{X} = \frac{\Sigma x}{n} = \frac{120}{10} = 12$$

$$\text{S.D. } (\sigma) = \sqrt{\frac{\Sigma d^2}{n}}$$

$$= \sqrt{\frac{274}{10}} = \sqrt{27.4}$$

$$= \mathbf{5.23}$$

Method 2. (*By using directly the values of the variable*)

Value (x)	x^2
8	64
9	81
15	225
23	529
5	25
11	121
19	361
8	64
10	100
12	144
120	1714

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\frac{1714}{10} - \left(\frac{120}{10}\right)^2}$$

$$= \sqrt{171.4 - 144}$$

$$= \sqrt{27.4}$$

$$= \mathbf{5.23.}$$

Method 3. ***(By taking deviations from the assumed mean)***

Value (x)	$d = x - 10$	d^2
8	–2	4
9	–1	1
15	5	25
23	13	169
5	–5	25
11	1	1
19	9	81
8	–2	4
10	0	0
12	2	4
	20	314

Let the assumed mean be 10.

$$\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{314}{10} - \left(\frac{20}{10}\right)^2}$$

$$= \sqrt{31.4 - 4}$$

$$= \sqrt{27.4}$$

$$= \mathbf{5.23.}$$

Ex. 10. Calculate the mean and standard deviation of first *n* natural numbers.

Sol. Here, $x_i = 1, 2, ... n$. Let $\bar{X}$ be the mean and σ the S.D. Then

$$\bar{X} = \frac{\Sigma x}{n} = \frac{1+2+3.....+n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

$$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = \frac{1^2+2^2+.....+n^2}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{\mathbf{n^2-1}}{\mathbf{12}}.$$

Type : Discrete Frequency Distribution

I. By direct method

II. By using deviations from actual mean

III. By Step Deviation Method

I. By direct method

Ex. 11. Compute the standard deivation for the following distribution.

Variable (x)	10	15	18	20	25
Frequency (f)	3	2	5	8	2

Sol.

Variable (x)	*Frequency* (f)	$f.x$	$d = x - \bar{X}$	d^2	f^2
10	3	30	–8	64	192
15	2	30	–3	9	18
18	5	90	0	0	0
20	8	160	2	4	32
25	2	50	7	49	98
	$n = 20$	$\Sigma fx = 360$			$\Sigma fd^2 = 340$

Arithmetic mean $(\bar{X}) = \frac{\Sigma fd}{\Sigma f} = \frac{360}{20} = 18$ $\quad \therefore$ S.D. $(\sigma) = \sqrt{\frac{\Sigma fd^2}{n}} = \sqrt{\frac{340}{20}} = \sqrt{17} = \mathbf{4.1.}$

II. By taking deviations from assumed mean

Ex. 12. Calculate the standard deviation for the following distribution.

x	8	11	17	20	25	30	35
f	2	3	4	1	5	7	3

Sol. Let the assumed mean (A) be 20.

Variable (x)	*Frequency* (f)	$d = x - A$	d^2	fd	fd^2
8	2	–12	144	–24	288
11	3	–9	81	–27	243
17	4	–3	9	–12	36
20	1	0	0	0	0
25	5	5	25	25	125
30	7	10	100	70	700
35	3	15	225	45	675
Total	$n = 25$			77	2067

$$\text{S.D. } (\sigma) = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}$$

$$= \sqrt{\frac{2067}{25} - \left(\frac{77}{25}\right)^2} = \sqrt{\frac{2067}{25} - \frac{5929}{625}} = \sqrt{\frac{51675 - 5929}{625}} = \frac{\sqrt{45746}}{25} = \frac{213.9}{25} = \mathbf{8.56.}$$

III. By step deviation method

Ex. 13. Find the variance and standard deviation for the following distribution.

x	4.5	14.5	24.5	34.5	44.5	54.5	64.5
f	1	5	12	22	17	9	4

Sol. Let the assumed mean = 34.5.

x	f	$d = x - 34.5$	$u = \frac{x - 34.5}{10}$	$f.u$	u^2	fu^2
4.5	1	– 30	–3	–3	9	9
14.5	5	– 20	–2	– 10	4	20
24.5	12	– 10	– 1	–12	1	12
34.5	22	0	0	0	0	0
44.5	17	10	1	17	1	17
54.5	9	20	2	18	4	36
64.5	4	30	3	12	9	36
Total	$n = \Sigma f = 70$			$\Sigma fu = 22$		$\Sigma fu^2 = 130$

$$\text{Var } (X) = \sigma^2 = i^2 \left[\frac{\Sigma fu^2}{n} - \left(\frac{\Sigma fu}{n}\right)^2\right] = 100 \left[\frac{130}{70} - \left(\frac{22}{70}\right)^2\right]$$

$$= 100 \left[\frac{13}{7} - \left(\frac{11}{35}\right)^2\right] = 100\ (1.857 - 0.098) = 100 \times 1.759 = 175.9$$

$$\text{S.D. } (\sigma) = \sqrt{175.9} = \mathbf{13.26.}$$

Ex. 14. Calculate the standard deviation of the following distribution:

Age	20–25	25–30	30–35	35–40	40–45	45–50
Number of persons	170	110	80	45	40	35

Sol. Let the assumed mean be 32.5.

Age	*Mid-Values*	*Number of Persons (f)*	$d = x - 32.5$	d^2	fd	fd^2
20–25	22.5	170	–10	100	–1700	17000
25–30	27.5	110	– 5	25	–550	2750
30–35	32.5	80	0	0	0	0
35–40	37.5	45	+ 5	25	+ 225	1125
40–45	42.5	40	+10	100	+400	4000
45–50	47.5	35	+15	225	+525	7875
		$n = \Sigma f = 480$			$\Sigma fd = -1100$	$\Sigma fd^2 = 32750$

$$\text{S.D. } (\sigma) = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2} = \sqrt{\frac{32750}{480} - \left(\frac{-1100}{480}\right)^2} = \sqrt{68.23 - 5.25} = \sqrt{62.98} = \mathbf{7.936.}$$

Ex. 15. Calculate the standard deviation of the distribution given in the following table.

Length of wire (in cm)	*No. of wires*	*Length of wire*	*No. of wires*
72.0–73.9	**7**	**82.0–83.9**	**24**
74.0–75.9	**31**	**84.0–85.9**	**22**
76.0–77.9	**42**	**86.0–87.9**	**8**
78.0–79.9	**54**	**88.0–89.9**	**4**
80.0–81.9	**33**		

Sol. Let the assumed mean be 80.95.

Length of wire	*No. of wire*	x	$u = \frac{x - 80.95}{2}$	u^2	fu	fu^2
72.0–73.9	7	72.95	–4	16	–28	112
74.0–75.9	31	74.95	–3	9	–93	279
76.0–77.9	42	76.95	– 2	4	–84	168
78.0–79.9	54	78.95	–1	1	–54	54
80.0–81.9	33	80.95	0	0	0	0
82.0–83.9	24	82.95	+1	1	24	24
84.0–85.9	22	84.95	+2	4	44	88
86.0–87.9	8	86.95	+3	9	24	72
88.0–89.9	4	88.95	+4	16	16	64
Total	$n = 225$				$\Sigma fu = -151$	$\Sigma fu^2 = 861$

$$\text{S.D. } (\sigma) = \sqrt{\frac{\Sigma fu^2}{n} - \left(\frac{\Sigma fu}{n}\right)^2} \times i = \sqrt{\frac{861}{225} - \left(\frac{-151}{225}\right)^2} \times 2$$

$$= \sqrt{3.8267 - 0.4504} \times 2 = \sqrt{3.3763} \times 2 = 1.8375 \times 2 = \mathbf{3.68}.$$

Ex. 16. A purchasing agent obtained samples of 60 watt bulbs from a standard company. He had the samples tested in his own laboratory for length of life with the following results:

Length of life (in hours)	*Number*
1700 and under 1900	**10**
1900 and under 2100	**16**
2100 and under 2300	**20**
2300 and under 2500	**8**
2500 and under 2700	**6**

Calculate the standard deviation for these samples. ***[ISC 1993]***

Sol. Let the assumed mean be 2200.

Length of life (in hours)	x	f	$d = x - 2200$	$u = \frac{d}{200}$	$f.u$	$f.u^2$
1700 and under 1900	1800	10	– 400	– 2	– 20	40
1900 and under 2100	2000	16	– 200	– 1	– 16	16
2100 and under 2300	2200	20	0	0	0	0
2300 and under 2500	2400	8	200	1	8	8
2500 and under 2700	2600	6	400	2	12	24
		$\Sigma f = 60$			$\Sigma fu = -16$	$\Sigma fu^2 = 88$

By step deviation method,

$$\sigma = i\sqrt{\frac{\Sigma fu^2}{\Sigma f} - \left(\frac{\Sigma fu}{\Sigma f}\right)^2} = 200\sqrt{\frac{88}{60} - \left(\frac{-16}{60}\right)^2} = 200\sqrt{\frac{22}{15} - \left(\frac{-4}{15}\right)^2}$$

$$= 200\sqrt{\frac{22}{15} - \frac{16}{225}} = 200 \times \frac{\sqrt{314}}{15} = \frac{40}{3} \times 17.720 = \mathbf{236.27 \text{ hours}}$$

Ex. 17. (*i*) Given the following information, find standard deviation.

(*ii*) The mean of 200 items is 48 and their standard deviation is 3. Find the sum, and sum of squares of all items.

Sol. (*i*) $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$. Substituting the values, we get

$$\sigma^2 = \frac{1000}{10} - \left(\frac{60}{10}\right)^2 = 100 - 36 = 64 \qquad \therefore \sigma = \sqrt{64} = 8.$$

(*ii*) It is given that $n = 200$, $\bar{x} = 48$ and $\sigma = 3$ and it is required to find the value of Σx and Σx^2.

Now $\bar{x} = \dfrac{\Sigma x}{n} \Rightarrow \Sigma x = n\,\bar{x} = 200 \times 48 = 9600$

Also $\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$. Substituting the values, we get

$$9 = \frac{\Sigma x^2}{200} - \left(\frac{9600}{200}\right)^2 \Rightarrow \frac{\Sigma x^2}{200} = 9 + (48)^2 = 9 + 2304 = 2313 \Rightarrow \Sigma x^2 = 200 \times 2313 = \mathbf{4,62,600.}$$

Ex. 18. For the distribution given in the following table, find the mean and the standard deviation by the step deviation method.

Table

Marks	*Frequency*	*Marks*	*Frequency*
0 – 4	2	20 – 24	21
5 – 9	5	25 – 29	16
10 – 14	7	30 – 34	8
15 – 19	13	35 – 39	3

Sol. We rewrite the table by taking the mid-value of each class as the value of x:

x	f	$d = x - 22$	$u = \dfrac{d}{5}$	fu	fu^2
2	2	– 20	– 4	– 8	32
7	5	– 15	– 3	– 15	45
12	7	– 10	– 2	– 14	28
17	13	– 5	– 1	– 13	13
22	21	0	0	0	0
27	16	5	1	16	16
32	8	10	2	16	32
37	3	15	3	9	27
Total	75			– 9	193

The assumed mean $A = 22$.

$$\bar{x} = A + \frac{\Sigma fu}{\Sigma f} = 22 + 5 \times \frac{-9}{75} = 22 - 0.6 = 21.4.$$

$$\sigma = i\sqrt{\frac{\Sigma fu^2}{\Sigma f} - \left(\frac{\Sigma fu}{\Sigma f}\right)^2} = 5\sqrt{\frac{193}{75} - \left(\frac{-9}{75}\right)^2}$$

$$= 5 \times \sqrt{\frac{193}{75} - \frac{81}{5625}} = 5 \times \sqrt{\frac{193 \times 75 - 81}{5625}} = 5 \times \frac{1}{75}\sqrt{14475 - 81}$$

$$= \frac{1}{15}\sqrt{14394} = \frac{1}{15} \times 119.975 = 7.998 = 8 \text{ (to the nearest whole number)}$$

Ex. 19. A student obtained the mean and standard deviation of 100 observations as 40 and 5.1 respectively. Later it was discovered that he had wrongly copied down an observation as 50 instead of 40. Calculate the true mean and standard deviation. ***[ISC]***

Sol. Let the hundred observations be

$$x_1, x_2, x_3, \ldots, x_{99}, x_{100}$$

Suppose that the correct value of x_{100} was 40 and that by mistake the student had copied it as 50. Then

$$\frac{x_1 + x_2 + x_3 + \ldots + x_{99} + 50}{100} = 40$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + \ldots + x_{99} = 4000 - 50 = 3950$$

which is the true mean.

Also $$\sigma^2 = \frac{\Sigma x^2}{n} - \bar{X}^2$$

$$\text{or } (5.1)^2 = \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x^2{}_{99} + x^2{}_{100}}{100} - 40^2 = \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{99}^2}{100} + \frac{50^2}{100} - 40^2$$

$$\therefore \quad \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{99}^2}{100} = (5.1)^2 + 40^2 - 25$$

Now $$\sigma^2 = \left(\frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{99}^2}{100} + \frac{x_{100}^2}{100} \right) - (39.9)^2$$

$$= (5.1)^2 + 40^2 - 25 + \frac{40^2}{100} - (39.9)^2 = 26.01 + 1600 - 25 + 16 - (39.9)^2$$

$$= 17.01 + 1600 - (39.9)^2 = 1617.01 - 1592.01 = 25$$

$\therefore$ $\sigma = 5$, which is the true standard deviation.

Ex. 20. In two factories A and B engaged in the same industrial area, the average weekly wages (in rupees) and the standard deviations are as follows:

Factory	*Average*	*S.D.*	*No. of workers*
A	34.5	5	476
B	28.5	4.5	524

(*i*) Which factory, A or B, pays out a larger amount as weekly wages?

(*ii*) Which factory, A or B, has greater variability in individual wages?

Sol. (*i*) Total wages paid by factory $A = ₹\ 34.5 \times 476 = ₹\ 16{,}422$

Total wages paid by factory $B = ₹\ 28.5 \times 524 = ₹\ 14,\ 934$

Therefore, factory *A* pays out larger amounts as weekly wages.

(*ii*) C.V. of distribution of weekly wages of factory *A* is

$$\text{C.V.(A)} = \frac{\sigma}{\bar{x}} \times 100 = \frac{5}{34.5} \times 100 = \mathbf{14.49}$$

$$\text{C.V. (B)} = \frac{\sigma}{\bar{x}} \times 100 = \frac{4.5}{28.5} \times 100 = \mathbf{15.79}$$

Since C.V. of factory *B* is greater than that of factory *A*, therefore, the factory *B* has greater variability in individual wages.

21.11. Some important points regarding the different measures of variability

1. The range may be used:

(*a*) When the data are too scant or too scattered to justify the computation of a more precise measure of variability :

(*b*) When a knowledge of extreme scores or of total dispersion is all that is required.

2. The semi-interquartile range $Q\left(=\frac{Q_3 - Q_1}{2}\right)$ may be used :

(*a*) When the median is the measure of central tendency ;

(*b*) When the middle 50% of the scores are of primary importance ;

(*c*) When the distribution has extreme scores which would affect the standard deviation (σ) disproportionately.

3. The standard deviation (σ) may be used :

(*a*) When the arithmetic mean is used for the central tendency ;

(*b*) When the statistics having the greater stability is sought;

(*c*) When the coefficient of correlation (r) and other statistics which depend on the standard deviation are to be computed later on (see Chapter on Correlation in Book II.).

EXERCISE 21 (*b*)

1. (*i*) Five students secured marks as; 8, 10, 15, 30, 22. Find the standard deviation.

(*ii*) For a set of ungrouped values the following sums are found:

$$n = 15,\ \Sigma x = 480,\ \Sigma x^2 = 15735.$$

Find the standard deviation.

(*iii*) The standard deviation of the numbers 2, 3, 11, x is 3½. Calculate the values of x.

2. Calculate the standard deviation and variance for the integers 11, 12, 13, ..., 20. **[*ISC 1995*]**

3. Find the standard deviation of the following set of numbers:

25, 50 , 45, 30, 70, 42, 36, 48, 34, 60 **[*ISC 1996*]**

4. Calculate the possible values of x, if the standard deviation of the numbers 2, 3, $2x$ and 11 is 3.5. **[*ISC 1994*]**

5. Calculate the standard deviation for the following distribution:

Class interval	0–4	4–8	8–12	12–16
Frequency	4	8	2	1

[*ISC 2000*]

6. Calculate the standard deviation of the the following data:

Size	4	5	6	7	8	9	10
Frequency	6	12	15	28	29	14	15

7. Calculate the standard deviation of the following data:

Class interval	0–6	6–12	12–18	18–24	24–30	30–36	36–40
Frequency	19	25	36	72	51	43	28

8. Calculate the standard deviation for the following data giving the age distribution of persons.

Age in years	20–30	30–40	40–50	50–60	60–70	70–80	80–90
No. of persons	3	61	132	153	140	51	2

Calculate the mean of these differences and their standard deviation. **[*ISC*]**

9. The heights, to the nearest cm, of 30 men are given below:

159	170	174	173	175	160	161	164	163	165
164	171	162	170	177	185	181	180	175	165
186	174	168	168	176	176	165	175	167	180

Using class intervals 155 – 160, 160 – 165, ... draw up a grouped frequency distribution and use this to estimate the Arithmetic mean and standard deviation. **[*ISC*]**

10. Find the mean and the standard deviation from the following:

Wages (in Rupees)	120 – 200	200 – 210	210 – 220	220 – 230
No. of workers	10	12	18	20
Wages (in Rupees)	230 – 240	240 – 250	250 – 260	260 – 270
No. of workers	25	18	16	5

[*ISC*]

11. The following table shows the I.Q. of 480 schoolchildren. Find

(*i*) the mean.

(*ii*) the standard deviation using the step deviation method. Use Charlier's check to verify the computation of the standard deviation.

x	70	74	78	82	86	90	94	98	102	106	110	114	118	122	126
f	4	9	16	28	45	66	85	72	54	38	27	18	11	5	2

12. In a certain test, the 30 scores were grouped as follows:

30 – 34	35 – 39	40 – 44	45 – 49	50 – 54	55 – 59	60 – 64
2	2	7	10	6	2	1

Calculate the mean and the standard deviation. **[*ISC*]**

13. The number of faults on the surface of each of 1000 tiles were distributed as follows:

No. of faults	0	1	2	3	4	5
Frequency	760	138	67	25	8	2

Calculate the mean and the standard deviation. **[*ISC*]**

14. The mean and the standard deviation of 25 observations are 60 and 3. Later on it was decided to omit an observation which was incorrectly recorded as 50. Calculate the mean and the standard deviation of the remaining 24 observations. **[*ISC 1991*]**

15. The scores of two golfers for 10 rounds each are:

A	58	59	60	54	65	66	52	75	69	52
B	84	56	92	65	86	78	44	54	78	68

Which may be regarded as the more consistent player?

16. Goals scored by two teams A and B in a football season were as follows:

Number of goals scored in a match	*Number of matches*	
	A	*B*
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

By calculating the coefficient of variation in each case find which team may be considered more consistent.

17. The mean of the numbers a, b, 8, 5, 10 is 6 and the variance is 6.80. Then which one of the following gives possible values of a and b ?

(*a*) $a = 0, b = 7$ (*b*) $a = 5, b = 2$ (*c*) $a = 1, b = 6$ (*d*) $a = 3, b = 4$ **[*AIEEE 2008*]**

ANSWERS

1. (*i*) 5.44 marks (*ii*) 5 (*iii*) $6,4\frac{2}{3}$. **2.** $\sigma = 2.87$, variance $= \sigma^2 = 8.27$

3. 13.1 **4.** $x=3$ or $\frac{7}{3}$ **5.** 3.3 approx **6.** 1.54 **7.** 9.99 approx

9.

155 – 160	160 – 165	165 – 170	170 – 175	175 – 180	180 – 185	185 – 190
1	6	6	6	6	3	2

Mean = 171.5 cm, $\sigma = 7.9$ cm

10. Mean = 226.77 approx., $\sigma =$ ₹ 25.49 approx.

11. $\bar{X} = 95.97$, $\sigma = 10.47$ **12.** $\bar{X} = 46\frac{1}{3}$, $\sigma = 6.8$ **13.** $\bar{X} = 0.389$, $\sigma = 0.810$

15. A **16.** B **17.** (*d*)

HINTS TO SELECTED QUESTIONS

1. (*c*) Use $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

2. $\Sigma x = 11+12+13+...+20 = \frac{10}{2}(11+20) = 5\times 31 = 155$ [Using $s = \frac{n}{2}(a+l)$]

$\Sigma x^2 = 11^2 + 12^2 + 13^2 + ... + 20^2$

$= (1^2 + 2^2 + 3^2 + ... + 20^2) - (1^2 + 2^2 + 3^2 + ... + 10^2)$

$= \frac{20(20+1)(40+1)}{6} - \frac{10(10+1)(20+1)}{6}$ $\left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}\right]$

Apply $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

3. You may take assumed mean = 45 and use $\sigma^2 = \frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2$

8. Age is a continuous variable. Take assumed mean = 54.5.

Age in years	*Actual limits*	x	f	$d = x - 54.5$	$u = \frac{d}{10}$	fu	fu^2	*c (c.f)*
20 – 30	19.5 – 29.5	24.5	3	–30	–3	–9	27	3
30 – 40	29.5 – 39.5	34.5	61	–20	–2	–122	244	64
40 – 50	39.5 – 49.5	44.5	132	–10	–1	–132	132	196
50 – 60	49.5 – 59.5	54.5	153	0	0	0	0	349
60 – 70	59.5 – 69.5	64.5	140	10	1	140	140	489
70 – 80	69.5 – 79.5	74.5	51	20	2	102	204	540
80 – 90	79.5 – 89.5	84.5	2	30	3	6	18	542
			$\Sigma f = 542$			$\Sigma fu = -15$	$\Sigma fu^2 = 765$	

$$\sigma = i\sqrt{\frac{\Sigma fu^2}{\Sigma f} - \left(\frac{\Sigma fu}{\Sigma f}\right)^2} = 10.\sqrt{\frac{765}{542} - \left(\frac{-15}{542}\right)^2} = \mathbf{11.87\ years.}$$

14. Let the 25 observations be $x_1, x_2, x_3, \ldots, x_{23}, x_{24}, x_{25}$

Then, $\frac{x_1 + x_2 + x_3 + \ldots + x_{23} + x_{24} + x_{25}}{25} = 60$

If $x_{25} = 50$, then $\frac{x_1 + x_2 + x_3 + \ldots + x_{23} + x_{24} + 50}{25} = 60$

or $x_1 + x_2 + x_3 + \ldots + x_{23} + x_{24} = 60 + 25 - 50 = 1450$

The mean of the remaining 24 observations $= \frac{x_1 + x_2 + x_3 + \ldots + x_{23} + x_{24}}{24} = \frac{1450}{24} = \mathbf{60\frac{5}{12}}.$

Now, $\sigma^2 = \frac{\Sigma x^2}{n} - \bar{x}^2$ or $3^2 = \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{25}^2}{25} - 60^2$

$\Rightarrow \quad 9 = \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{24}^2}{25} + \frac{50^2}{25} - 60^2$ [We suppose that $x_{25} = 50$.]

$\Rightarrow \quad \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{24}^2}{25} = 9 + 3600 - 100 = 3509$

$\Rightarrow \quad x_1^2 + x_2^2 + x_3^2 + \ldots + x_{24}^2 = 25 \times 3509$

If σ_1 is the standard deviation of the remaining 24 observations, then

$$\sigma_1^2 = \frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{24}^2}{24} - \left(\frac{1450}{24}\right)^2$$

$$= \frac{25 \times 3509}{24} - \frac{(1450)^2}{24 \times 24} = \frac{24 \times 25 \times 3509 - (1450)^2}{24 \times 24} = \frac{2105400 - 2102500}{24 \times 24} = \frac{29 \times 100}{24 \times 24}$$

$\Rightarrow \quad \sigma_1 = \frac{10}{24} \times \sqrt{29} = \frac{10}{24} \times 5.385 = \mathbf{2.24}$

15. Compare the coefficients of variation.

16. You will find that the coeff. of variation for the team B is less, so team B is more consistent.

17. $\frac{a+b+8+5+10}{5} = 6 \Rightarrow a+b=7$ .... (i)

Given : Variance of $a, b, 8, 5, 10, \Rightarrow 6.80$

$\Rightarrow \frac{(a-6)^2+(b-6)^2+(8-6)^2+(5-6)^2+(10-6)^2}{5} = 6.80$

$\Rightarrow a^2-12a+36+(1-a)^2+21=34$ [Using (i)]

$\Rightarrow a=3$ or $4 \Rightarrow b=4$ or 3.

CHAPTER SUMMARY

1. Mean deviation

(i) If $x_1, x_2, ..., x_n$ are n numbers and $\bar{X}$ is their arithmetic mean or median or mode, then the mean deviation of these numbers is given by M.D. $= \frac{\Sigma | d_i |}{x}$, where $d_i = x_i - \bar{X}$.

(ii) If $x_1, x_2, x_3, ..., x_n$ occur with frequencies $f_1, f_2, ..., f_n$ then the mean deviation is given by

$$\text{M.D.} = \frac{\Sigma f_i | d_i |}{N}, \text{ where } N = \Sigma f_i.$$

2. Formulas for evaluating S.D.

(i) $\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{\Sigma x^2}{n} - \bar{X}^2}$ (For a simple distribution)

(ii) $\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \bar{X}^2}$ (For a frequency distribution)

(iii) $\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$

(iv) $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

} When deviations are taken from any assumed mean

3. Step deviation method

$$\sigma = i\sqrt{\frac{\Sigma fu^2}{\Sigma f} - \left(\frac{\Sigma fu}{\Sigma f}\right)^2}$$

4. Variance $= \sigma^2$ and coefft. of variance (C.V.) $= \frac{\sigma}{\bar{X}} \times 100$

CHAPTER TEST

1. Find the mean deviation from the mean for the following data:

38, 70, 48, 40, 42, 55, 63, 46, 54, 44

2. Find the mean deviation from the mean for the following data:

x_i	3	5	7	9	11	13
f_i	6	8	15	25	8	4

3. Find the mean deviation for the mean for the following data:

Classes	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Frequencies	6	8	14	16	4	2

4. Find the mean deviation about the median for the following data:

11, 3, 8, 7, 5, 14, 10, 2, 9

5. Find the variance and standard deviation of the following data:

x_i	92	93	97	98	102	104	109
f_i	3	2	3	2	6	3	3

6. Calculate the mean and variance after the following data:

Classes	0 – 30	30 – 60	60 – 90	90 – 120	120 – 150	150 – 180	180 – 210
Frequency (f)	2	3	5	10	3	5	2

ANSWERS

1. 8.4 **2.** 2.09 **3.** 10.24 **4.** 3

5. Variance = 29.09, S.D. = 5.395 **6.** Mean = 107, Variance = 2276.1

CHAPTER 22

PROBABILITY

22.01. Random experiment

Experiment. A process which results in some well defined outcomes, is known as an experiment.

Random experiment. An estimate of probability is often made experimentally by conducting a special type of experiment called *random experiment.*

Definition. ***A random experiment is an experiment in which (i) all the outcomes of the experiment are known in advance and (ii) the exact outcome of any specific performance of the experiment is unpredictable, i.e., not known in advance.***

Thus (*a*) Tossing a coin is a random experiment because: (*i*) Assuming that the coin does not land on the edge there are two possible outcomes of the experiment: heads and tails; (*ii*) On any performance of the experiment it is not possible to know or predict what the outcome will be.

(*b*) Throwing an unbiased die is a random experiment because : (*i*) we know in advance that the possible outcomes are the showing of the faces 1, 2, 3, 4, 5 and 6; (*ii*) for any specific throw of the die, it is not possible to know or predict the outcome.

(*c*) Drawing a card from a well shuffled pack of 52 cards is a random experiment because: (*i*) we know in advance about the 52 possible outcomes; (*ii*) we do not know in advance which card is drawn in any specific draw.

Outcome. An outcome is the result of an experiment, such as scoring a 3 or a 6 on throwing a die.

22.02. Sample space

The set of all possible outcomes of a random experiment is known as a sample space associated with the random experiment. It is generally denoted by $\boldsymbol{S}$.

Each element of a sample space is called a **sample point.**

Illustrations :

1. When a coin is tossed there are only two possible outcomes, 'heads' or 'tails'. The sample space consists of these two sample points.

2. When a die is thrown once, the upper most face may show any one of the numbers '1', '2', '3', '4', '5' or '6'. The sample space of the experiment is $S = \{1, 2, 3, 4, 5, 6\}$.

3. Two marbles are to be drawn successively from a bag known to contain only red marbles and blue marbles. List a sample space for the experiment, and then list the event that both marbles are of the same colour.

Sol. Since the marbles must be either red or blue, a sample space is $\{(r, r), (r, b), (b, r), (b, b)\}$. The event that both are of the same colour is $\{(r, r), (b, b)\}$.

Remark. It is convenient to discuss the sample space of an experiment involving two distinct occurrences, such as two marbles, as the Cartesian product of the sample space of each of the occurrences. In the preceding example, the sample point for the drawing of the first marble

red and second blue is $\{r, b\}$. The sample space for the experiment is simply $\{r, b\} \times \{r, b\}$, *i.e.*, $\{(r, r), (r, b), (b, r), (b, b)\}$

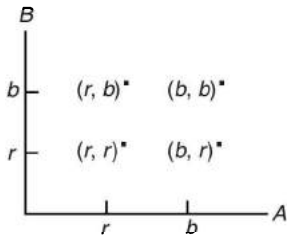


Fig. 22.01

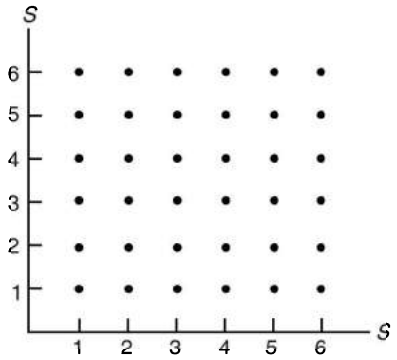


Fig. 22.02

4. A sample space associated with the random experiment '*throwing a coin thrice*' is

$$*S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same.

$$A = \{HHH, HHT, THH, HTH\} \text{ and } B = \{HHH, TTT\}$$

22.03. Definitions of basic terms and concepts

1. Event: An event is something that happens. An event associated with a random experiment is defined as a *subset* of the sample space associated with that random experiment. Getting a head or a tail is an event of the sample space when a coin is tossed.

2. Occurrence of an event: If a die is thrown and the desired event is getting an even number on the top, then, the event is said to *have occurred* if the number appearing on the top of the die is either 2 or 4 or 6. If 1 or 3 or 5 appears on top, then, we say that the event A has *not occurred.*

3. Elementary or simple event: An event containing only a single sample point is called an *elementary event*. An elementary event cannot be further decomposed in simpler events. Each elementary event is a subset of a sample space S containing a simple event.

Illustrations. (*i*) If a coin is tossed, getting a head or tail are two elementary events.

(*ii*) If a die is thrown, getting any one of 1, 2, 3, 4, 5 and 6 on the top are six elementary events.

4. Compound or composite event: An event containing more than one sample point is called a *compound or composite* event. A compound event can be further decomposed into simpler events. Thus, a compound event is a set of elementary events of the sample space S and **it is a subset of *S*.**

Illustrations:

(*i*) If a die is thrown, then the statement 'getting an odd number on the top' is a compound event consisting of the sample points 1, 3 or 5. We may denote the event by A and write $A = \{1, 3, 5\}$. Clearly, A is a subset of the sample space $S = \{1, 2, 3, 4, 5, 6\}$, *i.e.*, $A \subset S$. (*ii*) If 2 coins are tossed, then the sample space S is given by

$$S = \{(H, H), (H, T), (T, H), (T, T)\} \quad \vdots \quad \{H, T\} \times \{H, T\}$$

If A be the desired event that one head and one tail will turn up, then

$$A = \{(H, T), (T, H)\} \text{ is a compound event.}$$

and A is a subset of S, *i.e.*, $A \subset S$.

---

* $S = \{H, T\} \times \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\} \times \{H, T\}$
$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

If the outcome is (H, T) or (T, H), we say that the event A has occurred, but if the outcome is (H, H) or (T, T), then, we say that the event A has not occurred.

Conclusion:

Every event associated with a random experiment can be represented by a subset E of the sample space S. Conversely, every subset E of S represents an event associated with the random experiment. The event represented by E is said to have occurred if the outcome 'a' of the random experiment $\in$ ***E.***

If a $\notin$ ***E, we say that the event represented by the subset E has not occurred.***

5. Sure Event: Let S be a sample space. Since the set S is a subset of itself, it represents an event. Since all the outcomes of a random experiment belong to its sample space S, the event S represents the *sure event.*

6. Impossible event: Since the null set ϕ is also a subset of S, therefore, it also represents an event. As no outcome of the experiment belongs to ϕ, the event ϕ can never occur when the experiment is performed. For this reason, the event ϕ is called the impossible event.

7. Meaning of equally likely outcomes: The outcomes of a random experiment are said to be equally likely if none of them are expected to occur in preference to the other.

When we look at a coin, we have no reason to believe that it is any more likely to show heads than tails. These two outcomes are the only possible outcomes of tossing a coin, and they are equally likely, (In saying this we have totally discounted the possibility of the coin landing on its edge).

When we roll a die the outcome is the number on the top face when it comes to rest. There are 6 faces, numbered 1 to 6. We have no reason to believe that any face is more likely to be on top than any of the others. So there are 6 equally likely outcomes : 1, 2, 3, 4, 5 and 6.



Fig. 22.03

8. Complementary event: Let S be the sample space associated with a random experiment and let E be an event. The event of non-occurrence of E is called the complementary event of E. It is represented by $\overline{E}$ or E'. We also call the event $\overline{E}$ as **not** E.

Illustration:

If in the throw of a die, getting a prime number is a favourable event A, then the event B of getting a number which is not prime, *i.e.,* 1, 4, 6 is the complementary event of A.

If $A = \{2, 3, 5\}$, then, $A' = \{1, 4, 6\}$.

9. Exhaustive events: The total number of possible outcomes of a random experiment is called exhaustive events.

For example,

(*i*) In tossing a coin, exhaustive events are two.

(*ii*) In throwing a die, the exhaustive number of cases is 6, since any one of the six faces marked with 1, 2, 3, 4, 5, and 6 may come uppermost.

(*iii*) In drawing 4 cards from a well-shuffled pack of 52 cards, the exhaustive number of cases is ${}^{52}C_4$, since 4 cards can be drawn out of 52 cards in ${}^{52}C_4$ ways.

10. Mutually exclusive: The events are said to be mutually exclusive if they cannot occur simultaneously in a single draw. The occurrence of one of them automatically excludes the possibility of the occurrence of the other in a single observation. All simple events are mutually exclusive.

Consider the experiment of drawing either a red ball or a green ball or a black ball from a bag that contains red, green and black balls. In drawing a ball three events may take place. We may draw a red ball, we may draw a green ball, we may draw a black ball. If any one occurs, the other two cannot. Such events are mutually exclusive, because the occurrence of one prevents the occurrence of any of the others.

Again suppose a card is taken from a pack of cards and we define events as below:

(*a*) A — the card taken is a spade;

(*b*) B — the card taken is a heart;

(*c*) C — the card taken is a king.

A and B are mutually exclusive since if the card is a spade it cannot be a heart.

$\therefore$ If A occurs then B cannot occur and if B occurs then A cannot occur.

On the other hand, A and C are not mutually exclusive, because the card can be the king of spades when both A and C will have occurred.

Similarly, B and C are not mutually exclusive.

Two events, E and F are said to be mutually exclusive when through the occurrence of one of them, say F, the other event E cannot take place and vice versa.

Other examples of mutually exclusive events are:

(*i*) A coin cannot come up both heads and tails on the same toss, so the event of a coin coming up head and the event of the same coin coming up tail on a single toss are mutually exclusive. However, on two different tosses of a coin the result of the first toss in no way affects the result of the second, so the events of coming up heads on the first throw and coming up tails (or heads) on the second throw are not mutually exclusive.

(*ii*) If a die comes up 6 it cannot come up any other number on the same throw.

(*iii*) If you are 170 cm tall at the present time you are not of any other height.

22.04. Algebra of events: Set theoretic approach

Set theory proves very useful in building up a sound approach to probability since the elements in a set are analogous to the events in a sample space.

Let A and B be any two events associated with a random experiment, then

(*i*) Union of two events:

$A \cup B$ is the event that occurs if A occurs or B occurs (or both). Thus, if A is the event that a person will join Engineering course and B is the event that he will join MBA course, then $A \cup B$ is the event that this person will join either Engineering or MBA or both.

(*ii*) Intersection of two events:

$A \cap B$ is the event that occurs if A occurs and B occurs. It is the event set which consists of all the outcomes (sample points) which the two event sets A and B have in common.

With reference to the above example, $A \cap B$ is the event that the person will join both Engineering and MBA courses.

(*iii*) Complement of an event set:

The complement of A is the event that occurs if A does not occur. It is denoted by A^c or A', or $\overline{A}$. It is the set of all sample points of the sample space S that are not contained in A.

The complement of $A \cup B$ is $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ [A result from Set Theory.]

For example, if we toss a die and observe the number that appears on top, then the sample space consists of the six possible numbers : $S = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event that an even number occurs, B that an odd number occurs and C that a prime number occurs :

$A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{2, 3, 5\}$ then

$A \cup B = \{1, 2, 3, 4, 5, 6\}$ is the event that an even or an odd number occurs.

$A \cap C = \{2\}$ is the event that an even prime number occurs.

$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs.

$A' = \{1, 3, 5\}$ is the event that an even number does not occur.

(*iv*) $A \subseteq B$ means that event B implies event A.

(*v*) Mutually exclusive events:

Two or more events are mutually exclusive if the events cannot occur together. In this case A and B are disjoint sets since they have no elements in common, *i.e.,* $\boldsymbol{A \cap B = \phi}$.

For example, a sample space comprising 6 possible outcomes in a single throw of a die can have three subsets (events) one comprising odd numbers $A = \{1, 3, 5\}$, the second of even numbers, $B = \{2, 4, 6\}$ and the third of multiples of 3, $C = \{3, 6\}$. Then

$A \cap B = \phi$ and $A \cap C \neq \phi$. So, A and B are mutually exclusive events but A and C are not mutually exclusive events.

(*vi*) Not mutually exclusive or partially overlapping events:

Two or more events are considered partially overlapping (also called not mutually exclusive) if part of one event and part of another event can occur together.

For example, the events comprising spade cards and face cards are not mutually exclusive.

(*vii*) Mutually exclusive and exhaustive system of events:

Let S be the sample space associated with a random experiment. If $E_1, E_2, \ldots\ldots, E_n$ are elementary events associated with a random experiment.

If (*i*) $E_i \cap E_j = \phi$ for $i \neq j$ and (*ii*) $E_1 \cup E_2 \ldots \cup E_n = S$,

Then $E_1, E_2, .., E_n$ form a mutually exclusive and exhaustive system of events.

For example, since $A \cap A' = \phi$ and $A \cup A' = S$, so A and A' form mutually exclusive and exhaustive set of events.

Let 1 ball be drawn from a bag containing 12 balls of which 4 balls are white, 4 red and 4 green. Let A be the event "the ball drawn is white", B the event, "the ball drawn is red", and C the event, "the ball drawn is green", then it is obvious that one of these three events must occur, since the ball drawn is either white or red or green.

Hence, A, B, C form a mutually exclusive and exhaustive set of events.

In this case, $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$ and $A \cup B \cup C = S$.

22.05. Set theoretic notations

Various set theoretic notations discussed so far and which will be found useful in probability theory are summarised in the following table. The equivalent forms of probability statements and the corresponding statements in set theoretic notations are given.

Notations and Operations on event sets

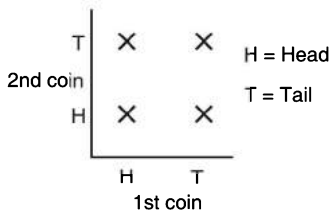
Recall the following from chapter Due.

Terms/statement of probability theory	Set notations
Sample space	S
Outcome of the random experiment	a
Event A	$A \subset S$
Event A has occurred	$a \in A$
Event A has not occurred	$a \notin A$
Event "A or B" (at least one of A or B)	$A \cup B$
Event "A and B"	$A \cap B$
Event "not A"	A^c or $\overline{A}$ or A'
Event A' implies event B	$A' \subset B$
Events A and B are mutually exclusive	$A \cap B = \phi$
Events A and B are not mutually exclusive	$A \cap B \neq \phi$

22.06. Diagrammatic representations – space diagram and tree diagrams

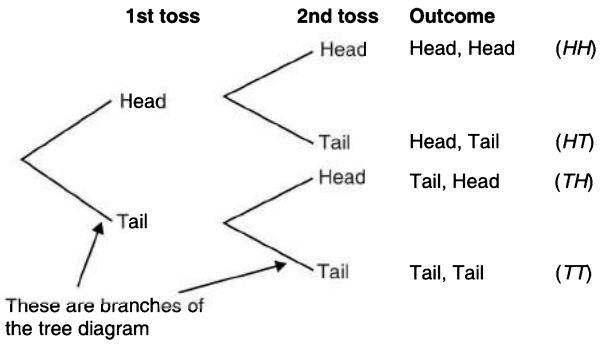
One way of showing possible outcomes in probability is in a diagram, called a sample space diagram. Tree diagrams are another method of showing all the possible outcomes.

For example, all the possible combined outcomes of tossing two fair coins can be displayed and found using a space diagram like this:

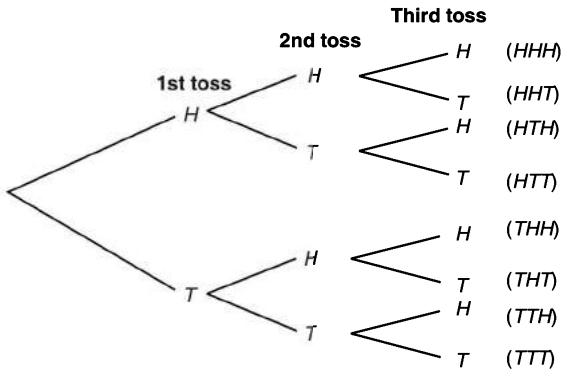


Each cross represents a different outcome. There are 4 different outcomes, all equally likely. These four outcomes are : *HH*, *HT*, *TH*, *TT*. You find these by reading off the 'coordinates' of each cross.

Another way of displaying and finding the possible combined outcomes by tossing of two coins are by using a tree diagram.



In case of three tosses of the coins, all the possible outcomes can be obtained by drawing a tree diagram as under:



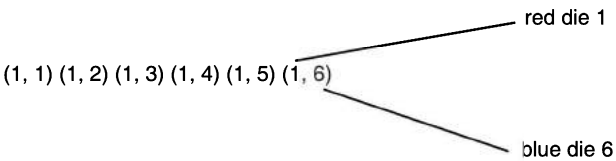
From the first two tosses we get: *HH*, *HT*; *TH*, *TT*.

Combining each one of these with the third toss, we get

HHH, *HHT*; *HTH*, *HTT*; *THH*, *THT*; *TTH*, *TTT*, which are all the possible outcomes.

Note. In case of a single toss, the number of possible outcomes is 2, in case of two tosses it is $2 \times 2 = 2^2 = 4$, in case of three tosses it is $2 \times 2 \times 2 = 2^3 = 8$, and in case of n tosses, it would be 2^n.

If you roll a red die and a blue die what are all the outcomes? If the red die shows 1, the blue die could show 1, 2, 3, 4, 5, or 6. You can record these outcomes as ordered pairs putting the red result first, then the blue result like this:



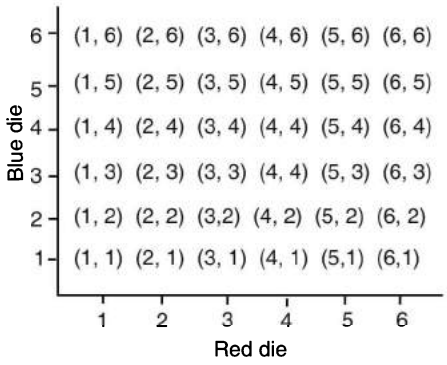


Fig. 22.04

If the red die showed 2 first the outcomes could be :

(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)

You can represent all the outcomes on a **sample space diagram :**

The sample space diagram shows that there are 36 possible outcomes.

22.07. Venn diagrams

It is often easier to visualise a problem using a **Venn diagram** which is helpful in seeing how the events relate to each other. The sample space is represented by a rectangle and the elements of the sample space by points within it. Subsets (or events) of the sample space are represented by regions within the rectangle, usually circles or parts of circles.

For example, consider various events which may be of interest when a die is thrown.

A = {even numbers} ={2, 4, 6}

B = {odd numbers} = {1, 3, 5}

C = {prime numbers} = {2, 3, 5}

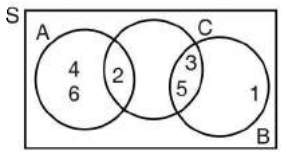


Fig. 22.05

If two or more events can never happen together they are called mutually exclusive events. These events have no outcomes in common.

The Venn diagram shows A and B are mutually exclusive, since if the outcome is even it cannot be odd.

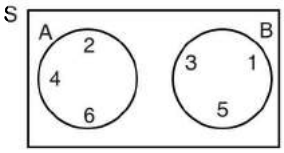


Fig. 22.06

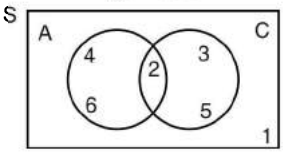


Fig. 22.07

A and C are not mutually exclusive, since the outcome {2} is in both events.

If one event contains certain outcomes in a sample space, then the event which contains all the other outcomes in the sample space and no others is called the **complementary** event. The complement of event E is the event $\overline{E}$, or E', and is read 'not E.'

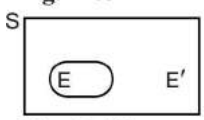


Fig. 22.08

For the above events,

* A and B are complementary, since outcomes not in A are all in B.

* A and C are not complementary, since the outcome $\{1\}$ is in neither event and, as well as, the outcome $\{2\}$ is in both.

If two or more events combine to form the complete sample space then they are called **exhaustive** events. In the Venn diagram X and Y are exhaustive, but X and Z are not.

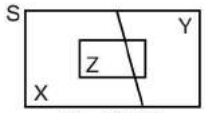


Fig. 22.09

Exhaustive events are not necessarily mutually exclusive.

For the above events, A and B are exhaustive, A and C are not, but A, B, C are.

Ex. 1. State which of the following events are mutually exclusive. Give reasons for your answer.

(*i*) E_1 : a total of 8, E_2 : a total of 12 in a single throw of two dice.

(*ii*) E_1 : a total of 11, E_2 : an odd number on each die in a simultaneous throw of two dice.

(*iii*) E_1 : a total of 8, E_2 : an even number on the toss in two successive tosses of a die.

(*iv*) E_1 : a red card, E_2 : an ace, in a draw of a card from a deck.

(*v*) E_1 : 2 heads, E_2 : at least one head in two successive tosses of a coin.

(*vi*) E_1 : a mother, E_2 : a woman in selecting the president of a ladies club.

(*vii*) E_1 : owns a house, E_2 : owns a *T.V.* set, in getting a response from an individual selected in a survey.

Sol. (*i*) E_1 and E_2 are mutually exclusive as a total of 8 and 12 cannot be obtained at the same time in a single throw of two dice.

(*ii*) E_1 and E_2 are mutually exclusive as when an odd number appears on each die, the sum must be an even number which is different from 11.

(*iii*) E_1 and E_2 are not mutually exclusive as (2, 6), (4, 4) and (6, 2) are the events which are favourable to both E_1 and E_2.

(*iv*) E_1 and E_2 are not mutually exclusive as 'red ace' is favourable to both the events.

(*v*) E_1 and E_2 are not mutually exclusive as the event '*HH*' is favourable to both E_1 and E_2.

(*vi*) E_1 and E_2 are not mutually exclusive as a woman can be a mother also.

(*vii*) E_1 and E_2 are not mutually exclusive as an individual may own both a house and a T.V. set.

Ex. 2. A die is rolled and a coin is tossed.

(*i*) Show all the possible combined outcomes in a space diagram.

(*ii*) List all the possible outcomes.

(*iii*) Show all the possible combined outcomes in a tree diagram.

Sol. (*i*) *Space diagram*

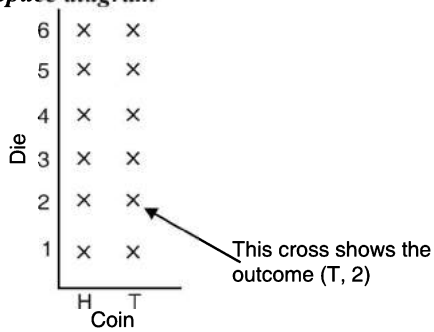




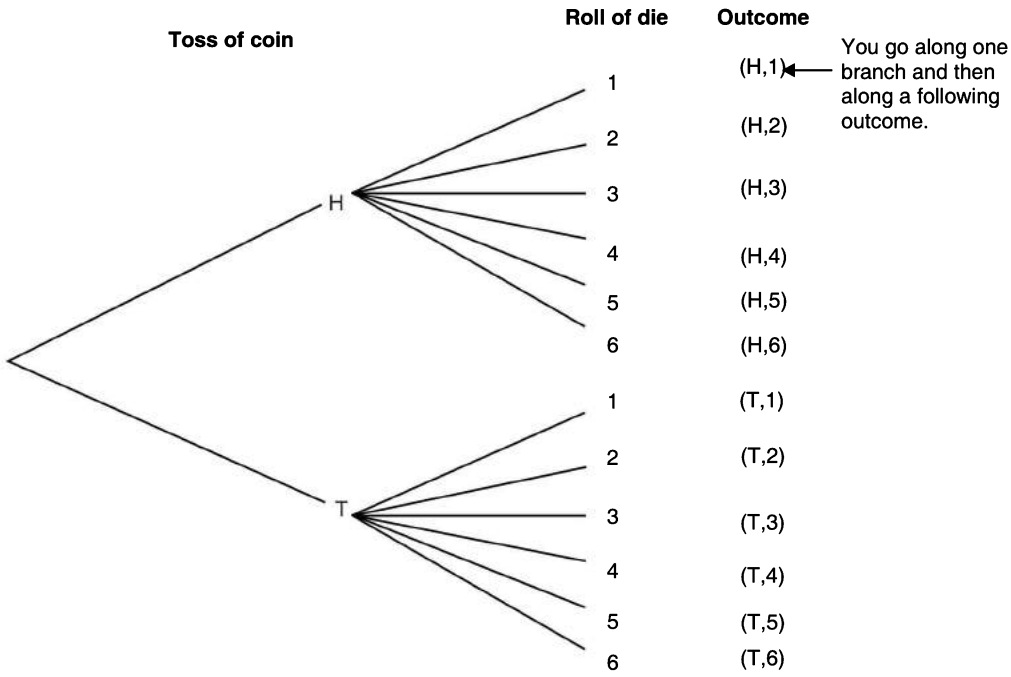


Fig. 22.10

(*ii*) There are 12 outcomes:

$(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6); (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$.

(*iii*) *Tree diagram*



Ex. 3. Three coins are tossed.

(*i*) Describe two events *A* and *B* which are mutually exclusive.

(*ii*) Describe three events *A, B* and *C* which are mutually exclusive and exhaustive.

(*iii*) Describe two events *A* and *B* which are not mutually exclusive.

(*iv*) Describe three events, *A, B* and *C* which are not mutually exclusive.

(*v*) Describe two events, *A* and *B* which are mutually exclusive but not exhaustive.

(*vi*) Describe three events *A, B* and *C* which are mutually exclusive but not exhaustive.

Sol. When three coins are tossed, the sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(*i*) Let A = Getting at least 2 heads = $\{HHT, HTH, THH, HHH\}$

B = Getting at least 2 tails = $\{TTH, THT, HTT, TTT\}$

Since $A \cap B = \phi$, therefore, A and B are mutually exclusive.

(*ii*) Let A = Getting at most one head = $\{HTT, THT, TTH, TTT\}$

B = Getting two heads = $\{HHT, HTH, THH\}$

C = Getting three heads = $\{HHH\}$

We see that $A \cap B = \phi$, $A \cap C = \phi$, $B \cap C = \phi$ and $A \cup B \cup C = S$.

$\therefore$ A, B and C are mutually exclusive and exhaustive.

(*iii*) Let A = Getting at most 2 heads = $\{HHT, HTH, HTT, THH, THT, TTH, TTT\}$

B = Getting 2 tails = $\{HTT, THT, TTH\}$

We find that $A \cap B \neq \phi \Rightarrow A$ and B are not mutually exclusive.

(*iv*) Let A = Getting one head = $\{HTT, THT, TTH\}$

B = Getting at most two heads = $\{HHT, HTH, HTT, THH, THT, TTH, TTT\}$

C = Getting two tails = $\{HTT, THT, TTH\}$

We find that $A \cap B \neq \phi$, $A \cap C \neq \phi$, $B \cap C \neq \phi \Rightarrow A, B$ and C are not mutually exclusive.

(*v*) Let A = Getting one head = $\{HTT, THT, TTH\}$

B = Getting two heads = $\{HHT, HTH, THH\}$

Since $A \cap B = \phi$ and $A \cap B \neq S$, therefore, A and B are mutually exclusive but not exhaustive.

(*vi*) Let A = Getting one head = $\{HTT, THT, TTH\}$

B = Getting two heads = $\{HHT, HTH, THH\}$

C = Getting three heads = $\{HHH\}$

Since $A \cap B \cap C \neq S$ and $A \cap B = \phi$, $A \cap C = \phi$, $B \cap C = \phi$, therefore, A, B and C are mutually exclusive but not exhaustive.

EXERCISE 22 (*a*)

1. What do you mean by *Random Experiment*? Give two illustrations. Define *sample space* associated with a random experiment. Give an example.

2. What is the resulting sample space if

(*i*) one coin is tossed;

(*ii*) two coins are tossed simultaneously;

(*iii*) three coins are tossed simultaneously?

3. Describe the sample space of this experiment:

(*i*) One die is rolled.

(*ii*) Two dice are rolled.

4. Describe the sample space:

(*i*) A coin is tossed twice. If the second throw results in a tail, a die is thrown.

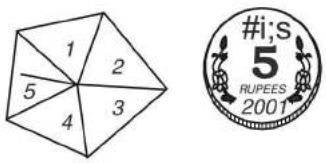
(*ii*) A coin is tossed twice. If the second throw results in a head, a die is thrown, otherwise a coin is tossed.

(*iii*) A coin is tossed. If it results in a head, a die is thrown. If the die shows up an even number, the die is thrown again.

5. A five-sided spinner is spun and a coin is tossed.

(*i*) Show the combined outcomes in a space diagram and in a tree diagram.

(*ii*) List the combined outcomes and state the number of equally likely combined outcomes.



The score on the spinner is 5

Fig. 22.11

6. In a bag there are three balls; one red, one blue and one yellow. A ball is selected, the colour is recorded and the ball is replaced. A second ball is then selected and the colour is recorded.

(*i*) Show in a space diagram and in a tree diagram all the possible combined outcomes.

(*ii*) List these combined outcomes and state the number of equally likely combined outcomes.

7. Satish and Mukesh who live in London wish to go on a holiday to France. They can travel to the coast by car, coach or train, and then cross the channel by ferry, train, helicopter or hovercraft.

(*i*) In a space diagram and in a tree diagram show all the combined outcomes of the different ways they could travel to France.

(*ii*) How many different ways could they travel?

8. From a group of 2 men and 3 women, two persons are selected. Describe the sample space of the experiment. If E is the event in which one man and one woman are selected, then which are the cases favourable to E?

9. A coin is tossed. If it results in a head, a coin is tossed, otherwise a die is thrown. Describe the following events:
(i) A : getting at least one head; (ii) B : getting an even number;
(iii) C : Getting a tail; (iv) D : getting a tail and an odd number.

10. A coin and a die are tossed. Describe the following events.
(i) A : getting a head and an even number ;(ii) B : getting a prime number ;
(iii) C : getting a tail and an odd number ; (iv) D : getting a head or a tail.

11. A fair coin is tossed. If it shows a head, we draw a ball from a bag consisting of 3 distinct red and 4 distinct black balls, if it shows a tail, we throw a fair die. Draw a tree diagram to show all the possible outcomes and obtain the sample space. What are sets representing the following events: (i) the ball drawn is black; (ii) the coin shows tail.

12. Two dice are rolled. A is the event that the sum of the numbers shown on the two dice is 5. B is the event that at least one of the dice shows up a 3. Are the two events A and B (i) mutually exclusive, (ii) exhaustive ? Give arguments in support of your answer.

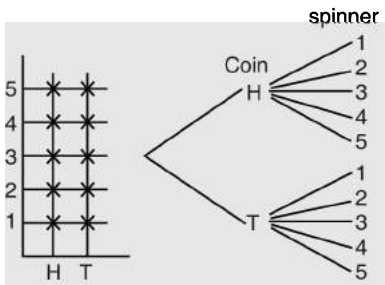
ANSWERS

2. (i) $S = \{H, T\}$
(ii) $S = \{HH, HT, TH, TT\}$
(iii) $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

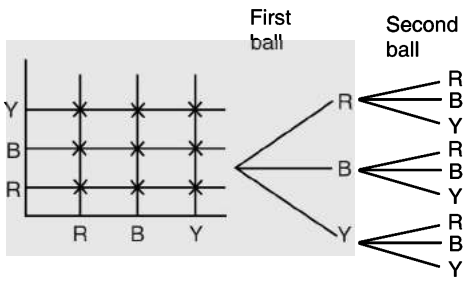
3. (i) $S = \{1, 2, 3, 4, 5, 6\}$
(ii) $S = \begin{Bmatrix} (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & (6,6) \end{Bmatrix}$

4. (i) $S = \{HH, HT1, HT2, HT3, HT4, HT5, HT6, TT1, TT2, TT3, TT4, TT5, TT6\}$
(ii) $S = \{HH1, HH2, HH3, HH4, HH5, HH6, TH1, TH2, TH3, TH4, TH5, TH6, HTH, HTT, TTH, TTT\}$
(iii) $S = \{T, H1, H3, H5, H21, H22, H23, H24, H25, H26, H41, H42, H43, H44, H45, H46, H61, H62, H63, H64, H65, H66\}$

5. (i)



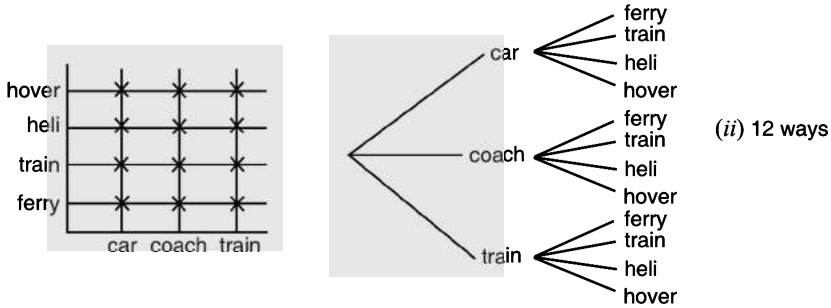
6. (i)



(ii) $(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5)$
10 outcomes

(b) $(R, R), (R, B), (R, Y), (B, R), (B, B), (B, Y)$
$(Y, R), (Y, B), (Y, Y)$
9 outcomes

7. (i)



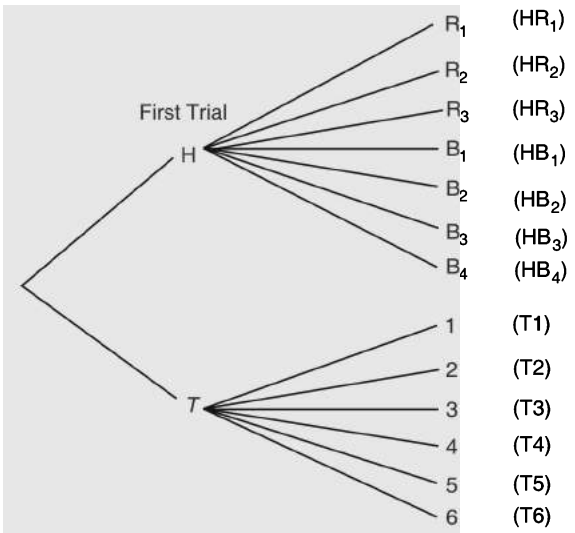
(ii) 12 ways

8. $S = \{M1\ M2, M1\ W1, M1\ W2, M1\ W3, M2\ W1, M2\ W2, M2\ W3, W1\ W2, W2\ W3, W3\ W1\}$
$E = \{M1\ W1, M1\ W2, M1\ W3, M2\ W1, M2\ W2, M2\ W3\}$

9. (*i*) $A = \{HH, HT\}$ (*ii*) $B = \{T2, T4, T6\}$
(*iii*) $C = \{T1, T2, T3, T4, T5, T6, HT\}$ (*iv*) $D = \{T1, T3, T5\}$

10. (*i*) $A = \{H2, H4, H6\}$ (*ii*) $B = \{H2, H3, H5, T2, T3, T5\}$
(*iii*) $C = \{T1, T3, T5\}$ (*iv*) $D = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

11. The possible outcomes are $HR_1, HR_2, HR_3, HB_1, HB_2, HB_3, HB_4, T_1, T_2, T_3, T_4, T_5, T_6$



(*i*) $\{(H, B_1), (H, B_2), (H, B_3), (H, B_4)\}$
(*ii*) $\{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$.

HINT

12. $A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$,
$B = \{(3, 1), (3, 2), (3,3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$
$A \cap B = \{(2, 3), (3, 2)\} \neq \phi \quad \therefore$ A and B are not mutually exclusive.
$n(A \cup B) = 13$ and $n(S) = 36$. Since $n(A \cup B) \neq S$, therefore, A and B are not exhaustive events.

22.08. Introduction – understanding the language of probability

If we put a lump of sugar in a glass of water, it is sure to dissolve in it. Similarly, if we hold a piece of paper over a fire, it is sure to burn. These two actions or certain other actions in everyday life are such that we are sure about their outcome. On the other hand, there are events about the outcome of which we are not sure or certain. Thus, if we toss a coin we cannot say with certainty which one, the head or the tail, will show up. The outcome of such actions, as tossing a coin, throwing up a die, or drawing a card from a well shuffled pack of cards, depends on ***chance***.

Chance factor plays an important part in life and many times important decisions depend on the result of such actions. Thus in a cricket match which team will have the right to decide whether to bat or field first is decided by the toss of a coin. Quality of goods purchased, for example, mangoes, are determined by checking a sample, like tasting a single mango. Gamblers always depend on chance and make or ruin fortunes.

In fact, that is where the theory of probability was first applied.

22.09. Historical background

The first known publication in the western world dealing with probability, dated 1494, is concerned with the division of the stakes in a game of chance between two gamblers. The two men were to play for a given number of points, but they were forced to stop before either attained his number. Because each gambler had attained a different score, the problem arose as to what portion of the stakes each one deserves. Obviously, the player with the greater chance of winning should receive the greater share.

The problem was posed by Chevalier de Me're, a gambler and a member of the aristocracy to two French mathematicians, Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662). They solved the problem, each by a different method. The correspondence that took place between these two mathematicians dealing with this problem led to the development of a theory of probability. As a result these men are considered as its founders.

Probability theory today has applications far beyond games of chance. It has contributed to the solution of problems in Biology, Physics, Chemistry, in other natural sciences, and in the social sciences as well. Business, industry and government are finding each year solutions to some of their problems through the use of probability theory. Life insurance companies have long made use of the principles of probability to help themselves remain solvent. Probability theory is used to estimate whether a missile will hit its target, to determine premium on insurance policies, and to make important business decisions such as where to locate a supermarket and how many check-out clerks to employ so that customers will not be kept waiting in line too long. Various sampling techniques, which are used in opinion polls and in the quality control of mass produced items, are based on the theory of probability.

22.10. Measurement of probability

Ordinarily speaking the probability of an event denotes the likelihood of its happening.

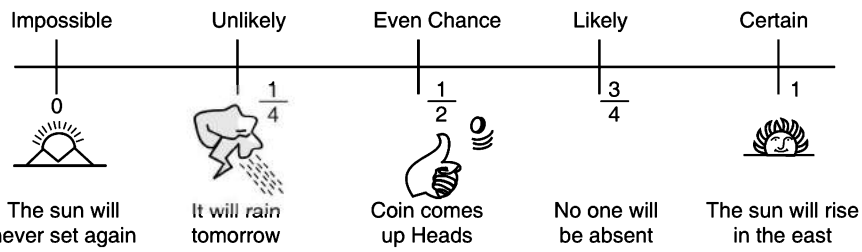
(A) The probability scale.

(*a*) ***Certainty*** **:** An event which is certain to happen has a probability of 1 (or 100%).
For example, the probability that the sun will rise in the east is 1.

(*b*) ***Impossibility*** **:** An event which is impossible has a probability of 0 (or 0%).
For example, the probability that the sun will never set again is 0 (or 0%).

All probabilities must have a value greater than or equal to 0 and less than or equal to 1.

This can be shown on a **probability scale.**



(B) Defining probability of an event

Definition. If in a random experiment there are *n* mutually exclusive and equally likely elementary events and *m* of them are favourable to an event *A*, then the probability *P* of happening of *A*, denoted by *P* (*A*), is defined as the ratio $\frac{m}{n}$.

$$p = P(A) = \frac{m}{n} = \frac{\text{Number of elementary events in } A, \textit{ i.e., favourable events}}{\text{Total number of elementary events in the random experiment}}$$

$$\textit{i.e., } P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

We can also write

$P(A) = \frac{m}{n} = \frac{n(A)}{n(S)}$, **where** $n(A)$ **and** $n(S)$ **denote the number of sample points in** A **and** S **respectively.**

Working rule for calculating probability

Step 1. *Find the number of elements in the sample space S of the given random experiment.*

Step 2. *Out of the sample space S, identify the elements which are favourable to the event A, whose probability is required.*

Step 3. *Then, required probability* $= \frac{n(A)}{n(S)}$.

The probability q **of failure of the event** A **is**

$$q = P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A) = 1 - p$$

It is obvious that $P(A) + P(\bar{A}) = p + q = 1$.

Remarks. From the definition we deduce the following results:

1. Since $p + q = 1$, therefore, the maximum value of p is 1 (when $q = 0$) and the minimum value of p is 0 (when $q = 1$). Thus, $0 \le p \le 1$ and $0 \le q \le 1$.
2. When $m = 0$, $p = 0$, there is no event.
   For example, the probability of drawing a black ball out of a bag containing all white balls is zero, therefore, the event cannot take place.
3. When $m = n$, $q = 0$ and $p = 1$, the event is certain.
   For example, the probability of drawing a white ball out of a bag containing all white balls is 1, therefore, the event is certain.
4. When $m = \frac{n}{2}$, $p = \frac{1}{2}$, $q = \frac{1}{2}$, the event is as likely to happen as to fail. In other words, the chances of success or failure are even. For example, when a coin is tossed, a head or a tail is equally likely to fall, therefore $p = \frac{1}{2}$, $q = \frac{1}{2}$.
5. **Probability of an event not happening.**
   We know that the probability of a tossed coin landing heads is $\frac{1}{2}$. Similarly, the probability of its landing tails is $\frac{1}{2}$. The sum of these two probabilities, $\frac{1}{2} + \frac{1}{2}$, equals unity. The sum of probabilities of all mutually exclusive outcomes for any event must be unity.
   The tossed cube must land with A up or without A up. The probability that the A lands up is $\frac{1}{6}$. Therefore, the probability that A does not land up must be $1 - \frac{1}{6}$, or $\frac{5}{6}$.

In general, if p is the probability of an event happening then the probability of the event not happening must be 1 – p.

22.11. What are odds?

Probabilities are often expressed in terms of "odds". Suppose an outcome A occurs m times out of n equally likely outcomes in an experiment. Then A does not occur in $n - m$ cases.

$$\text{Odds in } \textit{favour} \text{ of } A = \frac{\text{number of times } A \text{ occurs}}{\text{number of times } A \text{ does not occur}} = \frac{m}{n-m}$$

$$\text{Odds } \textit{against } A = \frac{\text{number of times } A \text{ does not occur}}{\text{number of times } A \text{ occurs}} = \frac{n-m}{m}$$

So, the odds in favour of A are m to $n - m$. The odds against A are $n - m$ to m.

Aid to Memory: ***Odds in favour of an event = prob. (success) : prob. (failure)***
Odds against an event = prob. (failure) : prob. (success)

Ex. 4. (*i*) Find the probability of the occurrence of the digit 3 when an unbiased die is thrown.

(*ii*) A card is drawn from a pack of 100 cards numbered 1 to 100. Find the probability of drawing a number which is a square. (*EAMCET*)

(*iii*) A die is thrown. Find the probability of getting a prime number.

(*iv*) A die is rolled. If the outcome is an odd number, what is the probability that it is a prime number.

(*v*) A die is tossed once. What is the probability of the number '8' coming up ? What is the probability of a number "less than 8" coming up ?

(*vi*) A lot consists of 12 good pencils, 6 with minor defects and 2 with major defects. A pencil is chosen at random. Find the probability that this pencil is not defective. (*EAMCET 1991*)

(*vii*) A single letter is selected at random from the word "Probability" . Find the probability that it is a vowel. (*MNR*)

(*viii*) What are the odds in favour of getting a '3' in a throw of a die ? What are the odds against getting a '3'?

(*ix*) If the odds in favour of an event are 4 to 5, find the probability that it will occur.

Sol. (*i*) A : getting digit 3 $\therefore$ $n(A) = 1$, also $n(S) = 6$

$\therefore$ Required probability $= P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{6}$.

Note. We may also write as under :
Number of favourable outcomes, *i.e.*, getting *a* '3' = 1.
Total number of possible outcomes = 6

$\therefore$ Reqd. probability $P(A) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \dfrac{1}{6}$.

(*ii*) A : Getting a number which is a square

$A = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$
$n(A) = 10$, also $n(S) = 100$

$\therefore$ Required probability $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{10}{100} = \dfrac{1}{10}$

Note. We may also write as under :
Number of favourable outcomes = 10
Total number of possible outcomes = 100

$\therefore$ $P(A) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \dfrac{10}{100} = \dfrac{1}{10}$.

(*iii*) A : Getting a prime number

$A = \{2, 3, 5\}$ $\therefore$ $n(A) = 3$, also $n(S) = 6$

$\therefore$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$.

(*iv*) Given, on rolling the die, we get an odd number.

$\therefore$ Total number of cases $n(S) = 3$ (These are 1 or 3 or 5)

Out of these 3 and 5 are prime numbers.

$\therefore$ Number of favourable cases $n(A) = 2$

$\therefore$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{2}{3}$.

(*v*) Since no face of the die is marked with the number 8,

$$P(\text{number '8'}) = \frac{0}{6} = 0$$

$\therefore$ Since the face of a die is marked with either of the numbers 1 to 6 so there are 6 favourable cases

$$P(\text{number less than 8}) = \frac{6}{6} = 1.$$

(*vi*) A : Pencil is not defective

$n(A) = 12$, $n(S) = 20$

$\therefore$ Required probability $= P(A) = \dfrac{n(A)}{n(S)} = \dfrac{12}{20} = \dfrac{3}{5}$.

(*vii*) A : selecting a vowel. The vowels in the word are o, a, i, i.

$n(A) = 4$, $n(S) = 11$

$\therefore$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{4}{11}$.

(*viii*) Odds in favour of getting a '3' $= \dfrac{\text{Number of favourable outcomes}}{\text{Number of unfavourable outcomes}} = \dfrac{1}{5}$ or 1 to 5.

$$\text{Odds against getting a '3'} = \frac{\text{Number of unfavourable outcomes}}{\text{Number of favourable outcomes}} = \frac{5}{1} \text{ or 5 to 1.}$$

Remark : Odds in favour of $E = \dfrac{1}{5} = \dfrac{\frac{1}{6}}{\frac{5}{6}} = \dfrac{P(E)}{P(E')} = \dfrac{P(E)}{1 - P(E)}$

Odds against $E = \dfrac{5}{1} = \dfrac{\frac{5}{6}}{\frac{1}{6}} = \dfrac{P(E')}{P(E)} = \dfrac{1 - P(E)}{P(E)}$

In general,

Odds in favour of an event E $= \dfrac{P(E)}{1 - P(E)} = \dfrac{p}{1 - p}$

Odds against event E $= \dfrac{1 - P(E)}{P(E)} = \dfrac{1 - p}{p}$.

(*ix*) The odds in favour of the event are $\dfrac{4}{5}$.

$\Rightarrow$ $\dfrac{P(E)}{1 - P(E)} = \dfrac{4}{5} \Rightarrow 4[1 - P(E)] = 5P(E) \Rightarrow P(E) = \dfrac{4}{9}$.

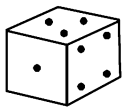
Hence, the probability that it will occur $= \dfrac{4}{9}$.

EXERCISE 22 (*b*)

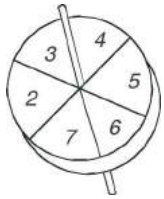
1. What is the probability of getting:



(*i*) a head

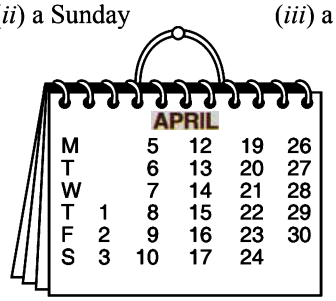


(*ii*) a 6

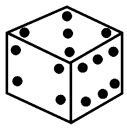


(*iii*) a prime number?

2. Ramesh chooses a date at random in April for a party. Calculate the probability that he chooses:
(*i*) a Saturday (*ii*) a Sunday (*iii*) a Saturday or a Sunday.



3. A normal die is rolled. Calculate the probability that the number on the uppermost face when it stops rolling will be



(*i*) 5 (*ii*) not 5 (*iii*) an odd number
(*iv*) a prime number (*v*) a 3 or a 4 (*vi*) a 1 or a 2 or a 3 or a 4.

4. Nine playing cards are numbered 2 to 10. A card is selected from them at random. Calculate the probability that the card will be:



(*i*) an odd number (*ii*) a multiple of 4.

5. Nine counters numbered 2 to 10 are put in a bag. One counter is selected at random. What is the probability of getting a counter with :
(*i*) a number 5 (*ii*) an odd number (*iii*) not an odd number
(*iv*) a prime number (*v*) a square number (*vi*) a multiple of 3?

6. A die is rolled. If the outcome is an even number, what is the probability that it is a prime number.

7. A bag contains 20 coloured balls. 8 are red, 6 are blue, 3 are green, 2 are white and 1 is brown. A ball is chosen at random from the bag. What is the probability that the ball chosen is:
(*i*) blue (*ii*) not blue (*iii*) brown (*iv*) not brown
(*v*) blue or red (*vi*) red or green (*vii*) green or white or brown?

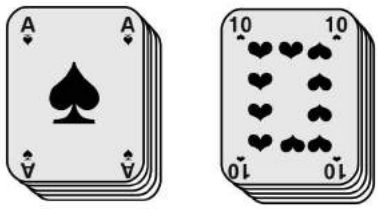
8. A bag contains 20 balls. These are of three different colours: green, red and blue. A ball is chosen at random from the bag. The probability of a green ball is $\frac{1}{4}$. The probability of a red ball is $\frac{2}{5}$.
(*i*) What is the probability of a blue ball? (*ii*) How many balls are red?
(*iii*) How many balls are green? (*iv*) How many balls are blue?

9. A pair of dice is thrown. Find the probability of getting a sum of 10 or more, if 5 appears on the first die.

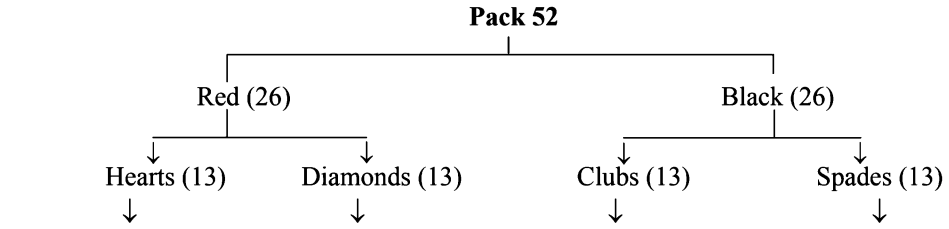
10. A match can be won, drawn or lost. One week a school is to play two matches. Draw a tree diagram to show all the possible outcomes and list them. If the outcomes are equally likely, calculate the probability that:
(*i*) both matches are won; (*ii*) one match is drawn; (*iii*) at least one match is drawn ;
(*iv*) no match is lost; (*v*) both matches are not lost.

11. The ace, king, queen, jack and ten from both the spades and hearts suits are placed in two separate piles and one card is taken from each pile : Draw the sample space diagram and find the probability that:

(*i*) both cards will be kings;
(*ii*) both of the cards could be either ace or a king;
(*iii*) both cards will be a pair;
(*iv*) at least one card will be an ace;
(*v*) neither card will be a 10;
(*vi*) neither card will be a king or jack;
(*vii*) one card will be a spade;
(*viii*) both cards will be hearts.



Note. Description of normal pack of cards (52)



Each of the colours, hearts, diamonds, clubs, spades, is called a suit. Kings, queens and jacks are called **face cards**. There are 12 **face cards** in a normal pack of cards.

ANSWERS

1. (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{6}$ (*iii*) $\frac{4}{6}$ or $\frac{2}{3}$

2. (*i*) $\frac{4}{30}$ or $\frac{2}{15}$ (*ii*) $\frac{4}{30}$ or $\frac{2}{15}$ (*iii*) $\frac{8}{30}$ or $\frac{4}{15}$

3. (*i*) $\frac{1}{6}$ (*ii*) $\frac{5}{6}$ (*iii*) $\frac{3}{6}$ (*iv*) $\frac{3}{6}$ (*v*) $\frac{2}{6}$ (*vi*) $\frac{4}{6}$

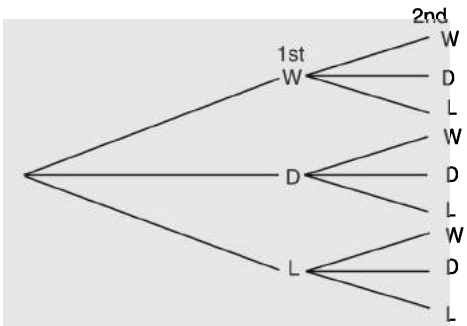
4. (*i*) $\frac{4}{9}$ (*ii*) $\frac{2}{9}$

5. (*i*) $\frac{1}{9}$ (*ii*) $\frac{4}{9}$ (*iii*) $\frac{5}{9}$ (*iv*) $\frac{4}{9}$ (*v*) $\frac{2}{9}$ (*vi*) $\frac{3}{9}$

6. $\frac{1}{3}$ **7.** (*i*) $\frac{6}{20}$ (*ii*) $\frac{14}{20}$ (*iii*) $\frac{1}{20}$ (*iv*) $\frac{19}{20}$ (*v*) $\frac{14}{20}$ (*vi*) $\frac{11}{20}$ (*vii*) $\frac{6}{20}$

8. (*i*) $\frac{7}{20}$ (*ii*) 8 (*iii*) 5 (*iv*) 7 **9.** $\frac{2}{6} = \frac{1}{3}$

10. 1st match and 2nd match outcome

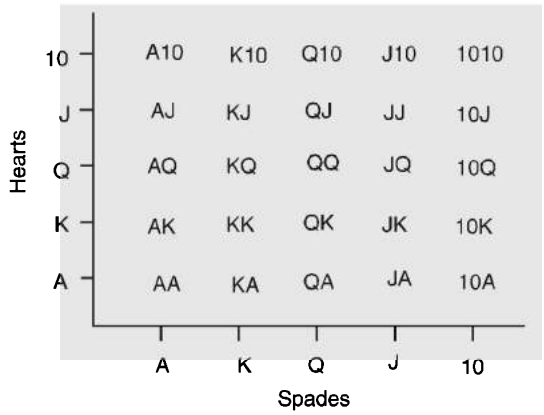


(*i*) $\frac{1}{9}$ (*ii*) $\frac{4}{9}$

(*iii*) $\frac{5}{9}$ (*iv*) $\frac{4}{9}$

(*v*) $\frac{8}{9}$

11.



(*i*) $\frac{1}{25}$ (*ii*) $\frac{2}{25}$

(*iii*) $\frac{2}{25}$ (*iv*) $\frac{9}{25}$

(*v*) $\frac{16}{25}$ (*vi*) $\frac{9}{25}$

(*vii*) 1 (*viii*) 0

HINTS TO SELECTED QUESTIONS

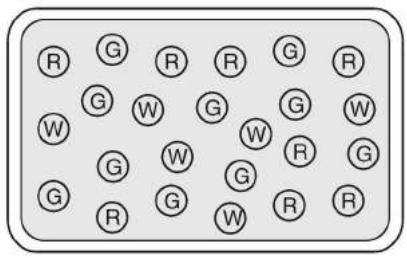
6. Type Solved Ex. 4 (*iv*)

9. Favourable cases are (5, 5) and (5, 6); $S = \{(4, 6), (5, 5), (6, 4), (5,6), (6, 5), (6,6)\}$

$\therefore \quad n(S) = 6, n(A) = 2.$

Ex. 5. A ball is drawn at random from a box containing 6 white, 8 red and 10 green balls. Determine the probability, that the ball drawn is (*i*) white (*ii*) red, (*iii*) green, (*iv*) not red, (*v*) red or green.

Sol. Total number of equally likely and exhaustive cases = 6 + 8 + 10 = 24



(*i*) Probability of drawing a white ball $= \frac{6}{24} = \frac{1}{4}$

(*ii*) Probability of drawing a red ball $= \frac{8}{24} = \frac{1}{3}$

(*iii*) Probability of drawing a green ball $= \frac{10}{24} = \frac{5}{12}$

(*iv*) Probability of drawing a ball which is not red

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

(*v*) Probability of drawing a red or a green ball

$$= \frac{\text{ways of choosing a red or a green ball}}{\text{Total number of cases}} = \frac{18}{24} = \frac{3}{4}.$$

Ex. 6. A card is drawn from a pack of cards. Find the probability that it is (*i*) a black card, (*ii*) a red card, (*iii*) a club, (*iv*) an ace, (*v*) a red ace, (*vi*) ace of spades, (*vii*) not a spade and (*viii*) a king or a queen.

Sol. In each of the events, total number of cases = 52, for any of the 52 cards may be drawn.

(*i*) There are 26 black and 26 red cards in a pack and any one of the black cards can be drawn.

$\therefore$ Probability of a black card $P(B) = \frac{26}{52} = \frac{1}{2}$.

(*ii*) Similarly, probability of a red card $P(R) = \frac{26}{52} = \frac{1}{2}$.

Note. $P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2} = P(R)$

and also $P(\overline{R}) = 1 - P(R) = \frac{1}{2} = P(B)$

(*iii*) There are 13 clubs in a pack. $\therefore$ Probability of a club = $\frac{13}{52} = \frac{1}{4}$.

(*iv*) There are 4 aces in all, one of each suit. $\therefore$ Probability of an ace = $\frac{4}{52} = \frac{1}{13}$.

(*v*) The ace of diamonds and the ace of hearts are two red aces.

$\therefore$ Probability of a red ace = $\frac{2}{52} = \frac{1}{26}$.

(*vi*) There is only one card of an ace of spade. $\therefore$ Probability of ace of spades = $\frac{1}{52}$.

(*vii*) In a pack of 52 cards, there are 13 spades and the remaining 39 are not spades.

$\therefore$ Probability of drawing a card that is not spade = $\frac{39}{52} = \frac{3}{4}$

(*viii*) There are 4 kings and 4 queens and we may select any one of the 8 cards to get a king or a queen.

$\therefore$ Probability of a king or a queen = $\frac{8}{52} = \frac{2}{13}$.

Ex. 7. If two coins are tossed once, what is the probability of getting (*i*) both heads, (*ii*) both heads or both tails (*iii*) at least one head?

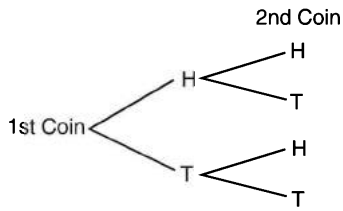
Sol. When two coins are tossed once there are four possible cases, *viz.*,

HH : Head on the first coin and head on the second

HT : Head on the first coin and tail on the second

TH : Tail on the first coin and head on the second

TT : Tail on the first and tail on the second



$\therefore$ $S = \{HH, HT, TH, TT\}$ $\therefore$ $n(S) = 4$.

(*i*) A : getting both heads $\therefore$ $A = \{HH\}$ and $n(A) = 1$

$\therefore$ $P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$.

(*ii*) A : getting both heads or both tails

$\therefore$ $A = \{HH, TT\}$ and $n(A) = 2$

$\therefore$ $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$.

(*iii*) A : getting at least one head $\therefore$ $A = \{HH, HT, TH\}$ and $n(A) = 3$

$\therefore$ $P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$.

Alternatively, P (at least one head) = $1 - P(TT) = 1 - \frac{1}{4} = \frac{3}{4}$.

Ex. 8. Three unbiased coins are tossed. What is the probability of obtaining (*i*) all heads, (*ii*) two heads, (*iii*) one head, (*iv*) at least one head, (*v*) at least two heads, (*vi*) all tails.

Sol. As shown in the tree diagram in Art. 22.06, (Page Ch 22-8) there are 2^3 or 8 possible cases. They are *HHH*, *HHT*, *HTH*, *HTT*, *THH*, *THT*, *TTH* and *TTT*. $\therefore$ $n = 8$.

In the above the three letters in each case denote the results on the 1st, 2nd and 3rd coins respectively. These are mutually exclusive, exhaustive and equally likely cases.

The favourable outcomes are as under:

Event	*Favourable*	*Number of favourable outcomes (m)*
(*i*) All heads	*HHH*	1
(*ii*) Two heads	*HHT, HTH, THH*	3
(*iii*) One head	*HTT, THT, TTH*	3
(*iv*) At least one head	*HTT, THT, TTH, HHT, HTH, THH, HHH*	7
(*v*) At least two heads	*HHT, HTH, THH, HHH*	4
(*vi*) All tails	*TTT*	1

Hence, (*i*) P (all heads) $= \frac{1}{8}$, (*ii*) P (two heads) $= \frac{3}{8}$

(*iii*) P (one head) $= \frac{3}{8}$, (*iv*) P (at least one head) $= \frac{7}{8}$

(*v*) P (at least two heads) $= \frac{4}{8} = \frac{1}{2}$ (*vi*) P (all tails) $= \frac{1}{8}$.

Note. (*vi*) is the same as getting no heads.

Ex. 9. Two unbiased dice are thrown. Find the probability that

(*i*) six may be obtained as product;

(*ii*) both the dice show the same number;

(*iii*) the first die shows 6;

(*iv*) the total of the numbers on the dice is 8;

(*v*) the total of the numbers on the dice is greater than 8;

(*vi*) sum of the numbers on the two faces is neither 9 nor 11;

(*vii*) the total of the numbers on the dice is 13 and

(*viii*) the total of the numbers on the dice is any number from 2 to 12, both inclusive.

Sol. The possible outcomes are found from sample space diagram given on Page Ch. 22-9 in Art 22.06.

The expression (2, 5) means the first die shows 2 and the second die 5. The total number of possible outcomes is 36, *i.e.*, $n = 36$

(*i*) The favouring event A consists of (1, 6), (2, 3), (3, 2), (6, 1)

$\therefore \quad n(A) = 4, \; n(S) = 36.$ So $P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$.

(*ii*) **Both the dice show the same number:** The favourable outcomes are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6). $\therefore \quad n(A) = 6.$

$\therefore$ Probability (Both the dice show the same number) $= \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$.

(*iii*) **The first die shows 6:** The favourable outcomes are (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) and (6, 6).

$\therefore \quad n(A) = 6.$

$\therefore$ Probability (the first die shows '6') $= \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$.

(*iv*) **The total of the numbers on the dice is 8:** The favourable outcomes are (2, 6), (3, 5), (4, 4), (5, 3), (6, 2). $\therefore \quad n(A) = 5.$

$\therefore$ Required probability $= \frac{n(A)}{n(S)} = \frac{5}{36}$.

(*v*) **The total of the numbers on dice is greater than 8 :** The favourable outcomes are (3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5) and (6, 6). $\therefore \quad n(A) = 10$.

$\therefore$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{10}{36} = \dfrac{5}{18}$.

(*vi*) **Sum of the numbers on the two faces is neither 9 or 11:**

The sum of 9 or 11 on the two dice can be obtained as (4, 5), (5, 4), (6, 3), (3, 6), (5, 6), (6, 5), *i.e.*, 6 ways.

$\therefore \quad n(A) = 30$, where A is the event of getting the sum which is neither 9 or 11.

$\therefore \qquad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{30}{36} = \dfrac{5}{6}$. [See also Solved Ex. 35 (*i*)]

(*vii*) **The total of the numbers on the dice is 13:** The event is impossible since the maximum total can be 6 + 6 = 12. $\therefore$ Required probability = 0.

(*viii*) **The total of the numbers on the dice is any number from 2 to 12 :** Since the total of the numbers on the dice is bound to be any number between 2 and 12, therefore, it is a **certain event**.

$\therefore$ Required probability = 1.

Ex. 10. (*i*) Two unbiased dice are rolled. Find the probability of

(*a*) obtaining a total of at least 10.

(*b*) getting a multiple of 2 on one die and a multiple of 3 on the other die.

(*c*) getting a multiple of 3 as the sum.

(*ii*) a die is thrown twice. What is the probability that at least one of the two numbers is 4?

(*iii*) three identical dice are rolled. Find the probability that the same number will appear on each of them. ***(IIT (SP) 1994)***

Sol. (*i*) Let A : obtaining a total of at least 10. Then

(*a*) $A = \{(6, 4), (5, 5), (6, 5), (4, 6), (5, 6), (6, 6)\}$

$\therefore \quad n(A) = 6, n(S) = 36 \quad \therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$.

(*b*) A : getting a multiple of 2 on one die and a multiple of 3 on the other die.

$A = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4)\}$

$n(A) = 11, \ n(S) = 36 \quad \therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{11}{36}$.

(*c*) A : getting a multiple of 3 as sum

$A = \{(1, 2), (2, 1), (3, 3), (5, 1), (1, 5), (4, 2), (2, 4), (6, 3), (3, 6), (4, 5), (5, 4), (6, 6)\}$

$n(A) = 12, \ n(S) = 36 \quad \therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{12}{36} = \dfrac{1}{3}$.

(*ii*) A : getting at least one of the two numbers as 4.

A : $\{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)\}$

$n(A) = 11, n(S) = 36 \quad \therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{11}{36}$.

(*iii*) A : getting same number on all the three dice.

$A = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$

$n(A) = 6, \ n(S) = 6^3 = 216 \quad \therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{216} = \dfrac{1}{36}$.

Ex. 11. (*i*) What is the probability that there are 53 Sundays in a leap year ? (*ISC 2004*)
(*ii*) What is the chance that a non leap year, selected at random, will contain 53 Sundays ?

Sol. (*i*) In a leap year there are 366 days; 366 days = 52 full weeks + 2 days.

For these two days left, there are seven possible combinations. These are :

1. Mon, Tues, 2. Tues, Wed, 3. Wed, Thu, 4. Thu, Fri, 5. Fri, Sat, 6. Sat, Sun, 7. Sun, Mon. The last two combinations are in favour of the event that there will be 53 Sundays.

Hence, required probability = $\frac{2}{7}$.



(*ii*) A non-leap year consists of 365 days. Therefore, in a non-leap year, there are 52 complete weeks and *1 day over* which can be one of the seven days of a week but there is only one favourable case that it is Sunday.

Number of favourable cases (m) = 1. Total number of possible cases (n) = 7.

The required probability = $\frac{m}{n} = \frac{1}{7}$.

Ex. 12. A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet ?

Sol. Total number of cards (outcomes) = 52.

Number of spade cards = 13. Number of aces = 4.

$\therefore$ Number of favourable outcomes = $n(A) = 13 + 4 - 1 = 16$

(Since ace of spade is counted in both)

$$n(S) = 52 \quad \therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{16}{52} = \frac{4}{13}$$

$\therefore$ The probability of losing the bet = $1 - \frac{4}{13} = \frac{9}{13}$.

Hence, the odds against winning are $\frac{9}{13} : \frac{4}{13}$, *i.e.*, 9 : 4.

Note. This is a case of *non-mutually exclusive case* because a card can be an ace as well as a spade.

Ex. 13. Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has

(*i*) an even number? (*ii*) a number 5 or multiple of 5?
(*iii*) a number which is greater than 75? (*iv*) a number which is a square?

Sol. Total number of tickets = 100

$\therefore$ Total number of exhaustive, mutually exclusive and equally likely cases $n(S) = 100$

(*i*) A : getting an even number. Since there are 50 even numbered tickets, $n(A) = 50$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{50}{100} = \frac{1}{2}$$

(*ii*) A : getting 5 or a multiple of 5

The favourable outcomes are 5, 10, 15, 20, 25, ..., 95, 100 [A.P.]

$$\therefore \quad n(A) = 20 \quad \therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{20}{100} = \frac{1}{5}$$

(*iii*) A : getting a number greater than 75

The favourable outcomes are 76, 77, ..., 100

$\therefore \quad n(A) = 25$

$\therefore \quad P(A) \quad \frac{n(A)}{n(S)} = \frac{25}{100} = \frac{1}{4}$

(*iv*) A : getting a square number

The favourable outcomes are 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100

$\therefore \quad n(A) = 10 \quad \therefore \qquad P(A) \;=\; \frac{n(A)}{n(S)} = \frac{10}{100} = \frac{1}{10}.$

Ex. 14. From a set of 17 cards numbered 1, 2, 3, 4, ..., 16, 17, one card is drawn at random. Show that the chance that its number is divisible by 3 or 7 is $\frac{7}{17}$. ***(ISC 2005 Type)***

Sol. A : getting a number divisible by 3 or 7. Total number of possible outcomes = 17.

Numbers divisible by 3 are 3, 6, 9, 12, 15 and those divisible by 7 are 7, 14.

$\therefore \qquad$ Number of favourable outcomes $= 5 + 2 = 7$

$\therefore \qquad P(A) \;=\; \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{7}{17}.$

Ex. 15. An integer is chosen at random from the first two hundred positive integers. What is the probability that the integer chosen is divisible by 6 or 8?

Sol. Total number of exhaustive, mutually exclusive and equally likely cases, *i.e.*, $n(S) = 200$,

Integers from 1 to 200 divisible by 6 are 6, 12, ..., 198. Their number $= \frac{198}{6} = 33$

Integers from 1 to 200 divisible by 8 are 8, 16, ..., 200. Their number $= \frac{200}{8} = 25$

Integers from 1 to 200 divisible by both 6 and 8 (*i.e.*, divisible by their L.C.M 24) are 24, 48, ... 192. Their number $= \frac{192}{24} = 8$

These 8 numbers have been counted twice, in multiples of 6 and also in multiples of 8

$\therefore$ Favourable number of cases $= 33 + 25 - 8 = 50$

$\therefore$ Required probability $= \frac{50}{200} = \frac{1}{4}.$ [A non-mutually exclusive case]

Ex. 16. The chance of one event happening is the square of the chance of a second event, but the odds against the first are the cube of the odds against the second. Find the chance of each.

Sol. Let the two events be E_1 and E_2. Let the chances of their happening be p_1 and p_2 respectively.

Then $\qquad p_1 = p_2^2 \qquad \ldots(1)$

The chances of not happening of the events are $1 - p_1$ and $1 - p_2$ respectively.

Odds against the first event are $1 - p_1 : p_1 = \frac{1 - p_1}{p_1}$

Odds against the second event are $1 - p_2 : p_2 = \frac{1 - p_2}{p_2}$

By the given condition $\frac{1 - p_1}{p_1} \;=\; \left(\frac{1 - p_2}{p_2}\right)^3$ or $\frac{1 - p_2^2}{p_2^2} = \frac{(1 - p_2)^3}{p_2^3}$ [Using (1)]

$\Rightarrow \qquad 1 + p_2 \;=\; \frac{(1 - p_2)^2}{p_2} \quad (\because \; p_2 \neq 1 \text{ and } p_2 \neq 0)$

or $p_2 + p_2^2 = 1 - 2p_2 + p_2^2 \Rightarrow 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3} \quad \therefore p_1 = \frac{1}{9}$

Hence, the chances of happening of E_1 and E_2 are $\frac{1}{9}$ and $\frac{1}{3}$ respectively.

Ex. 17. There are three events E_1, E_2 and E_3, one of which must and only one can happen. The odds are 7 to 4 against E_1 and 5 to 3 against E_2. Find the odds against E_3.

Sol. Since one and only one of the three events E_1, E_2, and E_3 can happen, *i.e.*, they are mutually exclusive, therefore,

$$p_1 + p_2 + p_3 = 1 \quad \text{where } p_1 = P(E_1), p_2 = P(E_2), p_3 = P(E_3) \qquad ...(i)$$

Odds against E_1 are $7 : 4 \Rightarrow p_1 = \frac{4}{7+4} = \frac{4}{11}$ ...(*ii*)

Odds against E_2 are $5 : 3 \Rightarrow p_2 = \frac{3}{5+3} = \frac{3}{8}$ ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*), we have $\frac{4}{11} + \frac{3}{8} + p_3 = 1 \Rightarrow p_3 = \frac{23}{88}$

Hence, the odds against E_3 are $\frac{1-p_3}{p_3} = \frac{1-\frac{23}{88}}{\frac{23}{88}} = \frac{65}{23}$.

EXERCISE 22 (*c*)

1. A die is thrown once. Find

(*i*) P(an ace), (*ii*) P(an even number), (*iii*) P(a number < 3),

(*iv*) P(a number ≥ 4), (*v*) P(a number < 7), (*vi*) P(a number > 8).

2. A card is drawn from a well shuffled pack of 52 cards. Find the probability of (*i*) an ace (*ii*) a spade (*iii*) a black card (*iv*) a face card (*v*) Jack, queen or king (*vi*) 3 of heart or diamond.

3. One card is drawn from a pack of 52 cards being equally likely to be drawn. Find the probability of (*i*) the card drawn to be red (*ii*) the card drawn to be a king (*iii*) the card drawn to be red and a king (*iv*) the card drawn to be either red or a king.

4. A book contains 100 pages. A page is chosen at random. What is the chance that the sum of digits on the page is equal to 9?

5. From 25 tickets, marked with the first 25 numerals, one is drawn at random. Find the probability that (*i*) it is a multiple of 5 or 7 (*ii*) it is a multiple of 3 or 7.

6. What is the probability that a number selected from the numbers 1, 2, 3, ..., 25 is a prime number? You may assume that each of the 25 numbers is equally likely to be selected.

7. In a simultaneous throw of two coins find the probability of (*i*) two heads, (*ii*) exactly one tail, (*iii*) at least one tail.

Problems of throwing simultaneously two dice, three dice.

8. (*i*) In a single throw of two dice, find the probability of getting a total of 10 or 11.

(*ii*) Two dice are thrown. Find the probability of getting an odd number on one die and a multiple of 3 on the other.

(*iii*) Find the probability of getting a sum as 6 when two dice are thrown simultaneously.

(*iv*) Two dice are thrown simultaneously. Find the probability of getting a multiple of 3 as the sum.

(*v*) Find the probability of getting the sum as a prime number when two dice are thrown together.

(*vi*) In a single throw of two dice, find the probability of throwing

(*a*) a number > 4 on each die;
(*b*) an odd number on one die and 5 on the other;
(*c*) a multiple of 2 on one and a multiple of 3 on the other;
(*d*) an even number as the sum;
(*e*) an odd number as the sum.

(*vii*) In a single throw of two dice, what is the probability of
(*a*) two aces (*b*) at least one ace (*c*) doublets
(*d*) a total less than 10 (*e*) a total of 11 (*f*) a total of 12
(*g*) a total of at least 10 (*h*) a doublet of even number

(*viii*) Find the probability of getting the product a perfect square (square of a natural number), when two dice are thrown together.

9. In a single throw of three dice, find the probability of getting a total of 17 or 18.

10. In a single throw of three dice, find the probability of getting
(*i*) a total of 5 (*ii*) a total of at most 5 ***(Pb 1991 C)***
(*iii*) a total of at least 5 (*iv*) the same number on all the dice
(*v*) not getting the same number on all the dice.

11. There are four events E_1, E_2, E_3, and E_4 one of which must and only one can happen. The odds are 2 : 5 in favour of E_1, 3 : 4 in favour of E_2 and 1 : 3 in favour of E_3. Find the odds against E_4.

12. In a simultaneous toss of 4 coins, what is the probability of getting exactly 3 heads?

ANSWERS

1. (*i*) $\frac{1}{6}$ (*ii*) $\frac{3}{6}$, *i.e.*, $\frac{1}{2}$ (*iii*) $\frac{2}{6}$, *i.e.*, $\frac{1}{3}$ (*iv*) $\frac{3}{6}$, *i.e.*, $\frac{1}{2}$ (*v*) $\frac{6}{6}$, *i.e.*, 1 (*vi*) 0

2. (*i*) $\frac{4}{52}$, *i.e.*, $\frac{1}{13}$ (*ii*) $\frac{13}{52}$, *i.e.*, $\frac{1}{4}$ (*iii*) $\frac{26}{52}$, *i.e.*, $\frac{1}{2}$ (*iv*) $\frac{16}{52}$, *i.e.*, $\frac{4}{13}$ (*v*) $\frac{12}{52}$, *i.e.*, $\frac{3}{13}$ (*vi*) $\frac{2}{52}$, *i.e.*, $\frac{1}{26}$

3. (*i*) $\frac{26}{52}$, *i.e.*, $\frac{1}{2}$ (*ii*) $\frac{4}{52}$, *i.e.*, $\frac{1}{13}$ (*iii*) $\frac{2}{52}$, *i.e.*, $\frac{1}{26}$ (*iv*) $\frac{28}{52}$, *i.e.*, $\frac{7}{13}$

4. $\frac{10}{100}$, *i.e.*, $\frac{1}{10}$ **5.** (*i*) $\frac{8}{25}$ (*ii*) $\frac{2}{5}$ **6.** $\frac{9}{25}$ **7.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{2}{4}$, *i.e.*, $\frac{1}{2}$ (*iii*) $\frac{3}{4}$

8. (*i*) $\frac{5}{36}$ (*ii*) $\frac{11}{36}$ (*iii*) $\frac{5}{36}$ (*iv*) $\frac{12}{36}$, *i.e.*, $\frac{1}{3}$ (*v*) $\frac{15}{36}$, *i.e.*, $\frac{5}{12}$

(*vi*) (*a*) $\frac{4}{36}$, *i.e.*, $\frac{1}{9}$ (*b*) $\frac{5}{36}$ (*c*) $\frac{11}{36}$ (*d*) $\frac{18}{36}$, *i.e.*, $\frac{1}{2}$ (*e*) $\frac{18}{36}$, *i.e.*, $\frac{1}{2}$

(*vii*) (*a*) $\frac{1}{36}$ (*b*) $\frac{11}{36}$ (*c*) $\frac{6}{36}$, *i.e.*, $\frac{1}{6}$ (*d*) $\frac{30}{36}$, *i.e.*, $\frac{5}{6}$ (*e*) $\frac{2}{36}$, *i.e.*, $\frac{1}{18}$

(*f*) $\frac{1}{36}$ (*g*) $\frac{6}{36}$, *i.e.*, $\frac{1}{6}$ (*h*) $\frac{3}{36}$, *i.e.*, $\frac{1}{12}$ (*viii*) $\frac{8}{36}$, *i.e.*, $\frac{2}{9}$

9. $\frac{4}{216}$, *i.e.*, $\frac{1}{54}$ **10.** (*i*) $\frac{6}{216}$, *i.e.*, $\frac{1}{36}$ (*ii*) $\frac{10}{216}$, *i.e.*, $\frac{5}{108}$ (*iii*) $\frac{212}{216}$, *i.e.*, $\frac{53}{54}$

(*iv*) $\frac{6}{216}$ or $\frac{1}{36}$ (*v*) $\frac{35}{36}$ **11.** 27 : 1 **12.** $\frac{1}{4}$

HINTS TO SELECTED QUESTIONS

1. (*i*) In a die, ace means the number '1'.
(*v*) All the outcomes are favourable to the event 'a number < 7'. ∴ Reqd. probability = 1.
(*vi*) None of the outcomes is favourable to the event 'a number > 8'. ∴ Reqd. probability = 0. It is an impossible event.

3. (*iv*) There are 26 red cards and 4 kings of which 2 are common, namely, the kings of diamonds and hearts, so there are 26 + 4 – 2 = 28 ways in which a red card or a king can be drawn.

∴ Number of favourable outcomes = 28.

4. The pages numbered 9, 18, 27, 36, 45, 54, 63, 72, 81, 90 form favourable events.

8. (*ii*) The outcomes favourable to the event are (1, 3), (1, 6), (3, 3), (3, 6), (5, 3), (5, 6), (3, 1), (6, 1), (6, 3), (3, 5), (6, 5)

(*vii*) (*g*) Favourable outcomes are getting a sum of 10, 11, 12, are (6, 4), (4, 6), (5, 5), (6, 5), (5, 6), and (6, 6)

(*h*) Favourable outcomes are (2, 2), (4, 4), (6, 6).

(*viii*) Favourable outcomes are (1, 1), (1, 4), (2, 2), (3, 3), (4, 1) (4, 4), (5, 5), (6, 6).

9. The favourable cases are (6, 6, 6), (6, 6, 5), (6, 5, 6), (5, 6, 6).

10. (*i*) The favourable outcomes are (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).

(*ii*) The sum is 3, 4 or 5. The favourable outcomes are (1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).

(*iii*) The event a total of at least 5 means that sum of numbers appearing on each of three dice is ≥ 5, *i.e.*, the sum of three numbers appearing on three dice, which can vary from 3 to 18, should neither be 3 nor 4. Since there are only 4 outcomes namely (1, 1, 1), (1, 1, 2), (1, 2, 1) and (2, 1, 1) for which the sum is either 3 or 4, therefore, there are $(6 \times 6 \times 6) - 4$ *i.e.*, $216 - 4 = 212$ outcomes favourable to the event 'a total of at least 5.

11. See Solved Ex. 17.

Using permutations and combinations

Ex. 18. Four cards are drawn from a full pack of cards. Find the probability that

(*i*) all are diamonds,

(*ii*) there is one card of each suit,

(*iii*) there are two spades and two hearts,

(*iv*) all the four are kings,

(*v*) all the four are spades, and one of them is a king, and

(*vi*) at least one of the four cards is an ace.

Sol. 4 cards can be drawn from a pack of 52 cards in ${}^{52}C_4$ ways.

$\therefore$ Exhaustive number of cases $= {}^{52}C_4 = \dfrac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$

(*i*) There are 13 diamonds in the pack and 4 can be drawn out of them in ${}^{13}C_4$ ways.

$\therefore$ Favourable number of cases $= {}^{13}C_4 = \dfrac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715$

$\therefore$ Required probability $= \dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \dfrac{{}^{13}C_4}{{}^{52}C_4} = \dfrac{715}{270725} = \dfrac{11}{4165}.$

(*ii*) There are 4 suits, each containing 13 cards.

$\therefore$ Favourable number of cases $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 \times 13 \times 13 = 28561$

$$\text{Required probability} = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

$$= \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{28561}{270725} = \frac{2197}{20825}.$$

(*iii*) 2 spades out of 13 can be drawn in ${}^{13}C_2$ ways, 2 hearts out of 13 can be drawn in ${}^{13}C_2$ ways.

$\therefore$ Favourable number of cases = ${}^{13}C_2 \times {}^{13}C_2 = 78 \times 78$

$\therefore$ Required probability = $\dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4} = \dfrac{78 \times 78}{270725} = \dfrac{468}{20825}$.

(*iv*) Four kings out of 4 can be drawn in ${}^4C_4$ ways.

$\therefore$ Favourable number of cases = ${}^4C_4 = 1$

$\therefore$ Required probability = $\dfrac{{}^4C_4}{{}^{52}C_4} = \dfrac{1}{270725}$.

(*v*) Since there is only one king of spades, we can select the three other spades from the remaining 12 spade cards in ${}^{12}C_3$ ways.

$\therefore$ Favourable number of cases = $1 \times {}^{12}C_3 = 220$

Required probability = $\dfrac{1 \times {}^{12}C_3}{{}^{52}C_4} = \dfrac{220}{270725} = \dfrac{44}{54145}$.

Note. If E denotes the happening of an event, then $\overline{E}$ is the complement of E and denotes the non-happening of the event. Also,

$$P(E) + P(\overline{E}) = 1 \quad \Rightarrow \quad P(E) = 1 - P(\overline{E}).$$

(*vi*) Since there are four aces, therefore, the number of ways of drawing 4 cards so that no card is an ace = ${}^{48}C_4$.

$\therefore$ Probability of drawing 4 cards so that none of these is an ace = $\dfrac{{}^{48}C_4}{{}^{52}C_4} = \dfrac{6486}{54145} = P(\overline{E})$.

$\therefore$ Probability $P(E)$ of drawing 4 cards so that at least one of them is an ace

$$= 1 - \frac{6486}{54145} = \frac{47659}{54145}. \qquad [P(E) = 1 - P(\overline{E})]$$

Remark. In problems, involving probability of **at least one**, it is better to find the probability of the complementary event, *i.e.,* the probability of **'none'.**

Ex. 19. In a hand at whist, what is the chance that the four kings are held by a specified player.

Sol. In a hand at whist, the specified player has 13 cards including 4 kings. Four kings can be drawn out of 4 in ${}^4C_4$ ways. The remaining 9 cards held by him can be drawn out of the remaining 48 cards in the pack in ${}^{48}C_9$ ways. The total number of ways in which 4 kings and 9 other cards can be drawn is ${}^4C_4 \times {}^{48}C_9$.

Number of favourable cases = ${}^4C_4 \times {}^{48}C_9$. Number of exhaustive cases = ${}^{52}C_{13}$

$\therefore$ Required probability = $\dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \dfrac{{}^4C_4 \times {}^{48}C_9}{{}^{52}C_{13}} = \dfrac{11}{4165}$.

Note. *'Whist' is a game of cards played by two pairs of players with a pack of 52 cards. The cards are dealt out equally.*

Ex. 20. A bag contains 7 white, 5 black and 4 red balls. Four balls are drawn without replacement. Find the probability that at least three balls are black.

Sol. A : getting at least 3 black balls

Number of favourable outcomes $= {}^5C_3 \times {}^{11}C_1 + {}^5C_4 = {}^5C_2 \times {}^{11}C_1 + {}^5C_1$

$$= \frac{5\times 4}{1\times 2}\times 11 + 5 = 115$$

Total number of possible outcomes $= {}^{16}C_4 = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 1820$

$\therefore$ Required probability $= P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{115}{1820} = \frac{23}{364}.$

Ex. 21. In an urn there are 4 white and 4 black balls. What is the probability of drawing the first ball white, the second black, the third white, and fourth black, and so on.

Sol. 4 white balls are to occupy four odd places, *viz.,* 1, 3, 5 and 7, and can be arranged in 4! ways. The black balls are to occupy four even places, *viz.,* 2, 4, 6 and 8 and can be arranged in 4! ways.

$\therefore$ Number of favourable cases = 4 ! × 4!.

Number of exhaustive cases = 8!

$\therefore$ Required probability $= \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$

$$= \frac{4!\times 4!}{8!} = \frac{(4\times 3\times 2\times 1)\times(4\times 3\times 2\times 1)}{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1} = \frac{1}{70}.$$

Ex. 22. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn at random from the box. What is the probability that

(*i*) all will be blue?

(*ii*) at least one will be green?

Sol. The box contains 60 marbles of which 5 marbles are drawn

$\therefore$ Total number of possible outcomes $= {}^{60}C_5$.

(*i*) 5 blue marbles can be drawn from 20 blue marbles in ${}^{20}C_5$ ways.

$\therefore$ P (all are blue marbles) $= \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$

$$= \frac{{}^{20}C_5}{{}^{60}C_5} = \frac{20\times 19\times 18\times 17\times 16}{60\times 59\times 58\times 57\times 56} = \frac{\mathbf{34}}{\mathbf{11977}}.$$

(*ii*) P (at least one marble is green) = 1 – P (5 marbles are not green)

$$= 1 - \frac{{}^{30}C_5}{{}^{60}C_5} = 1 - \frac{30\times 29\times 28\times 27\times 26}{60\times 59\times 58\times 57\times 56} = 1 - \frac{117}{4484} = \frac{\mathbf{4367}}{\mathbf{4484}}.$$

Ex. 23. (*i*) A bag contains 30 tickets numbered from 1 to 30. Five tickets are drawn at random and arranged in ascending order. Find the probability that the third number is 20.

(*ii*) A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). What is the probability that $x_3 = 30$?

Sol. (*i*) Total number of cases $= {}^{30}C_5$

The tickets are arranged in the form $T_1 \quad T_2 \quad T_3\ (=20) \quad T_4 \quad T_5$

where $T_1, T_2 \in \{1, 2, 3,....., 19\}$ and $T_4, T_5 \in \{21, 22,....., 30\}$

$\therefore$ Number of favourable cases $= {}^{19}C_2 \times {}^{10}C_2$

$\therefore$ Required probability $= \dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$

$$= \frac{{}^{19}C_2 \times {}^{10}C_2}{{}^{30}C_5} = \frac{\dfrac{19\times 18}{1\times 2}\times\dfrac{10\times 9}{1\times 2}}{\dfrac{30\times 29\times 28\times 27\times 26}{1\times 2\times 3\times 4\times 5}} = \frac{285}{5278}.$$

(*ii*) Exhaustive number of cases $= {}^{50}C_5$

1, 2, 3, ..., 29, x_3, 31, 32, 33, ..., 50.

If $x_3 = 30$, then the two tickets with numbers x_1 and x_2 must come out of 29 tickets numbered 1 to 29 and this can be done in ${}^{29}C_2$ ways. The other two tickets with numbers x_4 and x_5 must come out of the 20 tickets numbered 31 to 50 and this can be done in ${}^{20}C_2$ ways.

$\therefore$ Favourable number of cases $= {}^{29}C_2 \times {}^{20}C_2$

$\therefore$ Required probability $= \dfrac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$

$$= \frac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5} = \frac{551}{15134}.$$

Ex. 24. What is the probability of getting 9 cards of the same suit in one hand at a game of bridge?

Sol. One hand in a game of bridge consists of 13 cards.

$\therefore$ Exhaustive number of cases $= {}^{52}C_{13}$

Number of ways in which, in one hand, a particular player gets 9 cards of one suit are ${}^{13}C_9$ and the number of ways in which the remaining 4 cards are of some other suit are ${}^{39}C_4$. There are four suits, *viz.,* club, spade, heart and diamond, each of 13 cards. All the four cards of the same suit mean that either nine cards drawn are club cards or spade cards or heart cards or diamond cards. So, the total number of ways of getting all the nine cards of the same suit $= {}^{13}C_9 + {}^{13}C_9 + {}^{13}C_9 + {}^{13}C_9 = 4\,({}^{13}C_9)$.

$\therefore$ Total number of favourable cases $= 4 \times {}^{13}C_9 \times {}^{39}C_4$

$\therefore$ Required probability $= \dfrac{4 \times {}^{13}C_9 \times {}^{39}C_4}{{}^{52}C_{13}}.$

Ex. 25. There are 10 persons who are to be seated around a circular table. Find the probability that two particular persons will always sit together. **(*ISC 2002*)**

Sol. Total number of ways in which 10 men can sit around circular table = 9 !

If two particular persons (say A and B) sit together, then the number of ways in which 9 persons (considering the pair of A and B as one unit) can sit around the table = 2! × 8!. In other words, the favourable cases are (2 × 8!).

$\therefore$ p = (probability that A and B will sit together)

$$= \frac{\text{Favourable cases}}{\text{Total number of cases}} = \frac{2\times 8!}{9!} = \frac{2\times 8!}{9\times 8!} = \frac{2}{9}.$$

Ex. 26. A party of *n* men is to be seated round a circular table. Find the odds against the event that two particular men sit together. **(*MP PET 2002*)**

Sol. Total number of ways in which n men can sit round a circular table = $(n-1)$!.

Now, if two particular men (say A and B) sit together, then the number of ways in which $(n-1)$ men sit round the table = $(n-2)$! but the two men can exchange their seats in two ways.

$\therefore$ Number of ways that two particular men sit together = 2 $[(n-2)!]$

$\therefore$ p = probability that the 2 particular men sit together

$$= \frac{\text{Favourable cases}}{\text{Total number of cases}} = \frac{2[(n-2)!]}{(n-1)!} = \frac{2}{n-1}$$

$$\therefore \quad q = 1 - p = 1 - \frac{2}{n-1} = \frac{n-1-2}{n-1} = \frac{n-3}{n-1}$$

$$\therefore \quad \text{Odds against the event} = q : p = \frac{n-3}{n-1} : \frac{2}{n-1} = (n-3) : 2.$$

Ex. 27. *A* and *B* stand in a ring with 10 other persons. If the arrangement of the twelve persons is at random; find the chance that there are exactly three persons between *A* and *B*.

Sol. Let A occupy any seat at the round table, then there are to be 3 persons between A and B, then B has only 2 ways to sit as shown in the figure. Thus, the probability of the required event $= \frac{2}{11!}$.

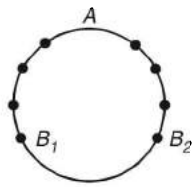


Fig. 22.12

Ex. 28. A five figured number is formed by the digits 0, 1, 2, 3, 4, (without repetition). Find the probability that the number formed is divisible by 4.

Sol. The digits 0, 1, 2, 3, 4 can be arranged among themselves in 5! ways out of which 4! numbers will start with 0.

Exhaustive number of cases = total number of five figured numbers = 5! – 4! = 96.

Now, only those numbers are divisible by 4 in which the number formed by their last two digits is divisible by 4.

Thus, the numbers ending with 04, 12, 20, 24, 32, 40 will be divisible by 4.

(*i*) If the numbers end in 04, the remaining three numbers 1, 2, 3 can be arranged in 3! = 6 ways.

(*ii*) If the numbers end in 20, the remaining three numbers 1, 3, 4 can be arranged in 3! = 6 ways.

(*iii*) If the numbers end in 40, the remaining three numbers 1, 2, 3 can be arranged in 3! = 6 ways.

(*iv*) If the numbers end in 12, the remaining three numbers 0, 3, 4 can be arranged in 3! ways but these cases in which 0 is the extreme left digit are to be discarded. The number of such cases = 2!.

$\therefore$ The number of numbers ending in 12 are 3! – 2 ! = 4.

(*v*) Similarly, the number of numbers ending in 24 are 4 and those ending in 32 is also 4.

$\therefore$ Total number of favourable cases = 6 + 6 + 6 + 4 + 4 + 4 = 30

$\therefore$ Required probability = $\frac{30}{96} = \frac{5}{16}$.

Ex. 29. The first twelve letters of the alphabet are written down at random. What is the probability that there are four letters between the *A* and the *B*?

Sol. Let the twelve positions be denoted by 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

When A is kept at 1, B must occupy place 6 to have four letters in between; when A is kept at 2, B must occupy place 7 etc. Lastly A may occupy a place 7 (so that B is at 12) just to have four letters between them.

Also A and B can interchange their positions in 2! ways.

Thus, A and B can take the required positions in $2\,! \times 7 = 14$ ways.

Now out of the remaining 10 alphabets, four can be selected in ${}^{10}C_4$ ways. These 4 letters and 6 other can give 4! 6! arrangements.

$\therefore$ The number of arrangements in which there are four letters between A and B

$$= 14 \times {}^{10}C_4 \times 4!\,.\,6!$$

Total number of arrangements in which 12 letters can be put = 12!

Hence the required probability $= \dfrac{\text{Favourable cases}}{\text{Total number of possible cases}} = \dfrac{14 \times {}^{10}C_4 \times 4!\,.\,6!}{12!} = \dfrac{7}{66}$.

Ex. 30. Let there be a word "REGULATIONS". If the letters of the given word be arranged at random, what is the chance that there will be exactly 4 letters between *R* and *E* ?

Sol. In the given word "REGULATIONS", there are 11 letters. The letters R and E can occur in ${}^{11}P_2$, *i.e.*, 110 positions.

The number of ways in which there will be exactly 4 letters between R and E, are given below.

(*i*) R in the Ist place and E in the 6th place.

$$\underset{1}{R}\ \underset{2}{_}\ \underset{3}{_}\ \underset{4}{_}\ \underset{5}{_}\ \underset{6}{E}\ \underset{7}{_}\ \underset{8}{_}\ \underset{9}{_}\ \underset{10}{_}\ \underset{11}{_}$$

(*ii*) R in the 2nd place and E in the 7th place.

(*iii*) R in the 3rd place and E in the 8th place.

(*iv*) R in the 4th place and E in the 9th place.

(*v*) R in the 5th place and E in the 10th place.

(*vi*) R in the 6th place and E in the 11th place.

But these two letters can also interchange their positions.

$\therefore$ The number of favourable cases $= 2 \times 6 = 12$

Thus the chance that there will be exactly 4 letters between R and E

$$= \frac{\text{Favourable cases}}{\text{Total number of possible cases}} = \frac{12}{110} = \frac{6}{55}.$$

EXERCISE 22 (*d*)

1. Four digit numbers are formed by using the digits 1, 2, 3, 4 and 5 without repeating the digit. Find the probability that a number, chosen at random, is an odd number.

2. What is the probability of getting 3 white balls in a draw of 3 balls from a box containing 6 white and 4 red balls?

3. A bag contains 6 white, 7 red and 5 black balls. Three balls are drawn at random. Find the probability that they will be white.

4. A bag contains 2 white marbles, 4 blue marbles, and 6 red marbles. A marble is drawn at random from the bag. What is the probability that

(*i*) it is white? (*ii*) it is blue? (*iii*) it is red?

(*iv*) it is not white? (*v*) it is not blue? (*vi*) it is black?

5. In Q. 4, three marbles are drawn from the bag. What is the probability that

(*i*) they are all blue? (*ii*) they are all red? (*iii*) they are all white?

(*iv*) none of them is red? (*v*) not all of them are red? (*vi*) none is black?

6. Two balls are drawn from an urn containing 2 white, 3 red and 4 black balls, one by one without replacement. What is the probability that (*i*) both balls are of the same colour, (*ii*) at least one ball is red?

7. From a pack of cards, three are drawn at random. Find the chance that they are a king, a queen and a knave.

8. (*i*) Two cards are drawn at random from 8 cards numbered from 1 to 8. What is the probability that the sum of the numbers is odd, if the two cards are drawn together?

(*ii*) Two cards are drawn at random from a well shuffled pack of 52 cards, show that the chances of drawing two aces is $\frac{1}{221}$.

(*iii*) From a pack of 52 cards, 3 cards are drawn 'at random'. Find the probability of drawing exactly two aces.

(*iv*) Three cards are drawn at a time at random from a well shuffled pack of 52 cards. Find the probability that all the three cards have the same number.

(*v*) Two cards are drawn from a well shuffled pack of cards without replacement. Find the probability that neither a Jack nor a card of spades is drawn.

9. Three cards are drawn from a deck of 52 cards. What is the probability that

(*i*) they are all spades? (*ii*) they are all red cards?

(*iii*) none of them is a club? (*iv*) all of them are aces?

10. In Q. 7 what are the odds that

(*i*) all three cards are spades? (*ii*) they are all red cards?

(*iii*) none is a club? (*iv*) all of them are aces?

11. One card is drawn from a pack of 52 cards, each of the 52 cards being equally like to be drawn. Find the probability that the card drawn is (*i*) an ace, (*ii*) red, (*iii*) either red or king, (*iv*) red and a king.

12. Four cards are drawn at random from a pack of 52 playing cards, Find the probability of getting

(*i*) all the four cards of the same suit; (*ii*) all the four cards of the same number;

(*iii*) one card from each suit; (*iv*) two red cards and two black cards;

(*v*) all cards of the same colour; (*vi*) all face cards; (King, Queen, Jack)

(*vii*) four honours of the same suit.

13. In a lottery of 50 tickets numbered 1 to 50, two tickets are drawn simultaneously. Find the probability that

(*i*) both the tickets drawn have prime numbers;

(*ii*) none of the tickets drawn has prime number;

(*iii*) a ticket has prime number.

14. (*i*) Out of 9 outstanding students in a college, there are 4 boys and 5 girls. A team of 4 students is to be selected for a quiz programme. Find the probability that 2 are girls and 2 are boys.

(*ii*) Four people are chosen at random from a group consisting of 3 men, 2 women, and 3 children. Find the probability that out of four chosen people, exactly 2 are children?

15. A committee of 5 principals is to be selected from a group of 6 gent principals and 8 lady principals. If the selection is made randomly, find the probability that there are 3 lady principals and 2 gent principals.

16. A bag contains tickets numbered 1 to 20. Two tickets are drawn. Find the probability that both numbers are odd.

17. A bag contains tickets numbered 1 to 30. Three tickets are drawn at random from the bag. What is the probability that the maximum number of the selected tickets exceeds 25?

18. (*i*) A room has 3 lamps. From a collection of 10 light bulbs of which 6 are no good, a person selects 3 at random and puts them in a socket. What is the probability, that he will have light?

(*ii*) *A* has 3 shares in a lottery containing 3 prizes and 9 blanks; *B* has two shares in lottery containing 2 prizes and 6 blanks. Compare their chances of success.

19. (*i*) There are *n* letters and *n* addressed envelopes. If the letters are placed in the envelopes at random, what is the probability that all the letters are not placed in the right envelope?

(*ii*) Three letters are written to different persons, and the addresses on the three envelopes are also written. Without looking at the addresses, find the probability that the letters go into the right envelopes. **(*MNR*)**

20. The letters of word 'SOCIETY' are placed at random in a row. What is the probability that three vowels come together?

21. In a random arrangement of the letters of the word "COMMERCE", find the probability that all vowels come together.

22. Given a group of 4 persons, find the probability that

(*i*) no two of them have their birthdays on the same day,

(*ii*) all of them have same birthday. [Ignore the existence of a leap year]

ANSWERS

1. $\frac{3\times {}^4P_3}{{}^5P_4}=\frac{3}{5}$ or 0.6 **2.** $\frac{{}^6C_3}{{}^{10}C_3}=\frac{1}{6}$ **3.** $\frac{{}^6C_3}{{}^{18}C_3}=\frac{5}{204}$

4. (*i*) $\frac{1}{6}$ (*ii*) $\frac{1}{3}$ (*iii*) $\frac{1}{2}$ (*iv*) $\frac{5}{6}$ (*v*) $\frac{2}{3}$ (*vi*) 0

5. (*i*) $\frac{{}^4C_3}{{}^{12}C_3}=\frac{1}{55}$ (*ii*) $\frac{{}^6C_3}{{}^{12}C_3}=\frac{1}{11}$ (*iii*) 0 (*iv*) $\frac{{}^6C_3}{{}^{12}C_3}=\frac{1}{11}$ (*v*) $\frac{10}{11}$ (*vi*) 1 (certainty)

6. (*i*) $\frac{5}{18}$ (*ii*) $\frac{7}{12}$ **7.** $\frac{{}^4C_1\times {}^4C_1\times {}^4C_1}{{}^{52}C_3}=\frac{16}{5525}$

8. (*i*) $\frac{4}{7}$ (*iii*) $\frac{72}{5525}$ (*iv*) $\frac{1}{425}$ (*v*) $\frac{105}{221}$

9. (*i*) $\frac{{}^{13}C_3}{{}^{52}C_3}=\frac{11}{850}$ (*ii*) $\frac{{}^{26}C_3}{{}^{52}C_3}=\frac{2}{17}$ (*iii*) $\frac{{}^{39}C_3}{{}^{52}C_3}=\frac{703}{1700}$ (*iv*) $\frac{{}^4C_3}{{}^{52}C_3}=\frac{1}{5525}$

10. (*i*) $\frac{11}{839}$; 11 to 839 (*ii*) $\frac{2}{15}$; 2 to 15 (*iii*) $\frac{703}{997}$; 703 to 997 (*iv*) 1 to 5524

11. (*i*) $\frac{{}^4C_1}{{}^{52}C_1}=\frac{1}{13}$ (*ii*) $\frac{{}^{26}C_1}{{}^{52}C_1}=\frac{1}{2}$ (*iii*) $\frac{{}^{28}C_1}{{}^{52}C_1}=\frac{7}{13}$ (*iv*) $\frac{{}^2C_1}{{}^{52}C_1}=\frac{1}{26}$

12. (*i*) $\frac{4.{}^{13}C_4}{{}^{52}C_4}=\frac{198}{20825}$ (*ii*) $\frac{13}{{}^{52}C_4}=\frac{13}{270725}$ (*iii*) $\frac{({}^{13}C_1)^4}{{}^{52}C_4}=\frac{2197}{20825}$ (*iv*) $\frac{{}^{26}C_2\times {}^{26}C_2}{{}^{52}C_4}$

(*v*) $\frac{2.({}^{26}C_4)}{{}^{52}C_4}$ (*vi*) $\frac{{}^{12}C_4}{{}^{52}C_4}$ (*vii*) $\frac{{}^4C_1}{{}^{52}C_4}=\frac{4}{270725}$

13. (*i*) $\frac{{}^{15}C_2}{{}^{50}C_2}=\frac{3}{35}$ (*ii*) $\frac{{}^{35}C_2}{{}^{50}C_2}=\frac{17}{35}$ (*iii*) $\frac{{}^{15}C_1\times {}^{35}C_1}{{}^{50}C_2}=\frac{3}{7}$

14. (*i*) $\frac{{}^4C_2\times {}^5C_2}{{}^9C_4}=\frac{10}{21}$ (*ii*) $\frac{3}{7}$ **15.** $\frac{{}^6C_2\times {}^8C_3}{{}^{14}C_5}=\frac{60}{143}$ **16.** $\frac{{}^{10}C_2}{{}^{20}C_2}=\frac{9}{38}$

17. $\frac{88}{203}$ **18.** (*i*) $\frac{5}{6}$ (*ii*) 952 : 715 **19.** (*i*) $1-\frac{1}{n!}$ (*ii*) $\frac{1}{6}$

20. $\frac{5!\times 3!}{7!}=\frac{1}{7}$

HINTS TO SELECTED QUESTIONS

6. (*i*) Reqd. probability $P(A) = \dfrac{{}^2C_1}{{}^9C_1}\cdot\dfrac{{}^1C_1}{{}^8C_1} + \dfrac{{}^3C_1}{{}^9C_1}\cdot\dfrac{{}^2C_1}{{}^8C_1} + \dfrac{{}^4C_1}{{}^9C_1}\cdot\dfrac{{}^3C_1}{{}^8C_1}$

(*ii*) $P(B) = 1 -$ Prob. (getting no red ball) $= 1 - \dfrac{{}^6C_1}{{}^9C_1}\cdot\dfrac{{}^5C_1}{{}^8C_1}$

8. (*iv*) $n(A) = 13.\ {}^4C_3,\ n(S) = {}^{52}C_3$

(*v*) Total number of cases $= {}^{52}C_2$. Since neither a Jack nor a spade card is to be drawn, so the number of cards from which two cards are to be drawn $= 52 - (4 + 13 - 1) = 36$.

$\therefore$ Number of favourable cases $= {}^{36}C_2$.

11. (*iii*) There are 26 red cards including 2 red kings and there are 2 more kings. So there are 28 cards which are red or king, and out of these one card may be drawn in ${}^{28}C_1$ ways.

12. (*i*) See Solved Ex. 18;

(*ii*) The favourable outcomes are (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3),....., (13, 13, 13, 13);

(*iii*) Number of ways of selecting 4 cards, ace from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = ({}^{13}C_1)^4$;

(*v*) Out of 26 red colour cards, 4 cards can be drawn in ${}^{26}C_4$ ways, 4 black cards can be drawn in ${}^{26}C_4$ ways. Therefore, 4 red or 4 black cards can be drawn in ${}^{26}C_4 + {}^{26}C_4 = 2\ .\ {}^{26}C_4$ ways;

(*vi*) **There are 12 face cards** (4 kings, 4 queens and 4 jacks). Out of these, 4 cards can be selected in ${}^{12}C_4$ ways;

(*vii*) **There are four suits** containing a set of four honours. Therefore, four honours of the same suit can be drawn in ${}^4C_1$ ways. (*Ace, King, Queen, Jack*)

16. Out of 20 tickets numbered 1 to 20, 10 are odd numbered.

$\therefore$ Number of favourable outcomes $= {}^{10}C_2$.

17. Negation of 'at least one is' none and therefore, it is easier to find the probability that none of the selected tickets bear number exceeding 25. If A be the event that none of the selected tickets bear number exceeding 25, then $P(A) = \dfrac{{}^{25}C_3}{{}^{30}C_3} = \dfrac{115}{203}$. $\therefore$ Required probability $= P(\bar{A}) = 1 - P(A)$

18. (*i*) Reqd. probability $=$ P (at least one good bulb)

$= 1 - P$ (no good bulb) $= 1 - \dfrac{{}^6C_3}{{}^{10}C_3}$.

(*ii*) A : A wins, B : B wins

Then, $P(A) = 1 - \dfrac{{}^9C_3}{{}^{12}C_3}$, $P(B) = 1 - \dfrac{{}^6C_2}{{}^8C_2}$.

19. (*i*) The total number of ways of placing n letters in n envelopes $= n\ !$
All the letters can be placed correctly in only 1 way.

$\therefore$ Probability of placing the letters in right envelopes $= \dfrac{1}{n!}$.

Hence, the probability that all letters are not placed in right envelope $= 1 - \dfrac{1}{n!}$

21. Required probability $= \dfrac{6!\ 3!}{2!\ 2!\ 2!} \Big/ \dfrac{8!}{2!\ 2!\ 2!} = \dfrac{3}{28}$.

22. Let us number the days of the year from 1 to 365. The first person can have his birthday on any of 365 days. The second person can have his birthday on any of 365 days. And so on.

Thus, the total number of ways of choosing the birthday for the four persons is

$365 \times 365 \times 365 \times 365 = 365^4$.

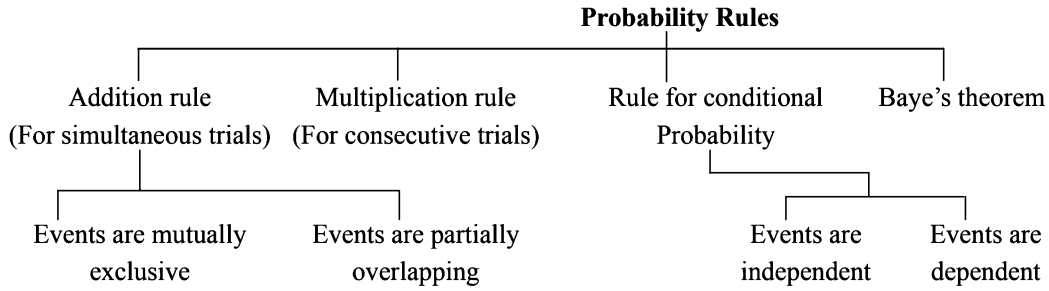
(*i*) Since no two persons have their birthday on the same day, then for the first person we have 365 choices, for the second we have 365 – 1 = 364 choices; for the third person we have 365 – 2 = 363 choices and for the fourth person we have 365 – 3 = 362 choices. Thus, the number of ways in which all the four persons have their birthdays on different days in 365 × 364 × 363 × 362. Thus,

$$P\text{ (distinct birthday)} = \frac{365\times364\times363\times362}{365^4} = \frac{364\times363\times362}{365^3} = \frac{{}^{364}P_3}{365^3}.$$

(*ii*) If all the four persons have to have their birthday on the same day, then we have to choose just one day out of 365. This can be done in 365 ways. Thus,

$$P\text{ (all the birthdays on the same day)} = \frac{365}{365^4} = \frac{1}{365^3}.$$

COMBINING PROBABILITIES



In this book we will study only the addition rule.

22.12. The addition rule

When two events are mutually exclusive, we can work out the probability of either of them occurring by **adding together the separate probabilites.**

Illustration. *A bag contains 15 red balls, 5 green balls, 4 blue balls and 16 black balls. A ball is drawn at random. What is the probability that it is*

(*i*) *red* (*ii*) *black* (*iii*) *red or black* (*iv*) *not green* (*v*) *neither green nor blue* ?

Sol. **(*i*)** $P(\text{red}) = \frac{15}{40} = \frac{3}{8}$

(*ii*) $P(\text{black}) = \frac{16}{40} = \frac{2}{5}$

(*iii*) $P(\text{red or black}) = P(\text{red}) + P(\text{black}) = \frac{3}{8} + \frac{2}{5} = \frac{31}{40}$

(*iv*) $P(\text{not green}) = \frac{35}{40} = \frac{7}{8}$

(*v*) $P(\text{neither green nor blue}) = P(\text{red or black}) = \frac{31}{40}$

The last part is another example of how confusing probability can be. You might say

$$P\text{ (neither green nor blue)} = P(\text{not green}) + P(\text{not blue}) = \frac{35}{40} + \frac{36}{40} = \frac{71}{40}$$

This cannot be correct, as $\frac{71}{40}$ is greater than 1. In fact, the events 'not green' and 'not blue' are not mutually exclusive, as there are lots of balls that are true for both events.

EXERCISE 22 (e)

1. Sameer throws an ordinary die. What is the probability that he throws
(*i*) 2 (*ii*) 5 (*iii*) 2 or 5?

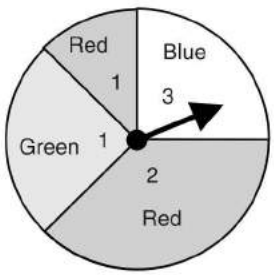
2. Kavita draws a card from a pack of cards. What is the probability that she draws
(*i*) a heart (*ii*) a club (*iii*) a heart or a club?

3. Anurag draws a card from a pack of cards. What is the probability that he draws one of the following numbers?
(*i*) 3 (*ii*) 7 (*iii*) 3 or 7

4. A letter is chosen at random from the letters in the word PROBABILITY. What is the probability that the letter will be
(*i*) B (*ii*) a vowel (*iii*) B or a vowel?

5. A bag contains 7 white balls, 9 green balls and 10 yellow balls. A ball is drawn at random from the bag. What is the probability that it will be
(*i*) white (*ii*) green (*iii*) green or white
(*iv*) not yellow (*v*) neither yellow nor green?

6. Suyash needs his calculator for his mathematics lesson. It is either in his pocket, bag or locker. The probability it is in his pocket is 0.20, the probability it is in his bag is 0.58. What is the probability that
(*i*) he will have the calculator for the lesson,
(*ii*) it is in his locker?



7. A spinner has numbers and colours on it, as shown in the diagram. Their probabilities are given in the tables. When the spinner is spun what is the probability of each of the following?

Red	0.5
Green	0.25
Blue	0.25

1	0.4
2	0.35
3	0.25

(*i*) red or green (*ii*) 2 or 3 (*iii*) 3 or green
(*iv*) 2 or green (*v*) Explain why the answer to P(1 or red) is not 0.9.

ANSWERS

1. (*i*) $\frac{1}{6}$, (*ii*) $\frac{1}{6}$, (*iii*) $\frac{1}{3}$ **2.** (*i*) $\frac{1}{4}$, (*ii*) $\frac{1}{4}$ (*iii*) $\frac{1}{2}$ **3.** (*i*) $\frac{1}{13}$, (*ii*) $\frac{1}{13}$, (*iii*) $\frac{2}{13}$

4. (*i*) $\frac{2}{11}$, (*ii*) $\frac{4}{11}$, (*iii*) $\frac{6}{11}$ **5.** (*i*) $\frac{7}{26}$, (*ii*) $\frac{9}{26}$, (*iii*) $\frac{16}{26}$ or $\frac{8}{13}$, (*iv*) $\frac{8}{13}$, (*v*) $\frac{7}{26}$

6. (*i*) 0.78 (*ii*) 0.22 **7.** (*i*) 0.75 (*ii*) 0.6 (*iii*) 0.5 (*iv*) 0.6
(*v*) They are not mutually exclusive.

22.13. Theorems

Theorem 1. ***If A and B are mutually exclusive events, then P*** $(A \cap B) = 0$.

Proof. Since A and B are mutually exclusive events, they have no outcome in common,
i.e., $A \cap B = \phi \quad \therefore \quad P(A \cap B) = P(\phi) = 0.$

Theorem 2. ***Let A and B be two events associated with a random experiment such that the event A implies the event B, then P*** $(A) \geq P(B)$.

Proof. Let S be the sample space. Since the event A implies the event B, the number of outcomes favourable to A are also favourable to B.

Let n be the total number of outcomes to the experiment. Let m_1 and m_2 be the number of outcomes favourable to A and B respectively.

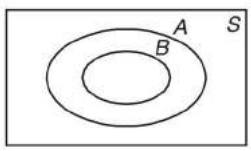


Fig. 22.13

Then $P(A) = \frac{m_1}{n}$ and $P(B) = \frac{m_2}{n}$

Clearly, $m_1 \geq m_2 \Rightarrow \frac{m_1}{n} \geq \frac{m_2}{n} \Rightarrow P(A) \geq P(B)$.

Theorem 3. ***If A and B be two mutually exclusive events connected with a random experiment, then the probability of occurrence of the event "A or B" is the sum of the probabilities of the events A and B. Thus,***

i.e., $P(A \text{ or } B) = P(A) + P(B)$

In set notation, $A \cap B = \phi$, ***then*** $P(A \cup B) = P(A) + P(B)$.

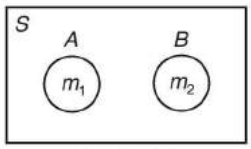


Fig. 22.14

Proof. Let n be the total number of outcomes in a sample space of a random experiment. Let m_1 be the outcomes favourable to A and m_2 be the outcomes favourable to B. Then

$$P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

Since outcomes A and B are mutually exclusive, the sets representing A and B are disjoint. Therefore, the number of outcomes in the set representing A or B (*i.e.*, $A \cup B$) is $m_1 + m_2$.

Hence, $P(A \cup B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$.

That is, $P(A \text{ or } B) = P(A) + P(B)$

Remark. The above theorem is known as addition theorem for mutually exclusive events.

Theorem 4. ***If A and B are mutually exclusive and exhaustive events, then*** $P(A) + P(B) = 1$.

Proof. Let S be the sample space.

Since A and B are exhaustive events, so $A \cup B = S$

$\therefore \quad P(A \cup B) = P(S) = 1$

Also since A and B are mutually exclusive, so

$$P(A \cup B) = P(A) + P(B)$$

$\Rightarrow \quad 1 = P(A) + P(B)$ or $P(A) + P(B) = 1$.

Cor. If $E_1, E_2 \ldots E_k$ are mutually exclusive events, then,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

Events that are not mutually exclusive *i.e.*, $A \cap B \neq \phi$.

Suppose we want to find probability of one card selected from a pack being either a heart or a queen. These events are not mutually exclusive as the card could be both a heart and a queen. *i.e.*, it could be the queen of hearts. We consider the possibility space diagram :

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♣	x	x	x	x	x	x	x	x	x	x	x	⊗	x
♦	x	x	x	x	x	x	x	x	x	x	x	⊗	x
♥	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗
♠	x	x	x	x	x	x	x	x	x	x	x	⊗	x

Thus $P(\heartsuit) = \frac{13}{52} = \frac{1}{4}$, $P(Q) = \frac{4}{52} = \frac{1}{13}$ and counting the circled entries in the diagram

$P(\heartsuit \text{ or } Q) = \frac{16}{52} = \frac{4}{13}$.

Therefore, P (♥ or Q) $\neq P$ (♥) + P (Q) because, as we can see from the diagram, the right-hand side of such a calculation would include the queen of hearts twice.

In general, for two events A and B

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or using set notation

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The reader should notice that for mutually exclusive events $P(A \cap B) = 0$ and the above rule reduces to $P(A \cup B) = P(A) + P(B)$ as required.

Theorem 5. Not mutually exclusive events (*i.e.*, $A \cap B \neq \phi$) .

If A and B are two events associated with a random experiment then

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

i.e., $$\boldsymbol{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).}$$

Proof. Let n be the total number of events in a sample space of random experiment. Let m_1 be events favourable to A and m_2 be events favourable to B.

$$\therefore \quad P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

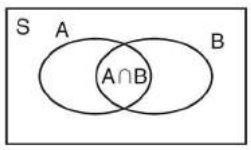


Fig. 22.15

Since A and B are any two events, therefore it may be possible that they are not mutually exclusive. Thus they may have some common events. Suppose m_3 are the events favourable to $A \cap B$.

$$\therefore \quad P(A \cap B) = \frac{m_3}{n}$$

It is obvious that the set $A \cup B$ representing the event A or B has $m_1 + m_2 - m_3$ events.

$$\therefore \quad P(A \text{ or } B) = P(A \cup B) = \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Cor. 1. If the events have no outcome in common *i.e.*, they are mutually exclusive, then $A \cap B = \phi$ and, therefore, $P(A \cap B) = P(\phi) = 0$

$$\therefore \quad P(A \cup B) = P(A) + P(B). \qquad \text{[Theorem 3]}$$

Cor 2. To show that $P(A \text{ or } B) \leq P(A) + P(B)$.

We have proved that

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B).$$

Since $P(A \cap B)$ is greater than or equal to 0, therefore it follows that $P(A \text{ or } B) < P(A) + P(B)$.

Note. Equality in the above result holds when A and B are mutually exclusive.

Theorem 6. If *A, B, C be any three events associated with a random experiment, then*

$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).}$$

Proof. Let $A \cup B$ be one event and C be other event. Then $P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P[(A \cup B) \cup C]$ (*Applying addition rule*)

$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)]$

(*Applying addition rule*)

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

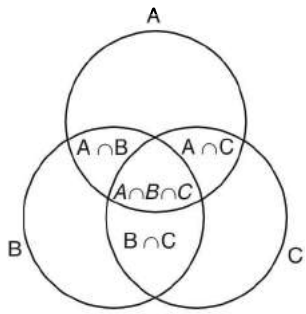


Fig. 22.16

Cor 1. *If A, B, C are* **mutually exclusive** *events, then*

$P(A \cap B) = P(B \cap C) = P(A \cap C)$

$= P(A \cap B \cap C) = 0$

$\therefore \quad \boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C).}$

This is the addition theorem for three mutually exclusive events.

Theorem 7. ***For every event associated with a random experiment, we have***

$$\boldsymbol{P(\text{not } A) = 1 - P(A);} \text{ ***or,***} \quad \boldsymbol{P(\bar{A}) = 1 - P(A)}$$

Proof. Event A and 'not A', *i.e.*, $\bar{A}$ are mutually exclusive. Hence,

$P(A \text{ or not } A) = P(A) + P(\text{not } A)$

Event A or not A is a sure event since one of the two events must occur.

$\therefore \quad P(A) + P(\text{not } A) = 1 \quad \Rightarrow \quad P(\text{not } A) = 1 - P(A)$ *i.e.*, $P(\bar{A}) = 1 - P(A)$

Theorem 8. ***Probability of an impossible event is zero, i.e., P (ϕ) = 0.***

Proof. Impossible event contains no sample point. Therefore, the sample space S and the impossible event ϕ are mutually exclusive.

$\Rightarrow \quad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S) \quad \Rightarrow \quad P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0$

Remarks. If A and B are two events associated with a random experiment, then

(*i*) $P(A)$ denotes the probability to happen.

(*ii*) $P(\bar{A})$ denotes the probability not to happen. It is also denoted by $P(A^c)$

(*iii*) $P(A + B)$ or $P(A \cup B)$ denotes the probability of happening of at least one of A or B.

(*iv*) $P(AB)$ or $P(A \cap B)$ denotes the probability of happening of both A and B.

Ex. 31. The probability that Sunil will receive a *D* in his mathematics course is 0.61 and the probability that he will receive a *C* is 0.28. What is the probability that Sunil will receive a *C* or *D* in mathematics?

Sol. The probability of receiving a D is $p_1 = 0.61$

The probability of receiving a C is $p_2 = 0.28$

Since the two events are mutually exclusive, therefore, the probability of receiving a C or D in mathematics $= p_1 + p_2 = 0.61 + 0.28 = 0.89$.

Ex. 32. Find the chance of throwing a total of 3 or 5 or 11 with two dice.

Sol. The three events are mutually exclusive.

Exhaustive number of cases $= 6^2 = 36$.

Let p_1, p_2, p_3 be the probability of throwing a total of 3, 5, 11 respectively.

Favourable cases of throwing a total of 3 are (1, 2), (2, 1). $\quad \therefore \quad p_1 = \frac{2}{36}$

Favourable cases of throwing a total of 5 are (1, 4), (4, 1), (2, 3), (3, 2) $\quad \therefore \quad p_2 = \frac{4}{36}$

Favourable cases of throwing a total of 11 are (5, 6), (6, 5) $\therefore\ p_3 = \frac{2}{36}$

If P is the probability of throwing a total of 3 or 5 or 11, then

$$P = p_1 + p_2 + p_3 = \frac{2}{36} + \frac{4}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}.$$

Ex. 33. There are 2 red and 3 black balls in a bag. 3 balls are taken out at random from the bag. Find the probability of getting 2 red and 1 black balls or 1 red and 2 black balls.

Sol. Given, 2 red ($2R$) and 3 black ($3B$) balls

$\therefore\ P$ ($2R$ and $1B$ or $1R$ and $2B$)

$$= P(2R \text{ and } 1B) + P(1R \text{ and } 2B)$$

$$= \frac{{}^2C_2 \cdot {}^3C_1}{{}^5C_3} + \frac{{}^2C_1 \cdot {}^3C_2}{{}^5C_3} = \frac{1 \times 3}{10} + \frac{2 \times 3}{10} = \frac{9}{10}.$$

Ex. 34. In a given race: the odds in favour of four horses *A, B, C,* and *D* are 1 : 3, 1 : 4, 1 : 5, and 1 : 6 respectively. Assuming that a dead heat is impossible, find the chance that one of them wins the race. (*ISC 1998*)

Sol. Let p_1, p_2, p_3, and p_4 be the probabilities of winning of the horses A, B, C, D respectively. Since a *dead heat* (in which all the four horses win *i.e.*, they cover same distance in same time) is not possible, the events are mutually exclusive. Odds in favour of A are 1 : 3.

$\therefore\quad p_1 = \frac{1}{1+3} = \frac{1}{4}$. Similarly, $p_2 = \frac{1}{5}, p_3 = \frac{1}{6}, p_4 = \frac{1}{7}$

If P is the chance that one of them wins, then

we have $P = p_1 + p_2 + p_3 + p_4 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}$.

Ex. 35. (*a*) A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or a bolt? (*Type ISC 2012*)

(*b*) (*i*) Two dice are thrown together. What is the probability that the sum of the numbers on two faces is neither 9 nor 11?

(*ii*) Two unbiased dice are thrown. Find the probability that the sum of the numbers obtained on the two dice is neither a multiple of 2 nor a multiple of 3.

Sol. (*a*) Let A be the event that the item chosen is rusted and B be the event that the item chosen is a bolt.

There are 200 items in all, of which 100 are rusted.

$$\therefore P(A) = \frac{100}{200},\ P(B) = \frac{50}{200} \text{ and } P(A \cap B) = \frac{25}{200}$$

Reqd. probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{100}{200} + \frac{50}{200} - \frac{25}{200} = \frac{125}{200} = \frac{5}{8}.$$

(*b*) (*i*) **Method I.**

A = Getting total of 9

B = Getting total of 11

A = {(5, 4), (4, 5), (6, 3) (3, 6)}

B = {(6, 5), (5, 6)}, $A \cap B = \phi$

$$n(A) = 4,\ n(B) = 2,\ n(S) = 36$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9},\ P(B) = \frac{2}{36} = \frac{1}{18}$$

$$\text{Required probability} = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left(\frac{1}{9} + \frac{1}{18} - 0\right) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Method II.

$$\text{Favourable number} = n(S) - n(A) - n(B) = 36 - 4 - 2 = 30$$

$$\therefore \quad \text{Required probability} = \frac{30}{36} = \frac{5}{6}.$$ [See Solved Ex. 9 (*f*)]

(*ii*) A = Getting sum of two numbers as a multiple of 2.

B = Getting sum of two numbers as a multiple of 3.

$A = \{(1, 1), (3, 1), (1, 3), (2, 2), (5, 1), (1, 5), (4, 2), (2, 4), (3, 3), (6, 2),$
$(2, 6), (5, 3), (3, 5), (4, 4), (6, 4), (4, 6), (5, 5), (6, 6)\}$

$B = \{(2, 1), (1, 2), (5, 1), (1, 5), (4, 2), (2, 4), (3, 3), (3, 6), (6, 3), (5, 4), (4, 5), (6, 6)\}$

$$A \cap B = \{(5, 1), (1, 5), (4, 2), (2, 4), (3, 3), (6, 6)\}$$

$$n(A) = 18,\ n(B) = 12,\ n(A \cap B) = 6,\ n(S) = 36$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2},\ P(B) = \frac{n(B)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$$\text{Required probability} = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{6}\right] = 1 - \frac{4}{6} = \frac{2}{6} = \frac{1}{3}.$$

Ex. 36. A ticket is drawn from two hundred tickets numbered from 1 to 200. Find the probability that the number is divisible by 2 or 3 or 5.

Sol. A = number divisible by 2, B = number divisible by 3, C = number divisible by 5

$A = \{2, 4, 6, ..., 200\}$, $B = \{3, 6, 9, ..., 198\}$, $C = \{5, 10, 15, ..., 200\}$

$A \cap B = \{6, 12, 18, 24, ..., 198\}$, $B \cap C = \{15, 30, 45, ..., 195\}$, $C \cap A = \{10, 20, 30, ..., 200\}$

$$A \cap B \cap C = \{30, 60, 90, 120, 150, 180\}$$

$$n(A) = 100,\ n(B) = 66,\ n(C) = 40,\ n(A \cap B) = 33,\ n(B \cap C) = 13,$$

$$n(C \cap A) = 20,\ n(A \cap B \cap C) = 6,\ n(S) = 200$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{100}{200},\ P(B) = \frac{n(B)}{n(S)} = \frac{66}{200},\ P(C) = \frac{n(C)}{n(S)} = \frac{40}{200}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{33}{200},\ P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{13}{200},$$

$$P(C \cap A) = \frac{n(C \cap A)}{n(S)} = \frac{20}{200},\ P(A \cap B \cap C) = \frac{n(A \cap B \cap C)}{n(S)} = \frac{6}{200}$$

$\therefore$ Required probability $= P(A \cup B \cup C)$

$$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

$$= \frac{100}{200}+\frac{66}{200}+\frac{40}{200}-\frac{33}{200}-\frac{13}{200}-\frac{20}{200}+\frac{6}{200}=\frac{146}{200}=\frac{73}{100}.$$

Ex. 37. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace or both. ***(ISC 1996)***

Sol. Let A be the event of drawing a spade and B the event of drawing an ace. A and B are not mutually exclusive as one ace is a spade also.

$A \cap B$ = the event of drawing the ace of spades.

$$P(A) = \frac{13}{52}, P(B)=\frac{4}{52}, \; P(A\cap B)=\frac{1}{52}$$

$$\therefore \quad P(A\cap B) = P(A)+P(B)-P(A\cup B)=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}.$$

Ex. 38. In a group there are 3 women and 3 men. 4 people are selected at random from this group. Find the probability that 3 women and 1 man or 1 woman and 3 men are selected.

Sol. A : selecting 3 women and 1 man

B : selecting 1 woman and 3 men

$$n(A) = {}^3C_3 . {}^3C_1 = 1\times 3 = 3, \quad n(B)={}^3C_1 . {}^3C_3 = 3\times 1 = 3, \quad n(S)={}^6C_4={}^6C_2=\frac{6\times 5}{1\times 2}=15$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)}=\frac{3}{15}=\frac{1}{5}, P(B)=\frac{n(B)}{n(S)}=\frac{3}{15}=\frac{1}{5}$$

Required probability $= P(A)+P(B)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}.$

Ex. 39. Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings?

Sol. 2 cards out of 52 cards can be drawn in ${}^{52}C_2$ ways.

$$n(S) = {}^{52}C_2$$

Let A be the event: 'Two cards drawn are red cards', and B be the event : 'Two cards drawn are kings'. Then

$$n(A) = {}^{26}C_2, n(B) = {}^4C_2$$

Required probability $= P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{{}^{26}C_2}{{}^{52}C_2} \text{ and } P(B)=\frac{n(B)}{n(S)}=\frac{{}^4C_2}{{}^{52}C_2}$$

Also there are 2 cards which are both red and kings.

$\therefore$ $P(A\cap B)$ = Probability of getting 2 red kings $= \dfrac{{}^2C_2}{{}^{52}C_2}$

$\therefore$ Reqd. probability $= P(A)+P(B)-P(A\cap B)$

$$= \frac{{}^{26}C_2}{{}^{52}C_2}+\frac{{}^4C_2}{{}^{52}C_2}-\frac{{}^2C_2}{{}^{52}C_2}=\frac{325}{1326}+\frac{1}{221}-\frac{1}{1326}=\frac{55}{221}.$$

Ex. 40. (*i*) If $P(A\cup B)=0.65$ and $P(A\cap B)=0.15$, find $P(\bar{A})+P(\bar{B})$. ***(EAMCET)***

(*ii*) If $P(A)=\frac{1}{4}, P(B)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{8}$, find (*a*) $P(A\cup B)$, (*b*) $P[(A\cup B)']$.

Sol. (*i*) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A) + P(B) = P(A \cup B) + P(A \cap B) = 0.65 + 0.15 = 0.80.$

Now $P(\overline{A}) = 1 - P(A)$ and $P(\overline{B}) = 1 - P(B)$

$\therefore \quad P(\overline{A}) + P(\overline{B}) = 1 - P(A) + 1 - P(B) = 2 - P(A) - P(B)$

$= 2 - [P(A) + P(B)] = 2 - 0.80 = 1.2$

(*ii*) (*a*) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$

(*b*) $P[(A \cup B)'] = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}.$

Ex. 41. When two dice are thrown, calculate the probability of throwing a total of (*i*) a 7 or an 11; (*ii*) a doublet or a total of 6.

Sol. A : Getting a total of 7, B : getting a total of 11

$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$\therefore \quad n(A) = 6, \; P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$

$B = \{(5, 6), (6, 5)\} \qquad \therefore \quad n(B) = 2, P(B) = \frac{n(B)}{n(S)} = \frac{2}{36}$

Clearly, the two events are mutually exclusive, therefore,

$P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$

(*ii*) The sample space consists of 36 sample points $\therefore \quad n(S) = 36$

A : Getting a double

$\therefore \quad A = \{(1, 1,), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}. \quad n(A) = 6$

B : Getting a total of 6.

$\therefore \quad B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \quad \therefore \quad n(B) = 5$

$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}, \; P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$

Clearly, the two events are not mutually exclusive since (3, 3) is one common sample point, therefore, $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$

Ex. 42. A bag contains 20 tickets with marked numbers 1 to 20. One ticket is drawn at random. Find the probability that it will be a multiple of 2 or 5.

Sol. The sample space consists of 20 sample points.

$\therefore \quad n(S) = 20$

A = Getting a multiple of 2, $\qquad B$ = Getting a multiple of 5

$\therefore \quad A = \{2, 4, 6, ..., 20\} \qquad \therefore \quad n(A) = 10$ and $P(A) = \frac{n(A)}{n(S)} = \frac{10}{20}$

$B = \{5, 10, 15, 20\} \qquad \therefore \quad n(B) = 4$ and $P(B) = \frac{n(B)}{n(S)} = \frac{4}{20}$

Clearly, the two events are not mutually exclusive since the two points 10, 20 are common.

$\therefore \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{20}$

Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{10}{20} + \frac{4}{20} - \frac{2}{20} = \frac{12}{20} = \frac{3}{5}$.

Ex. 43. The probability that a person will win a game is $\frac{2}{3}$, and the probability that he will not win a horse race is $\frac{5}{9}$. If the probability of getting success in at least one of the events is $\frac{4}{5}$, what is the probability that he will be successful in both the events?

Sol. Let A : winning a game $\therefore \quad P(A) = \frac{2}{3}$

and B denote the event of winning a horse race. Then,

$$P(B) = 1 - P(\bar{B}) = 1 - \frac{5}{9} = \frac{4}{9} \qquad \left[\because P(\bar{B}) = P \text{ (not winning horse race)} = \frac{5}{9} \text{ (Given)}\right]$$

Also, $P(A \cup B) = \frac{4}{5}$

This is a non-mutually exclusive case as both events can happen together.

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad \frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(A \cap B) \Rightarrow P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{42}{135} = \frac{14}{45}$.

Ex. 44. Supposing that it is 8 to 7 against a person who is now 30 years of age living till he is 60, and 2 to 1 against a person who is now forty living till he is 70; find the probability that at least one of these persons will be alive 30 years hence.

Sol. This is a non-mutually exclusive case as both events can happen simultaneously.

Method I. Let A : person of age 30 years living till 60 years.

B : person of age 40 years living till 70 years.

Then, $P(A) = \frac{7}{15}, \; P(B) = \frac{1}{3}, P(A \cap B) = \frac{7}{15} \times \frac{1}{3}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{7}{15} + \frac{1}{3} - \left(\frac{7}{15} \times \frac{1}{3}\right) = \frac{29}{45}$.

Method II. $p_1 = \frac{7}{15}, q_1 = \frac{8}{15}; \; p_2 = \frac{1}{3}, q_2 = \frac{2}{3}$

The required probability $= p_1q_2 + p_2q_1 + p_1p_2 = 1 - q_1q_2$

$$= 1 - \frac{8}{15} \cdot \frac{2}{3} = 1 - \frac{16}{45} = \frac{29}{45}.$$

Note. In Method II, p denotes the probability for the event, and q denotes against the event.

EXERCISE 22 (*f*)

1. (*i*) In a single throw of two coins, find the probability of getting both heads or both tails.

(*ii*) A die is thrown twice. Find the probability that the sum of the two numbers obtain is 5 or 7?

(*iii*) Two dice are tossed once. Find the probability of getting an even number on first die or a total of 8.

2. If the probability of a horse A winning a race is $\frac{1}{5}$ and the probability of horse B winning the same race is $\frac{1}{4}$, what is the probability that one of the horses will win?

3. In a single throw of two dice, what is the probability of obtaining a total of 9 or 11?

4. In a group there are 2 men and 3 women. 3 persons are selected at random from the group. Find the probability that 1 man and 2 women or 2 men and 1 woman are selected.

5. In a class of 25 students with roll numbers 1 to 25, a student is picked up at random to answer a question. Find the probability that the roll number of the selected student is either a multiple of 5 or 7.

6. If chance of A, winning a certain race be $\frac{1}{6}$ and the chance of B winning it is $\frac{1}{3}$, what is the chance that neither should win?

7. Discuss and criticise the following : $P(A)=\frac{2}{3}, P(B)=\frac{1}{4}, P(C)=\frac{1}{3}$ are the probabilities of three mutually exclusive events A, B and C.

8. E and F are two events associated with a random experiment for which $P(F) = 0.35$, $P(E \text{ or } F) = 0.85$, $P(E \text{ and } F) = 0.15$. Find $P(E)$.

9. (*i*) Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14. Find the probability that neither A nor B occurs. ***(NMOC 1995, EAMCET)***

(*ii*) The probability of an event A occurring is 0.5 and of B is 0.3. If A and B are mutually exclusive events, then find the probability of neither A nor B occurring. ***(IIT)***

10. (*i*) A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even?

(*ii*) A bag contains 7 white, 5 black and 4 red balls. Four balls are drawn without replacement. Find the probability that at least three balls are black. ***(CBSE)***

11. (*i*) A and B are two mutually exclusive events of an experiment. If $P(\text{not } A) = 0.65$, $P(A \cup B) = 0.65$ and $P(B) = p$, find the value of p.

(*ii*) A and B are two mutually exclusive events. If $P(A) = 0.5$ and $P(\bar{B}) = 0.6$, find $P(A \text{ or } B)$.

12. (*i*) A, B, and C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$ given that $P(B)=\frac{3}{2}P(A)$ and $P(C)=\frac{1}{2}P(B)$.

(*ii*) If A, B, C are three mutually exclusive and exhaustive events and $P(A) = 2P(B) = 3P(C)$, then find $P(A)$. ***(NMOC 1995)***

13. (*i*) In a single throw of two dice, find the probability that neither a doublet nor a total of a 10 will appear.

(*ii*) Two unbiased dice are thrown. Find the probability that the sum of the numbers obtained on the two dice is neither a multiple of 3 nor a multiple of 4. ***(CBSE 2006)***

(*iii*) Two dice are thrown together; what is the probability that the sum of the numbers on the two faces is neither divisible by 3 nor by 5.

14. In a given race, the odds in favour of horses A, B, C and D are 1 : 3, 1 : 4, 1 : 5 and 1 : 6 respectively. Find the probability that one of them wins the race.

15. 100 students appeared for two examinations. 60 passed the first, 50 passed the second and 30 passed both. Find the probability that a student selected at random has failed in both examinations.

16. (*i*) A card is drawn from a deck of 52 cards. Find the probability of getting an ace or a spade card.

(*ii*) From a well shuffled deck of 52 cards, 4 cards are drawn at random. What is the probability that all the drawn cards are of the same colour.

17. (*i*) A card is drawn at random from a well shuffled pack of cards. What is the probability that it is a heart or a queen?

(*ii*) A card is drawn at random from a well shuffled pack of 52 cards. Find the probability it is neither a king nor a heart? ***(ISC 2006, NMOC 1994)***

18. A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

19. There are three events *A, B, C* one of which must and only one can happen. The odds are 8 to 3 against *A,* 5 to 2 against *B*; find the odds against *C*.

20. In a group of students, there are 3 boys and 3 girls. Four students are to be selected at random from the group. Find the probability that either 3 boys and 1 girl or 3 girls and 1 boy are selected.

ANSWERS

1. (*i*) $\frac{1}{2}$ (*ii*) $\frac{5}{18}$ (*iii*) $\frac{5}{9}$ **2.** $\frac{9}{20}$ **3.** $\frac{1}{6}$ **4.** $\frac{9}{10}$

5. $\frac{8}{25}$ **6.** $\frac{1}{2}$ **8.** 0.65 **9.** (*ii*) 0.2 **10.** (*i*) $\frac{37}{50}$ (*ii*) $\frac{23}{364}$

11. (*i*) $p = 0.30$ (*ii*) 0.9 **12.** (*i*) $P(A) = \frac{4}{13}$ (*ii*) $\frac{6}{11}$ **13.** (*i*) $\frac{7}{9}$ (*ii*) $\frac{4}{9}$ (*iii*) $\frac{17}{36}$

14. $\frac{319}{420}$ **15.** 0.2 **16.** (*i*) $\frac{4}{13}$ (*ii*) $\frac{92}{833}$ **17.** (*i*) $\frac{4}{13}$ (*ii*) $\frac{9}{13}$

18. $\frac{7}{13}$ **19.** 43 to 34 **20.** $\frac{2}{5}$

HINTS TO SELECTED QUESTIONS

1. (*iii*) 18 outcomes favourable to E are (2, 1), (2, 2), ... (2, 6), (4, 1) etc.

$P(E) = P$ (an even number on the first die), $P(F) = P$ (a total of eight) $= \frac{5}{36}$ [outcomes are (2, 6), (6, 2), ... etc.] (2, 6), (6, 2), (4, 4) are the outcomes favourable to both E and F.

$\therefore \quad P(A \cap B) = \frac{3}{36}$. Reqd. prob. $= P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

7. Since A, B, and C are mutually exclusive events,

$$P(A + B + C) = P(A) + P(B) + P(C) = \frac{2}{3} + \frac{1}{4} + \frac{1}{3} = \frac{5}{4} > 1$$

But $P(A + B + C)$ must be ≤ 1 $\quad \therefore$ The given statement is false.

9. (*i*) Probability that either A occurs or B occurs

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.50 - 0.14 = 0.61$$

$\therefore$ Probability that neither A occurs nor B occurs

$$= P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - 0.61 = 0.39.$$

10. (*i*) The product of the numbers will be even when either both of them are even or one is even and the other is odd. There are 12 even numbers and 13 odd numbers.

$\therefore$ Reqd. probability $= P(A) + P(B) = \frac{^{12}C_2}{^{25}C_2} + \frac{^{12}C_1 \times {}^{13}C_1}{^{25}C_2} = \frac{37}{50}$.

12. (*i*) Since A, B, and C are mutually exclusive, therefore,

$A \cup B \cup C = S \quad \therefore \; P(A \cup B \cup C) = P(S) \Rightarrow P(A) + P(B) + P(C) = 1$

13. (*ii*) $n(S) = 36$. Let the events A: multiple of 3, B: multiple of 4.

$A = \{(1, 2), (1, 5), ..., (6, 6)\} \Rightarrow n(A) = 12$

$B = \{(1, 3), (2, 2), (2, 6), ..., (6, 6)\} \Rightarrow n(B) = 9$

$A \cap B = \{(6, 6)\}$. Now, use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(\overline{A} \cap \overline{B}) = P\overline{(A \cup B)} = 1 - P(A \cup B)$

14. $P(A)=\frac{1}{1+3}=\frac{1}{4},\ P(B)=\frac{1}{1+4}=\frac{1}{5}, P(C)=\frac{1}{1+5}=\frac{1}{6}, P(D)=\frac{1}{1+6}=\frac{1}{7}$

These events being mutually exclusive, $P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$.

15. $P(A)=\frac{60}{100},\ P(B)=\frac{50}{100}, P(A \cap B)=\frac{30}{100}$. Reqd. prob. $= 1 - P(A \cup B)$.

16. See Solved Ex. 6.

(*i*) The events are not mutually exclusive. Reqd. prob. $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(*ii*) the events are mutually exclusive. Reqd. prob. $= P(A \cup B) = P(A) + P(B)$.

17. (*i*) Let A = drawing a heart, B = drawing a queen, then

$P(A)=\frac{13}{52}, P(B)=\frac{4}{52}$, there is only one case favourable to the event $P(A \cap B)$, *viz.*, queen of heart.

Hence $P(A \cap B)=\frac{1}{52}$. Reqd. prob. $= P(A \cup B)$.

18. A : the card drawn is a king; B : the card drawn is heart; C : the card drawn is a red card. Then A, B, C are not mutually exclusive $P(A)=\frac{4}{52}, P(B)=\frac{13}{52},\ P(C)=\frac{26}{52},.\ P(A \cap B \cap C)=\frac{1}{52},\ P(B \cap C)=\frac{13}{52}$,

$P(A \cap C)=\frac{2}{52},\ P(A \cap B)=\frac{1}{52}$. Reqd. prob. $= P(A \cup B \cup C)$.

19. $P(A)=\frac{3}{8+3},\ P(B)=\frac{2}{5+2},\ P(A) + P(B) + P(C) = 1$.

CHAPTER TEST

1. In rolling two fair dice, what is the probability of obtaining a sum greater than 3 but not exceeding 6? **[*VITEEE 2006*]**

2. Find the probability of obtaining a sum 8 in a single throw of two dice.

[*Raj PET 2009, Kerala PET 2004*]

3. A bag contains 4 red, 6 white and 5 balck balls. 2 balls are drawn at random. Find the probability of getting one red and one white ball. **[*UP SEE 2001*]**

4. Out of 26 cards numbered from 1 to 26, one card is chosen. Find the probability that it is not divisible by 4. **[*Orissa JEE 2010*]**

5. If A and B are mutually exclusive events with $P(A) = \frac{1}{2} = P(B)$ and $A \cup B = S$, the sample. space. Find $P(A)$. **[*J&K CET 2007*]**

[Hint. $A \cup B = S \Rightarrow P(A \cup B) = P(S) = 1$**]**

6. A and B are two events, such that $P(A) = 0.42$, $P(B) = 0.48$, P (A and B) $= 0.16$

Determine (*i*) P (not A), (*ii*) P (not B), (*iii*) P (A or B) **[*NCERT*]**

ANSWERS

1. $\frac{1}{3}$ **2.** $\frac{5}{36}$ **3.** $\frac{8}{35}$ **4.** $\frac{10}{13}$

5. $\frac{1}{3}$ **6.** (*i*) 0.58, (*ii*) 0.52 (*iii*) 0.74

CHAPTER 23

PARABOLA

23.01. History of the conic sections

The curves called 'conic sections' were so named after their historical discovery as inter-sections of a plane and a surface called 'a right circular cone'. A *right circular cone* is the surface generated by a line moving about a circle and containing a fixed point on the normal to the plane of the circle at the centre of the circle. The fixed point V, called the **vertex**, separates the surface into two parts called **nappes.** Each line determined by the vertex V and a point like A of the circle is called a **generator*** or **element** of the cone. The normal to the plane of the circle at its centre C and passing through V is called the **axis** of the cone. The proper conic sections are circles, parabolas, ellipses and hyperbolas as explained later on.

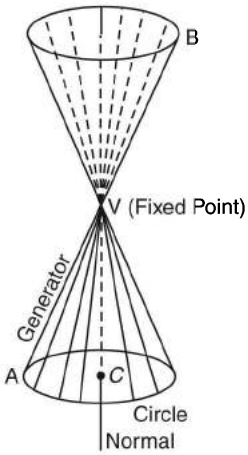


Fig. 23.01

The discovery of the conic sections is attributed to the Greek mathematician Manaechmus (circa 375–N325 B.C.) who was a tutor to Alexander the Great. He apparently used them in an attempt to solve three famous problems, the trisection of an angle, the duplication of a cube and the squaring of a circle. Although the Greek mathematicians were primarily interested in the mathematical application of the conic sections, they did know some of the optical properties of the curves. The definition of conic sections, which we shall use, is attributed to Apollonius who flourished before 200 B.C.

Further discoveries of the physical applications of the conic sections did not occur until the conjectures of the German scientist and mathematician Johannes Kepler (1571 – 1630), who hypothesised that the planets moved in elliptic orbits with the sun as focus. The theoretical development of Kepler's conjectures followed the gravitation theory and calculus developed by Isaac Newton (1642 – 1727). In fact, it may be shown that any physical object subjected to a force which is described by what is called an inverse square law will move in an orbit which is a conic section. Gravity is such a force; the electrical force between charged bodies was found to be another such described by Charles Augustin de Coulomb (1736 – 1806).

23.02. Modern Applications

Today, we find applications of the theory of conic sections in the orbits of planets, comets and artificial satellites. The theory also applied to the lenses of telescopes, microscopes and other optical instruments, weather prediction, communication by satellites, geological surveying and the construction of buildings and bridges. Conics also occur in the study of atomic structure, the long range guidance systems for ships and aircrafts, the location of hidden gun emplacements and the detection of approaching enemy ships and aircrafts. The surfaces of revolution formed by the conic

* All straight lines, like AB passing through the vertex V and lying on the surface of the cone are called *generators*. A vendor often uses as a support for the container of his wares a double cone, as in Fig. 23.01 made of sticks. The sticks are the generators.

sections, such as a paraboloid, ellipsoid, and hyperboloid find application in the sciences dealing with light, sound, and radio waves.

It is helpful to visualise the conic sections formed by the intersections of a plane and a right circular cone, as shown in the next article. The cone is thought of as extending indefinitely on both sides of its vertex, that is a double right circular cone of infinite extent in both directions.

In studying conic sections you will use the knowledge and techniques acquired so far on analytical geometry. Only rectangular coordinate axes will be used; often parametric representation will be helpful. Ideas of locus and curve sketching will be used.

23.03. Conics as sections of a plane and a right circular cone

Take a point P on a cone and consider the sections of the double cone by the planes passing through P.

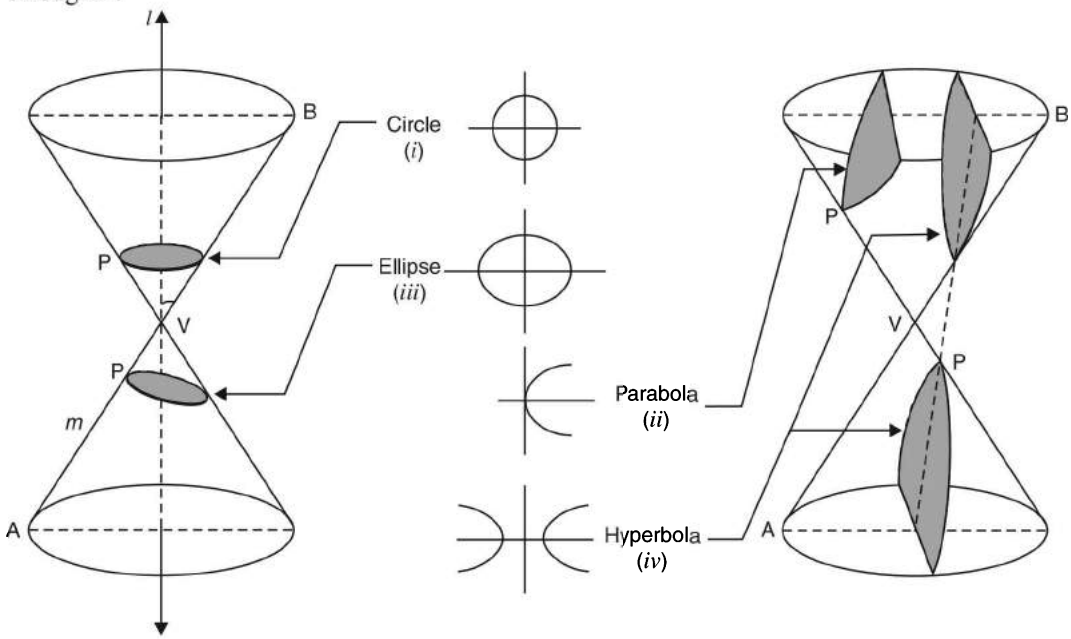


Fig. 23.02

When the cutting plane is at right angles to the axis of the cone, the curve of intersection is a **circle** [Fig. 23.02 (*i*)]

When the cutting plane is somewhat inclined to this direction and intersects only one nappe, the curve of intersection is an **ellipse**. (Fig. 23.02 (*iii*))

As the inclination increases, the ellipse gets more and more elongated till finally, when the cutting plane is parallel to the generator AB, which is diametrically opposite to P, the ellipse is infinitely long. This is then really a **parabola** (Fig. 23.02 (*ii*)).

When the cutting plane is still more inclined, it intersects both nappes, the curve of intersection is a **hyperbola.** A hyperbola consists of two branches [Fig. 23.02 (*iv*)]

When P is the vertex V of the cone, the section is a *pair of straight lines*, which become coincident when the cutting plane touches the cone along a generator like AB. (Two parallel lines cannot be obtained as a plane section of a cone.)

Finally, the locus is a single point if the plane contains the vertex and does not intersect either nappe of the cone.

Note. The circle, ellipse, parabola, and hyperbola are the main conic sections. Two straight lines, intersecting or coincident and a single point are called **degenerate** cases.

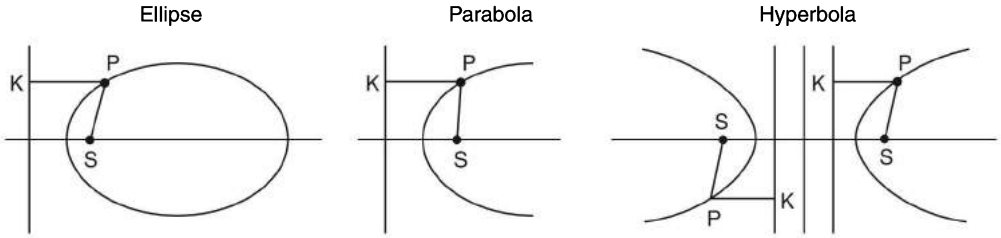
23.04. Definitions

A conic section or conic, is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called a **focus**, the fixed straight line is called a **directrix,** and the constant ratio is called the **eccentricity** which is denoted by 'e'.

From Fig. 23.03, we have $\frac{|PS|}{|PK|}$ = constant = e.

If $e = 1$, the curve is a **parabola.**
If $e < 1$, the curve is an **ellipse.**
If $e > 1$, the curve is a **hyperbola.**



$0 < \frac{|PS|}{|PK|} < 1$ $\qquad$ $\frac{|PS|}{|PK|} = 1$ $\qquad$ $\frac{|PS|}{|PK|} > 1$

Fig. 23.03

The straight line passing through the focus and perpendicular to the directrix is called the **axis** of the conic.

A point of intersection of a conic with its axis is called **vertex** of the conic.

Note. It is assumed that the point P moves in the plane in which the fixed point and the fixed straight line lie.

The definition of the conics is purely geometric in the sense that it says nothing about the coordinates, that is the values of x and y. Recall that the definition of the circle as the locus of points at a given distance from a fixed point is also a geometric one. It will turn out, of course, that parabolas, ellipses, and hyperbolas are always given by second degree equations, as is true in the case of the circle.

We proceed to find the equation of the simplest of these curves, namely that in which the eccentricity is equal to *unity*. This curve, as you know, is called a parabola.

PARABOLA

23.05. Standard equation of a parabola

Let S be the fixed point, called the focus, and LM the fixed line, called the directrix. Draw SZ perpendicular to LM. Bisect SZ at A and let $SZ = 2a$ so that $|SA| = |AZ| = a$.

Take A as the origin, AS the positive x-axis and AY the positive y-axis perpendicular to the x-axis AS or AX.

Then the coordinates of the focus S are $(a, 0)$, and the equation of the directrix LM, being parallel to the y-axis, is $x = -a$ or $(x + a) = 0$.

Let $P(x, y)$ be any point on the locus. By definition of the parabola

$$|SP| = |PT|$$

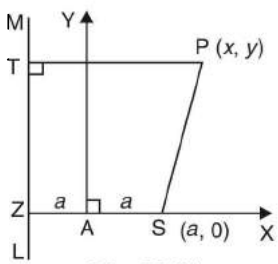


Fig. 23.04

or $\sqrt{(x-a)^2+(y-0)^2} = \dfrac{|x+a|}{1}$ [Perp. from (x, y) to the line $x + a = 0$]

or $(x-a)^2 + y^2 = (x+a)^2$

or $\boldsymbol{y^2 = 4ax.}$ **...(Form I)**

which is the required equation.

Since this is the simplest form of the equation of a parabola, it is called the **standard form.** Since $|AS| = |AZ|$, therefore by definition, A is a point of the parabola. Again, since A is the point of intersection of the parabola and its axis, therefore A is the **vertex** of the parabola.

Hence, $\boldsymbol{y^2 = 4ax}$ **is the equation of a parabola where origin is the vertex and the x-axis is the axis of the parabola.**

Note. It should be carefully noted that

(*i*) the coordinates of the vertex are (0, 0); (*ii*) the coordinates of the focus are (a, 0);

(*iii*) the equation of the axis is $y = 0$; (*iv*) the equation of the directrix is $x = -a$.

The parabola $y^2 = 4ax$ is symmetrical about x-axis because on changing y to $-y$ in $y^2 = 4ax$, the equation remains unchanged.

23.06. Sketching the parabola $y^2 = 4ax$

We shall now try to have a rough sketch of the curve on the assumption that A is the origin, AS the x-axis, and AY the y-axis.

1. Origin. We know that the origin A lies on the curve. Or, in other words the curve passes through the point (0, 0).

2. Axes intersections. The curve intersects the x-axis where $y = 0$. Putting $y = 0$ in $y^2 = 4ax$, we get $x = 0$. Thus, the curve intersects the x-axis only at the origin A (0, 0) and at no other point.

The curve intersects the y-axis where $x = 0$. Putting $x = 0$ in $y^2 = 4ax$, we get $y^2 = 0$ which gives two equal values of $y = 0$. This shows that the y-axis intersects the curve at two coincident points at A. Thus, the y-axis, by definition, is a tangent to the curve at A.

3. Symmetry. $y^2 = 4ax$ gives $y = \pm 2\sqrt{a}\,\sqrt{x}$. Therefore for every positive value of x, there are two equal and opposite values of y. This means that the chords perpendicular to the axis of the curve are bisected by the axis and hence the curve is symmetrical about the x-axis.

4. Imaginary values. From $y = \pm 2\sqrt{a}\,\sqrt{x}$ we find that if x is negative, y is imaginary. Hence, there is no point of the curve to the left of the y-axis. In other words, no part of the curve lies to the left of the y-axis.

5. Space occupied. From $y = \pm 2\sqrt{a}\,\sqrt{x}$ we find that as x increases, y increases positively and decreases negatively, and that there is no limit to the increase in x. This means that the curve increases without limit above the x-axis, and decreases without limit below the x-axis. The same fact can be expressed in symbols thus: as $x \to \infty$, $y \to \pm\infty$.

Therefore, the curve is not a closed one.

The shape of the parabola $y^2 = 4ax$ is as drawn in Fig. 23.05.

The curve is **concave** to the right. Since it lies wholly to the right-handed side of the y-axis, it is called the ***right-handed parabola.***

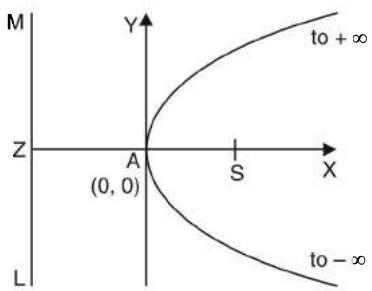


Fig. 23.05

23.07. Other forms of the parabola

If the focus and the directrix are placed in other positions, the equation will of course be different. There are four standard positions for the focus-directrix combination, of which the first has been described in Art. 23.05. Now we shall describe the other three combinations.

(*a*) If AS is taken as the negative x-axis as shown in Fig. 23.06, the focus S is to left of the origin and the directrix to its right. Instead of the positive x-axis, the negative x-axis is the axis of the parabola. The coordinates of S are $(-a, 0)$ and the equation of the directrix is $x = a$ or $x - a = 0$. Either by using the definition or changing x into $-x$ in $y^2 = 4ax$, the equation of the ***left handed parabola*** will be found as

$$y^2 = -4ax \qquad \text{...(Form II)}$$

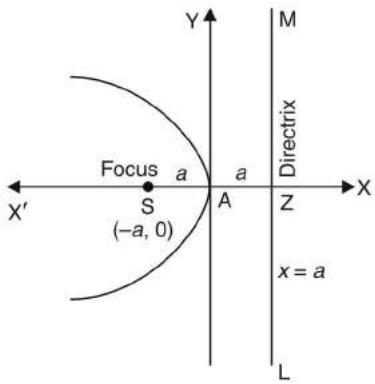


Fig. 23.06

For this parabola,

(*i*) the coordinates of the vertex are (0, 0);

(*ii*) the coordinates of the focus are $(-a, 0)$;

(*iii*) the equation of the axis is $y = 0$;

(*iv*) the equation of the directrix is $x - a = 0$.

(*b*) As shown in Fig. 23.07, if AS is taken as the positive y-axis the focus S is above the origin and the directrix below the origin. The coordinates of S are $(0, a)$ and the equation of the directrix is

$$y = -a, \text{ or } y + a = 0.$$

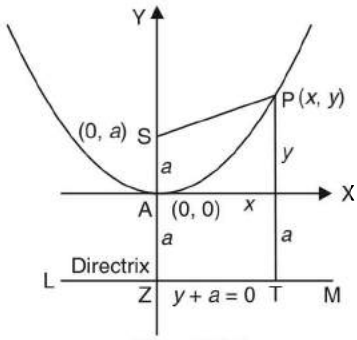


Fig. 23.07

By definition, $|SP| = |PT|$

or $\sqrt{(x-0)^2 + (y-a)^2} = |y + a|$

or $x^2 + (y-a)^2 = (y+a)^2$

or $\mathbf{x^2 = 4ay}$ **... (Form III)**

which is the equation of the curve. This curve is called the ***upward parabola.***

The equation $x^2 = 4ay$ could also be found by interchanging x and y in the equation $y^2 = 4ax$.

For this parabola,

(*i*) the coordinates of the vertex are (0, 0);

(*ii*) the coordinates of the focus are $(0, a)$;

(*iii*) the equation of the axis is $x = 0$;

(*iv*) the equation of the directrix is $y + a = 0$.

(*c*) If AS is taken as the negative y-axis as shown in Fig. 23.08, the focus S is below the origin and the directrix above the origin. The coordinates of S are $(0, -a)$ and the equation of the directrix is

$$y = a \quad \text{or} \quad y - a = 0$$

Either by definition or by changing y into $-y$ in $x^2 = 4ay$, the equation of the ***downward parabola*** will be found as $\mathbf{x^2 = -4ay}$. **(Form IV)**

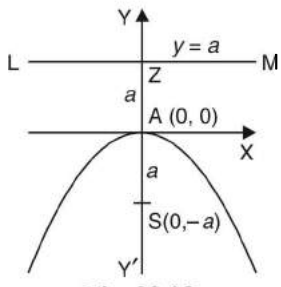


Fig. 23.08

For this parabola,

(*i*) the coordinates of the vertex are (0, 0);
(*ii*) the coordinates of the focus are $(0, -a)$;
(*iii*) the equation of the axis is $x = 0$;
(*iv*) the equation of the directrix is $y = a$.

(*d*) (*i*) As in Art. 23.10 it can be easily shown that the length of the latus rectum in each of these three parabolas, namely $y^2 = -4ax$, $x^2 = 4ay$, $x^2 = -4ay$ is also $4a$ in length.

(*ii*) A little observation will convince you that the length of the latus rectum is the coefficient of x in $y^2 = 4ax$ and $y^2 = -4ax$; and that it is the coefficient of y in $x^2 = 4ay$ and $x^2 = -4ay$.

(*iii*) In all the four parabolas, **the distance between the focus and the vertex is one-quarter of the length of the latus rectum.**

Aid of memory

Remark. In all the forms, the variable with exponent 2 appears on the left side of the equation and the other variable and constants on the right side of the equations.

The equation of a parabola may be of the standard form I, II, III or IV depending on the exponent and sign (+ ve or – ve) of x and y.

1. If exponent of y is 2 and x has a positive coefficient, it is of the standard form I.
2. If exponent of y is 2 and x has a negative coefficient, it is of the standard form II.
3. If exponent of x is 2 and y has a positive coefficient, it is of the standard form III.
4. If exponent of x is 2 and y has a negative coefficient, it is of the standard form IV.

23.08. A few important observations (Summarising)

We observe the following points from a close examination of the four standard equations of the parabola and the results derived above relating to them.

1. **Vertex of each parabola is at the origin (0, 0).**
2. Focus of each type of parabola lies on one of the coordinate axes and the directrix is parallel to the other coordinate axis. Thus, **directrix is perpendicular to the line segment joining the vertex and the focus.**
3. Each parabola is symmetrical about its axis.
4. (*i*) If the standard equation of a parabola contains y^2, then its axis is $y = 0$, *i.e.*, the x-axis.
   (*ii*) If the standard equation of a parabola contains x^2, then its axis is $x = 0$, *i.e.*, the y-axis.
5. (*a*) If the axis of parabola is along the x-axis, then it is a horizontal parabola and its equation is $y^2 = 4ax$ or $y^2 = -4ax$ $(a > 0)$.

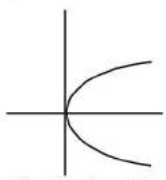


Fig. 23.09

(*i*) If the coefft. of x is positive, the parabola opens to the right, *i.e.*, its concavity is in the direction of positive x-axis. It is, therefore, called a **right-handed parabola**.

(*ii*) If the coefft. of x is negative, the parabola opens to the left, *i.e.*, its concavity is in the negative direction of x-axis. That is, why it is called a **left-handed parabola**.

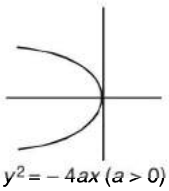


Fig. 23.10

(*b*) If the axis of the parabola is along the *y*-axis, then it is a vertical parabola. The equation of the parabola is $x^2 = 4ay$ or $x^2 = -4ay$ $(a > 0)$.

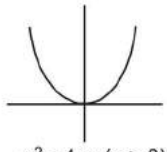


Fig. 23.11

(*i*) If the coefft. of *y* is positive, then the parabola opens upwards, *i.e.*, its concavity is in the positive direction of *y*-axis. Therefore, it is called an **upward parabola**.

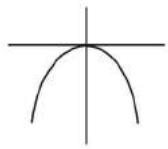


Fig. 23.12

(*ii*) If the coefft. of *y* is negative, then the parabola opens downwards, *i.e.*, its concavity is in the negative direction of *y*-axis. Therefore, it is called a **downward parabola**.

23.09. Definitions

(1) Chord. *The straight line joining any two points on the parabola is called a* **chord** *of the parabola.*

(2) Focal chord. *A chord of a parabola which passes through its focus is called a* **focal chord** *of the parabola.*

(3) Focal distance. *The distance of a point on a parabola from the focus is called its* **focal distance.**

If *P* be any point (x_1, y_1) on the parabola $y^2 = 4ax$, then the focal distance of *P*.

$$= |SP| = |PT| = |ZN|$$
$$= |ZA| + |AN| = a + x_1.$$
$$\therefore \quad |SP| = \boldsymbol{a + x_1}.$$
$$= \boldsymbol{a} + \textbf{abscissa of the point } \boldsymbol{P}.$$

Note. This result should be committed to memory.

(4) Double ordinate. *A straight line drawn perpendicular to the axis of a parabola and terminated at both ends by the curve is a* **double ordinate.**

In Fig. 23.13, PP' is a double ordinate.

(5) Latus rectum. *The double ordinate passing through the focus of a parabola is called the* **Latus rectum.**

In Fig. 23.13, LSL' is the latus rectum.

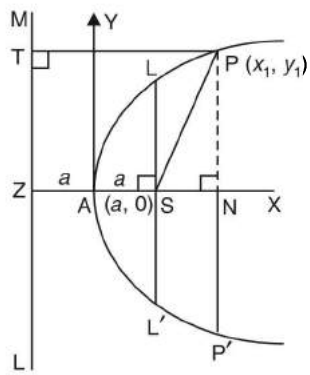


Fig. 23.13

Another name for the latus rectum is **the principal parameter.**

(6) The point '*t*'. *The coordinates $(at^2, 2at)$ satisfy the equation of the parabola $y^2 = 4ax$.*

$$(2at)^2 = 4a(at^2) \Rightarrow 4a^2t^2 = 4a^2t^2.$$

Hence, $\boldsymbol{(at^2, 2at)}$ is a point on the parabola $y^2 = 4ax$ for all real values of *t*. This point is also described as the point '*t*' on the parabola.

23.10. Length of the latus rectum

In Fig 23.13, if $SL = l$, then the coordinates of *L* are (a, l). Since *L* lies on the parabola $y^2 = 4ax$, the coordinates should satisfy this equation.

$$l^2 = 4a \times a \text{ or } l = 2a \quad \therefore \quad LL' = 2l = 4a$$

Hence, the length of the latus rectum of the parabola is $\boldsymbol{4a}$.

Cor. 1. Equation of the latus rectum. Since LSL' in Fig. 23.13, is a straight line parallel to the y-axis and at a distance 'a' from it, therefore its equation is $\boldsymbol{x = a}$.

Cor. 2. Coordinates of L and L'. The coordinates of L are $(a, 2a)$ and those of L' are $(a, -2a)$.

23.11. Summary of the main facts about four parabolas in standard form

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

Ex. 1. For the following parabolas, find the coordinates of the focus, length of the latus rectum, equation of the axis and the equation of the directrices.

(*i*) $y^2 = 18x$ (*ii*) $y^2 = -16x$ (*iii*) $x^2 = 10y$ (*iv*) $x^2 = -7y$

(*v*) $3x^2 = 8y$ (*vi*) $4y^2 = 15x$

Sol. (*i*) $y^2 = 18x$

This equation is of the standard form I and can be written as $y^2 = 4 . \frac{18}{4} x$. Here, $a = \frac{18}{4} = \frac{9}{2}$.

Hence, coordinates of the focus are $\left(\frac{9}{2}, 0\right)$. $[(a, 0)]$

Length of the latus rectum = coefft. of $x = 18$. Equation of the axis is $y = 0$.

Equation of the directrix is $x = \frac{-18}{4} \Rightarrow x = -\frac{9}{2} \Rightarrow 2x + 9 = 0$. $[x = -a]$

(*ii*) $y^2 = -16x$

This equation is of the form II can be written in the form

$$y^2 = 4 . (-4) x. \text{ Here, } a = -4.$$

Hence, coordinates of the focus are $(-4, 0)$. Length of the latus rectum is 16. Equation of the axis is $y = 0$ and equation of the directrix is $x = -(-4)$ or $x = 4$.

(*iii*) $x^2 = 10y$

This equation is of the standard form III and can be written in the form

$$x^2 = 4\left(\frac{10}{4}\right) y. \text{ Here, } a = \frac{10}{4} = \frac{5}{2}.$$

Hence, coordinates of the focus are $\left(0, \frac{5}{2}\right)$. Length of the latus rectum = 10. Equation of the axis is $x = 0$ and equation of the directrix is $y = \frac{-5}{2}$ or $2y + 5 = 0$.

(*iv*) $x^2 = -7y$

This equation is of the standard form IV and can be written as $x^2 = 4 . \left(\frac{-7}{4}\right) y$. Here, $a = \frac{-7}{4}$.

Hence, coordinates of the focus are $\left(0, \frac{-7}{4}\right)$. Length of the latus rectum is 7. Equation of the axis is $x = 0$ and equation of the directrix is $y = -\left(\frac{-7}{4}\right) \Rightarrow y = \frac{7}{4}$ or $4y - 7 = 0$.

(*v*) $3x^2 = 8y$

This equation can be written as $x^2 = \frac{8}{3} y = 4 . \left(\frac{8}{12}\right) y = 4 . \left(\frac{2}{3}\right) y$. Here, $a = \frac{2}{3}$.

It is of the standard form III. Coordinates of the focus are $\left(0, \frac{2}{3}\right)$; length of the latus rectum is $\frac{8}{3}$; equation of the axis is $x = 0$; and equation of the directrix is $y = \frac{-2}{3}$ or $3y + 2 = 0$.

(*vi*) $4y^2 = 15x$

This equation can be written as $y^2 = \frac{15}{4} x = 4 . \left(\frac{15}{16}\right) x$. Here, $a = \frac{15}{16}$.

It is of the standard form I. Coordinates of the focus are $\left(\frac{15}{16}, 0\right)$; length of the latus rectum is $\frac{15}{4}$; equation of the axis is $y = 0$ and equation of the directrix is $x = \frac{-15}{16}$, or $16x + 15 = 0$.

Note. Coordinates of the vertex in all of the above cases are (0, 0).

Ex. 2. Find the equation of the parabolas with vertices at the origin and satisfying the following conditions.

(*i*) Focus at (*a*, 0) **(*ii*) Focus at (0, *a*)**

(*iii*) Focus at (0, – *a*) **(*iv*) Directrix *x* = 7**

(*v*) Directrix *y* = 5 **(*vi*) Passing through (–3, 7) and axis along the *x*-axis**

(*vii*) Passing through (4, 9) and axis along the *y*-axis.

Sol. (*i*) Focus is at $(a, 0)$. Therefore, axis of the parabola is along the x-axis. It is of the standard form I. Hence, equation of the parabola is $y^2 = 4ax$.

(*ii*) Focus is at $(0, a)$. Therefore, axis of the parabola is along the y-axis. It is of the standard form III. Hence, equation of the parabola is $x^2 = 4ay$.

(*iii*) Focus is at $(0, -a)$. Therefore, axis of the parabola is along the y-axis. It is of the standard form IV. Hence, equation of the parabola is $x^2 = -4ay$.

(*iv*) Directrix is $x = 7 \Rightarrow x - 7 = 0 \therefore$ Focus of the parabola is $(-7, 0)$.

We know that equation of the parabola with focus at $(a, 0)$ and axis along the x-axis is $y^2 = 4ax$.

Hence, required equation of the parabola is $y^2 = 4 (-7) x$ or $y^2 = -28x$.

(*v*) Directrix $y = 5$ or $y - 5 = 0$.

Focus of the parabola is $(0, -5)$. It shows that the axis of the parabola is along the y-axis. We know that the equation of the parabola with focus at $(0, a)$ and axis along the y-axis is $x^2 = 4ay$.

Hence, required equation of the parabola is $x^2 = 4 (-5) y$ or $x^2 = -20y$.

(*vi*) Equation of the parabola with vertex at origin, focus at $(a, 0)$ and axis along the x-axis is $y^2 = 4ax$.

Since the parabola passes through $(-3, 7)$, therefore,

$$(7)^2 = 4 \times a \times (-3) \Rightarrow 49 = -12\, a \Rightarrow a = \frac{-49}{12}$$

$\therefore$ Required equation of the parabola is

$$y^2 = 4\left(\frac{-49}{12}\right)x \text{ or } 3y^2 = -49x.$$

(*vii*) Equation of the parabola with vertex at the origin, focus at $(0, a)$ and axis along y-axis is $x^2 = 4ay$.

Since, that parabola passes through the point (4, 9),

therefore, $(4)^2 = 4 \times a \times 9 \Rightarrow a = \frac{16}{36} = \frac{4}{9}$.

$\therefore$ Required equation of the parabola is $x^2 = 4 \cdot \left(\frac{4}{9}\right) \cdot y$ or $9x^2 = 16y$.

23.12. General equation of a parabola

So far we have found the equations of parabola in standard forms when the directrix is parallel to either of the axes and the focus is also either on the x-axis or on the y-axis. We shall now obtain the *equation of a parabola when the coordinates of focus are already given, the vertex is any point, and the directrix any line.*

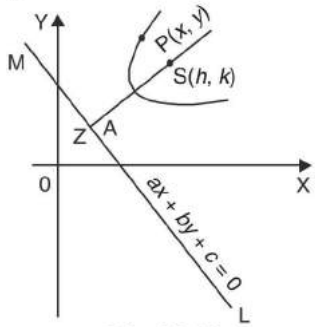


Fig. 23.14

Let (h, k) be the focus S and $ax + by + c = 0$, the equation of the directrix LM of a parabola. Let $P(x, y)$ be any point on the parabola. Then, by definition,

$$|PS| = \text{distance of } P \text{ from } LM$$

or $$\sqrt{(x-h)^2 + (y-k)^2} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

or $$(x-h)^2 + (y-k)^2 = \frac{(ax + by + c)^2}{a^2 + b^2}$$

or $(a^2 + b^2)(x-h)^2 + (a^2 + b^2)(y-k)^2 = a^2x^2 + b^2y^2 + c^2 + 2abxy + 2acx + 2bcy$

or $(a^2 + b^2)x^2 + (a^2 + b^2)y^2 - a^2x^2 - b^2y^2 - 2abxy + x$ term $+ y$ term + constant = 0

or $(b^2x^2 + a^2y^2 - 2abxy) + x$ term $+ y$ term + constant = 0

This is of the form $(bx - ay)^2 + 2gx + 2fy + d = 0$

This equation is the general equation of a parabola. It will be noticed that

(*i*) *the equation of a parabola is a second degree equation;*

(*ii*) *the second degree terms form a perfect square.*

Ex. 3. Find the equation of the following parabolas.

(*i*) Directrix $x = 0$, focus at (6, 0)

(*ii*) Focus at (2, 3), directrix $x - 4y + 3 = 0$. Also, find the equation of the axis.

(*iii*) Focus (a, b), directrix $\frac{x}{a} + \frac{y}{b} = 1$ **(*ISC 1988*)**

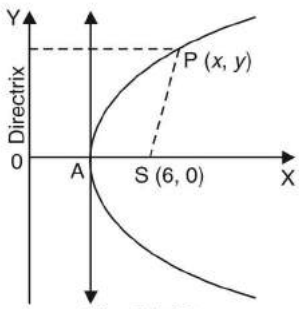


Fig. 23.15

Sol. (*i*) Let $P(x, y)$ be any point on the parabola. Then, by definition of parabola,

distance of P from the focus = its distance from the directrix

$\therefore$ $$\sqrt{(x-6)^2 + (y-0)^2} = |x|$$

$\Rightarrow$ $x^2 - 12x + 36 + y^2 = x^2 \Rightarrow y^2 = 12x - 36$

which is the required equation.

(*ii*) Let $P(x, y)$ be any point on the parabola. By definition, the distance of P from (2, 3) is equal to the perpendicular distance of P from $x - 4y + 3 = 0$.

Therefore, $$\sqrt{(x-2)^2 + (y-3)^2} = \frac{|x - 4y + 3|}{\sqrt{17}}$$

$$\Rightarrow \quad (x-2)^2 + (y-3)^2 = \frac{(x - 4y + 3)^2}{17}$$

$\Rightarrow 17(x^2 - 4x + 4 + y^2 - 6y + 9) = x^2 + 16y^2 + 9 - 8xy + 6x - 24y$,

which simplifies to $16x^2 + 8xy + y^2 - 74x - 78y + 212 = 0$

which is the required equation.

The axis of the parabola is the line passing through the focus (2, 3) and perpendicular to the directrix $x - 4y + 3 = 0$.

Equation of any line perp. to the line $x - 4y + 3 = 0$ is $4x + y + k = 0$ ...(1)

where k is any arbitrary constant

If it passes through the point (2, 3), then

$$4 \times 2 + 3 + k = 0 \Rightarrow k = -11$$

Hence, from (1), the equation of the axis is $4x + y - 11 = 0$.

Note. It can be easily verified that the second degree terms $16x^2 + 8xy + y^2$ form a perfect square, namely, $(4x + y)^2$.

(*iii*) Let $P(x, y)$ be any point on the parabola. Then,

Distance of P from the focus $= \sqrt{(x-a)^2 + (y-b)^2}$.

Distance of P from the directrix $\frac{x}{a} + \frac{y}{b} = 1$, *i.e.*, $bx + ay - ab = 0$ is $\left|\frac{bx + ay - ab}{\sqrt{a^2 + b^2}}\right|$.

By the definition of parabola,

distance of P from the focus = its distance from the directrix

$$\therefore \quad \sqrt{(x-a)^2 + (y-b)^2} = \frac{|bx + ay - ab|}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = \frac{(bx + ay - ab)^2}{a^2 + b^2}$$

$\Rightarrow \quad (a^2 + b^2)(x^2 + y^2 - 2ax - 2by + a^2 + b^2) = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2a^2by - 2ab^2x$

$\Rightarrow a^2x^2 - 2abxy + b^2y^2 - 2a^3x - 2b^3y + (a^4 + a^2b^2 + b^4) = 0$

which is the required equation of the parabola.

Ex. 4. Find the equation of the parabola whose vertex is at the point (– 2, 2) and whose focus is (– 6, – 6).

Sol. See Fig. 23.14 for general equation of the parabola.

Let A be the vertex (– 2, 2) and S the focus (– 6, – 6). The axis is the line joining the vertex and the focus whose slope is

$$\frac{-6-2}{-6-(-2)} = \frac{-8}{-4} = 2 \qquad \left[\frac{y_2 - y_1}{x_2 - x_1}\right]$$

The directrix is the line perpendicular to the axis and passing through the vertex.

$\therefore$ Slope of directrix $= -\frac{1}{2}$ $\left[m_2 = -\frac{1}{m_1}\right]$

Let (x_1, y_1) be the coordinates of the point of intersection Z of the axis and the directrix. Then, vertex A $(-2, 2)$ is the middle point of the line segment ZS where $Z \equiv (x_1, y_1)$ and $S \equiv (-6, -6)$.

$$\left.\begin{aligned} \therefore \quad -2 &= \frac{x_1 - 6}{2} \Rightarrow x_1 = 2 \\ \text{and} \quad 2 &= \frac{y_1 - 6}{2} \Rightarrow y_1 = 10 \end{aligned}\right\} \quad \therefore \; Z(x_1, y_1) \equiv (2, 10)$$

$\therefore$ Equation of the directrix (slope $-\frac{1}{2}$ and passing through $(2, 10)$) is

$$y - 10 = -\frac{1}{2}(x - 2) \qquad [y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad x + 2y - 22 = 0$$

Let $P(x, y)$ be any points on the required parabola. Then, by definition,
distance of P from the focus = its distance form the directrix, *i.e.*,

$$(x + 6)^2 + (y + 6)^2 = \frac{|x + 2y - 22|}{\sqrt{1^2 + 2^2}}$$

$$\Rightarrow \quad 5[(x + 6)^2 + (y + 6)^2] = (x + 2y - 22)^2$$

which is the required equation of the parabola.

Explanation. The student may note that we have made use of the following three properties in solving the above problem.

(1) The distance of any point on the parabola from the focus = its distance from the directrix.

(2) Vertex A is the middle point of the line segment joining a point Z on the directrix and the focus S.

(3) Directrix is a line perpendicular to the axis of the parabola, that is, the line joining A and S.

So we have solved the given problem in the following steps.

Step 1. Found the equation of the directrix by using the formula $y - y_1 = m(x - x_1)$. The values of x_1, y_1 were found by using property (2) mentioned above.

The values of slope m was found by using the property 3 mentioned above.

Step 2. Apply the property (1) above to find the required equation of the parabola.

Now try these: Find the equation of the parabola whose

(*a*) vertex is at the point $(-2, 2)$ and focus is $(-6, 6)$

(*b*) vertex is at the point $(3, -2)$ and focus is $(6, 2)$ ***(ISC BM 1994 Type)***

(*c*) vertex is at $(2, 1)$ and focus is $(1, -1)$.

Ans. (*a*) $2[(x + 6)^2 + (y - 6)^2] = (x + y - 4)^2$

(*b*) $25[(x - 6)^2 + (y - 2)^2] = (3x + 4y + 24)^2$

(*c*) $5[(x - 1)^2 + (y + 1)^2 = (x + 2y - 9)^2$

23.13. Equation of the parabola when the vertex is shifted to another point say (α, β), while its axes remain parallel to x-axis or y-axis

Let the vertex of the parabola be shifted to the point (α, β), then in each of the four forms, x and y are replaced by $(x - \alpha)$ and $(y - \beta)$ respectively. Thus, the parabola whose vertex is at (α, β) and which opens out along the positive direction of x-axis is

$(y - \beta)^2 = 4A\,(x - \alpha)$, where $4A$ = length of latus rectum

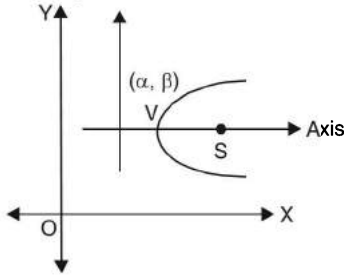


Fig. 23.16

The above equation can be re-written as

$$y^2 - 2\beta y + \beta^2 = 4Ax - 4A\alpha$$

$$\Rightarrow \quad 4Ax = y^2 - 2\beta y + \beta^2 + 4A\alpha$$

$$\Rightarrow \quad x = \frac{1}{4A}y^2 + \frac{-2\beta}{4A}y + \frac{\beta^2 + 4A\alpha}{4A}$$

$$\Rightarrow \quad x = ay^2 + by + c, \text{ where } a, b, c \text{ are constants.}$$

Thus, **the equation to the parabola whose axis is parallel to the x-axis can be taken as $x = ay^2 + by + c$,** where a, b, c are constants.

Similarly, the equation to the parabola whose axis is parallel to the y-axis can be taken as

$$y = ax^2 + bx + c, \text{ where } a, b, c \text{ are constants.}$$

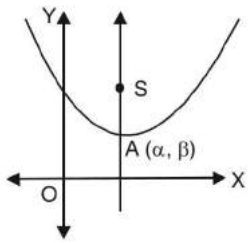


Fig. 23.17

Note. Other forms:

(1) If the axis is taken as the x-axis, the directrix as the y-axis, the equation of the parabola is of the form $y^2 = 4a\,(x - a)$.

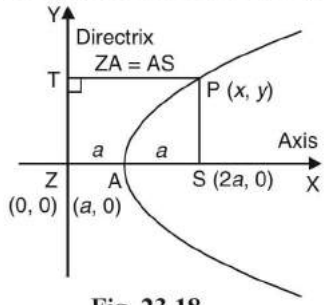


Fig. 23.18

(2) If the focus is taken as the origin, and the axis along the x-axis, the equation of the parabola is of the form $y^2 = 4a\,(x + a)$.

You can derive the above equations yourself.

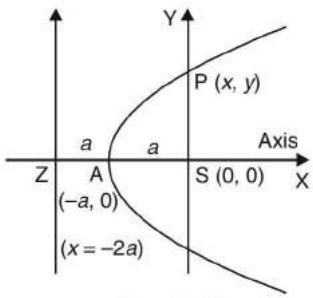


Fig. 23.19

Ex. 5. Derive the equation of the parabola with its vertex at (3, 2) and its focus at (5, 2).

Sol. Since the axis of the parabola is parallel to the x-axis, the equation is of the form

$$(y - \beta)^2 = 4a\,(x - \alpha) \qquad \text{[Art. 23.14]}$$

or $\quad (y - 2)^2 = 4a\,(x - 3)$

as (α, β) is the vertex (3, 2), a = distance between the focus and the vertex

$$= 5 - 3 = 2$$

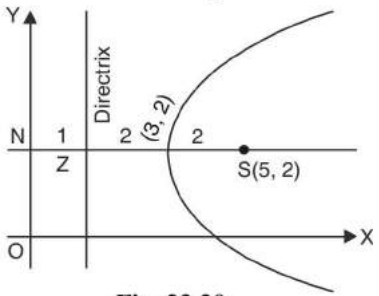


Fig. 23.20

Hence, the required equation is

$$(y-2)^2 = 8(x-3), \text{ which simplifies to}$$

$$y^2 - 4y - 8x + 28 = 0$$

Method II. Since $ZN = 1$, therefore the equation of the directrix is

$$x = 1, \text{ or } x - 1 = 0$$

By definition, if $P(x, y)$ is any point on the parabola then PS is equal to the perpendicular distance of P from the directrix.

$$\therefore \quad \sqrt{(x-5)^2 + (y-2)^2} = \frac{|x-1|}{1}$$

or $(x-5)^2 + (y-2)^2 = (x-1)^2$ or $y^2 - 4y - 8x + 28 = 0$

Ex. 6. Determine the equation of the parabola with its vertex at the point (2, 3), its axis parallel to the *y*-axis and which passes through the point (4, 5).

Sol. The equation of the parabola is of the form

$$(x-\alpha)^2 = 4a(y-\beta) \text{ or } (x-2)^2 = 4a(y-3) \qquad \text{[Art. 23.14]}$$

Since (4, 5) lies on this, therefore $(4-2)^2 = 4a(5-3)$ or $a = \frac{1}{2}$

Hence, the required equation is $(x-2)^2 = 2(y-3)$ or $x^2 - 4x - 2y + 10 = 0$.

Ex. 7. Find the equation of the parabola with its axis parallel to the *x*-axis and which passes through the points (–2, 1), (1, 2) and (–1, 3).

Sol. The equation of the parabola is $x = Ay^2 + By + C$. (Art. 23.14)

Substituting for x and y the coordinates of the given points, we have

$$-2 = A + B + C, \qquad 1 = 4A + 2B + C, \qquad -1 = 9A + 3B + C.$$

Solving these three equations simultaneously, we get $A = -\frac{5}{2}$, $B = \frac{21}{2}$, $C = -10$

Hence, the required equation of the parabola is $x = -\frac{5}{2}y^2 + \frac{21}{2}y - 10$.

or $5y^2 + 2x - 21y + 20 = 0$

Ex. 8. Derive the equation of the parabola with latus rectum joining the points (3, 5) and (3, –3). ***(Type ISC 2014)***

Sol. Since the axis is parallel to the x-axis, the equation of two possible parabolas will be of the form $(y-\beta)^2 = \pm 4a(x-\alpha)$.

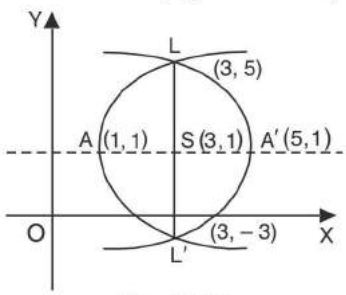


Fig. 23.21

Since the length of the latus rectum is 8, therefore $4a = 8$, *i.e.*, $a = 2$. The equation of the parabola is thus

$$(y-\beta)^2 = \pm 8(x-\alpha)$$

Since (3, 5) and (3, –3) lie on the parabolas, therefore

$$(5-\beta)^2 = \pm 8(3-\alpha) \text{ and } (-3-\beta)^2 = \pm 8(3-\alpha)$$

Solving these equations simultaneously we get (α, β) to be (1, 1) and (5, 1).

The required equations are $(y-1)^2 = 8(x-1)$ and $(y-1)^2 = -8(x-5)$.

Ex. 9. Find the equation of the parabola whose vertex and focus lie on the *y*-axis at distance *b* and *b′* respectively from the origin. ***(ISC 1997)***

Sol. The vertex A is $(0, b)$ and the focus is $(0, b')$. The distance between the vertex and the focus is $b' - b$.

It is an upward parabola

$|AS| = |OS| - |OA| = b' - b$

The equation of the parabola is $(x - \alpha)^2 = 4a\,(y - \beta)$

where $4a$ = length of latus rectum

$\Rightarrow (x - 0)^2 = 4\,(b' - b)\,(y - b)$

$\Rightarrow x^2 = 4\,(b' - b)\,(y - b)$ [See Solved Ex. 6]

$\Rightarrow x^2 = 4\,(b' - b)\,y - 4\,b\,(b' - b)$.

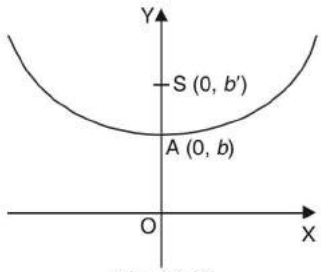


Fig. 23.22

23.14. To find the vertex, focus, directrix, axis, etc.

When the equation of a parabola is given in a form other than standard form

Method. *We first express the given equation in standard form by shifting the origin.*

Type I. $(y \pm \beta)^2 = \lambda\,(x \pm \alpha)$ or $(x \pm \alpha)^2 = \lambda\,(y \pm \beta)$, λ, α, β being constants.

Method. Shift the origin to $(\mp\alpha, \mp\beta)$ by putting $X = x \pm \alpha$ and $Y = y \pm \beta$. The equation of the parabola in the new coordinate system is $Y^2 = \lambda X$ or $Y^2 = 4\left(\frac{\lambda}{4}\right)X$ and $X^2 = \lambda Y$ or $X^2 = 4\left(\frac{\lambda}{4}\right)Y$.

Ex. 10. Transform the following parabolas to the standard forms:

(*i*) $(y - 2)^2 = 2\,(x + 1)$ (*ii*) $(x + 3)^2 = 8\,(y - 5)$

Sol. (*i*) Shifting the origin to the point $(-1, 2)$ by putting $X = x + 1$ and $Y = y - 2$, we have

$$Y^2 = 2X \text{ or } Y^2 = 4\left(\frac{1}{2}\right)X,$$

which is the equation in standard form.

(*ii*) Shifting the origin to $(-3, 5)$ by putting $X = x + 3$ and $Y = y - 5$, we have

$$X^2 = 8Y \text{ or } X^2 = 4\,(2)\,Y$$

which is the equation in the standard form.

Ex. 11. Find the vertex, focus and directrix and latus rectum of the parabola.

$$(y + 3)^2 = 2(x + 2)$$

Sol. To shift the origin to the point $(-2, -3)$, let $x + 2 = X$ and $y + 3 = Y$. Then the given equation of the parabola takes up the form (Fig. 23.23)

$$Y^2 = 2X = 4 \cdot \frac{1}{2} \cdot X$$

Therefore, we have

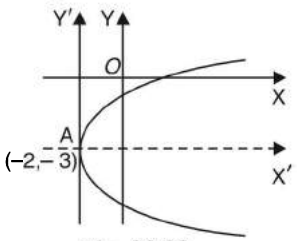


Fig. 23.23

In new coordinate system	*In old coordinate system*
(*i*) Vertex is $(0, 0)$.	(*i*) $X = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$
$\therefore$ $X = 0$ and $Y = 0$	$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$
	$\therefore$ Vertex in the old system is $(-2, -3)$.
(*ii*) Focus is $(a, 0)$.	(*ii*) $X = \frac{1}{2} \Rightarrow x + 2 = \frac{1}{2} \Rightarrow x = \frac{-3}{2}$
$\therefore$ $X = \frac{1}{2}$ and $Y = 0$	$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$
	$\therefore$ Focus in the old system is $\left(\frac{-3}{2}, -3\right)$.

(*iii*) Directrix is $x = -a$. $\therefore\ X = \frac{-1}{2}$ (*iv*) Length of the latus rectum = 2.	(*iii*) $X = \frac{-1}{2} \Rightarrow x + 2 = \frac{-1}{2}$ $\Rightarrow$ $2x + 5 = 0$ which is the equation of the directrix in the old system.

Type II. $\left.\begin{array}{l} ay^2 + by + cx + d = 0 \\ \text{or} \\ ax^2 + bx + cy + d = 0 \end{array}\right\}$ a, b, c, and d are constants.

Method. *First make the second degree terms a perfect square. On doing so the given equation reduces to Type 1.*

Note. It is convenient to divide by a and make the coefficient of second degree term unity.

Ex. 12. Find the vertex, focus, directrix and axis of the parabola $x^2 + 4y + 3x = 2$. Sketch the curve.

Sol. The given equation can be written as $x^2 + 3x = -4y + 2$

or $$x^2 + 3x + \frac{9}{4} = -4y + 2 + \frac{9}{4}$$ (Completing the square on L.H.S.)

or $$\left(x + \frac{3}{2}\right)^2 = -4\left(y - \frac{17}{16}\right) \quad ...(1)$$

Transferring the origin to $A\left(-\frac{3}{2}, \frac{17}{16}\right)$ we put $X = x + \frac{3}{2}$ and $Y = y - \frac{17}{16}$ ...(2)

Substituting in (1) we get $X^2 = -4Y = 4(-1)\,Y$

which is a downward parabola in X and Y. Here, $a = -1$.

The vertex is (0, 0), focus (0, −1). Directrix $Y = 1$, and the axis $X = 0$, referred to the new axes AX and AY.

Substituting the values for X and Y in (2), we get the corresponding coordinates and equations referred to the original axes Ox and Oy as given below.

For the vertex, $X = 0$, $Y = 0$

$\Rightarrow$ $x + \frac{3}{2} = 0,\ y - \frac{17}{16} = 0$

$\Rightarrow$ $x = -\frac{3}{2}, y = \frac{17}{16}$

$\therefore$ The vertex is $\left(-\frac{3}{2}, \frac{17}{16}\right)$

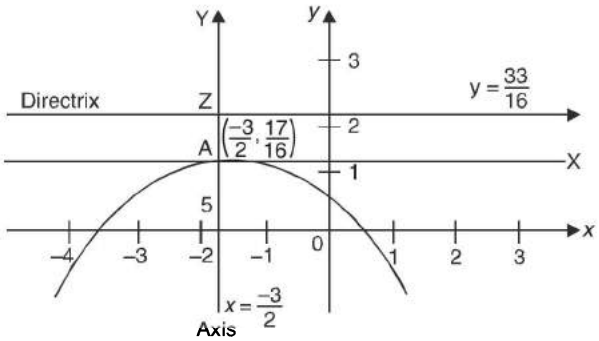


Fig. 23.24

For the focus, $X = 0$, $Y = -1$ $\Rightarrow$ $x + \frac{3}{2} = 0,\ y - \frac{17}{16} = -1$

$\Rightarrow$ $x = -\frac{3}{2}, y = -1 + \frac{17}{16} = \frac{1}{16}$ $\therefore$ The focus is $\left(-\frac{3}{2}, \frac{1}{16}\right)$.

For the directrix, $Y = -a \Rightarrow y - \frac{17}{16} = 1 \Rightarrow y = \frac{33}{16}$ $\therefore$ The directrix is $y = \frac{33}{16}$.

For the axis, $X = 0$ $\Rightarrow$ $x + \frac{3}{2} = 0$ $\Rightarrow$ $x = \frac{-3}{2}$.

Ex. 13. Find the vertex, focus, directrix and axis of the parabola and length of its latus rectum $5x^2 + 30x + 2y + 59 = 0$.

Sol. $5x^2 + 30x = -2y - 59 \Rightarrow 5(x^2 + 6x) = -2y - 59$

$\Rightarrow 5(x^2 + 6x + 9) = -2y - 14$ [Completing the square in the bracket on L.H.S.]

$\Rightarrow 5(x + 3)^2 = -2(y + 7)$ [Coefficient of y in the bracket is made unity] ...(1)

Transferring the origin to $(-3, -7)$ we put $x + 3 = X, y + 7 = Y$

Then from (1), equation of the parabola referred to the new axes is

$$5X^2 = -2Y \text{ or } X^2 = \frac{-2}{5}Y \Rightarrow X^2 = 4\left(\frac{-2}{4 \times 5}\right)Y \Rightarrow X^2 = 4\left(\frac{-1}{10}\right)Y \quad ...(2)$$

Equation (2) represents a downward parabola in X and Y.

After this, by proceeding as in Ex. 13, the student will find that, with reference to the new axes $O'X'$, $O'Y'$ vertex is $(0, 0)$, focus is $\left(0, -\frac{1}{10}\right)$, directrix is $Y' = \frac{1}{10}$, and the axis is $X' = 0$.

Then, with reference to the original axes Ox, Oy it will be found that the vertex is $(-3, -7)$; the focus is $\left(-3, -\frac{71}{10}\right)$; the directrix is $y = -\frac{69}{10}$; the axis is $x + 3 = 0$.

Length of the latus rectum is $\frac{2}{5}$.

EXERCISE 23

Find the equation of the parabola in problems 1 through 12.

1. The focus at $(10, 0)$ the directrix $x = -10$.
2. The focus at $(0, 5)$, the directrix $y = -5$.
3. The focus at $(-3, 0)$, the directrix $x + 5 = 0$.
4. The focus at $(2, -3)$, the directrix $x + 5 = 0$.
5. The focus at $(1, 1)$, the directrix $x - y = 3$.
6. The vertex at the origin, the axis along the x-axis, and passes through $(-3, 6)$.
7. The focus at $(-2, -1)$ and the latus rectum joins the points $(-2, 2)$ and $(-2, -4)$.
8. The vertex at $(-2, 3)$ and the focus at $(1, 3)$.
9. The vertex at $(0, 0)$ and the focus at $(0, 1)$.
10. The vertex at $(0, a)$ and the focus at $(0, 0)$.
11. The axis parallel to the x-axis, and the parabola passes through $(3, 3)$, $(6, 5)$, and $(6, -3)$.
12. The axis parallel to the y-axis and the parabola passes through the points $(4, 5)$, $(-2, 11)$ and $(-4, 21)$.
13. The parabola $y^2 = 4px$ passes through the point $(3, -2)$. Obtain the length of the latus rectum and the coordinates of the focus.
14. Prove that the equation $y^2 + 2ax + 2by + c = 0$ represents a parabola whose axis is parallel to the axis of x. Find its vertex.
15. Of the parabola, $4(y - 1)^2 = -7(x - 3)$ find
    (*i*) the length of the latus rectum.
    (*ii*) the coordinates of the focus and the vertex.

16. Find the vertex, focus, and directrix of the following parabolas:

(*i*) $y^2 - 2y + 8x - 23 = 0$ (*ii*) $x^2 + 8x + 12y + 4 = 0$

17. Find the vertex, focus, and directrix of the parabola $(x - h)^2 + 4a(y - k) = 0$.

18. Find the equation to the parabola whose axis is parallel to the y-axis and which passes through the points (0, 4), (1, 9), and (–2, 6) and determine its latus rectum.

19. Find the coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 8.

20. If the ordinate of a point on the parabola $y^2 = 4ax$ is twice the latus rectum, prove that the abscissa of this point is twice the ordinate.

21. Find the equation of the parabola whose focus is at the origin, and whose directrix is the line $y - x = 4$. Find also the length of the latus rectum, the equation of the axis, and the coordinates of the vertex.

22. The directrix of a conic section is the straight line $3x - 4y + 5 = 0$ and the focus is (2, 3). If the eccentricity e is 1, find the equation to the conic section. Is the conic section a parabola?

(ISC 1980)

23. Find the equation to the parabola whose focus is (–2, 1) and directrix is $6x - 3y = 8$. ***(ISC)***

24. The length of the latus rectum of the parabola whose focus is (3, 3) and directrix is $3x - 4y - 2 = 0$ is

(*a*) 2 (*b*) 1 (*c*) 4 (*d*) None of these

(UP SEAT 2001)

ANSWERS

1. $y^2 = 40x$ **2.** $x^2 = 20y$ **3.** $y^2 = 4(x + 4)$ **4.** $y^2 - 14x + 6y - 12 = 0$

5. $x^2 + 2xy + y^2 + 2x - 10y - 5 = 0$ **6.** $y^2 = -12x$

7. $y^2 + 2y - 6x - 20 = 0, y^2 + 2y + 6x + 4 = 0$ **8.** $y^2 - 6y - 12x - 15 = 0$

9. $x^2 = 4y$ **10.** $x^2 = 4a(a - y)$ **11.** $y^2 - 2y - 4x + 9 = 0$

12. $x^2 - 4x - 2y + 10 = 0$ **13.** $\frac{4}{3}, \left(\frac{1}{3}, 0\right)$ **14.** $\left(\frac{b^2 - c}{2a}, -b\right)$

15. (*i*) $\frac{7}{4}$ (*ii*) Focus $\left(\frac{41}{16}, 1\right)$, Vertex (3, 1)

16. (*i*) Latus rectum = 8, Vertex (3, 1), Focus (1, 1), Axis $y = 1$, Directrix $x = 5$.

(*ii*) Latus rectum = 12, Vertex (–4, 1), Focus (–4, –2), Axis $x + 4 = 0$, Directrix $y = 4$.

17. $(h, k), (h, k - a), y = a + k$ **18.** $y = 2x^2 + 3x + 4$; $\frac{1}{2}$ **19.** $(6, \pm 4\sqrt{3})$

21. $x^2 + 2xy + y^2 - 8x + 8y - 16 = 0$, Latus rectum = $4\sqrt{2}$,
Axis $x + y = 0$, Vertex (–1, 1)

22. $16x^2 + 24xy + 9y^2 - 130x - 110y + 300 = 0$; Yes

23. $9x^2 + 36xy + 36y^2 + 276x - 138y + 161 = 0$ **24.** a

HINTS TO SELECTED QUESTIONS

7. See Solved Ex. 9.

16. See Solved Ex. 13.

21. (*i*) Length of the latus rectum = 2 × (Perpendicular from the focus (0, 0) on the directrix).

(*ii*) The axis is the line through the focus (0, 0) perpendicular to the directrix, *i.e.*, $y + x = 0$.

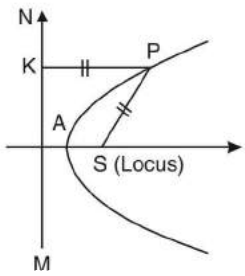
(*iii*) The vertex A is the mid-point of SZ, where Z is the point of intersection of the directrix and the axis. [Solving $y - x = 4$ and $y + x = 0 \Rightarrow Z \equiv (-2, 2)$].

24. Latus rectum = 2 (Length of the ⊥ from (3, 3) to $3x - 4y - 2 = 0$).

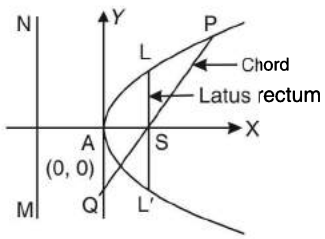
CHAPTER SUMMARY

1. Definition

1. (*i*) A parabola is the locus of a point P which moves such that its distance from a fixed point S is equal to its distance from a fixed straight line m, *i.e.*, $|PS| = |PK|$.



(*ii*) The fixed point is called the **focus** and the fixed line is called the **directrix**. The point at which the parabola meets its axes is called the **vertex** of the parabola. The line segment joining any two points of a parabola is called the **chord** of the parabola.



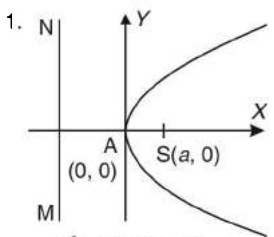
(*iii*) The chord PQ of a parabola through its focus S is called its **focal chord**.

(*iv*) The chord LL' of the parabola passing through its focus and perpendicular to the axis of the parabola is called its **latus rectum**.

(*v*) The distance $|PS|$ of any point P on the parabola from its focus S is called its **focal distance**.

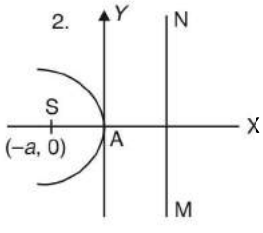
2. Standard forms of the equation of a parabola.

1.



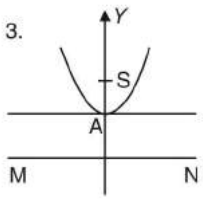
$y^2 = 4ax\ (a > 0)$
Axis along x-axis
(Right-handed parabola)

2.



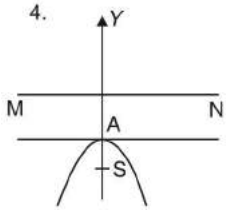
$y^2 = -4ax\ (a > 0)$
Axis along x-axis
coeff. of x is – ve
(left-handed parabola)

3.



$x^2 = 4ay\ (a > 0)$
The axis of the parabola is along y-axis.
(upward parabola)

4.



$x^2 = -4ay\ (a > 0)$
coeff. of y is – ve
axis along y-axis.
(downward parabola)

3. Summary of the main facts about four parabolas in standard form

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

CHAPTER TEST

1. The equation of the directrix of the parabola is $3x + 2y + 1 = 0$. The focus is $(2, 1)$. Find the equation of the parabola. **[*ISC 2002*]**

2. The points $(0, 4)$ and $(0, 2)$ are the vertex and focus of a parabola. Find the equation of the parabola. **[*ISC 2005*]**

3. Find the equation of the parabola with latus rectum joining points $(4, 6)$ and $(4, -2)$. **[*ISC 2014*]**

4. Find the equation of the parabola whose focus is $(-1, -2)$ and the equation of the directrix is given by $4x - 3y + 2 = 0$. Also find the equation of the axis.

5. Find the equation of the parabola if its vertex is at $(0, 0)$, passes through $(5, 2)$ and is symmetric w.r.t. y-axis.

6. The parabola $y^2 = 4ax$ passes through the point $(2, -6)$. Find the length of its latus rectum.

7. Find the coordinates of the vertex and the focus of the parabola $y^2 = 4(x + y)$.

8. Find the focus, the equation of the directrix and the length of the latus rectum of the parabola $y^2 + 12 = 4x + 4y$.

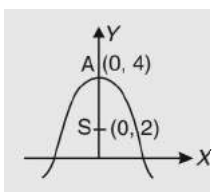
ANSWERS

1. $4x^2 + 9y^2 - 12xy - 58x - 30y + 64 = 0$

2. $x^2 + 8y = 32$ [**Hint:** $AS = a = \sqrt{0 + 2^2} = 2$

Eqn. of the parabola is

$(x - 0)^2 = -4a(y - 4)$]



3. $(y - 2)^2 = 8x$

4. $25(x + 1)^2 + (y + 2)^2 = (4x - 3y + 2)^2$; Axis: $3x + 4y + 11 = 0$

[**Hint.** The axis of the parabola is the line passing through the focus $(-1, -2)$ and perpendicular to the directrix $4x - 3y + 2 = 0$]

5. $x^2 = \frac{25}{2}y$ [**Hint:** Equation of the parabola with vertex at the origin, focus at $(0, a)$ and axis along y-axis is $x^2 = 4ay$.

Since, that parabola passes through the point $(4, 9)$, therefore,

$$(4)^2 = 4 \times a \times 9 \Rightarrow a = \frac{16}{36} = \frac{4}{9}$$

$\therefore$ Required equation of the parabola is $x^2 = 4 \cdot \left(\frac{4}{9}\right) \cdot y$ or $9x^2 = 16y$.

6. 18 units **7.** $y^2 = 4x + 4y \Rightarrow (y - 2)^2 = 4(x + 1)$; vertex $(-1, 2)$, focus $(0, 2)$.

8. Eqn. of directrix $x - 1 = 0$, length of latus rectum = 4 units.

CHAPTER 24

ELLIPSE

24.01. Definition

An ellipse is the locus of a point which moves so that its distance from a fixed point, called the focus, bears a constant ratio, which is less than unity, to its distance from a fixed line, called the directrix.

The focus is denoted by S, the first letter of the word 'Sun', because all planets move in elliptical orbits with the sun as focus.

The constant ratio, as you know is called the eccentricity, which is denoted by e. For the ellipse $e < 1$.

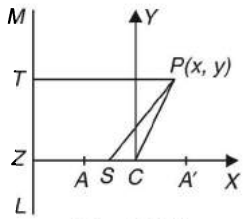


Fig. 24.01

Note. It is assumed that the point moves in the plane in which the fixed point and the fixed straight line lie.

24.02. Standard equation of an ellipse

Let S be the focus and LM the directrix.

Draw SZ perpendicular to LM.

Since $e < 1$, we can divide ZS both internally and externally in the given ratio, suppose $1 : e$.

Let ZS be divided internally at A and externally at A', such that

$$ZA : AS = 1 : e$$

and $$ZA' : SA' = 1 : e$$

or $$AS = e \times ZA \text{ and } SA' = e \times ZA' \quad ...(1)$$

Then by definition, A and A' lie on the ellipse.

Bisect AA' at C and let $AA' = 2a$, so that $AC = CA' = a$.

Adding the equations in (1) we have $AS + SA' = e\,(ZA + ZA')$

$$\Rightarrow \quad AA' = e \times (\overline{CZ - CA} + \overline{CZ + CA'})$$

or $$2a = 2e \times CZ \quad \Rightarrow CZ = \frac{a}{e} \quad ...(2) \quad [\because CA = CA']$$

Subtracting the equations in (1), we have $SA' - AS = e\,(ZA' - ZA)$

or $(CS + CA') - (CA - CS) = e \times AA'$ or $2 \times CS = e \times 2a \Rightarrow CS = ae$ ...(3) $[\because CA = CA']$

Now, choose C as the origin, CA' as the axis of x, and a line perpendicular to CA' as the axis of y.

Let $P(x, y)$ be any point on the curve. Since $CS = ae$ and $CZ = \frac{a}{e}$, therefore, the coordinates of the focus S are $(-ae, 0)$ and the equation of the directrix LM is

$$x = -\frac{a}{e} \text{ or } x + \frac{a}{e} = 0$$

By definition, $$|SP| = e \times |PT| \text{ or } \sqrt{(x + ae)^2 + (y - 0)^2} = e \times \frac{|x + \left(\frac{a}{e}\right)|}{1}$$

$$\Rightarrow \quad (x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2 \Rightarrow x^2 + 2aex + a^2e^2 + y^2 = e^2\left(x^2 + \frac{2a}{e}x + \frac{a^2}{e^2}\right)$$

$$\Rightarrow \qquad x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2 \text{ or } x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\Rightarrow \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \qquad \ldots(4)$$

Putting $a^2(1 - e^2) = b^2$, we get $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,}$

which is the required equation of the ellipse in the simplest form, called the standard form.

Cor. $\boldsymbol{CS = ae, CZ = \frac{a}{e}, b^2 = a^2(1 - e^2).}$

From $b^2 = a^2(1 - e^2)$, we get $e^2 = 1 - \dfrac{b^2}{a^2}$, which gives eccentricity e.

Since $e < 1$, therefore from $b^2 = a^2(1 - e^2)$ we deduce that $b < a$.

Note. The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$ has its locus only the point (0, 0). Such a locus is spoken of as ***degenerate ellipse*** or ***a point ellipse***, since its equation resembles that of an ellipse.

24.03. Sketching the ellipse

The equation of the curve is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ ...(1)

1. Origin. Since the coordinates (0, 0) do not satisfy (1) therefore, the curve does not pass through the origin.

2. Axes intersection. The curve intersects the x-axis where $y = 0$. Putting $y = 0$ in (1), we get $x = \pm a$. Thus the curve intersects the x-axis at $A'(a, 0)$ and $A(-a, 0)$.

The curve intersects the y-axis where $x = 0$. Putting $x = 0$ in (1), we get $y = \pm b$. Thus, the curve intersects the y-axis at $B(0, b)$ and $B'(0, -b)$. Since $b < a$, therefore each of CB and CB' is less than CA or CA'.

3. Symmetry. The equation (1) can be re-written as

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad x = \pm a\sqrt{1 - \frac{y^2}{b^2}}.$$

The first equation shows that for each value of x, $(-a < x < a)$, there are two equal and opposite values of y. This means that the curve is symmetrical about the x-axis. Similarly, from the second equation we deduce that the curve is symmetrical about the y-axis as for each value of y, $(-b < y < b)$, there are two equal and opposite values of x.

Symmetry about the x-axis could also be established by changing y into $-y$ in (1). If the equation remains unchanged, then the curve is symmetrical about the x-axis. Similarly, symmetry about the y-axis could be established by changing x into $-x$ in the equation of the curve.

4. Imaginary values. The equation (1) is the same as

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2} \quad \text{or } x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$$

y is imaginary if $x^2 > a^2$ *i.e.*, if $x > a$ or $x < -a$. Therefore, no part of the curve lies to the right of A' or to the left of A. Similarly, x is imaginary if $y > b$ or $y < -b$, which shows that no part of the curve lies above B or below B'.

The line $x = a$ intersects the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

where $y^2 = 0$. Since this gives 2 conincident values of y, therefore $x = a$ is a tangent to the curve at A'.

Similarly, it can be shown that the line $y = b$ is a tangent to the curve at B. Since the curve is symmetrical about the axes, therefore $x = -a$ and $y = -b$ are also tangents to the curve.

5. Space occupied. The equation (1) is the same as

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

As x increases from $-a$ to 0, y increases from 0 to b above the x-axis and decreases from 0 to $-b$ below the x-axis.

As x increases from 0 to a, y decreases from b to 0 above the x-axis and increases from $-b$ to 0 below the x-axis. The curve is thus a closed one.

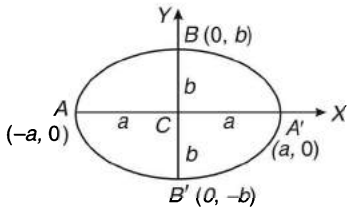


Fig. 24.02

With the above data, the shape of the curve is as sketched in Fig. 24.02.

24.04. Definitions

The points A and A′ are called the **vertices** *of the ellipse. AA′ is called the* **major axis** *of the ellipse and is equal to 2a in length.*

BB′ is called the **minor axis** of the ellipse and is equal to $2b$ in length.

The two together are called the **axes** *of the ellipse. C is called the* **centre** *of the ellipse.*

A chord through the centre is called a **diameter**.

A curve is said to be **central curve** *when the centre bisects all chords passing through it.*

Note. *AA′* is called the major and *BB′* the minor axis because *AA′* is longer than *BB′* ($a > b$).

The relation $b^2 = a^2(1 - e^2)$ connects the length of the semi-major axis, the semi-minor axis, and the eccentricity e. If any two of the quantities a, b, e are given, the third can be found from this relation.

24.05. Second focus and second directrix

Since an ellipse is symmetrical, about the y-axis, therefore, if on the x-axis the points S' and Z' are taken such that then S' will also be a focus, and $L'M'$, the line through Z' perpendicular to CZ', will be the corresponding directrix of the ellipse.

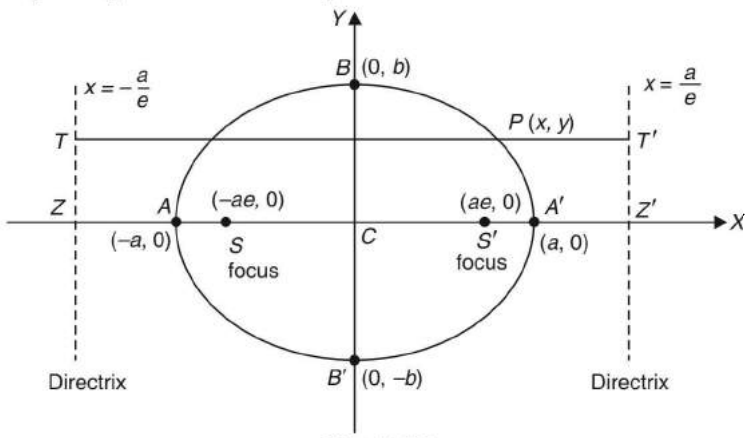


Fig. 24.03

$$|CS'| = |CS| = ae \text{ and } |CZ'| = |CZ| = \frac{a}{e},$$

Thus, an ellipse has two foci $(\pm ae, 0)$ and two directrices $x = \pm \frac{a}{e}$.

Just as $SP = e \times PT$, so is $S'P = e \times PT'$.

Note 1. As two directrices of an ellipse are parallel lines their equations differ by a constant.
Note 2. The foci of an ellipse always lie on the major axis.

24.06. Ellipse as a central curve

Let $P(x_1, y_1)$ be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

so that $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \qquad ...(2)$$

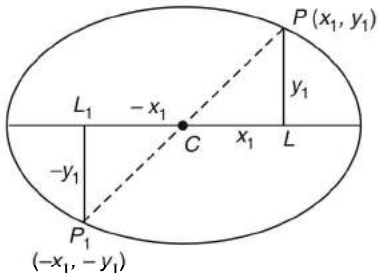


Fig. 24.04

(2) shows that $P_1(-x_1, -y_1)$ also lies on (1) because on substituting in (1) we get,

$$\frac{(-x_1)^2}{a^2} + \frac{(-y_1)^2}{b^2} = 1, \quad i.e., \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Now the middle point of the chord joining the points (x_1, y_1) and $(-x_1, -y_1)$ is (0, 0) which is C. Therefore, C bisects every chord passing through it. Hence, the ellipse is a central curve and when (1) is its equation, the origin is its centre.

24.07. Latus rectum

A chord through a focus at right angles to the major axis is called a latus rectum. Let LSL' be a latus rectum of length $2l$ so that

$$|SL| = |SL'| = l.$$

The coordinates of L are $(-ae, l)$. Since L lies on the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

therefore $$\frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1$$

or $l^2 = b^2(1 - e^2)$

$$= b^2 \times \frac{b^2}{a^2} \left[\because b^2 = a^2(1 - e^2) \therefore 1 - e^2 = \frac{b^2}{a^2}\right]$$

or $$l = \frac{b^2}{a} \text{ or } 2l = \frac{2b^2}{a}.$$

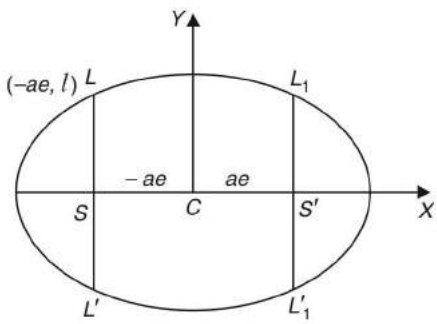


Fig. 24.05

Thus, the latus rectum = $\dfrac{2b^2}{a}$.

Note. The plural of 'latus rectum' is 'latera recta'.

Cor. 1. The equations of the two latera recta LL' and L_1L_1' are respectively $x = -ae$ and $x = ae$.

Cor. 2. The coordinates of the extremities of the latera recta are

$$L\left(-ae, \frac{b^2}{a}\right),\ L'\left(-ae, \frac{-b^2}{a}\right),\ L_1\left(ae, \frac{b^2}{a}\right),\ L_1'\left(ae, \frac{-b^2}{a}\right).$$

24.08. Focal property of the ellipse

We shall now prove that ***the sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis.***

The distances of any point on the ellipse from its foci are known as its focal distances or **focal radii.**

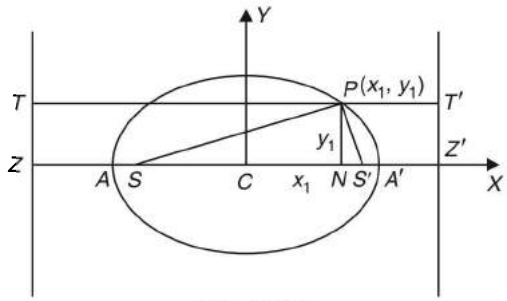


Fig. 24.06

Let $P(x_1, y_1)$ be any point on the ellipse

$PN \perp AA'$.

Since $|SP| = e \times |PT|$

We have $|SP| = e \times |ZN| = e\,|(ZC + CN)|$

or $$|\boldsymbol{SP}| = e\left(\frac{a}{e} + x_1\right) = \boldsymbol{a + ex_1} \quad ...(1)$$

Similarly, $|\boldsymbol{S'P}| = e \times |PT'|$

$$= e \times NZ' = e\ (CZ' - CN) = e\left(\frac{a}{e} - x_1\right)$$

$$\therefore \quad |\boldsymbol{S'P}| = \boldsymbol{a - ex_1} \quad ...(2)$$

Adding (1) and (2), we have $|\boldsymbol{SP}| + |\boldsymbol{S'P}| = \boldsymbol{2a}$, which is constant and $> 2ae$ ($\because\ e < 1$), *i.e.*, $|SP| + |S'P| > |SS'|$.

Note 1. The results (1) and (2) should be committed to memory.

Note 2. Another definition of the ellipse. *An* **ellipse** *is sometimes defined as the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

24.09. Other forms of the ellipse

(*a*) If, the equation is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$.

$a > b$, then the major axis is of length $2a$ and lies along the y-axis and the minor axis is of length $2b$ and lies along the x-axis.

The foci are now $(0, \pm ae)$; the directrices are $y = \pm \dfrac{a}{e}$, and the eccentricity e is given by the equation

$$b^2 = a^2(1 - e^2)$$

or $$e = \sqrt{1 - \frac{b^2}{a^2}}$$

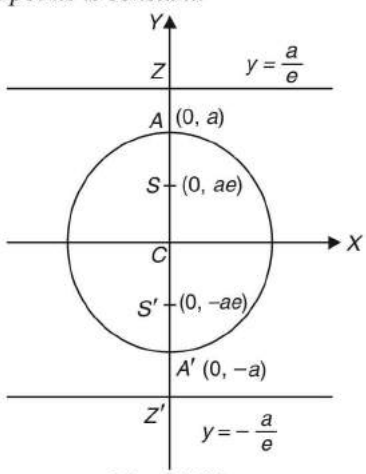


Fig. 24.07

The shape of the ellipse is as shown in Fig. 24.07.

Note 1. In order to avoid any error, the student should keep in mind that in the equations of the ellipse:

(*i*) if the denominator of x^2 > the denominator of y^2, it is a horizontal ellipse and resembles an egg lying on table.

(*ii*) if the denominator of y^2 > the denominator of x^2, it is a vertical ellipse and resembles an egg standing on a table.

Note 2. Remember that the larger of the two numbers under x^2 and y^2 is to be called a^2.

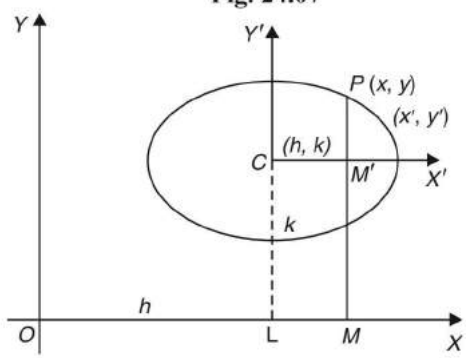


Fig. 24.08

(*b*) **Axes parallel to the axes of coordinates.** Let *OX* and *OY* be the axes of coordinates. Let the coordinates of the centre *C* of the ellipse be (h, k). (Fig. 24.08).

Let *C′* and *CY′* along the major and the minor axes be the new pair of axes of coordinates. Let the coordinates of any point *P* on the ellipse be (x, y) referred to *OX* and *OY* as the axes of coordinates, and (X, Y) referred to *CX′* and *CY′* as the axes of coordinates.

Then, the equation of the ellipse referred to *CX′* and *CY′* as the axes of coordinates is

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \qquad \ldots(1)$$

Since $x = X + h$ and $y = Y + k$,
i.e., $X = x - h$ and $Y = y - k$,

the equation (1) changes to $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$,

which is the required equation of the ellipse.

Hence, the equation of the ellipse with centre (h, k) and the axes parallel to the axes of coordinates is of the form

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.}$$

The equation, when simplified, reduces to the form $lx^2 + my^2 + 2gx + 2fy + c = 0$, where *l* and *m* are two different positive numbers.

If the major axis is parallel to *y*-axis, then the equation of the ellipse would be

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a > b.$$

Conversely, given an equation of the form $lx^2 + my^2 + 2gx + 2fy + c = 0$, we at once conclude that it represents an ellipse whose axes are parallel to the axes of coordinates.

Note. The equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 0$ has its locus only the point (h, k). Such a locus is called a degenerate ellipse or a point ellipse, since its equation resembles that of an ellipse.

(*c*) **General equation**

Given the coordinates of the focus, the equation of the corresponding directrix, and the eccentricity '*e*', the equation of the ellipse can be easily found by using the definition, namely, the distance of any point *P* on the ellipse from the focus

$= e \times$ (the distance of P from the directrix).

Let the focus *S* be (α, β), and the equation of the corresponding directrix be $lx + my + n = 0$, and eccentricity be '*e*'. Let $P(x, y)$ be any point on the ellipse. Then, by definition of the ellipse,

$$\left|\sqrt{(x-\alpha)^2 + (y-\beta)^2}\right| = e \times \frac{|lx + my + n|}{\sqrt{l^2 + m^2}} \qquad \left(l^2 + m^2 \neq 0\right)$$

or $(l^2 + m^2)\,[(x-\alpha)^2 + (y-\beta)^2] = e^2\,(lx + my + n)^2$

The equation, when simplified, reduces to the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is the general equation of an ellipse.

It will be noticed that

(*i*) *the equation of an ellipse is a second degree equation;*

(*ii*) *the constants a, h, b are such that* $h^2 < ab$.

Hence, ***If in an equation of the second degree, the constants a, h, b are such that $h^2 < ab$, then the equation represents an ellipse, unless it represents two imaginary straight lines.***

Note. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola, then $h^2 = ab$.

24.10. Solved Examples

Type 1. *To find the equation of the ellipse from the given data.*

Ex. 1. Obtain the equation of the ellipse whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$, the axes of the ellipse being the axes of coordinates. ***(ISC 2000, 1993 Type)***

Sol. Let $2a$ and $2b$ be the lengths of the axes of the ellipse.

Then $\frac{2b^2}{a} = 5$ and $b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{4}{9}\right) = \frac{5}{9}a^2$

or $b^2 = \frac{5a}{2}$ and $\frac{5a}{2} = \frac{5a^2}{9}$, *i.e.*, $a = \frac{9}{2}$ or $a^2 = \frac{81}{4}$ $\therefore$ $b^2 = \frac{5}{2} \times \frac{9}{2} = \frac{45}{4}$

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{45}{4}} = 1$, *i.e.*, $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$.

Ex. 2. Find the equation of the ellipse with axes along the *x*-axis and the *y*-axis, which passes through the points *P* (4, 3) and *Q* (6, 2).

Sol. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We don't know whether 'a' is greater than 'b' or 'b' is greater than 'a'. Since the points P and Q lie on the ellipse, therefore,

$$\frac{16}{a^2} + \frac{9}{b^2} = 1 \text{ and } \frac{36}{a^2} + \frac{4}{b^2} = 1$$

Solving them simultaneously, we have $a^2 = 52$, $b^2 = 13$.

Hence the required equation of the ellipse is $\frac{x^2}{52} + \frac{y^2}{13} = 1$

The major axis is along the x-axis as $a > b$.

Ex. 3. Find the equation of the ellipse whose vertices are (2, –2), (2, 4) and whose eccentricity is $\frac{1}{3}$.

Sol. The centre of the ellipse is the mid-point of the line joining the vertices, therefore its coordinates are (2, 1).

Also, $2a = AA' = 6$ $\therefore$ $a = 3$

The equation of the ellipse whose centre is (2, 1) is

$$\frac{(x-2)^2}{b^2} + \frac{(y-1)^2}{9} = 1$$

Moreover, $b^2 = a^2(1 - e^2) = 9\left(1 - \frac{1}{9}\right) = 8$.

Hence, the equation of the ellipse is $\frac{(x-2)^2}{8} + \frac{(y-1)^2}{9} = 1$.

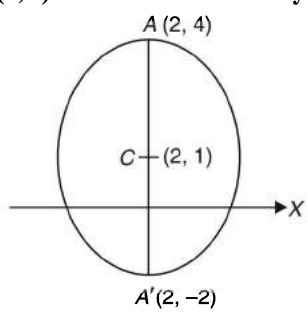


Fig. 24.09

Ex. 4. Find the equation of the ellipse with its centre at (1, 2), focus at (6, 2) and containing the point (4, 6). ***(Type ISC 2012)***

Sol. Let C be the centre and S and S' the foci. Let the coordinates of the second focus S' be (h, k). Then, C being the mid-point of SS', $\frac{h+6}{2} = 1$ and $\frac{k+2}{2} = 2$

$\Rightarrow \quad h = -4, k = 2.$

Since the ordinates of S and S' are equal, the transverse axis of the ellipse is parallel to x-axis, *i.e.*, it is a horizontal ellipse.

Let the equation of the ellipse be

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1.$$

$$|CS| = \sqrt{(6-1)^2 + (2-2)^2} = 5$$

$\therefore \quad ae = |CS| = 5$

and $\quad b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = a^2 - 25$

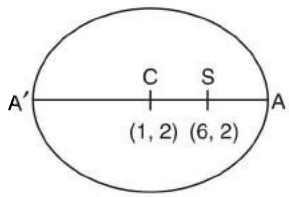


Fig. 24.10

$\therefore \quad$ The ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{a^2 - 25} = 1.$

Since (4, 6) lies on it, therefore $\frac{(4-1)^2}{a^2} + \frac{(6-2)^2}{a^2 - 25} = 1$, which gives $a^2 = 45$ and $b^2 = 20$.

Hence, the required equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1.$

Ex. 5. Obtain the equation of the ellipse whose focus is the point (–1, 1), and the corresponding directrix is the line $x - y + 3 = 0$, and the eccentricity is $\frac{1}{2}$.

Sol. Let P (x, y) be any point on the ellipse. Then, by definition of the ellipse, distance of P from the focus = $e \times$ distance of P from the directrix.

$$\therefore \quad \sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2}\frac{|x - y + 3|}{\sqrt{2}}$$

Squaring and simplifying, we have $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$, which is the required equation.

Note. It can be easily verified that $h^2 - ab < 0$.

Here, $a = 7$, $h = 1$, $b = 7$. Therefore, $h^2 - ab = 1 - 49 = -48 < 0$.

Ex. 6. A point $P(x, y)$ moves so that the sum of its distances from the points S (4, 2) and S' (–2, 2) is 8. Find the equation of its locus and show that it is an ellipse. (*ISC 1990 Type*)

Sol. Required locus is an ellipse having foci (4, 2) and (–2, 2) and length of major axis = 8. **[*Art. 24.07*]**

Since y-coordinates are same, therefore, major axis is parallel to x-axis.

Let the equation of the ellipse be $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$, where (α, β) = mid-point of SS'

$\therefore \quad \alpha = \frac{4-2}{2} = 1, \beta = \frac{2+2}{2} = 2$ and $2a = 8 \Rightarrow a = 4$

Distance between foci $= 2ae = 6 \Rightarrow ae = 3$

$\therefore \quad b^2 = a^2 - a^2e^2 = 16 - 9 = 7.$

Hence, the equation of the ellipse is $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{7} = 1.$

Aliter. $|SP| + |S'P| = 8$

or $\sqrt{(x-4)^2 + (y-2)^2} + \sqrt{(x+2)^2 + (y-2)^2} = 8$

or $\sqrt{(x-4)^2 + (y-2)^2} = 8 - \sqrt{(x+2)^2 + (y-2)^2}$

Squaring both sides, we have

$$(x-4)^2 + (y-2)^2 = 64 - 16\sqrt{(x+2)^2 + (y-2)^2} + (x+2)^2 + (y-2)^2$$

$$\Rightarrow \quad 16\sqrt{(x+2)^2 + (y-2)^2} = 12x + 52$$

$$\Rightarrow \quad 4\sqrt{(x+2)^2 + (y-2)^2} = 3x + 13$$

Squaring again and simplifying, we have $7x^2 + 16y^2 - 14x - 64y - 41 = 0$, which is the required equation of the locus.

To check whether it represents an ellipse we will find whether $e < 1$ and also $H^2 < AB$. The equation obtained above can be written in the form

$$7(x^2 - 2x) + 16(y^2 - 4y) = 41 \quad \Rightarrow \quad 7(x^2 - 2x + 1) + 16(y^2 - 4y + 4) = 41 + 7 + 64$$

$$\Rightarrow \quad 7(x-1)^2 + 16(y-2)^2 = 112 \quad \Rightarrow \quad \frac{(x-1)^2}{16} + \frac{(y-2)^2}{7} = 1$$

So $a^2 = 16, b^2 = 7 \ \therefore \ e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{7}{16} = \frac{9}{16} \Rightarrow e = \frac{3}{4}$ which is < 1.

Also, $A = 7, H = 0, B = 16$. Now, $H^2 - AB = -112 < 0$. Hence, the locus is an ellipse.

Ex. 7. Find the equation of the ellipse whose foci are at the points S (2, 0) and S' (–2, 0), and whose latus rectum is 6. ***(ISC 1994 Type)***

Sol. Let a, b be the semi-major and semi-minor axes of the ellipse and e its eccentricity. Since the foci lie on the x-axis, therefore, the major axis of the ellipse is along the x-axis. The centre, being the mid-point of SS' is the origin. The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The foci are $(\pm 2, 0) \approx (\pm ae, 0) \Rightarrow ae = 2$

Also the latus rectum is 6, therefore

$$\frac{2b^2}{a} = 6 \quad \Rightarrow \quad b^2 = 3a$$

$\Rightarrow \quad a^2 - a^2e^2 = 3a \quad \Rightarrow a^2 - 4 = 3a \quad [\because b^2 = a^2(1 - e^2)]$ ...(2)

$\Rightarrow \quad a^2 - 3a - 4 = 0 \quad \Rightarrow (a-4)(a+1) = 0 \quad \Rightarrow \quad a = 4, -1 \quad \Rightarrow \quad a = 4$, rejecting $a = -1$

$\therefore$ Using $ae = 2$, $e = \frac{2}{a} = \frac{2}{4} = \frac{1}{2}$. Also, $b^2 = 3a = 3 \times 4 = 12$.

Hence, the required equation is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

Ex. 8. Find the equation of the ellipse, whose centre is at (2, –3), one focus at (3, –3) and vertex at (4, –3).

Sol. Distance of the vertex from the centre, $(a) = \sqrt{(4-2)^2 + (-3+3)^2} = 2$ or $(4 - 2 = 2)$

$\therefore \quad a = 2$

Distance of the focus from the centre, $(ae) = \sqrt{(4-3)^2 + (-3+3)^2} = 1$, (or $3 - 2 = 1$)

$\therefore \quad ae = 1$ and $b^2 = a^2 - a^2e^2 = 4 - 1 = 3$.

Since y-coordinates of the centre, focus and vertex are the same, the major axis is parallel to x-axis.

So its equation is of the form $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{b^2} = 1$

i.e., $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 - 12x + 24y + 36 = 0.$

Type 2. ***Equation of the ellipse being given to find centre, vertices, foci, directrices, latera recta, etc.***

Now we shall solve some examples to show how to find the centre, vertices, foci, directrices, etc., from the given equation of an ellipse.

The student should bear in mind that we know the formulae for all these for the standard form of the equation *viz.*, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$. These formulae have been summarised in the table given below. If the equation of the ellipse is given in any other form, then our first task would be to reduce that equation to any of the above standard forms.

Equation	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\ (a > b)$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1\ (a > b)$
1. Coordinates of the centre	$(0, 0)$	$(0, 0)$
2. Coordinates of foci	$(\pm ae, 0)$	$(0, \pm ae)$
3. Coordinates of vertices	$(\pm a, 0)$	$(0, \pm a)$
4. Equation of major axis	$y = 0$	$x = 0$
5. Equation of minor axis	$x = 0$	$y = 0$
6. Equations of directrices	$x = \pm \dfrac{a}{e}$	$y = \pm \dfrac{a}{e}$
7. Length of major axis	$2a$	$2a$
8. Length of minor axis	$2b$	$2b$
9. Length of latus rectum	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
10. Eccentricity	$e^2 = 1 - \dfrac{b^2}{a^2}$	$e^2 = 1 - \dfrac{b^2}{a^2}$

Case 1. ***The equation*** $\alpha x^2 + \beta y^2 = \gamma$ ***;*** α, β, γ ***are constants.***

We will work out in the following steps to render the given equation in standard form.

Step 1. ***Divide both sides by*** γ, $\dfrac{\alpha}{\gamma}x^2 + \dfrac{\beta}{\gamma}y^2 = 1$

Step 2. ***Write the coefficients of*** x^2 ***and*** y^2 ***as their denominators*** $\dfrac{x^2}{\frac{\gamma}{\alpha}} + \dfrac{y^2}{\frac{\gamma}{\beta}} = 1$

This is the standard form. If $\dfrac{\gamma}{\alpha} > \dfrac{\gamma}{\beta}$, it is a horizontal ellipse and if $\dfrac{\gamma}{\beta} > \dfrac{\gamma}{\alpha}$, it is a vertical ellipse.

Ex. 9. Find the eccentricity, the semi-major axis, the semi-minor axis, the coordinates of the foci, the equations of the directrices and the length of the latus rectum of the ellipse $3x^2 + 4y^2 = 12$.

Sol. The equation of the ellipse can be written as $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Since the denominator of x^2 is greater than the denominator of y^2, it is a horizontal ellipse.

$\therefore$ $a^2 = 4$, $b^2 = 3$. $\therefore$ Semi-major axis = $a = 2$. Semi-minor axis = $b = \sqrt{3}$

$\therefore$ Eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$.

The coordinates of the foci are $(\pm ae, 0)$ or $(\pm 1, 0)$.

The equations of the directrices are $x = \pm \frac{a}{e} = \pm 4$ or $x = \pm 4$.

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 3}{2} = 3$.

Hence, the eccentricity = $\frac{1}{2}$; semi-major axis = 2; semi-minor axis = $\sqrt{3}$; the foci are $(\pm 1, 0)$; the directrices are $x = \pm 4$; the length of the latus rectum = 3.

Ex. 10. Find the eccentricity, distance between the foci, equation of the directrices, and the length and the coordinates of the ends of the latera recta of the ellipse $25x^2 + 16y^2 = 400$.

Sol. The equation of the ellipse can be written as

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Since, the denominator of y^2 is greater than the denominator of x^2, it is a vertical ellipse.

$a^2 = 25$ and $b^2 = 16$, therefore, $a = 5$ and $b = 4$.

(*i*) Eccentricity (e) = $\sqrt{1 - \frac{16}{25}} = \frac{3}{5}$.

(*ii*) Distance between the foci = $2ae = 2 \times 5 \times \frac{3}{5} = 6$.

(*iii*) Equation of the directrices are $y = \pm \frac{a}{e} = \pm 5 \times \frac{5}{3} = \pm \frac{25}{3}$ or $3y \pm 25 = 0$.

(*iv*) Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$.

(*v*) $ae = 5 \times \frac{3}{5} = 3$ and $\frac{b^2}{a} = \frac{16}{5}$

$\therefore$ The coordinates of the ends of the latera recta are $\left(\pm \frac{b^2}{a}, \pm ae\right)$

or $\left(\frac{16}{5}, 3\right), \left(-\frac{16}{5}, 3\right), \left(\frac{16}{5}, -3\right), \left(\frac{-16}{5}, -3\right)$.

Case 2. ***Equations of the form***

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

If the given equation has either of the above forms or can be written as such, then transfer the origin to (h, k) by putting $x - h = X$ and $y - k = Y$.

Ex. 11. Find the centre, the equations of the major and the minor axes, the length and the equations of the latera recta, foci and the directrices of the ellipse $25x^2 + 9y^2 - 150x - 90y + 225 = 0$.

Sol. Writing the x terms and y terms separately, we have

$(25x^2 - 150x) + (9y^2 - 90y) = -225$

or $25(x^2 - 6x) + 9(y^2 - 10y) = -225$ (Coefficients of x^2 and y^2 in the parentheses are made unity)

or $25(x^2 - 6x + 9) + 9(y^2 - 10y + 25) = -225 + 225 + 225$ (Completing the squares)

or $25(x-3)^2 + 9(y-5)^2 = 225$ or $\dfrac{(x-3)^2}{9} + \dfrac{(y-5)^2}{25} = 1$ ...(1)

Shifting the origin to the point (3, 5) we have $x - 3 = X, y - 5 = Y$ ...(2)

Substituting for x and y in (1), we have $\dfrac{X^2}{9} + \dfrac{Y^2}{25} = 1$ ...(3)

(1) The centre of (3) is (0, 0) *i.e.*, $X = 0, Y = 0$.

Putting in (2), we get $x = 3$ and $y = 5$. $\therefore$ The centre of (1) is (3, 5).

(2) Lengths of the semi-major and semi-minor axes are respectively 5 and 3 as given by (3).

$\therefore$ Lengths of the major and minor axes are respectively 10 and 6.

The equations of the major and the minor axes of (3) are $X = 0$ and $Y = 0$

or using (2), they are $x = 3$ and $y = 5$.

(3) Length of the latus rectum $= 2\dfrac{b^2}{a} = \dfrac{2 \times 9}{5} = \dfrac{18}{5}$.

Eccentricity $(e) = \sqrt{1 - \dfrac{9}{25}} = \dfrac{4}{5}$.

The equation of the latera recta of (3) are $Y = \pm ae = \pm 5 \times \dfrac{4}{5} = \pm 4$.

Using (2), we get the equations of the latera recta referred to the original axes as $y - 5 = \pm 4$, *i.e.*, $y = 9$ and $y = 1$.

(4) Coordinates of the foci are $(0, \pm ae)$ or $\left(0, \pm 5 \times \dfrac{4}{5}\right)$ or $(0, \pm 4)$, *i.e.*, $X = 0, Y = \pm 4$ referred to the new axes OX' and OY'.

Using (2), the coordinates of the foci are $x - 3 = 0, y - 5 = \pm 4$, *i.e.*, (3, 9) and (3, 1).

(5) Equations of the directrices are $Y = \pm \dfrac{a}{e} = \pm \dfrac{25}{4}$.

Using (2), referred to the original axes the equations of the directrices are

$$y - 5 = \pm \frac{25}{4}, \text{ i.e., } y = 5 \pm \frac{25}{4} \text{ or } 4y - 45 = 0 \text{ and } 4y + 5 = 0.$$

EXERCISE 24

1. Find the eccentricity of the ellipse of which the major axis is double the minor axis.
2. If the minor axis of an ellipse is equal to the distance between its foci, prove that its eccentricity is $\dfrac{1}{\sqrt{2}}$.
3. Find the latus rectum and eccentricity of the ellipse whose semi-axes are 5 and 4.
4. Find the eccentricity of the ellipse whose latus rectum is (*i*) half its major axis, (*ii*) half its minor axis.

5. If the eccentricity is zero, prove that the ellipse becomes a circle.

6. Find the equation to the ellipse with axes as the axes of coordinates.

(*i*) major axis = 6, minor axis = 4;

(*ii*) which passes through the points (−3, 1) and (2, −2);

(*iii*) axes are 10 and 8 and the major axis along

(*a*) the axis of x, (*b*) the axis of y;

(*iv*) major axis $\frac{9}{2}$ and eccentricity $\frac{1}{\sqrt{3}}$, where the major axis is the horizontal axis;

(*ISC 1996, 95*)

(*v*) latus rectum is 5 and eccentricity $\frac{2}{3}$; **(*ISC 1993*)**

(*vi*) foci are (± 4, 0) and $e = \frac{1}{3}$;

(*vii*) distance between the foci is 10 and its latus rectum is 15;

(*vii*) distance of the focus from the corresponding directrix is 9 and eccentricity is $\frac{4}{5}$;

(*ix*) the minor axis is equal to the distance between the foci, and the latus rectum is 10.

7. Find the equation of the ellipse whose centre is at (−2, 3) and whose semi-axes are 3 and 2, when the major axis is (*i*) parallel to the axes of x; (*ii*) parallel to the axis of y.

8. Find the equation of the ellipse with its centre at (4, −1), focus at (1, −1), and passing through (8, 0).

9. Find the equation of the ellipse with its centre at (3, 1), vertex at (3, −2), and eccentricity equal to $\frac{1}{3}$.

10. Find the equation of the ellipse whose centre is at (0, 2) and major axis along the axis of y and whose minor axis is equal to the distance between the foci and whose latus rectum is 2.

11. Find the equation of the ellipse with

(*i*) focus at (1, −1), directrix $x = 0$, and $e = \frac{\sqrt{2}}{2}$; [Use definition]

(*ii*) focus at (0, 0), eccentricity is $\frac{5}{6}$, and directrix is $3x + 4y - 1 = 0$. [Use definition]

12. Find the equation of the ellipse from the following data: axis is coincident with $x = 1$, centre is (1, 5), focus is (1, 8) and the sum of the focal distances of a point on the ellipse is 12.

13. A point $P(x, y)$ moves so that the product of the slopes of the two lines joining P to the two points (−2, 1) and (6, 5) is −4. Show that the locus is an ellipse and locate its centre.

14. Find the eccentricity, the coordinates of the foci, and the length of the latus rectum of the ellipse $2x^2 + 3y^2 = 1$.

15. For the ellipse, $9x^2 + 16y^2 = 576$, find the semi-major axis, the semi-minor axis, the eccentricity, the coordinates of the foci, the equations of the directrices, and the length of the latus rectum.

16. Find the length of the axes, the co-ordinates of the foci, the eccentricity, and latus rectum of the ellipse $3x^2 + 2y^2 = 24$.

17. Find the eccentricity of the ellipse, $4x^2 + 9y^2 - 8x - 36y + 4 = 0$.

18. Find the centre of the ellipse, $\frac{x^2 - ax}{a^2} + \frac{y^2 - by}{b^2} = 0$.

19. Find the distance between a focus and an extremity of the minor axis of the ellipse

(*i*) $4x^2 + 5y^2 = 100$. (*ii*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

20. Given the ellipse $36x^2 + 100y^2 = 3600$, find the equations and the lengths of the focal radii drawn through the point $\left(8, \frac{18}{5}\right)$.

21. The focal distance of an end of the minor axis of the ellipse is k and the distance between the foci is $2h$. Find the lengths of the semi-axes.

22. Find the eccentricity of the ellipse whose latus rectum is 4 and distance of the vertex from the nearest focus is 1.5 cm.

23. The directrix of a conic section is the line $3x + 4y = 1$ and the focus S is $(-2, 3)$. If the eccentricity e is $\frac{1}{\sqrt{2}}$, find the equation to the conic section. ***(ISC)***

24. Find the equation to the conic section whose focus is $(1, -1)$, eccentricity is $\left(\frac{1}{2}\right)$ and the directrix is the line $x - y = 3$. Is the conic section an ellipse? ***(ISC)***

25. Find the equation of the ellipse whose foci are $(-1, 5)$ and $(5, 5)$ and whose major axis is 10. [See Solved Ex. 6] ***(ISC 1999)***

26. Find the ellipse if its foci are $(\pm 2, 0)$ and the length of the latus rectum is $\frac{10}{3}$. ***(ISC 1994)***

27. Find the eccentricity of the ellipse of minor axis 26, if the line segment joining the foci subtends an angle 2α at the upper vertex. Also, find the equation of the ellipse. ***(ISC 1991)***

ANSWERS

1. $\frac{\sqrt{3}}{2}$ **3.** Latus rectum $= \frac{32}{5}$, $e = \frac{3}{5}$ **4.** (*i*) $\frac{1}{\sqrt{2}}$ (*ii*) $\frac{1}{2}\sqrt{3}$

6. (*i*) $4x^2 + 9y^2 = 36$ (*ii*) $3x^2 + 5y^2 - 32 = 0$
(*iii*) (*a*) $16x^2 + 25y^2 = 400$ (*b*) $25x^2 + 16y^2 = 400$
(*iv*) $16x^2 + 24y^2 = 81$ (*v*) $20x^2 + 36y^2 = 405$ (*vi*) $8x^2 + 9y^2 - 1152 = 0$
(*vii*) $3x^2 + 4y^2 = 300$ (*viii*) $\frac{x^2}{400} + \frac{y^2}{144} = 1$ (*ix*) $x^2 + 2y^2 = 100$

7. (*i*) $4x^2 + 9y^2 + 16x - 54y + 61 = 0$ (*ii*) $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

8. $x^2 + 2y^2 - 8x + 4y = 0$ **9.** $9x^2 + 8y^2 - 54x - 16y + 17 = 0$

10. $2x^2 + y^2 - 4y = 0$

11. (*i*) $x^2 + 2y^2 - 4x + 4y + 4 = 0$ (*ii*) $27x^2 + 20y^2 - 24xy + 6x + 8y - 1 = 0$

12. $\frac{(x-1)^2}{27} + \frac{(y-5)^2}{36} = 1$ **13.** $4x^2 + y^2 - 16x - 6y - 43 = 0$, centre $(2, 3)$

14. $\frac{1}{\sqrt{3}}, \left(\pm\frac{1}{\sqrt{6}}, 0\right), \frac{2\sqrt{2}}{3}$ **15.** $a = 8$, $b = 6$, $e = \frac{\sqrt{7}}{4}$, Foci $(\pm 2\sqrt{7}, 0)$, Directrices $x = \pm\frac{32}{7}\sqrt{7}$

16. $4\sqrt{3}, 4\sqrt{2}, (0, \pm 2), \frac{1}{\sqrt{3}}, \frac{8}{\sqrt{3}}$ **17.** $\frac{\sqrt{5}}{3}$ **18.** $\left(\frac{a}{2}, \frac{b}{2}\right)$ **19.** (*i*) 5 (*ii*) a

20. Equations are $x - 8 = 0$, $9x - 40y + 72 = 0$; Lengths of the focal radii are $\frac{18}{5}, \frac{82}{5}$.

21. $k, \sqrt{k^2 - h^2}$ **22.** $\frac{1}{3}$ **23.** $41x^2 - 24xy + 34y^2 + 206x - 292y + 649 = 0$

24. $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$. Yes, the conic section is an ellipse, as the eccentricity is less than 1.

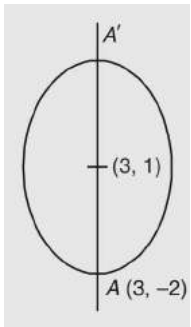
25. $16x^2 + 25y^2 - 64x - 250y + 289 = 0$ **26.** $\frac{x^2}{9} + \frac{y^2}{5} = 1$

27. Eccentricity $= \sin\alpha$. Eqn. of the ellipse is $x^2\cos^2\alpha + y^2 = b^2$.

HINTS TO SELECTED QUESTIONS

8. See Solved Ex. 4

9. Centre $\equiv (3, 1)$, Vertex $A \equiv (3, -2)$, $e \equiv \frac{1}{3}$. Let $A' \equiv (x_1, y_1)$. Then



$\frac{x_1+3}{2} = 3 \Rightarrow x_1 = 3$, $\frac{y_1-2}{2} = 1 \Rightarrow y_1 = 4$

$\therefore \quad A' = (3, 4)$, $2a = |AA'| = 4 + 2 = 6 \Rightarrow a = 3$

$b^2 = a^2(1 - e^2) \Rightarrow b^2 = 9\left(1 - \frac{1}{9}\right) = 8$

$\therefore$ Required equation is $\frac{(x-3)^2}{8} + \frac{(y-1)^2}{9} = 1$

10. $2ae = 2b \Rightarrow ae = b$. Now, $b^2 = a^2(1 - e^2) \Rightarrow b^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2$

$\Rightarrow 2a = a^2$ [$\because$ Latus rectum $= \frac{2b^2}{a} = 2$ (given) $\therefore b^2 = a$]

$\Rightarrow a^2 - 2a = 0 \Rightarrow a = 0$, since a cannot be 0, $a = 2$ in $b^2 = 2$

12. It is vertical ellipse. Its equation is $\frac{(x-1)^2}{b^2} + \frac{(y-5)^2}{a^2} = 1$

Sum of the focal distances $= 2a = 12 \Rightarrow a = 6$. Distance of the focus from the centre $= ae = 8 - 5 = 3$

$b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = 36 - 9 = 27$

13. $\left(\frac{y-1}{x+2}\right)\left(\frac{y-5}{x-6}\right) = -4 \Rightarrow (y-3)^2 + 4(x-2)^2 = 68$

16. $3x^2 + 2y^2 = 24 \Rightarrow \frac{x^2}{8} + \frac{y^2}{12} = 1 \Rightarrow b^2 = 8, a^2 = 12$

17. $4x^2 + 9y^2 - 8x - 36y + 4 = 0 \Rightarrow 4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 4 - 4 - 36 = 0 \Rightarrow \frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$

18. $\frac{x^2 - ax}{a^2} + \frac{y^2 - by}{b^2} = 0 \Rightarrow \frac{x^2 - ax + \frac{a^2}{4}}{a^2} + \frac{y^2 - by + \frac{b^2}{4}}{b^2} = \frac{1}{4} + \frac{1}{4}$

19. Reqd. distance $= \sqrt{(ae - 0)^2 + b^2}$

20. $a = 10$, $b = 6$, $e = \frac{4}{5}$. Focal radii are $a + ex_1$ and $a - ex_1$. Foci are $(\pm ae, 0) \equiv (\pm 8, 0)$.

Use $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ to obtain equations of focal radii which join the points $(\pm 8, 0)$ and $\left(8, \frac{18}{5}\right)$.

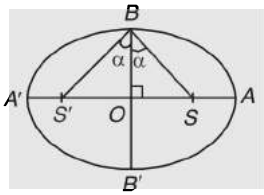
21. Ends of minor axis are $(0, b)$ $(0, -b)$. Focal distance of one end $(0, b) = a + ex_1 = a + e \times 0 = a = k$ (given).

Distance between foci $= 2ae = 2h$ (given) $\Rightarrow e = \frac{h}{a} = \frac{h}{k}$. Now $b^2 = a^2(1 - e)^2$.

26. See Solved Ex. 7

27. Let major axis $= 2a$ and O the centre of the ellipse. $SS' = 2ae$, $OS = b \tan\alpha$, $SS' = 2OS$

$\Rightarrow 2ae = 2b \tan\alpha \Rightarrow ae = b \tan\alpha$



Now, $b^2 = a^2(1 - e^2) = a^2 - b^2 \tan^2\alpha$

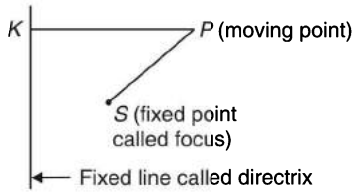
$\Rightarrow a^2 = b^2(1 + \tan^2\alpha) = b^2 \sec^2\alpha \Rightarrow a = b \sec\alpha$

$\therefore e = \frac{b}{a}\tan\alpha = \frac{b}{b\sec\alpha}.\tan\alpha = \sin\alpha$

CHAPTER SUMMARY

1. Definitions

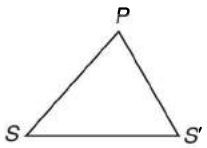
(*i*) An ellipse is the locus of a point which moves such that the ratio of its distance from a fixed point to its distance from a fixed line is constant e, this constant being less than unity, *i.e.*, $e < 1$.



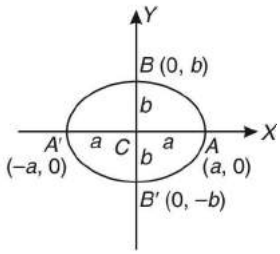
The constant 'e' is called the eccentricity of the ellipse.

Alternative definition

An ellipse is the locus of a point which moves such that the sum of its distances from two fixed points is constant, *i.e.*, $|PS| + |PS'| = \text{constant} = 2a$.



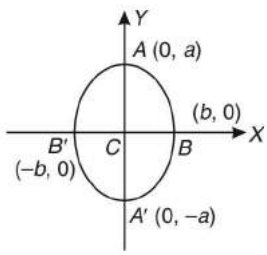
2. The standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ where a and b are constants ($a > b$) and $b^2 = a^2(1 - e^2)$. This equation represents a horizontal ellipse.



- The points A and A' are called the vertices of the ellipse. AA' ($= 2a$) is called the major axis and BB' ($= 2b$) the minor axis of the ellipse, C is called the centre of the ellipse and a chord through C, is diameter. An ellipse is called a central curve since it bisects all chords passing through its centre.
- As shown in Fig. 24.03, an ellipse has two vertices, two foci, two directrices and two letera recta.
- The foci of an ellipse always lie on the major axis.
- **Focal property**: $|SP| = a + ex_1$, $|S'P| = a - ex_1$ and $|SP| + |S'P| = 2a$, *i.e.*, the sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis.

3. Other forms of the ellipse

- If the equation of the ellipse is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$, *i.e.*, the denominator of y^2 > the denominator of x^2, then it represents a vertical ellipse.



Major axis along y –axis

- If the centre of the ellipse is at the point (h, k) and its axes are parallel to coordinate axis (Fig. 24.08), then its equation is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1 \ (a > b)$$

If the major axis is parallel to y-axis, then the equation of the ellipse would be

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2} = 1 \ (a > b)$$

4. Summary of the facts for the standard forms of ellipse

S. No.	Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1(a>b)$
1.	Coordinates of the centre	(0, 0)	(0, 0)
2.	Coordinates of foci	$(\pm ae, 0)$	$(0, \pm ae)$
3.	Coordinates of vertices	$(\pm a, 0)$	$(0, \pm a)$
4.	Equation of major axis	$y=0$	$x=0$
5.	Equation of minor axis	$x=0$	$y=0$
6.	Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
7.	Length of major axis	$2a$	$2a$
8.	Length of minor axis	$2b$	$2b$
9.	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
10.	Eccentricity	$e^2=1-\frac{b^2}{a^2}$	$e^2=1-\frac{b^2}{a^2}$

CHAPTER TEST

1. Find the eccentricity of the ellipse

$$\frac{(x-3)^2}{8}+\frac{(y-4)^2}{6}=1.$$ **(*ISC 2002*)**

2. The distance between the foci of the ellipse $5x^2 + 9y^2 = 45$ is

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5

3. Find the equation of an ellipse whose latus rectum is 8 and eccentricity is $\frac{1}{3}$ **(*ISC 2015*)**

4. Find the equation of the ellipse whose foci are at (–2, 4) and (4, 4) and major and minor axes are 10 and 8 respectively. Also, find the eccentricity of the ellipse. **(*ISC 2003*)**

5. Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$ and whose foci are at the points (±2, 0). **(*ISC 2006, 1995*)**

6. Find the equation of the ellipse whose centre is the origin, major axis $\frac{9}{2}$ and eccentricity $\frac{1}{\sqrt{3}}$ where the major axis is the horizontal axis. **(*ISC 2006, 1996*)**

7. Find the equation of the ellipse whose minor axis is 4 and which has a distance of 6 units between foci. **(*ISC 2007*)**

8. Find the equation of the ellipse with its centre at (4, –1), focus at (1, –1) and given that *i* passes through (8, 0). **(*ISC 2012*)**

9. Find the coordinates of the vertices and the foci and the length of the latus rectum of the ellipse $9x^2 + 25y^2 = 225$.

10. Find the eccentricity and the equations of the directrices of the ellipse $7x^2 + 16y^2 = 112$.

ANSWERS

1. $\frac{1}{2}$ **2.** (*c*) **3.** $8x^2 + 9y^2 = 162$

4. $\frac{(x-1)^2}{25} + \frac{(y-4)^2}{16} = 1$ [**Hint:** See Solved Ex. 6]

5. $\frac{x^2}{16} + \frac{y^2}{12} = 1$ [**Hint:** See Solved Ex. 7]

6. $16x^2 + 24y^2 = 81$

7. $\frac{x^2}{13} + \frac{y^2}{4} = 1$

8. $x^2 + 2y^2 - 8x + 4y = 0$ [**Hint:** See Solved Ex. 4]

9. Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$. Length of latus rectum = $\frac{18}{5}$ units.

10. $e = \frac{3}{4}$, $3x \pm 16 = 0$

CHAPTER 25

HYPERBOLA

25.01. Definition

A hyperbola is the locus of a point which moves so that its distance from a fixed point, called the **focus,** *bears a constant ratio, which is greater than unity, to its distance from a fixed line, called the* **directrix.**

The focus is denoted by S, and the constant ratio called the eccentricity is denoted by e. For the hyperbola, $e > 1$.

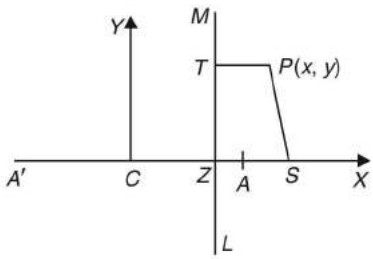


Fig. 25.01

25.02. Standard equation of the hyperbola

Let S be the focus and LM the directrix. Draw SZ perpendicular to LM. Divide ZS internally at A so that

$SA : AZ = e : 1$ or $SA = e \times AZ$ ...(1)

SZ can also be divided externally at A' so that

$SA' : ZA' = e : 1$ or $SA' = e \times ZA'$ ...(2)

Since $e > 1$, $SA' > A'Z$, and therefore A' lies on SZ produced and not on ZS produced.

Then, by definition, A and A' lie on the hyperbola. Bisect AA' at C.

Let $AA' = 2a$ so that $CA = CA' = a$

Adding (1) and (2), we have $SA + SA' = e\,(ZA + ZA')$

or $(CS - CA) + (CS + CA') = e(\overline{CA - CZ} + \overline{CZ + A'C})$ $(\because\ CA = CA')$

or $2 \times CS = 2e \times CA = 2ae$ or $CS = ae$ ...(3)

Subtracting (1) from (2), we have $SA' - SA = e(ZA' - AZ)$

or $AA' = e(\overline{CZ + A'C - CA - CZ})$ or $2a = 2e\ .\ CZ$ or $CZ = \dfrac{a}{e}$ $[\because\ CA = CA']$ ...(4)

Now, let C be taken as the origin, CA as the axis of x, and a line perpendicular to CA as the axis of y.

Then the co-ordinates of S are $(ae, 0)$ and the equation of the directrix LM is $x = \dfrac{a}{e}$.

Let $P\,(x, y)$ be any point on the curve. Then, by definition, $|SP| = e \times |PT|$

or $\left|\sqrt{(x - ae)^2 + (y - 0)^2}\right| = e \times \dfrac{\left|x - \frac{a}{e}\right|}{1}$ or $(x - ae)^2 + y^2 = e^2\left(x - \dfrac{a}{e}\right)^2$

or $x^2 - 2aex + a^2e^2 + y^2 = e^2\left(x^2 - \dfrac{2a}{e}x + \dfrac{a^2}{e^2}\right)$ or $x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$

or $x^2(1-e^2)+y^2=a^2(1-e^2)$ or $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$ or $\dfrac{x^2}{a^2}-\dfrac{y^2}{a^2(e^2-1)}=1$

Putting $a^2(e^2-1)=b^2$, we have $\boldsymbol{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$

which is the required equation of the hyperbola in the standard form.

Cor. $\boldsymbol{CS=ae,\ CZ=\dfrac{a}{e},\ b^2=a^2(e^2-1).}$

From $b^2=a^2(e^2-1)$, we get $e^2=1+\dfrac{b^2}{a^2}$, or $e=\sqrt{1+\dfrac{b^2}{a^2}}$,

which gives eccentricity e.

We can also write $e^2=1+\left(\dfrac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2=1+\left(\dfrac{2b}{2a}\right)^2=1+\dfrac{b^2}{a^2}$

Note. The y-axis meets the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ where $x=0$. This gives $y=\pm\sqrt{-b^2}$ which are imaginary values. Therefore, the y-axis does not meet the hyperbola (in real points).

25.03. Definitions

The points A and A′ are called the **vertices** *of the hyperbola.*

The line segment AA′ is called the **transverse axis** *of the hyperbola.*

If two points B and B' are taken on the y-axis such that $CB=CB'=b$, then *the line BB′ is called the* **conjugate axis** *of the hyperbola (Fig. 25.02)* segment.

The middle point C of AA′ is called the **centre** *of the hyperbola, because it bisects every chord that passes through it. This can be easily verified. If P (x_1, y_1) be any point on the curve, then*

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1.$$

This can be put as $\dfrac{(-x_1)^2}{a^2}-\dfrac{(-y_1)^2}{b^2}=1,$

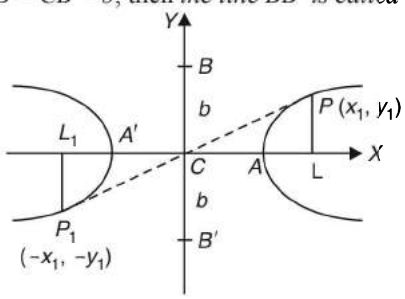


Fig. 25.02

which shows that $(-x_1, -y_1)$ also lies on the curve. Now, the middle point of the chord joining the points (x_1, y_1) and $(-x_1, -y_1)$ is $(0, 0)$ which is C. Therefore, C bisects every chord. Hence, the hyperbola is a **central curve** when $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is its equation and the origin is its centre.

25.04. Sketching the hyperbola

The equation of the curve is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots(1)$$

1. Origin. Since the co-ordinates (0, 0) do not satisfy (1), therefore the curve does not pass through the origin.

2. Axes intersections. The curve intersects the x-axis where $y=0$. Putting $y=0$ in (1), we get $x=\pm a$. Thus, the curve intersects the x-axis at $A\ (a, 0)$ and $A'\ (-a, 0)$.

The curve intersects the y-axis in imaginary points because by putting $x = 0$, we get $y = \pm\sqrt{-b^2}$ which are imaginary values. In other words, the curve does not meet the y-axis (in real points).

3. Symmetry. The equation (1) can be written as

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1} \qquad \text{or} \qquad x = \pm a\sqrt{1 + \frac{y^2}{b^2}}$$

The first equation shows that for each value of x ($x > a$ or $x < -a$), there are two values of y, equal and opposite. This means that the curve is symmetrical about the x-axis. Similarly, from the second equation we deduce that for all values of y, the curve is symmetrical about the y-axis as for each value of y there are two equal and opposite value of x.

4. Imaginary values. y is imaginary for $-a < x < a$, therefore no part of the curve lies between $x = a$ and $x = -a$.

The line $x = a$ or $x = -a$ intersects the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $y^2 = 0$. Since this gives two coincident values of y, therefore $x = a$ and $x = -a$ are tangents to the curve at A and A' respectively.

5. Space occupied. The equation (1) is the same as $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$.

As x increases ($x > a$) from a to ∞, y increases from 0 to $+\infty$ above the x-axis and 0 to $-\infty$ below the x-axis. Since the curve is symmetrical about the axes of x and y, therefore, it can be drawn as shown in Fig. 25.03.

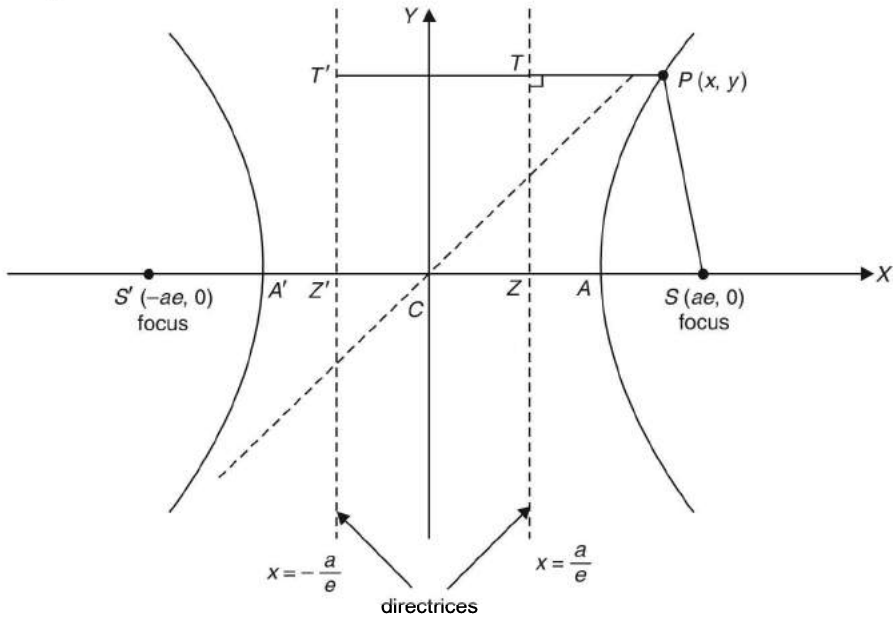


Fig. 25.03

25.05. Second focus and second directrix

Since a hyperbola is symmetrical about the y-axis, therefore, if on the x-axis the points S' and Z' are taken such that $CS' = CS = ae$ and $CZ' = CZ = \frac{a}{e}$, then S' will also be a focus and $L'M'$, the line through Z' perpendicular to CZ', will be the corresponding directrix of the hyperbola (Fig. 25.03).

Thus, a hyperbola has two foci ($\pm ae$, 0) and two directrices $x = \pm\frac{a}{e}$. Just as $SP = e \times PT$, so is

$$S'P = e \times PT'.$$

25.06. Latus rectum

If y_1 be the ordinate of the extremity L of the latus rectum, then the coordinates of L are (ae, y_1). These coordinates satisfy the equation of the hyperbola, namely,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{a^2e^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

or $\quad y_1^2 = b^2(e^2 - 1) = b^2 \times \frac{b^2}{a^2} \qquad [\because\ b^2 = a^2(e^2 - 1)]$

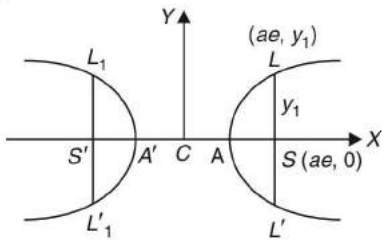


Fig. 25.04

or $\quad y_1 = \frac{b^2}{a}$ $\quad \therefore$ the semi-latus rectum $= \frac{b^2}{a}$ and $\therefore$ the latus rectum $= \frac{2b^2}{a}$.

Cor. 1. The equations of the two latera recta LL' and L_1L_1' are respectively $x = ae$ and $x = -ae$.

Cor. 2. The co-ordinates of the extremities of the latera recta are

$$L\left(ae, \frac{b^2}{a}\right);\ L'\left(ae, \frac{-b^2}{a}\right);\ L_1\left(-ae, \frac{b^2}{a}\right);\ L_1'\left(-ae, \frac{-b^2}{a}\right).$$

25.07. Focal property of the hyperbola

To prove that

The difference of the focal distances of any point on the hyperbola is constant and equal to the distance AA′, i.e., S′P – SP = 2a.

Let $P(x_1, y_1)$ be any point on the hyperbola. Let PN be perpendicular to the x-axis. Then

$$SP = e \times PT = e \times ZN \quad \text{[From Fig. 25.03]}$$

$$= e(CN - CZ) = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$$

$$S'P = e \times PT' = e \times Z'N = e(CZ' + CN)$$

$$= e\left(\frac{a}{e} + x_1\right) = ex_1 + a$$

$$\therefore \quad S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a = AA'.$$

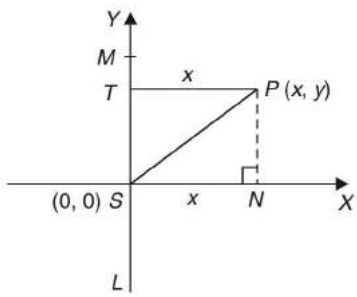


Fig. 25.05

Note. Remember that the focal distances of any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $ex_1 + a$ and $ex_1 - a$.

Note. This property has applications in range finding and LORAN (Long Range Navigation). Both of these use intersections of families of hyperbolas.

25.08. Other forms of the hyperbola

1. The equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is another standard form of the hyperbola whose transverse axis is along the y-axis. The foci are $(0, \pm ae)$. The directrices are $y = \pm \frac{a}{e}$, etc.

It should be remembered that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the coefficient of y^2 is negative has its transverse axis horizontal, *i.e.*, along the x-axis, and that $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ in which the co-efficient of x^2 is negative has its transverse axis vertical, *i.e.*, along the y-axis. [Fig. 25.06]

2. Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the **conjugate hyperbola** of the given hyperbola.

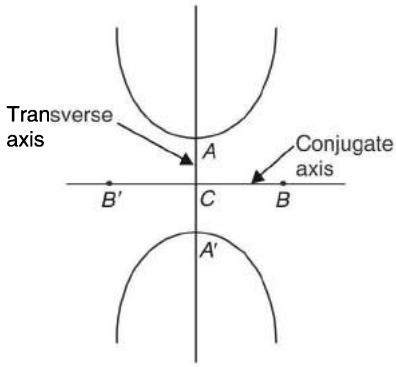


Fig 25.06

Thus, if $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the given hyperbola, then

$\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$, *i.e.*, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$,

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is its conjugate hyperbola.

3. Axes parallel to the axes of coordinates

As in the case of the ellipse so for the hyperbola it can be easily established that the equation of the hyperbola with centre (h, k) and the axes parallel to the axes of coordinates is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

This equation, when simplified, reduces to the form $lx^2 - my^2 + 2gx + 2fy + c = 0$, where l and m are two different positive numbers.

Conversely, given an equation of the form $lx^2 - my^2 + 2gx + 2fy + c = 0$, we at once conclude that it represents a hyperbola whose axes are parallel to the axes of coordinates.

4. General equation. Given the coordinates of the focus, the equation of the corresponding directrix, and the eccentricity 'e', the equation of the hyperbola can be easily found by using the definition, namely, the distance of any point P on the hyperbola from the

focus = $e \times$ (the distance of P from the directrix).

Let the focus S be (α, β) and the equation of the corresponding directrix be $lx + my + n = 0$, and eccentricity be 'e'. Let $P(x, y)$ be any point on the hyperbola. Then, by definition

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e \times \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

or $(l^2 + m^2)[(x-\alpha)^2 + (y-\beta)^2] = e^2 (lx + my + n)^2$

This equation, when simplified, reduces to the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

which is the general equation of the hyperbola.

It will be noticed that

(*i*) *the equation of a hyperbola is a second degree equation:*

(*ii*) *the constants a, h, b are such that* $h^2 > ab$.

The condition (*ii*) is merely stated without a proof. [For proof see Art. 25.10].

25.09. Solved Examples

Type 1. ***Finding equation of a hyperbola when its foci, eccentricity and directrix are given.***

Ex. 1. Find the equation of the hyperbola whose focus is (1, 2), directrix $2x + y = 1$ and eccentricity $\sqrt{3}$. ***(ISC 2000 Type)***

Sol. Let $P(x, y)$ be any point on the hyperbola. Then, by definition of the hyperbola, the distance of P from (1, 2) is equal to e times its distance from $2x + y = 1$.

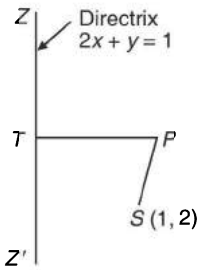


Fig. 25.07

$$\therefore \quad \sqrt{(x-1)^2+(y-2)^2} = \sqrt{3}\,.\frac{2x+y-1}{\sqrt{5}}$$

or $\quad 5\,[(x^2 - 2x + 1) + (y^2 - 4y + 4)] = 3\,(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$

or $\quad 7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0,$

which is the required equation of the hyperbola.

Check. Here, $a = 7$, $h = 6$, $b = -2$. Therefore, $h^2 - ab = 36 - 7\,(-2) = 50 > 0$.

Type 2. ***Finding the various parts of a given hyperbola shown below separately for your easy understanding.***

Case I : Hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which coeff. of y^2 is –ve. (Horizontal hyperbola)

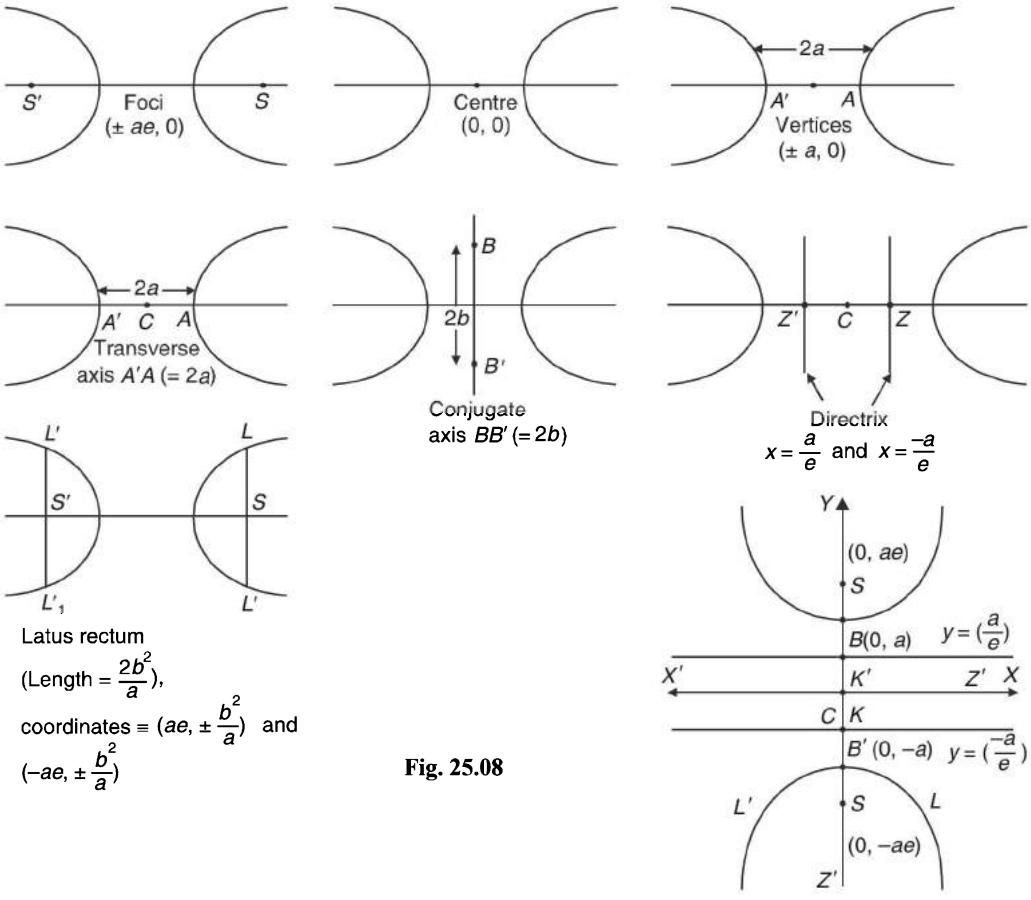


Fig. 25.08

Fig. 25.09

Case II : Vertical hyperbola

Form : $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in which coeff. of x^2 is –ve.

Table for ready reference

Parts \ Equation of the hyperbola	Horizontal hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Vertical hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
1. Coordinates of the centre	(0, 0)	(0, 0)
2. Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, \pm a)$
3. Coordinates of foci	$(\pm ae, 0)$	$(0, \pm ae)$
4. Length of the transverse axis	$2a$	$2a$
5. Length of the conjugate axis	$2b$	$2b$
6. Equations of the directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
7. Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$ or, $b^2 = a^2(e^2-1)$	$e=\sqrt{\frac{a^2+b^2}{a^2}}$ or, $b^2 = a^2(e^2-1)$
8. Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
9. Equation of the transverse axis	$y = 0$	$x = 0$
10. Equation of the conjugate axis	$x = 0$	$y = 0$
11. Focal distances of any point (x_1, y_1)	$ex \pm a$	$ey \pm a$

Ex. 2. Find the axes, vertices, foci, eccentricity, equations of the directrices, and length of the latus rectum of the hyperbola $9x^2 - 16y^2 = 144$.

Sol. The given equation can be written in the form

$$\frac{x^2}{16}-\frac{y^2}{9}=1.$$

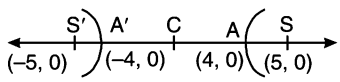


Fig. 25.10

Here, $a = 4$, $b = 3$, $e^2 = 1+\frac{b^2}{a^2}=1+\frac{9}{16}=\frac{25}{16}$ $\Rightarrow$ $e=\frac{5}{4}$, and $ae = 5$, $\frac{a}{e}=\frac{16}{5}$.

The transverse axis is along the x-axis. The foci are $(\pm ae, 0)$, *i.e.*, $(\pm 5, 0)$. Equations of the directrices are

$$x = \pm\frac{a}{e} \quad i.e., \quad x = \pm\frac{16}{5}.$$

The vertices are $(\pm a, 0)$, *i.e.*, $(\pm 4, 0)$. Latus rectum $= \frac{2b^2}{a} = 2\times\frac{9}{4}=\frac{9}{2}$.

Ex. 3. Find the axes, vertices, foci, eccentricity, equations of the directrices and length of the latus rectum of the hyperbola $5y^2 - 9x^2 = 36$.

Sol.

$$5y^2 - 9x^2 = 36$$

$$\Rightarrow \quad \frac{5y^2}{36}-\frac{x^2}{4}=1$$

$$\Rightarrow \quad \frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)^2}-\frac{x^2}{4}=1$$

Here, $a^2 = \left(\frac{6}{\sqrt{5}}\right)^2 \Rightarrow a=\frac{6}{\sqrt{5}}$, $b^2 = 4 \Rightarrow b = 2$

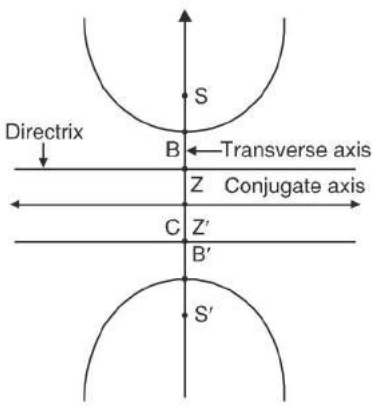


Fig. 25.11

The transverse axis is along the y-axis. Its equation is $x = 0$. The conjugate axis is along the x-axis. Its equation is $y = 0$.

$$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{4}{\frac{36}{5}} = \frac{56}{36} = \frac{14}{9} \Rightarrow e = \frac{\sqrt{14}}{3}.$$

Vertices and foci lie on the y-axis. So vertices are $(0, \pm a)$, *i.e.*, $\left(0, \pm\frac{6}{\sqrt{5}}\right)$ and foci are $(0, \pm ae)$, *i.e.*, $\left(0, \pm\frac{6}{\sqrt{5}}\cdot\frac{\sqrt{14}}{3}\right) \Rightarrow \left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$.

Directrices are parallel to the x-axis. Their equations are $y = \pm\frac{a}{e}$, *i.e.*, $y = \pm\frac{6}{\sqrt{5}}\cdot\frac{3}{\sqrt{14}}$ $\Rightarrow$ $y = \pm\frac{9\sqrt{2}}{\sqrt{35}}$. Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 4}{\frac{6}{\sqrt{5}}} = \frac{4\sqrt{5}}{3}$.

Type 3. ***Finding the equation of a hyperbola when some of its parts are given.***

Ex. 4. Find the equation of the hyperbola whose centre is (–4, 1), vertex (2, 1), and semi-conjugate axis equal to 4.

Sol. The distance between the centre and the vertex is 6.

Therefore, $a = 6$. Also semi-conjugate axis is 4, *i.e.*, $b = 4$.

Hence, the equation of the hyperbola is $\frac{(x+4)^2}{36} - \frac{(y-1)^2}{16} = 1$.

Ex. 5. Find the equation of the hyperbola whose

(*i*) foci are (± 3, 0) and vertices (± 2, 0)

(*ii*) foci are (0, ± 8) and vertices (0, ± 5).

Sol. Since the foci are of the form $(\pm ae, 0)$ and vertices $(\pm a, 0)$, it is a horizontal hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$a = 2, ae = 3 \quad \Rightarrow e = \frac{3}{2} \therefore \; b^2 = a^2(e^2-1) \Rightarrow b^2 = 4\left(\frac{9}{4}-1\right) = 5$$

$\therefore$ Equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$

(*i*) Since the foci are of the form $(0, \pm ae)$ and vertices $(0, \pm a)$, it is a vertical hyperbola of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

$$ae = 8, a = 5 \quad \Rightarrow \quad e = \frac{8}{5} \therefore \; b^2 = a^2(e^2-1) \Rightarrow b^2 = 25\left(\frac{64}{25}-1\right) = 39$$

So the equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{39} = 1$.

Ex. 6. Find the equation of the hyperbola whose foci are $(\pm 3\sqrt{5}, 0)$ and latus rectum is of length 8.

Sol. Since the foci are of the form $(\pm ae, 0)$, it is a horizontal hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$ae = 3\sqrt{5}, \; \frac{2b^2}{a} = 8 \quad \Rightarrow b^2 = 4a$$

$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = a^2e^2 - a^2 \Rightarrow 4a = 45 - a^2 \Rightarrow a^2 + 4a - 45 = 0$

$\Rightarrow (a + 9)(a - 5) = 0 \Rightarrow a = -9, 5$. Rejecting –ve value $a = 5 \Rightarrow b^2 = 4 \times 5 = 20$.

So, the equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1 \Rightarrow 4x^2 - 5y^2 = 100$.

Ex. 7. Find the equation of the hyperbola whose foci are $(0, \pm \sqrt{10})$ and which passes through (2,3). ***(ISC 2015)***

Sol. Since the foci are of the form $(0, \pm ae)$, it is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Foci $\equiv (0, \pm \sqrt{10}) \Rightarrow ae = \sqrt{10}$ $\quad \therefore b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 10 - a^2$

Point (2, 3) lies on the hyperbola $\Rightarrow \frac{9}{a^2} - \frac{4}{b^2} = 1$

$\Rightarrow \frac{9}{a^2} - \frac{4}{10 - a^2} = 1 \quad \Rightarrow 9(10 - a^2) - 4a^2 = a^2(10 - a^2)$

$\Rightarrow a^4 - 23a^2 + 90 = 0 \quad \Rightarrow (a^2 - 18)(a^2 - 5) = 0 \Rightarrow a^2 = 18, 5$

$a^2 = 18$ is not possible as it would make b^2 negative.

$\therefore a^2 = 5 \Rightarrow b^2 = 10 - 5 = 5$. So equation of the hyperbola is $\frac{y^2}{5} - \frac{x^2}{5} = 1$

$\Rightarrow y^2 - x^2 = 5$.

Ex. 8. Find the equation of the hyperbola whose foci are (8, 3), (0, 3) and eccentricity = $\frac{4}{3}$.

Sol. The centre of the hyperbola is the mid-point of the line joining the two foci.

$\therefore$ Centre $\equiv \left(\frac{8+0}{2}, \frac{3+3}{2}\right)$, *i.e.*, (4, 3).

Since the foci lie on the line $y = 3$ (parallel to the x-axis), the equation of the hyperbola is of the form $\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1$, where $2a$ and $2b$ are the transverse and conjugate axis respectively.

Now, distance between the foci $= \sqrt{(8-0)^2 + (3-3)^2} = 8 = 2ae \Rightarrow 8 = 2 \times a \times \frac{4}{3} \Rightarrow a = 3$

Also, $b^2 = a^2(e^2 - 1) = 9\left(\frac{16}{9} - 1\right) = 7$.

Hence, equation of the hyperbola is

$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1 \Rightarrow 7x^2 - 9y^2 - 56x + 54y - 32 = 0$.

Miscellaneous Problems

Ex. 9. If e and e' are the eccentricities of a hyperbola and its conjugate respectively, prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (1)

Then equation of its conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ ... (2)

$$e = \sqrt{1+\left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2} = \sqrt{1+\left(\frac{2b}{2a}\right)^2} \Rightarrow e^2 = \frac{a^2+b^2}{a^2} \qquad \text{(Cor. Art. 25.02)}$$

$$e' = \sqrt{1+\left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2} = \sqrt{1+\left(\frac{2a}{2b}\right)^2} \Rightarrow e'^2 = \frac{b^2+a^2}{b^2}$$

$$\therefore \quad \frac{1}{e^2}+\frac{1}{e'^2} = \frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

Ex. 10. Prove that the locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$, for different values of k, is a hyperbola whose eccentricity is 2. (*ISC 2009*)

Sol. Given straight lines are

$$\sqrt{3}x - y - 4\sqrt{3}k = 0 \quad \ldots (i)$$

$$\sqrt{3}kx + ky - 4\sqrt{3} = 0 \quad \ldots (ii)$$

To find the locus of the point of intersection of the above lines, we have to eliminate k. The eliminant will be the required locus.

From (i), we get $k = \frac{\sqrt{3}x - y}{4\sqrt{3}}$, substituting this value in (ii) we get $\left(\sqrt{3}x + y\right).\frac{\sqrt{3}x - y}{4\sqrt{3}} = 4\sqrt{3}$

$\Rightarrow \quad 3x^2 - y^2 = 48 \Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$, which represents a hyperbola of the first form (coefft. of y^2 is –ve).

Here, $a^2 = 16$, $b^2 = 48$. If e be the eccentricity of the hyperbola, then

$b^2 = a^2(e^2 - 1) \quad \Rightarrow \quad 48 = 16(e^2 - 1) \quad \Rightarrow \quad e^2 = 4 \quad \Rightarrow \quad e = 2$

Hence, the locus of the point of intersection of the given straight line is a hyperbola whose eccentricity is 2.

Ex. 11. A (–1, 0) and B (2, 0) are two given points. A point M is moving in such a way that the angle B in the triangle AMB remains twice as large as the angle A. Show that the locus of the point M is a hyperbola. Find the eccentricity of the hyperbola. (*ISC 1992*)

Sol. Let $\angle MAB = \theta$, and $\angle MBA = 2\theta$, so that $\angle MBX = 180° - 2\theta$

Let the coordinates of M be (x, y).

$$m_1 = \text{Gradient of } MA = \tan\theta = \frac{y-0}{x+1} = \frac{y}{x+1}$$

$$m_2 = \text{Gradient of } MB = \frac{y-0}{x-2} = \frac{y}{x-2}$$

$$\therefore \quad \tan(180° - 2\theta) = \frac{y}{x-2}$$

$$\Rightarrow \quad -\tan 2\theta = \frac{y}{x-2} \Rightarrow \tan 2\theta = \frac{y}{2-x}$$

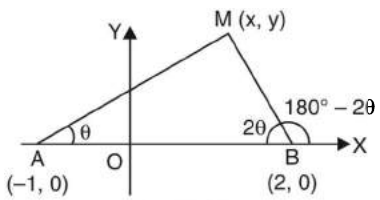


Fig. 25.12

$$\text{or} \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{y}{2-x} \Rightarrow \frac{\frac{2y}{x+1}}{1-\frac{y^2}{(x+1)^2}} = \frac{y}{2-x}$$

or $\frac{2y}{x+1} \times \frac{(x+1)^2}{x^2+2x+1-y^2} = \frac{y}{2-x}$

$\Rightarrow \quad 2(x+1)(2-x) = x^2 - y^2 + 2x + 1$

or $3x^2 - y^2 - 3 = 0$, which is the required locus of M.

Here, $a = 3, h = 0, b = -1 \quad \therefore \quad h^2 - ab = 0 - 3 \times (-1) = 3 > 0$

$\therefore$ The locus is a hyperbola.

The equation $3x^2 - y^2 - 3 = 0$ can be written as $\frac{x^2}{1} - \frac{y^2}{3} = 1$

Here, $a^2 = 1, b^2 = 3$. Also $b^2 = a^2(e^2 - 1)$, or $3 = 1(e^2 - 1)$, or $e = 2$

Hence, the eccentricity of the hyperbola is 2.

EXERCISE 25 (*a*)

1. Find the equation of the hyperbola whose focus is (1, 1), directrix $2x + 2y = 1$, and eccentricity $\sqrt{2}$.

2. Find the equation to the hyperbola whose eccentricity is 2, whose focus is (2, 0) and whose directrix is $x - y = 0$.

3. Find the equation to the conic section whose focus is $(ae, 0)$, directrix is the line $ex = a$, and eccentricity is e. State whether the conic section is an ellipse or a hyperbola. ***(ISC)***

4. Find the equation of the hyperbola whose axes are along the coordinate axes and which passes through (–3, 4), and (5, 6).

5. Find the eccentricity of the hyperbola whose equation is $2x^2 - 3y^2 = 15$.

6. Find the eccentricity and the coordinates of the foci of the curve $3x^2 - y^2 = 4$.

7. Find the coordinates of the foci, vertices, eccentricity and the length of the latus rectum of the hyperbola

(*i*) $16x^2 - 9y^2 = 576$ (*ii*) $\frac{y^2}{9} - \frac{x^2}{27} = 1$ (*iii*) $9y^2 - 4x^2 = 36$ (*iv*) $49y^2 - 16x^2 = 784$

8. In the hyperbola $x^2 - 4y^2 = 4$, find the length of the axes, the coordinates of the foci, the eccentricity, and the latus rectum, and the equations of the directrices.

9. Find (*a*) the eccentricities, (*b*) the co-ordinates of the foci (*c*) the equations of the directrices of the following hyperbolas (*i*) $\frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$ (*ii*) $\frac{(x+1)^2}{64} - \frac{(y-2)^2}{36} = 1$.

10. Find the equation of the hyperbola, referred to its axes as the axes of coordinates,

(*i*) whose transverse and conjugate axes are in length respectively 2 and 3;

(*ii*) whose foci are (2, 0) and (–2, 0) and eccentricity equal to $\frac{3}{2}$;

(*iii*) the distance between whose foci is 4 and whose eccentricity is $\sqrt{2}$;

(*iv*) whose conjugate axis is 3 and the distance between whose foci is 5; ***(Type ISC 2012)***

(*v*) vertices are $(0, \pm 3)$ and foci $(0, \pm 5)$;

(*vi*) foci are $(\pm 5, 0)$ and transverse axis of length 8;

(*vii*) foci are $(0, \pm 13)$ and conjugate axis of length 24;

(*viii*) vertices are $(\pm 7, 0)$ and $e = \frac{4}{3}$;

(*ix*) foci are (6, 4) and (–4, 4) and eccentricity is 2. ***(ISC 1998)***

11. Find the equation of the hyperbola, referred to its axes as the axes of coordinates, whose conjugate axis is 5 and which passes through the point (1, –2). Find also the equation of the conjugate hyperbola.

12. Find the locus of a point such that the difference of its distances from (4, 0) and (–4, 0) is always equal to 2. Name the curve. ***(ISC 2001)***

ANSWERS

1. $8xy + 4x + 4y = 7$ **2.** $x^2 - 4xy + y^2 + 4x - 4 = 0$

3. $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$; Ellipse if $e < 1$; Hyperbola if $e > 1$.

4. $4y^2 - 5x^2 = 19$ **5.** $\sqrt{\frac{5}{3}}$ **6.** See Solved Ex. 2. 2; $\left(\pm\frac{4}{\sqrt{3}}, 0\right)$

7. (*i*) Foci $(\pm 10, 0)$, Vertices $(\pm 6, 0)$, $e = \frac{5}{3}$, Latus rectum $= \frac{64}{3}$.

(*ii*) Foci $(0, \pm 6)$, Vertices $(0, \pm 3)$, $e = 2$, Latus rectum = 18

(*iii*) Foci $(0, \pm \sqrt{13})$, Vertices $(0, \pm 2)$, $e = \frac{\sqrt{13}}{2}$, Latus rectum = 9

(*iv*) Foci $(0, \pm \sqrt{65})$, Vertices $(0, \pm 4)$, $e = \frac{\sqrt{65}}{4}$, Latus rectum $= \frac{49}{2}$

8. Axes : 4, 2; Foci $(\pm \sqrt{5}, 0)$, $e = \frac{\sqrt{5}}{2}$, Latus rectum = 1;

Equations of the directrices : $\sqrt{5}x \pm 4 = 0$

9. (*i*) $e = \frac{\sqrt{13}}{3}$, Foci $(1 \mp \sqrt{13}, 0)$, equations of the directrices : $x = 1 \mp \frac{9}{\sqrt{13}}$

(*ii*) $e = \frac{5}{4}$, Foci (9, 2), (–11, 2), Directrices : $x = \pm\frac{32}{5}$

10. (*i*) $9x^2 - 4y^2 - 9 = 10$ (*ii*) $45x^2 - 36y^2 - 80 = 0$ (*iii*) $x^2 - y^2 - 2 = 0$

(*iv*) $9x^2 - 16y^2 - 36 = 0$ (*v*) See Solved Ex. 5. $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (*vi*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(*vii*) $144y^2 - 25x^2 = 3600$ (*viii*) $7x^2 - 9y^2 = 343$

(*ix*) See Solved Ex. 8. $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

11. $41x^2 - 4y^2 - 25 = 0$; Conjugate hyperbola : $41x^2 - 4y^2 + 25 = 0$ **12.** $15x^2 - y^2 = 15$, Hyperbola.

25.10. General Conic

General second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $h^2 = ab$, ellipse if $h^2 < ab$, and hyperbola if $h^2 > ab$, provided $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$.

Proof. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... (*i*)

Let the focus S be (α, β), and the equation of the corresponding directrix be $lx + my + n = 0$ $(l^2 + m^2 \neq 0)$, and eccentricity be 'e'. Let P be any point (x, y) on the conic. Then by Focus-Directrix property the equation of the conic is $SP^2 = e^2 . PM^2$, where M is the foot of the perpendicular from $P(x, y)$ on the directrix.

$$\therefore \quad \sqrt{(x-\alpha)^2 + (y-\beta)^2} = e.\frac{lx+my+n}{\sqrt{l^2+m^2}}, \qquad \left(l^2 + m^2 \neq 0\right)$$

$$\Rightarrow \quad (x-\alpha)^2+(y-\beta)^2 = e^2.\frac{(lx+my+n)^2}{l^2+m^2}$$

$$\Rightarrow \quad (l^2+m^2)\left[(x-\alpha)^2+(y-\beta)^2\right]-e^2(lx+my+n)^2=0 \qquad \ldots (ii)$$

Comparing the coefficients of x^2, y^2, and xy in (i) and (ii), we get

$$a = l^2+m^2-e^2l^2,\ b = l^2+m^2-e^2m^2,\ h = -lme^2 \qquad \ldots (iii)$$

Now, $h^2-ab = l^2m^2e^4-(l^2+m^2-e^2l^2)(l^2+m^2-e^2m^2)$

$$= l^2m^2e^4-\left[\left(l^2+m^2\right)^2-e^2m^2\left(l^2+m^2\right)^2-e^2l^2\left(l^2+m^2\right)+e^4l^2m^2\right]$$

$$= -(l^2+m^2)(l^2+m^2-e^2m^2-e^2l^2)$$

$$= -\left(l^2+m^2\right)\left[(l^2+m^2)-e^2(m^2+l^2)\right]$$

$$= -(l^2+m^2).(l^2+m^2)(1-e^2) = (e^2-1)(l^2+m^2)^2$$

$$\Rightarrow \quad h^2-ab = (e^2-1)(l^2+m^2)^2 \qquad \ldots (iv)$$

From (iv), we get

$e = 1 \Rightarrow h^2-ab = 0$ and (i) represents a parabola;

$e < 1 \Rightarrow h^2-ab < 0$ and (i) represents an ellipse;

$e > 1 \Rightarrow h^2-ab > 0$ and (i) represents a hyperbola.

Note : If $\Delta = 0$ then the general second degree equation represents a pair of straight lines.

Illustrative Examples

Ex. 12. Find the value of k, if the equation $8x^2 - 16xy + ky^2 - 22x + 34y = 12$ represents an ellipse. ***(ISC 2011)***

Sol. The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents an ellipse if $h^2 < ab$. Comparing the given equation with the general equation, we have $a = 8$, $b = k$, $h = -8$.

The given equation will represent an allipse if $h^2 < ab$.

$\Rightarrow \quad 64 < 8k \Rightarrow 8k > 64 \Rightarrow k > 8$.

1. The equation $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ represents

(a) Parabola (b) Ellipse (c) Hyperbola (d) Two straight lines

(Ans : At $\Delta \neq 0$, $h^2 > ab$ so it represents a hyperbola)

2. Curve $16x^2 + 8xy + y^2 - 74x - 78y + 212 = 0$ represents

(a) Parabola (b) Hyperbola (c) Ellipse (d) None of these

(Ans : $\Delta \neq 0$, $h^2 = ab$ so it represents a parabola)

CONDITIONS OF TANGENCY TO THE CONICS

25.11. To find the condition that the line y = mx + c may touch the parabola $y^2 = 4ax$

The points of intersection of the line $y = mx + c$ and the parabola $y^2 = 4ax$ are given by

$$(mx + c)^2 = 4ax$$

$$\Rightarrow \quad m^2x^2 + (2mc - 4a)x + c^2 = 0 \qquad \ldots (i)$$

Since (i) is a quadratic equation, we see that every straight line meets a parabola in two points, which may be a real, coincident, or imaginary.

From algebra we know that (i) has one root infinite if the coefficient of x is zero, *i.e.,* if $m = 0$. Hence, every straight line parallel to the axis of a parabola, *i.e.*, $y = c$ meets the curve in one point at a finite distance and in another at an infinite distance from the vertex.

If the line be a tangent, *i.e.*, if it cuts the parabola in two coincident points, the roots of (i) must be equal. The condition for this is that the discriminant is zero, *i.e.*, $(2mc - 4a)^2 - 4m^2c^2 = 0$

$\Rightarrow \quad 4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 = 0$

$\Rightarrow \quad -16mca + 16a^2 = 0$

$\Rightarrow \quad c = \dfrac{a}{m}.$

Hence, whatever m may be, the line

$$\boldsymbol{y = mx + \frac{a}{m}} \qquad \ldots (ii)$$

will touch the parabola.

Substituting $c = \dfrac{a}{m}$ in (i), we get

$$m^2x^2 + (2m \times \frac{a}{m} - 4a)x + \frac{a^2}{m^2} = 0$$

$$\Rightarrow \quad m^2x^2 - 2ax + \left(\frac{a}{m}\right)^2 = 0 \Rightarrow \left(mx - \frac{a}{m}\right)^2 = 0 \Rightarrow mx - \frac{a}{m} = 0 \Rightarrow x = \frac{a}{m^2}$$

Substituting in $y = mx + c$, we get $y = m \times \dfrac{a}{m^2} + \dfrac{a}{m} \Rightarrow y = \dfrac{a}{m} + \dfrac{a}{m} = \dfrac{2a}{m}$

Thus, the point of contact is $\boldsymbol{\left(\frac{a}{m^2}, \frac{2a}{m}\right)}$.

Cor. Two tangents to the parabola $y^2 = 4ax$, which intersect on the x-axis can be written as $y = mx + \dfrac{a}{m}$ and $y = mx - \dfrac{a}{m}$, so that the point of intersection lies on $y = 0$, which is obtained by adding the equations.

Note. The equation $y = mx + \dfrac{a}{m}$ and the point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ should be committed to memory.

Ex. 13. Find the equation of the tangent to the parabola $y^2 = 12x$ which makes an angle of 60° with the x-axis.

Sol. The equation of the tangent is $y = mx + \dfrac{a}{m}$ ... (i)

Here $m = \tan 60° = \sqrt{3}$, and $a = \dfrac{12}{4} = 3$

Substituting in (i), we get

$$y = \sqrt{3}x + \frac{3}{\sqrt{3}} = \sqrt{3}x + \sqrt{3}$$

$\Rightarrow \quad y = \sqrt{3}(x + 1)$, which is the required equation.

Ex. 14. Prove that the locus of the foot of the perpendicular drawn from the focus of the parabola $y^2 = 4ax$ upon any tangent to it is the tangent at the vertex.

Sol. Let any tangent to the parabola $y^2 = 4ax$ be

$$y = mx + \frac{a}{m} \qquad \ldots (i)$$

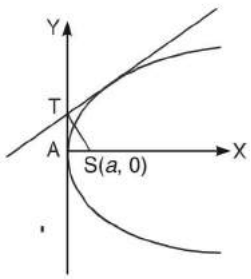


Fig. 25.13

The equation of the perpendicular ST from $S\,(a,\,0)$ to (i) is

$$y - 0 = \frac{-1}{m}(x - a)$$

or $$my = -x + a \qquad \ldots (ii)$$

To find the locus of T, the foot of the perpendicular, we have to eliminate m from (i) and (ii).

(i) and (ii) can be written down as

$$my = m^2x + a$$

and $$my = -x + a$$

Subtracting, we have

$$0 = m^2x + x \Rightarrow 0 = (m^2 + 1)x \Rightarrow x = 0,$$

which is the tangent at the vertex.

25.12. To find the condition that the line y = mx + c may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The points of intersection of the line and the ellipse are given by the equation

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1 \quad \Rightarrow \quad b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$\Rightarrow \quad x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0 \qquad \ldots (i)$$

This is a quadratic in x. It gives two values of x.

Substituting these values of x in $y = mx + c$ we get the corresponding two values of y.

The roots of (i) will be equal if the discriminant is zero, *i.e.*, if

$$4m^2c^2a^4 - 4(b^2 + a^2m^2)\,(c^2 - b^2)a^2 = 0$$

$$\Rightarrow \quad m^2c^2a^2 - (b^2 + a^2m^2)\,(c^2 - b^2) = 0$$

$$\Rightarrow \quad m^2c^2a^2 - b^2c^2 - a^2m^2c^2 + b^4 + a^2m^2b^2 = 0$$

$$\Rightarrow \quad -b^2c^2 + b^4 + a^2m^2b^2 = 0 \quad \Rightarrow \quad -c^2 + b^2 + a^2m^2 = 0$$

$$\Rightarrow \quad c^2 = a^2m^2 + b^2$$

If two values of x are equal, the two values of y must also be equal. Therefore the two points of intersection will be coincident if

$$c^2 = a^2m^2 + b^2, \qquad \text{or} \quad c = \pm\sqrt{a^2m^2 + b^2} \qquad \ldots (ii)$$

Substituting this value of c in $y = mx + c$, we get

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

which are the two tangents to the ellipse for every value of m. This means that there are two tangents to the ellipse parallel to any given straight line. These two parallel tangents are equidistant from the centre of the ellipse.

Point of contact

Substituting the value of c obtained in (ii) in (i), we get

$$x^2(a^2m^2 + b^2) \pm 2x \,.\, \sqrt{a^2m^2 + b^2}\,.\, ma^2 + a^2.\, a^2m^2 = 0$$

$$\Rightarrow \quad \left\{x\sqrt{a^2m^2+b^2} \pm a^2m\right\}^2 = 0 \Rightarrow x = \mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}$$

Substituting in $y = mx + c$, we have

$$y = m.\left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}\right) \pm \sqrt{a^2m^2+b^2}$$

$$= \frac{a^2m^2-(a^2m^2+b^2)}{\sqrt{a^2m^2+b^2}} = \frac{-b^2}{\sqrt{a^2m^2+b^2}}$$

$\therefore$ Point of contact is $\left(\pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Ex. 15. Find the equation of the tangent to the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9} = 1$ which makes an angle of 30° with the x-axis.

Sol. The equation of a tangent with gradient m to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ is

$$y = mx + \sqrt{a^2m^2+b^2} \qquad ...(i)$$

Here, we have $m = \tan 30° = \dfrac{1}{\sqrt{3}}$, $a^2 = 16$, $b^2 = 9$, putting these values in (i), we get

$$y = \frac{1}{\sqrt{3}}x + \sqrt{\frac{16}{3}+9} \Rightarrow \sqrt{3}y = x + \sqrt{43}$$

which is the required tangent.

25.13. To find the condition that the line $y = mx + c$ may touch the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$$y = mx + c \qquad ...(i) \qquad\qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(ii)$$

Solving them simultaneously, we get

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow \quad (b^2 - a^2m^2)\,x^2 - 2mc\,a^2\,x - a^2\,(c^2 + b^2) = 0 \qquad ...(iii)$$

This quadratic equation in x has two roots, real, coincident, or imaginary.

Hence, there are two values of x and correspondingly two values of y from (i).

The roots of (iii) will be equal if

$$(-2mca^2)^2 - 4\,(b^2 - a^2m^2)\,(-c^2 - b^2)\,a^2 = 0$$

$$\Rightarrow \quad 4m^2c^2a^4 + 4a^2\,(b^2 - a^2m^2)\,(c^2 + b^2) = 0$$

$$\Rightarrow \quad a^2m^2c^2 + b^2c^2 + b^4 - a^2m^2c^2 - a^2b^2m^2 = 0$$

$$\Rightarrow \quad c^2 + b^2 - a^2m^2 = 0 \Rightarrow c = \pm\sqrt{a^2m^2 - b^2}$$

which is the required condition that the line may touch the hyperbola.

Substituting the value of 'c' in (i), we get

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

which are the two tangents to the hyperbola for every value of m. This means that there are two tangents to the hyperbola parallel to any given straight line. These two tangents are equidistant from the centre of the hyperbola.

Ex. 16. Find the equations of the tangents to the hyperbola $4x^2 - 9y^2 = 36$, which are parallel to the line $5x - 3y = 2$.

Sol. The hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Slope of the line $5x - 3y = 2$ is $\frac{5}{3}$.

So slope of the tangent parallel to $5x - 3y = 2$ is $\frac{5}{3}$.

Thus, we have $a^2 = 9, b^2 = 4, m = \frac{5}{3}$.

$\therefore$ The equations of tangents are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad y = \frac{5}{3}x \pm \sqrt{9 \times \frac{25}{9} - 4} \Rightarrow y = \frac{5}{3}x \pm \sqrt{21},$$

which are the required tangents.

Ex. 17. Prove that the straight line $5x + 12y = 9$ touches the hyperbola $x^2 - 9y^2 = 9$ and find the point of contact.

Sol.
$$x^2 - 9y^2 = 9 \qquad ...(i)$$

$$5x + 12y = 9 \Rightarrow y = \frac{9 - 5x}{12} \qquad ...(ii)$$

Substituting $y = \frac{9 - 5x}{12}$ in (i), we get

$$x^2 - 9 \times \left(\frac{9 - 5x}{12}\right)^2 = 9 \Rightarrow x^2 - 9 \times \frac{81 - 90x + 25x^2}{144} = 9$$

$$\Rightarrow \quad 16x^2 - 81 + 90x - 25x^2 = 144$$

$$\Rightarrow \quad -9x^2 + 90x - 225 = 0$$

$$\Rightarrow \quad x^2 - 10x + 25 = 0 \Rightarrow (x - 5)^2 = 0 \Rightarrow x = 5, 5$$

Since the roots are equal, the line touches the hyperbola.

Substituting $x = 5$ in (ii), we get $y = \frac{9 - 25}{12} = \frac{-16}{12} = \frac{-4}{3}$.

$\therefore$ The point of contact is $\left(5, \frac{-4}{3}\right)$

EXERCISE 25 (*b*)

1. Find the tangent to the parabola $y^2 = 16x$, making an angle of 45° with the x-axis.

2. A tangent to the parabola $y^2 = 16x$ makes an angle of 60° with the x-axis. Find its point of contact.

3. (*i*) Find the equations of the tangents to the parabola $y^2 = 6x$ which pass through the point $\left(\frac{3}{2}, 5\right)$.

(*ii*) Find the equations of the tangents to the parabola $y^2 + 12x = 0$ from the point (3, 8) ***(ISC 2009)***

4. Show that the line $12y - 20x - 9 = 0$ touches the parabola $y^2 = 5x$.

5. Show that the line $x + y = 1$ touches the parabola $y = x - x^2$.

6. Show that the line $x + ny + an^2 = 0$ touches the parabola $y^2 = 4ax$ and find the point of contact.

7. Find the tangents to the ellipse $x^2 + 9y^2 = 3$, which are (*i*) parallel (*ii*) perpendicular to the line $3x + 4y = 9$.

8. Find the equations of the tangents to the ellipse $\frac{x^2}{2} + \frac{y^2}{7} = 1$ that make an angle of 45° with the x-axis.

9. Find the equation of the tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, which make equal intercepts on the axes.

10. Find the value of 'c' so that $2x - y + c = 0$ may touch the ellipse $x^2 + 2y^2 = 2$.

11. Show that the line $lx + my = 1$ will touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2l^2 + b^2m^2 = 1$.

12. Show that the following lines are tangents to the given hyperbolas and determine the points of contact.

(*i*) $x + 1 = 0, 4x^2 - 3y^2 = 4$ (*ii*) $x - 2y + 1 = 0, x^2 - 6y^2 = 3$

13. Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 6$, which are parallel to the line $x + y - 2 = 0$.

14. The tangents from P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are mutually perpendicular, show that the locus of P is the circle $x^2 + y^2 = a^2 - b^2$.

15. Show that the straight line $x + y = 1$ touches the hyperbola $2x^2 - 3y^2 = 6$. Also find the coordinates of the point of contact.

16. Find the equations of the tangents to the hyperbola $4x^2 - 9y^2 = 144$, which are perpendicular to the line $6x + 5y = 21$.

ANSWERS

1. $y = x + 4$ **2.** $\left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$

3. (*i*) $6y = 2x + 27, 2y = 6x + 1$ (*ii*) $3x - y = 1, x + 3y = 27$

6. $(an^2, -2an)$ **7.** $3x + 4y \pm \sqrt{\frac{97}{3}} = 0$ and $3y - 4x \pm \sqrt{51} = 0$

8. $y = x \pm 3$ **9.** $y - x \pm 5 = 0$

10. $c = \pm 3$ **12.** (*i*) $(-1, 0)$ (*ii*) $(-3, -1)$

13. $x + y \pm 1 = 0$ **15.** $(3, -2)$ **16.** $5x - 6y \pm 18 = 0$

HINTS TO SELECTED QUESTIONS

9. Slope (m) of tangent making equal intercepts on the axes = ± 1.

14. The equation of the tangents are $y = mx \pm \sqrt{a^2 m^2 - b^2} \Rightarrow (y - mx)^2 = a^2 m^2 - b^2$

$\Rightarrow m^2x^2 - 2mxy + (b^2 - a^2 m^2 + y^2) = 0$

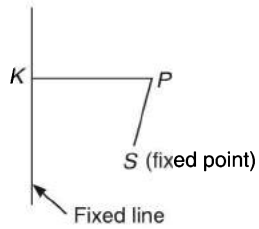
This gives two roots m_1 and m_2

Apply the condition $m_1 m_2 = -1$, etc.

CHAPTER SUMMARY

1. Definition

A hyperbola is the locus of a point which moves such that its distance from a fixed point bears a constant ratio to its distance from a fixed line, the constant being greater than unity, *i.e.*,



$$\left|\frac{PS}{PK}\right| = \text{Constant } (e),\ e > 1.$$

The fixed point is called the focus, the fixed line the directrix and the constant the eccentricity of the hyperbola.

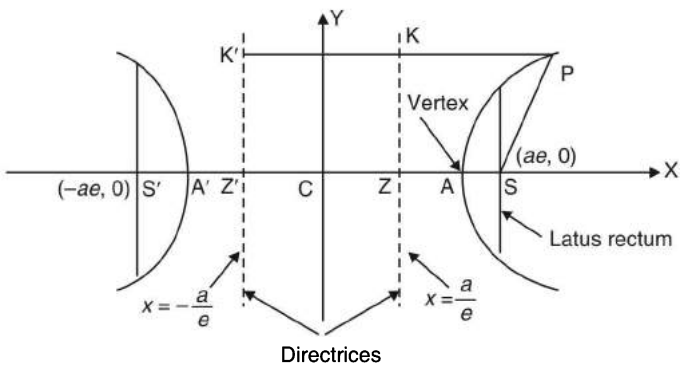
Alternative definition

A hyperbola is the locus of a point which moves such that the difference of its distances from the two foci, *i.e.*, its focal distances is constant. (Art. 25.07), *i.e.*, $|S'P| - |SP| = AA' = 2a$.

2. The standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

3. As shown in the figure, the hyperbola has two foci (S and S'), two directries, two vertices (A and A'). The mid-point C of AA' is called centre of the hyperbola.

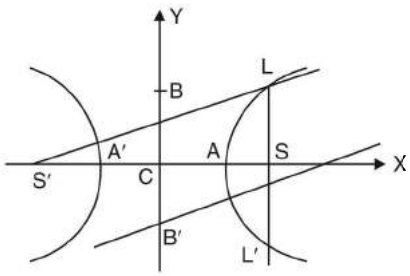
- A line joining any two points of a hyperbola is called its chord. A chord of the hyperbola through its focus is called a focal chord.
- The line segment AA' is called the **transverse axis.** Its length is $AA' = 2a$.

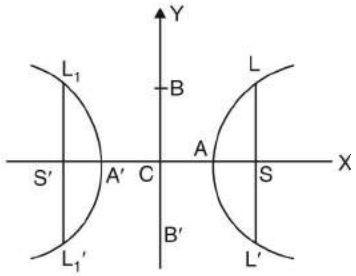


Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

If two points B and B' are taken on the y-axis such that $CB = CB' = b$, then the line segment BB' is called the conjugate axis of the hyperbola. Its length is $BB' = 2b$. The line segment LSL' through the focus and perpendicular to the transverse axis is called the **latus rectum**. There are two latera recta.

Length of each latus rectum = $\frac{2b^2}{a}$, the end points are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, \frac{-b^2}{a}\right)$, $L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.





$AA' = 2a \rightarrow$ Transverse axis
$BB' = 2b \rightarrow$ Conjugate axis
$LSL' \perp$ Transverse axis $\rightarrow$ Latus rectum

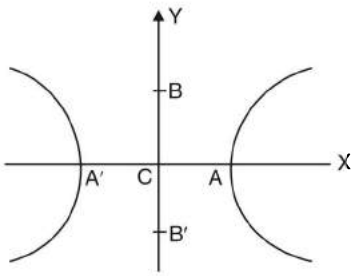
- The eccentricity 'e' is given by the relation

$$b^2 = a^2(e^2 - 1) \Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}}$$

i.e., $e = \sqrt{1 + \dfrac{(\text{length of conjugate axis})^2}{(\text{length of transverse axis})^2}}$

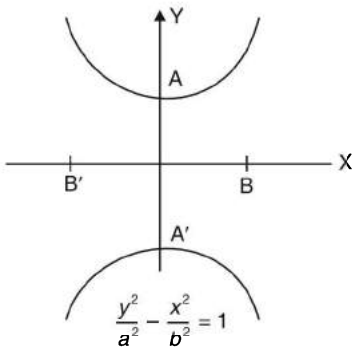
4. The equation $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ is another standard form of the hyperbola whose transverse axis is along the y-axis. Its foci are $(0, \pm ae)$ and the directrices are $y = \pm \dfrac{a}{e}$.

- The equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ in which the coefficient of y^2 is negative represents a hyperbola whose transverse axis is horizontal, *i.e.*, along the x-axis while the equation $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ in which the coefficient of x^2 is negative represents a hyperbola whose transverse axis is vertical *i.e.*, along the y-axis.



$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Coeffi. of y^2 is –ve
Transverse axis AA′ is horizontal, *i.e.*, along x-axis



$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Coeffi. of y^2 is +ve
Transverse axis AA′ is vertical, *i.e.*, along y-axis

- **Conjugate hyperbola.** The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The conjugate hyperbola of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $\dfrac{-x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *i.e.*, $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, *i.e.*, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$.

5. The equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, represents a hyperbola whose centre is (h, k) and whose axes are parallel to the axes of coordinates.

6.

	Equation of the hyperbola / Parts	*Horizontal hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	*Vertical hyperbola* $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
1.	Coordinates of the centre	$(0, 0)$	$(0, 0)$
2.	Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, \pm a)$
3.	Coordinates of foci	$(\pm ae, 0)$	$(0, \pm ae)$
4.	Length of the transverse axis	$2a$	$2a$
5.	Length of the conjugate axis	$2b$	$2b$
6.	Equations of the directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
7.	Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$	$e=\sqrt{\frac{a^2+b^2}{a^2}}$
		or $b^2=a^2(e^2-1)$	or $b^2=a^2(e^2-1)$
8.	Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
9.	Equation of the transverse axis	$y=0$	$x=0$
10.	Equation of the conjugate axis	$x=0$	$y=0$
11.	Focal distances of any point (x_1, y_1)	$ex \pm a$	$ey \pm a$

7. The general second degree equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which represents

(*i*) a parabola if $h^2 = ab$

(*ii*) an ellipse if $h^2 < ab$

(*iii*) a hyperbola if $h^2 > ab$

provided $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$.

Note. If $\Delta = 0$, then the general second degree equation represents a pair of straight lines.

8. The line $y = mx + c$ touches

(*i*) parabola $y^2 = 4ax$ if $c = \frac{a}{m}$, the equation of the tangent is $\boldsymbol{y = mx+\frac{a}{m}}$ and the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(*ii*) ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\boldsymbol{c=\pm\sqrt{a^2m^2+b^2}}$. The equation of the tangent is $y = mx \pm \sqrt{a^2m^2+b^2}$ and the point of contact is $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

(*iii*) hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c=\pm\sqrt{a^2m^2-b^2}$.

The equation of the tangent is $\boldsymbol{y=mx\pm\sqrt{a^2m^2-b^2}}$.

Rule:

Step 1. Eliminate y from the equation $y = mx + c$ of the line and the given equation of the curve and form a quadratic equation in x.

Step 2. Apply the condition that the roots of this equation be equal, *i.e.*, put discriminant $(b^2 – 4ax) = 0$ and get the value of c. This gives the condition for the given line to be tangent to the given curve.

Step 3. Substitute the value of c in $y = mx + c$ to obtain the equation of the tangent.

CHAPTER TEST

1. Find the eccentricity and the coordinate of foci of the hyperbola $25x^2 – 9y^2 = 225$. ***(ISC 2014)***

2. Find the value (s) of k so that the line $2x + y + k = 0$ may touch the hyperbola $3x^2 – y^2 = 3$ ***(ISC 2013)***

3. From the following information, find the equation of the hyperbola and the equation of the transverse axis.

Focus (–2, 1), Directrix: $2x – 3y + 1 = 0$, $e=\frac{2}{\sqrt{3}}$ ***(ISC 2013)***

4. Find the equation of the hyperbola whose eccentricity is $\sqrt{5}$ and the sum of whose semi-axes is 9. ***(ISC 2001)***

5. Find the equation of the hyperbola whose foci are (4, 1), (8, 1) and whose eccentricity is 2. ***(ISC 2005)***

6. Show that the line $y = x + \sqrt{7}$ touches the hyperbola $9x^2 – 16y^2 = 144$. ***(ISC 2010)***

7. Find the equation of the hyperbola whose foci are $(0, \pm 13)$ and the length of the conjugate axis is 20. ***(ISC 2011)***

8. Find the equation of the hyperbola whose transverse and conjugate axes are the x and y axes respectively, given that the length of conjugate axis is 5 and distance between the foci is 13. ***(ISC 2012)***

9. Find the equation to the conic whose focus is $(1, – 1)$, eccentricity is $\frac{1}{2}$ and the directrix is the line $x – y = 3$. Is the conic section an ellipse?

ANSWERS

1. $\left(e=\frac{\sqrt{34}}{3}, \text{foci } (\pm\sqrt{34}, 0)\right)$ **2.** $k = \pm 1$

3. $23x^2 + 48xy + 3y^2 + 140x – 54y + 191 = 0$; $3x + 2y + 4 = 0$ **4.** $\frac{x^2}{9}-\frac{y^2}{36}=1$

5. $3(x – 6)^2 – (y – 1)^2 = 3$ **7.** $\frac{y^2}{69}-\frac{x^2}{100}=1$ **8.** $25x^2 – 144y^2 = 900$

9. $7x^2 + 2xy + 7y^2 – 10x + 10y + 7 = 0$, yes it is an ellipse.

HINTS TO SELECTED QUESTIONS

3. Transverse axis is perp. to directrix and focus lies on it.

5. See Solved Ex. 8. **7.** $2b = 20$, $ae = 13$

CHAPTER 26

POINTS AND THEIR COORDINATES

In 3-Dimensions

26.01. Introduction

So far, our experience in analytical geometry has been in a plane, that is, in two dimensions. It is easier to analyse loci in a plane, but we live in a world of three dimensions. If we are to apply our geometric knowledge to physical problems, we must be able to extend our concepts to 3-space.

In this chapter, we shall consider the basic extension to 3-space of the ideas which we have already developed. We shall consider the analytical representations of lines and planes. One thing you might keep in mind is that the locus of a condition depends upon the space to which it is applied. We have already seen that the equation $x = a$ describes both a point on a line and a line in a plane. Here, we shall see that it also describes a plane in 3-space. In spaces of higher dimensions, it would be subject to still other interpretations. In general, analytical representations describe loci in any space which has at least as many dimensions as the analytic representation has independent variables. To describe the locus, we must first know the number of dimensions of the space in which it occurs.

26.02. A coordinate system for three-space

We begin by selecting an arbitrary point O in space and three mutually perpendicular lines $X'OX$, $Y'OY$, and $Z'OZ$ through O. The point O is called the origin of the coordinate system and the lines $X'OX$, $Y'OY$, $Z'OZ$ are called respectively the ***x* -axis,** the ***y* -axis,** and the ***z* -axis.** The positive directions of the axes are indicated by arrows in Fig. 26.01. The plane determined by the x-axis and the y-axis is called the XOY or xy-plane. The other two planes YOZ (or yz-plane) and XOZ (or xz-plane) are defined similarly. The three are called the **coordinate planes.**

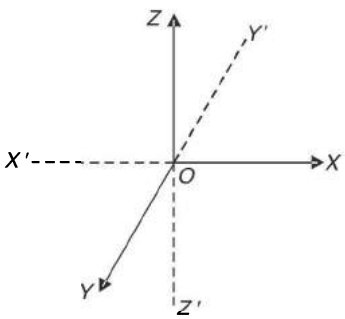


Fig. 26.01

26.03. Coordinates of a point in three-space*

Let P be any point in three-space (Fig. 26.02). Through P draw the three planes parallel to the three coordinate planes (which will automatically be also perpendicular to the corresponding axes, that is, the plane parallel to the yz-plane will be perpendicular to the x-axis, etc.) to meet the axes in A, B and C.

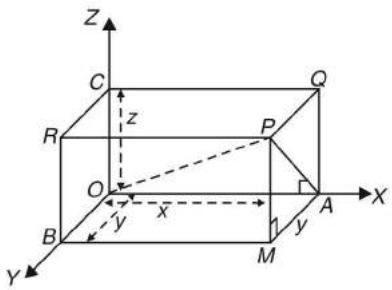


Fig. 26.02

* In future we shall write only 'space' for three-space.

If $OA = x$, $OB = y$, $OC = z$, then the three numbers x, y, z, taken with proper signs are called the coordinates of P, and are written as (x, y, z).

Thus, *the coordinates of a point in space are the distances of the origin from the points of axes in which the planes through that point parallel to the coordinate planes intersect the axes.*

Conversely, given three numbers x, y, z, we can find a point whose coordinates are (x, y, z). To do this we proceed as follows:

(*i*) Measure OA, OB, OC along the x-axis, the y-axis, and the z-axis equal to x, y, z respectively.

(*ii*) Through A, B, C draw respectively planes parallel to the coordinate planes YOZ, ZOX, XOY intersecting one another at P.

Then P is the required point whose coordinates are (x, y, z).

Note 1. The coordinate planes divide space into eight regions called, not unnaturally, **octants.** Usually one of them is numbered, and it is called the first and is one in which all the coordinates of every point are positive.

Note 2. The configuration of axes shown in Fig. 26.02 is called a **left handed system** because a 90° rotation of the positive side of the x-axis into the positive side of the y-axis will advance a left-handed screw along the positive side of the z-axis. If the locations of the x- and y-axis are interchanged, then the system is right-handed.

The following table gives the signs in the eight octants.

Table 26.01

Octant → / Coordinates ↓	$OXYZ$	$OX'YZ$	$OX'Y'Z$	$OXY'Z$	$OXYZ'$	$OX'YZ'$	$OX'Y'Z'$	$OXY'Z'$
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

26.04. Another explanation of the coordinates

In Art. 26.03, we learnt that in order to obtain the coordinates of a point P we have to draw three planes through P respectively parallel to the three coordinate planes. The three planes through P and the three coordinate planes form a rectangular parallelopiped (Fig. 26.02). Then,

(*i*) The x-coordinate of $P = OA = CQ = RP$

= the length of perpendicular P on the yz-plane

The y-coordinate of $P = OB = AM = QP$

= the length of the perpendicular from P on the zx-plane

The z-coordinate of $P = OC = AQ = MP$

= the length of the perpendicular from P on the xy-plane.

Thus, *the* **coordinates x, y, z of any point P** *are the perpendicular distances of P from the three rectangular coordinate planes yz, zx and xy respectively.*

(*ii*) Since OA is normal to the plane $PMAQ$, therefore, $PA \perp OA$ *at A.*

Similarly, $PB \perp OB$ at B and $PC \perp OC$ at C.

Thus, *the coordinates x,y, z of any point P are also the distances from the origin O of the feet A, B, C respectively of the perpendiculars from the point P to the coordinate axes X'OX, Y'OY, Z'OZ.*

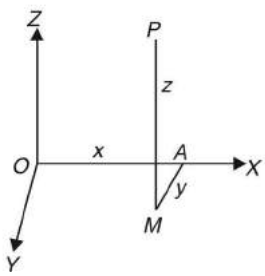


Fig. 26.03

Cor. The perpendicular distance of $P(x, y, z)$ from the x-axis

$$= PA = \sqrt{MA^2 + MP^2} = \sqrt{y^2 + z^2}.$$

Similarly, the perpendicular distances of P from the y-axis and z-axis respectively are $\sqrt{x^2 + z^2}$ and $\sqrt{x^2 + y^2}$.

(*iii*) $MP = AQ = OC = z$-coordinate of P; $AM = OB = y$-coordinate of P and $OA = x$-coordinate of P.

Thus, *if PM is drawn perpendicular to the xy-plane to meet it at M, and MA is drawn perpendicular to OX to meet it at A, then OA, AM, MP equal to x, y, z respectively are called the coordinates of P.*

Ex. 1. Locate the points

(*i*) (2, 3, 4) **(*ii*) (–2, –2, 3)** **(*iii*) (2, –2, –3)**

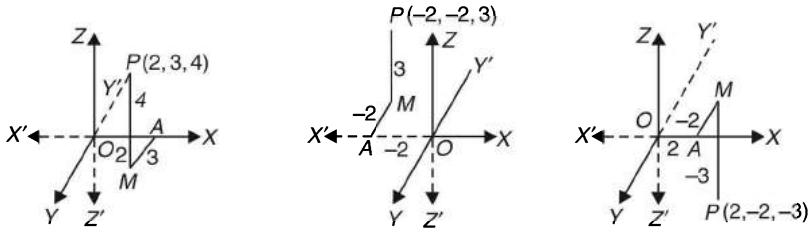


Fig. 26.04

(*i*) From the origin O move 2 units to A on OX. From A move 3 units to M parallel to OY, and then from M move 4 units to P parallel to OZ. P is the required point with coordinates (2, 3, 4).

(*ii*) From the origin O move 2 units to A on OX'. From A move 2 units to M parallel to OY', and then from M move 3 units to P parallel to OZ. P is the required point with coordinates (–2, –2, 3).

(*iii*) From the origin O move 2 units to A on OX. From A move 2 units to M parallel to OY' and then from M move 3 units to P parallel to OZ'. P is the required point with coordinates (2, –2, –3).

26.05. Distance between the two points

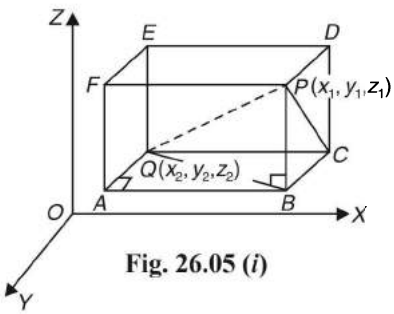


Fig. 26.05 (*i*)

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two given points. Through P and Q draw planes parallel to the coordinate planes to form a rectangular parallelopiped whose one diagonal is PQ.

AB is the distance between the planes drawn through the points P and Q parallel to yz-plane and is, therefore, equal to the difference between their x-coordinates.

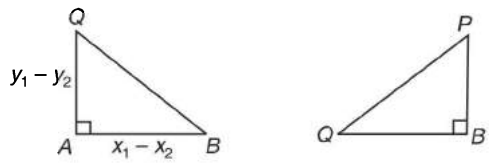


Fig. 26.05 (*ii*)

$\therefore$ $AB = x_1 - x_2$

Similarly, $AQ = y_1 - y_2$ and $BP = z_1 - z_2$

Now, since $\angle BAQ$ is a right angle, therefore

$BQ^2 = AB^2 + AQ^2$ (Pythagoras' Theorem)

Also, since $\angle QBP$ is a right angle, therefore

$PQ^2 = BQ^2 + BP^2$ [$\because$ BP is normal to the plane $ABCQ$ $\therefore$ $BP \perp BQ$]

$= (AB^2 + AQ^2) + BP^2 = AB^2 + AQ^2 + BP^2$

or $PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

or $\boldsymbol{PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$

which is the required distance formula.

Cor. Distance of *P* from the origin

When Q coincides with the origin, then $x_2 = 0 = y_2 = z_2$. Then

$OP = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$ or $\boldsymbol{OP = \sqrt{(x_1^2 + y_1^2 + z_1^2)}}$

Aid to Memory:

Distance between two points = $\sqrt{(\text{diff. of the } x's)^2 + (\text{diff. of the } y's)^2 + (\text{diff. of the } z's)^2}$

Note. The student must have noticed that the formula for distance in space is just an extension of the formula for distance in a plane.

Ex. 2. Prove by using the distance formula that the points *A* (1, 2, 3), *B* (–1, –1, –1) and *C* (3, 5, 7) are collinear.

Sol.

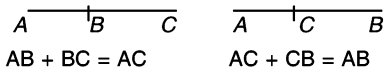


Fig. 26.06

Method : Step 1. *Find the three distances AB, BC and CA.*

Step 2. *If the sum of any two distances is equal to the third, the three given points are collinear.*

Here, $AB = \sqrt{(1+1)^2 + (2+1)^2 + (3+1)^2} = \sqrt{29}$

$BC = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{116} = 2\sqrt{29}$

$CA = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{29}$

Since $CA + AB = BC$, therefore the points A, B, C are collinear.

Ex. 3. Show that the points *A* (0, 1, 2), *B* (2, –1, 3), and *C* (1, –3, 1) are the vertices of an isosceles right-angled triangle.

Sol. $AB^2 = (2 - 0)^2 + (-1 - 1)^2 + (3 - 2)^2 = 9$

$BC^2 = (2 - 1)^2 + (-1 + 3)^2 + (3 - 1)^2 = 9$

$CA^2 = (1 - 0)^2 + (-3 - 1)^2 + (1 - 2)^2 = 18$

Since $AB^2 + BC^2 = CA^2$, and $AB = BC$, therefore ABC is an isosceles right-angled triangle, right-angled at B.

Ex. 4. Find the coordinates of the point equidistant from the four points *A* (0, 0, 0), *B* (*a*, 0, 0), *C* (0, *b*, 0) and *D* (0, 0, *c*).

Sol. Let the coordinates of the required point P be (x, y, z).

Since $PA = PB$, therefore

$x^2 + y^2 + z^2 = (x - a)^2 + y^2 + z^2$ or $a^2 - 2ax = 0$ or $x = \dfrac{a^2}{2a} = \dfrac{a}{2}$

Similarly, since $PA = PC$, therefore $x^2 + y^2 + z^2 = x^2 + (y - b)^2 + z^2$ or $y = \dfrac{b^2}{2b} = \dfrac{b}{2}$

Since $PA = PD$, therefore, $z = \dfrac{c}{2}$.

Hence, $\left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2}\right)$ is the required point.

Ex. 5. If A and B be the points (3, 4, 5) and (–1, 3, –7) respectively, find the locus of P such that $PA^2 + PB^2 = k^2$.

Sol. Let the coordinates of P be (x, y, z). Then since $PA^2 + PB^2 = k^2$,

Therefore, $\{(x-3)^2 + (y-4)^2 + (z-5)^2\} + \{(x+1)^2 + (y-3)^2 + (z+7)^2\} = k^2$

On simplifying, we get $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 - k^2 = 0$,

which is the required locus.

Ex. 6. What loci do the following equations represent in space?

(*i*) $z = 0$ (*ii*) $y = 0, z = 0$ (*iii*) $z = a, y = b$

Sol. (*i*) Since on the xy-plane the z coordinate of every point is zero, therefore $z = 0$ is the equation of the xy-plane.

(*ii*) Since $y = 0$ is the equation of the xz-plane and $z = 0$ is the equation of the xy-plane, therefore $y = 0$ and $z = 0$ both together represent the x-axis which is the common line of intersection of the planes $y = 0$ and $z = 0$.

(*iii*) Just as $z = 0$ is the equation of the xy-plane, similarly $z = a$ is the equation of the plane parallel to the xy-plane and at a distance a units from it. In the same way, $y = b$ is the equation of the plane parallel to the xz-plane and at a distance b units from it. Thus, the common line of intersection of these two planes is represented by $z = a, y = b$.

Note. It should be noted that in space, a plane is represented by a single equation whereas a line is represented by two equations.

EXERCISE 26 (*a*)

1. Find the distance from the origin to each of the points:

(*i*) (2, 2, 3) (*ii*) (4, –1, 2) (*iii*) (0, 4, –4) (*iv*) (–4, –3, –2)

2. Find the distance between each of the following pairs of points:

(*i*) (2, 5, 3) and (–3, 2, 1) ; (*ii*) (0, 3, 0) and (6, 0, 2); (*iii*) (–4, –2, 0) and (3, 3, 5).

3. Show that the triangle with vertices (6, 10, 10), (1, 0, – 5), (6, – 10, 0) is a right-angled triangle, and find its area.

4. Show that the triangle with vertices A (3, 5, –4), B (–1, 1, 2), C (– 5, – 5, – 2) is isosceles.

5. Show that (4, 2, 4), (10, 2, –2), and (2, 0, – 4) are the vertices of an equilateral triangle.

6. Show that the points (1, –1, 3), (2, – 4, 5) and (5, –13, 11) are collinear.

7. Derive the equation of the locus of a point equidistant from the points (1, –2, 3) and (–3, 4, 2).

8. Derive the equation of the locus of a point twice as far from (–2, 3, 4) as from (3, –1, – 2).

9. Find the equation of the locus of a point whose distance from the y-axis is equal to its distance from (2, 1, –1).

10. Find the equation of the locus of a point whose distance from the xy-plane is equal to its distance from the point (–1, 2, –3).

11. A point moves so that the difference of the squares of its distances from the x-axis and the y-axis is constant. Find the equation of its locus.

12. Find the equation of the locus of a point whose distance from the z-axis is equal to its distance from the xy-plane.

ANSWERS

1. (*i*) $\sqrt{17}$, (*ii*) $\sqrt{21}$, (*iii*) $4\sqrt{2}$, (*iv*) $\sqrt{29}$ **2.** (*i*) $\sqrt{38}$ (*ii*) 7 (*iii*) $3\sqrt{11}$

3. $25\sqrt{21}$ sq units. **7.** $8x - 12y + 2z + 15 = 0$

8. $3x^2 + 3y^2 + 3z^2 - 28x + 14y + 24z + 27 = 0$ **9.** $y^2 - 2y - 4x + 2z + 6 = 0$

10. $x^2 + y^2 + 2x - 4y + 6z + 14 = 0$ **11.** $y^2 - x^2 = a$

12. $x^2 + y^2 - z^2 = 0$

26.06. Division or section formulae

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Let the point $R(x, y, z)$ divide the line joining P and Q in the ratio $m_1 : m_2$ so that

$$\frac{PR}{RQ} = \frac{m_1}{m_2}$$

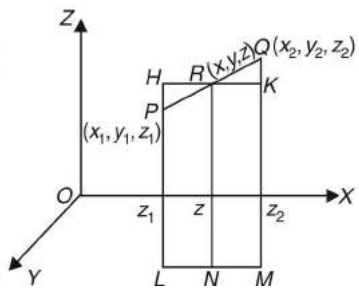


Fig. 26.07

We have to find expressions for x, y and z.

Draw PL, QM, RN perpendiculars to the xy-plane. The lines, being perpendicular to the xy-plane, are parallel to one another. Moreover, since they are cut by the line PRQ, therefore, they are coplanar. Points L, M, N lie on a straight line which is the intersection of this plane with the xy-plane. Through R draw the line HK parallel to LM to meet LP produced at H and MQ at K.

Since the triangles HRP and KRQ are similar, therefore

$$\frac{m_1}{m_2} = \frac{PR}{RQ} = \frac{HP}{QK} = \frac{HL - PL}{MQ - KM} = \frac{RN - PL}{MQ - RN} = \frac{z - z_1}{z_2 - z}$$

or $$m_1 (z_2 - z) = m_2 (z - z_1) \text{ or } z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

Similarly, by drawing perpendiculars to the xz-plane and yz-plane, we obtain

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \text{ and } x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

The point R divides PQ internally or externally according as the ratio $m_1 : m_2$ is positive or negative. Thus, *the coordinates of the point which divides the join of the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *in the ratio* $m_1 : m_2$ *are*

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \right) \quad \text{... (A)}$$

Aid to memory:

1. Multiply m_1 by the x-coordinate of the point remote from m_1 and m_2 by the x-coordinate of the point remote from m_2 as shown by the arrows.

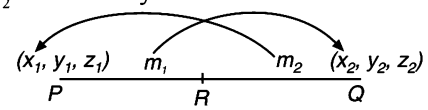


Fig. 26.08

2. Add these products and divide the sum by $m_1 + m_2$.

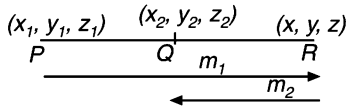
Cor. 1. Coordinates of the mid-point of *P* and *Q*. In case R is the mid-point of PQ, we have

$$m_1 : m_2 = 1 : 1,$$

and therefore the coordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right). \qquad \text{... (B)}$$

Cor. 2. External Division. If m_1 and m_2 are both positive, the division is internal, *i.e.*, the



$$\left[\frac{PR}{RQ} = \frac{m_1}{-m_2}, m_1 \text{ and } m_2 \text{ being of opposite sign}\right]$$

Fig. 26.09

point R lies on the segment PQ. However, if m_1 and m_2 are of opposite sign, the point of division R lies on the line PQ but outside the segment PQ, *i.e.*, the division is external. In this case the coordinates of R are obtained by changing the sign of one of the ratios, *i.e.*, by putting m_2 for (say) in the section formula for internal division. Thus, the coordinates of R are,

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}.\right)$$

Cor. 3. Coordinates of any point on the join of *P* and *Q*. Changing $\frac{m_1}{m_2}$ to k, we see that the coordinates of the point R which divides PQ in the ratio $k : 1$ are

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right), \qquad \text{....(C)}$$

provided $k \neq -1$.

To every value of k, except $k = -1$, there corresponds a point R on the line PQ and to every point R on the line PQ corresponds some value of k, namely $\frac{PR}{RQ}$. This idea is sometimes expressed by saying that (C) are the general coordinates of any point of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Ex. 7. Find the coordinates of a point which divides internally the points (1, 3, 7), (6, 3, 2) in the ratio 2 : 3.

Sol. Let the coordinates of the required point be (x, y, z).

$$x = \frac{2\times 6 + 3\times 1}{2+3} = \frac{12+3}{5} = 3,\ y = \frac{2\times 3 + 3\times 3}{2+3} = \frac{6+9}{5} = 3,$$

$$z = \frac{2\times 2 + 3\times 7}{2+3} = \frac{4+21}{5} = 5$$

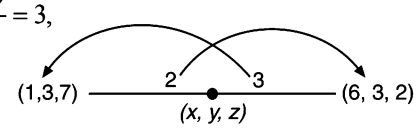


Fig. 26.10

$\therefore$ Reqd. point is (3, 3, 5).

Ex. 8. Find the ratio in which the *xy*-plane divides the join of (–3, 4, –8) and (5, – 6, 4). Also obtain the point of intersection of this line with the plane.

Sol. We know that the equation of the xy-plane is $z = 0$. Let it divide the join of (–3, 4, –8) and (5, –6, 4) in the ratio $k : 1$.

The coordinates of the point of intersection of the line with the plane are

$$\left[\frac{k\times 5 + 1\times(-3)}{k+1}, \frac{k\times(-6) + 1\times(4)}{k+1}, \frac{k\times 4 + 1\times(-8)}{k+1}\right] or \left(\frac{5k-3}{k+1}, \frac{-6k+4}{k+1}, \frac{4k-8}{k+1}\right)$$

This satisfies $z = 0$ if $\frac{4k-8}{k+1} = 0$ or $k = 2$.

Hence, the required ratio is 2 : 1.

Also, the coordinates of the point of intersection are

$$\left(\frac{5\times 2-3}{2+1}, \frac{-6\times 2+4}{2+1}, \frac{4\times 2-8}{2+1}\right), \textit{i.e.,} \left(\frac{7}{3}, \frac{-8}{3}, 0\right).$$

Ex. 9. Given that *P* (3, 2,–4), *Q* (5, 4, –6), *R* (9, 8, – 10) are collinear, find the ratio in which *Q* divides *PR*.

Sol. Let Q divide PR in the ratio k : 1. Then $5 = \frac{3\times 1+9\times k}{k+1}$ or $5k + 5 = 3 + 9k$ or $k = \frac{1}{2}$.

Hence, the required ratio is 1 : 2.

Ex. 10. *A* (3, 2, 0), *B* (5, 3, 2), *C* (–9, 6, –3) are three points forming a triangle. *AD*, the bisector of angle *BAC* meets *BC* in *D*. Find the coordinates of the point *D*.

Sol. Since AD is the internal bisector of $\angle BAC$, so $\frac{BD}{DC} = \frac{AB}{AC}$

(From Geometry) ....(*i*)

Now $AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} = 3$

$AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2} = 13$

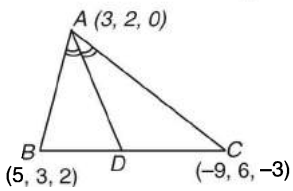


Fig. 26.11

$\therefore$ From (*i*) $\frac{BD}{DC} = \frac{AB}{AC} = \frac{3}{13}$, *i.e.,* D divides BC in the ratio 3 : 13.

$\therefore$ Coordinates of D are

$$\left(\frac{3(-9)+13(5)}{3+13}, \frac{3(6)+13(3)}{3+13}, \frac{3(-3)+13(2)}{3+13}\right), \textit{i.e.} \left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right).$$

Ex. 11. Show that the centroid of the triangle whose vertices are (x_r, y_r, z_r), $r = 1, 2, 3$, is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

Sol. Let the coordinates of A, B, C be respectively (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).
The coordinates of D, the mid-point of BC, are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

The centroid G divides the median AD in the ratio 1 : 2.

So, $\frac{DG}{GA} = \frac{1}{2}$.

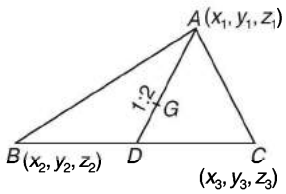


Fig. 26.12

By the Section Formula, the x-coordinate of G is

$$\frac{1\times x_1 + 2\times \frac{x_2+x_3}{2}}{1+2} \text{ or } \frac{x_1+x_2+x_3}{3}.$$

Similarly, the y-coordinate is $\frac{y_1+y_2+y_3}{3}$, and the z-coordinate is $\frac{z_1+z_2+z_3}{3}$. Hence, the coordinates of G, the centroid of the triangle ABC, are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

Ex. 12. The midpoints of the sides of a triangle are $\left(3, 2, \frac{3}{2}\right), \left(1, \frac{3}{2}, 3\right)$, and $\left(2, \frac{5}{2}, \frac{5}{2}\right)$. Find its vertices.

Sol. Let $A\ (x_1, y_1, z_1)$, $B\ (x_2, y_2, z_2)$ and $C\ (x_3, y_3, z_3)$ be the vertices of ΔABC. It is given that $D\left(3, 2, \frac{3}{2}\right), E\left(1, \frac{3}{2}, 3\right)$, and $F\left(2, \frac{5}{2}, \frac{5}{2}\right)$ are the midpoints of sides BC, CA and AB respectively.

Since $D\left(3, 2, \frac{3}{2}\right)$ is the midpoint of BC, so

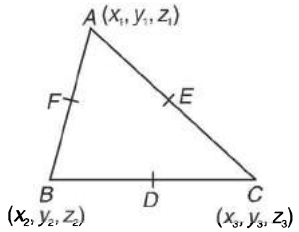


Fig. 26.13

$$\frac{x_2 + x_3}{2} = 3, \frac{y_2 + y_3}{2} = 2, \frac{z_2 + z_3}{2} = \frac{3}{2}$$

$\Rightarrow x_2 + x_3 = 6$ ... (1) $\quad y_2 + y_3 = 4$ ...(2) $\quad z_2 + z_3 = 3$ ...(3)

Similarly, for E we have $\frac{x_3 + x_1}{2} = 1, \frac{y_3 + y_1}{2} = \frac{3}{2}, \frac{z_3 + z_1}{2} = 3$

$\Rightarrow x_3 + x_1 = 2$ ...(4) $\quad y_3 + y_1 = 3$ ...(5) $\quad z_3 + z_1 = 6$ ...(6)

For F, we have $\frac{x_1 + x_2}{2} = 2, \frac{y_1 + y_2}{2} = \frac{5}{2}, \frac{z_1 + z_2}{2} = \frac{5}{2}$

$\Rightarrow x_1 + x_2 = 4$ ...(7) $\quad y_1 + y_2 = 5$ ...(8) $\quad z_1 + z_2 = 5$ ...(9)

Adding (1), (4) and (7), we get $2\ (x_1 + x_2 + x_3), = 12 \Rightarrow x_1 + x_2 + x_3 = 6$ ...(10)

Subtracting (1), (4) and (7) from (10) one by one respectively, we get $x_1 = 0, x_2 = 4, x_3 = 2$

Adding (2), (9) and (8), we get $2(y_1 + y_2 + y_3) = 12 \Rightarrow y_1 + y_2 + y_3 = 6$ ...(11)

Subtracting (2), (9) and (8) from (11), one by one respectively, we get $y_1 = 2, y_2 = 3, y_3 = 1$

(3) + (6) + (9) gives $2\ (z_1 + z_2 + z_3) = 14 \Rightarrow z_1 + z_2 + z_3 = 7$ ...(12)

Subtracting (3), (6), (9) from (12) one by one respectively, we get $z_1 = 4, z_2 = 1, z_3 = 2$.

Hence, the vertices of Δ ABC are A (0, 2, 4), B (4, 3, 1) and C (2, 1, 2).

Ex. 13. Using section formula, prove that the three points A (– 2, 3, 5), B (1, 2, 3) and C (7, 0, – 1) are collinear.

Sol. Let C divides AB in the ratio $\lambda : 1$. Then $\frac{\lambda - 2}{\lambda + 1} = 7, \frac{2\lambda + 3}{\lambda + 1} = 0, \frac{3\lambda + 5}{\lambda + 1} = -1$

Each of these equations gives $\lambda = -\frac{3}{2}$. This shows that C divides AB externally in the ratio 3 : 2.

So, C lies on the line joining A and B, *i.e.,* the given points A, B, C are collinear.

Ex. 14. Show that the point (1, 2, 3) is common to the lines which join A (4, 8, 12) to B (2, 4, 6), and C (3, 5, 4) to D (5, 8, 5).

Sol. Any point on AB is $P\left(\frac{2k + 4}{k + 1}, \frac{4k + 8}{k + 1}, \frac{6k + 12}{k + 1}\right)$ and any point on CD is

$$Q\left(\frac{5k' + 3}{k' + 1}, \frac{8k' + 5}{k' + 1}, \frac{5k' + 4}{k' + 1}\right).$$

If AB and CD have a common point, then P and Q coincide. In other words, for some values of k and k',

$$\frac{2k + 4}{k + 1} = \frac{5k' + 3}{k' + 1} \quad \text{... (1)}$$

$\frac{4k+8}{k+1}=\frac{8k'+5}{k'+1}$ ....(2) and $\frac{6k+12}{k+1}=\frac{5k'+4}{k'+1}$ ....(3)

Dividing (1) by (2), we get $\frac{1}{2}=\frac{5k'+3}{8k'+5}$ or $k'=-\frac{1}{2}$

From (1), $\frac{2k+4}{k+1}=\frac{\frac{-5}{2}+3}{\frac{-1}{2}+1}=1$ or $k=-3$

Substituting in (3) the values of k and k',we find that (3) is satisfied. Hence, (1), (2), (3) are consistent for $k=-3$ and $k'=-\frac{1}{2}$.

Substituting $k=-3$ in the coordinates of P, we get the required point as (1, 2, 3).

EXERCISE 26 (*b*)

1. Find the coordinates of the points which divide the join of the points (2, – 1, 3) and (4, 3, 1) in the ratio 3 : 4 internally.

2. Find the coordinates of the points which divide the line joining the points (2, – 4, 3), (– 4, 5, – 6) in the ratio (*i*) 1 : –4 (*ii*) 2 : 1.

3. Find the ratio in which the line joining the points (2, 4, 5), (3, 5, –4) is divided by the *yz*-plane.

4. The three points A (0, 0, 0), B (2, – 3, 3), C (–2, 3, –3) are collinear. Find in what ratio each point divides the segment joining the other two.

5. Find the coordinates of the points which trisect AB given that A (2, 1, – 3) and B (5, – 8, 3).

6. Find the coordinates of the point which is three-fifths of the way from (3, 4, 5) to (–2, – 1, 0).

7. Show that the point (1, – 1, 2) is common to the lines which join (6, –7, 0) to (16, – 19, – 4) and (0, 3, –6) to (2, –5, 10).

8. Find the lengths of the medians of the triangle whose vertices are A (2, –3, 1), B (– 6, 5, 3), C (8, 7, – 7).

9. Find the point of intersection of the medians of the triangle with vertices (–1, –3, –4), (4, – 2, –7), (2, 3, –8).

10. Find the ratio in which the join of A (2, 1, 5) and B (3, 4, 3) is divided by the plane $2x + 2y – 2z = 1$. Also, find the coordinates of the point of division.

11. The mid-points of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find its vertices.

12. Three vertices of a parallelogram $ABCD$ are A (3, –1, 2), B (1, 2, –4) and C (–1, 1, 2). Find the coordinates of the fourth vertex D.

13. What is the locus of a point for which

(*i*) $x = 0$ (*ii*) $y = 0$ (*iii*) $z = 0$ (*iv*) $x = a$ (*v*) $y = b$ (*vi*) $z = c$?

14. What is the locus of a point for which

(*i*) $x = 0, y = 0$ (*ii*) $y = 0, z = 0$ (*iii*) $z = 0, x = 0$

(*iv*) $x = a, y = b$ (*v*) $y = b, z = c$ (*vi*) $z = c, x = a$?

ANSWERS

1. $\left(\frac{20}{7},\frac{5}{7},\frac{15}{7}\right)$ **2.** (*i*) (4, – 7, 6), (*ii*) (–2, 2, –3) **3.** –2 : 3

4. $\frac{AB}{BC}=\frac{-1}{2},\frac{BC}{CA}=\frac{-2}{1},\frac{CA}{AB}=\frac{1}{1}$ **5.** (3, –2, 1), (4, –5, 1) **6.** (0, 1, 2)

8. $\sqrt{91}$, $\sqrt{166}$, $\sqrt{217}$ **9.** $\left(\frac{5}{3},-\frac{2}{3},\frac{-19}{3}\right)$ **10.** 5 : 7, $\left(\frac{29}{12},\frac{9}{4},\frac{25}{6}\right)$

11. (1, 2, 3), (3, 4, 5), (–1, 6, –7) **12.** (l, –2, 8)

13. (*i*) *yz*-plane ; (*ii*) *xz*-plane ; (*iii*) *xy*-plane ; (*iv*) the plane parallel to the *yz*-plane at a distance *a* units from it ; (*v*) the plane parallel to the *xz*-plane at a distance *b* units from it ; (*vi*) the plane parallel to the *xy*-plane at a distance *c* units from it.

14. (*i*) The *z*-axis ; (*ii*) The *x*-axis ; (*iii*) The *y*-axis ; (*iv*) The line of intersection of the given planes $x = a$ and $y = b$; (*v*) The line of intersection of the given planes $y = b$ and $z = c$ (*vi*) The line of intersection of the given planes $z = c$ and $x = a$.

HINTS TO SELECTED QUESTIONS

10. Let the given plane intersect *AB* at point *C* and let the required ratio be $k : 1$. Then

$C = \left(\frac{3k+2}{k+1}, \frac{4k+1}{k+1}, \frac{3k+5}{k+1}\right)$. Since *C* lies on the plane $2x + 2y - 2z = 1$, therefore,

$$2\left(\frac{3k+2}{k+1}\right) + 2\left(\frac{4k+1}{k+1}\right) - 2\left(\frac{3k+5}{k+1}\right) = 1$$

Note. You will study the **Plane** in Book 2.

12. Let *D* (*x, y, z*) be the required point. Find mid-points of diagonals AC and BD. Use the fact that they coincide.

CHAPTER SUMMARY

1. Distance formula

(*i*) Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by the formula

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$= \sqrt{(\text{diff. of the } x\text{'s})^2 + (\text{diff. of the } y\text{'s})^2 + (\text{diff. of the } z\text{'s})^2}$$

(*ii*) Distance of a point $P(x_1, y_1, z_1)$ from the origin is

$$OP = \sqrt{x_1^2 + y_1^2 + z_1^2}\ .$$

2. Section formula

(*i*) The coordinates of the point which divides the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$ are

(*a*) $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$, if the division is internal.

(*b*) $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$, if the division is external.

(*c*) Coordinates of the middle point of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

CHAPTER TEST

1. Find the point on y-axis which is equidistant from the points (3, 1, 2) and (5, 5, 2).

[**Hint.** Any point on y-axis is $(0, y, 0)$] (**Ans.** 0, 5, 0)

2. Show that the points (a, b, c) (b, c, a) and (c, a, b) are the vertices of an equilateral triangle.

3. Show that the points (0, 7, –10), (1, 6, 6) and (4, 9, – 6) are the vertices of a right angled triangle.

4. Show that the points A(–2, 3, 5), B(1, 2, 3) and C (7, 0, – 1) are collinear.
[**Hint.** Show that AB + BC = CA]

5. Find the lengths of the medians of the triangle A(0, 0, 6), B(0, 4, 0) and C(6, 0, 0) [***NCERT***]
[Ans. 7, $\sqrt{34}$, 7 units **]**

6. Using section formula, show that the points (2, – 3, 4), (–1, 2, 1) and (0, $\frac{1}{3}$, 2) are collinear.
[***NCERT***]

7. Find the ratio in which the yz-plane divides the line segment formed by joining the point (–2, 4, 7) and (3, –5, 8). **[Ans.** 2 : 3] [***NCERT***]

CHAPTER 27

MATHEMATICAL REASONING

27.01. Introduction

The word "**logic**" is derived from the Greek word ***logos***, which means reason or discourse, and the ancient Greeks are usually credited with initiating the study of man's reasoning processes. The principles discovered by the Greeks were first systematized by Aristotle (ca. 384–322 B.C.), and Aristotles' type of reasoning constitutes the traditional logic that has been studied and taught from his time to the present day. A simple illustration of Aristotelean logic goes on as follows:

All men are mortal.
Socrates was a man.
Therefore, Socrates was mortal.

Although, the modern study of symbolic logic finds its roots in the works of such men as the German mathematician, Leibnitz (1647–1716), modern symbolism and algebraic – type operations were first systematically applied to logic in the works of the English mathematician **George Boole** (1815–1864).

The study of logic has developed at an accelerated pace since Boole's analysis, and considerable progress toward an understanding of logical truth has been made. Whether we are trying to solve a problem, taking part in a debate, or working a crossword puzzle, we are engaged in a mental activity called logical reasoning. This reasoning is usually expressed in terms of **declarative sentences**, and it is to the study of these sentences that we turn our attention.

Declarative sentences are of the following types:

1. **Assertive sentences** are sentences that make an assertion.
   e.g. 'There is no atmosphere on moon' is an assertive sentence.
2. **Imperative sentences** are sentences that express a request or a command.
   e.g. (*i*) 'Complete your home work', (*ii*) 'Please post this letter' are imperative sentences.
3. **Interrogative sentences** are sentences that ask some questions.
   e.g. 'Who was the first Prime Minister of India'? is an interrogative sentence.
4. **Exclamatory sentences** are sentences that express some strong feelings.
   e.g. 'May you live a prosperous and healthy long life' and 'How smart and beautiful Priyanka is' are exclamatory sentences.

In the following sections, we shall be concerned with certain types of **declarative sentences** that are called ***statements*** and the manner in which we may combine such statements and arrive at valid conclusions.

27.02. Statements

These are assertions in words or symbols which are either true or false but not both simultaneously. The difference between an ordinary sentence and a logical statement is that it is not possible to say about truth or falsity of an ordinary sentence, whereas true or false is an essential requisite of a logical statement.

The following sentences are illustrations of statements.

(*i*) New Delhi is the capital of India. [True]

(*ii*) 66 is an odd number. [False]

(*iii*) $(x + 1)^2 = x^2 + 4x + 1$, for all values of x. [False]

(*iv*) Each angle of an equilateral triangle measures 60°. [True]

The following are illustrations of non-statements.

(*i*) What time is it? (Interrogative sentence)

(*ii*) Oh God! you should not drive so fast. (Exclamatory sentence)

(*iii*) Open the door. (Imperative sentence)

(*iv*) May God bless you. (Exclamatory sentence)

(*v*) This statement is false. (Assertive sentence)

The above sentences are not statements. Notice that if we assume that (*iv*) is true, then it is false, and if we assume that it is false, then it is true. Hence, the sentence cannot be classified as either true or false, so it is not a statement.

27.03. Notation

It is customary to represent statement by lower case letters like $p, q, r, s, \ldots$ Thus, we may represent the statement "Mumbai is a big city" by the letter 'p' and the statement '$5 + 7 = 12$' by the letter 'q'.

27.04. Simple and compound statements

A statement is said to be *simple*, if it cannot be broken down into two or more sentences. Each of the following is a *simple statement*.

1. 15 is an odd number. 2. 15 is divisible by 3.

New statements that can be formed by combining two or more simple statements are called ***compound statements***. Thus, the statement '15 is an odd number and 15 is divisible by 3' is a compound statement formed by combining the two simple statements (1) and (2) above. The two simple statements which are combined to form the compound statements are called **sub-statements** or **component statements** of the compound statement.

As a further example, "Sunil is 170 cm tall," is a simple statement. On the other hand, the statement, "Sunil is 170 cm tall and he plays basket ball," is a compound statement, because it is a combination of the two simple statements, "Sunil is 170 cm tall" and "He plays basket ball".

27.05. Truth and falsity of statements

Examine the following statements:

1. $5 + 4 = 9$
2. $3 + (10 \div 2) = 8$
3. The sun rises in the west.
4. $3 > 4$.

It is easy to see that statements 1 and 2 are true, while statements 3 and 4 are false. If a statement is true, we say that its truth value is true. The truth value of each of the statements 1 and 2 is ***true***. If a statement is false, we say that its truth value is ***false***. The truth value of each of the statements 3 and 4 is *false*.

It may be remarked that in the mathematical sense, ***true means true always and without exceptions.*** **If we can find even a *single* case in which a given statement is false, then even though it may be true in rest of the cases, the statement is adjudged as *false.***

In logic, we are primarily concerned with considering the truth and falsity for sentences. The importance is fully as great for compound sentences, and so now we will learn when to regard

compound statements as true and when to regard them as false depending upon the truth and falsity of its components.

A fundamental property of a compound statement is that its truth value is completely determined by the truth value of each of its sub-statements together with the way in which they are connected to form the compound statement.

EXERCISE 27 (*a*)

In the following problems, determine if the sentence is a statement. Classify each sentence that is a statement as *simple* or *compound*. If compound, give the *components*.

1. Open the door.
2. 5 is a prime number.
3. Do you like mathematics?
4. Every rectangle is a square.
5. Today is Sunday and tomorrow is Monday.
6. May you live long!
7. Rekha is studying in class eleven and she has to offer 5 subjects.
8. The earth revolves around the moon.
9. New Delhi is a big city and it is the capital of India.
10. 20 is a prime number and 20 is less than 21.

Write down whether the following statements are true (T) or false (F).

11. 8 is a prime number.
12. Every square is a rectangle.
13. The earth revolves around the moon.
14. The set of whole numbers is a finite set.
15. 32 is a multiple of 8.
16. $3 + 4i$ is a complex number.

ANSWERS

1. Not a statement **2.** Statement **3.** Not a statement **4.** Statement
5. Compound statement. The components are "Today is Sunday; Tomorrow is Monday".
6. Not a statement.
7. Compound statement. The component statements are: Rekha is studying in class eleven. She has to offer 5 subjects.
8. Simple statement
9. Compound statement. The component statements are: New Delhi is a big city. It is the capital of India.
10. Compound statement : 20 is a prime number. 20 is less than 21.
11. F **12.** T **13.** F **14.** F **15.** T **16.** T

27.06. Quantifiers

Quantifiers are phrases like 'There exists' and 'For all' and 'None'.

An open sentence with a quantifier becomes a statement, called a **quantified statement**.

The quantifier ***'For all'*** or ***'For every'*** is known as **universal quantifier** and is symbolised by the symbol '$\forall$'. Thus $\forall x$ is read as "for all x".

The quantifier ***'There exists'*** or ***'For some'*** or ***'For at least'*** is called **exitential quantifier** and is symbolised by the symbol '$\exists$'.

Remark. The inverted A, namely $\forall$, derives from the word 'all,' while the backward E, namely $\exists$, derives from the word 'exists'.

A special phrase is used for writing ***'all'*** sentences. This phrase contains the word 'all' (or *every*, *each* or *any*) and at least one variable and it names the replacement set for each variable.

For example, 'Every whole number is equal to itself' is stated as ***'For all' x in W, $x = x$.*** Using symbols, it is abbreviated as $\forall x \in W, x = x$.

Similarly, 'At least one whole number is less than 1' is stated as 'There exists an x in W such that $x < 1$'

It is abbreviated as $\exists x \in W, x < 1$.

The symbol $\in$ in the above abbreviations should be read as "in" rather than "is a member of".

Remark. It may be noted that whereas an open sentence "x is a bird" is neither true nor false, the statement by quantifying an open sentence may be judged either true or false and is therefore a statement. Also, it may be pointed out that it is possible for a quantifying phrase to contain more than one variable.

For example, the phrase 'For all x and y in W' contains two variables and may be symbolised as $\forall x, y \in W$. Similarly, the phrase 'For all x, y and z in W', contains three variables and its symbolic form is $\forall x, y, z \in W$.

Ex. 1. Using quantifiers convert each of the following open sentences defined on N, into a true statement.

(*i*) $x + 1 > x$ (*ii*) $x^2 > 0$ (*iii*) $x + 2 = 7$ (*iv*) $x + 5 < 8$

Sol. (*i*) $\forall x \in N, x + 1 > x$ is a true statement, since successor of every natural number is greater than itself.

(*ii*) $\forall x \in N, x^2 > 0$ is a true statement since square of every natural number is positive.

(*iii*) $\exists\, x \in N$, such that $x + 2 = 7$, is a true statement since $x = 5 \in N$ satisfies $x + 2 = 7$.

(*iv*) $\exists\, x \in N$, such that $x + 5 < 8$, is a true statement since $x = 1 \in N$ satisfies $x + 5 < 8$.

Ex. 2. Let $A = \{2, 4, 6, 8, 10\}$. Determine the truth value of each of the following:

(*i*) $\forall x \in A$, x is an even number.

(*ii*) $\exists\, x \in A$, x is a prime number.

(*iii*) $\forall x \in A, x^2 < 0$.

(*iv*) $\exists\, x \in A$, such that x is an odd number.

Sol. (*i*) True, since every number in set A is an even number.

(*ii*) True, since the number 2 in set A is a prime number.

(*iii*) False, there is no number whose square is negative.

(*iv*) False, since there is no number in set A which is an odd number.

EXERCISE 27 (*b*)

1. Identify the quantifier in the following statements.

(*i*) There exists a capital city for every state of India.

(*ii*) For every real number x, x is less than $x + 1$.

(*iii*) At least one natural number is not a prime number.

2. Symbolise the following statements.

(*i*) There is at least one number in the set of natural numbers which is equal to 'its' cube.

(*ii*) The square of every real number is positive.

(*iii*) There exists at least one number in $A = \{5, 7, 8, 9, 10\}$ which is an even number.

(*iv*) For every real number x, $x < x + 1$.

(*v*) The square roots of all prime numbers are irrational numbers. (Let P denote the set of prime numbers and Q that of irrational numbers).

ANSWERS

1. (*i*) There exists (*ii*) For every (*iii*) At least

2. (*i*) $\exists x \in N, x = x^3$ (*ii*) $\forall x \in R, x > 0$ (*iii*) $\exists x \in A$, x is even

(*iv*) $\forall x \in R, x < x + 1$. (*v*) $\forall x \in P, \sqrt{x} \in Q$.

27.07. Basic logical connectives

There are many ways in which simple statements can be combined to form compound statements. Such combinations are formed by using words called ***connectives*** to join the statements. Two of the most important connectives are the words **and** and **or**. Suppose that we use the letters p and q to represent statements as follows:

p : It is cold today.

q : It is raining.

Then we can form the following compound sentences.

p and q : It is cold today and it is raining.

p or q : It is cold today or it is raining.

In the study of logic, the word "and" is symbolized by $\wedge$ and the word "or" by $\vee$. Thus, p and q is written $p \wedge q$ and p or q is written $p \vee q$.

Definition 27.07 (*a*).

If two statements are combined by the word "and" (or an equivalent word), the resulting statement is called a ***conjunction.*** *If the two statements are symbolized by p and q, respectively, then the conjunction is symbolized by* $p \wedge q$.

Ex. 3. Symbolize the conjunction, "Promod is a boy and Savita is a girl".

Sol. We let p stand for "Promod is a boy," and q stand for "Savita is a girl." Then the given conjunction may be symbolized by $p \wedge q$.

Remark. Normally, the conjunction 'and' is used between two statements which has same kind of relation. Therefore, the compound statement 'Ravi is in the seventh grade and 2 is an even number' sounds odd. But in logic it is a perfectly acceptable statement.

Definition 27.07 (*b*).

If two statements are combined by the word "or" (or an equivalent word), the result is a ***disjunction****. If the two statements are symbolized by p or q, respectively, then the disjunction is symbolized by* $p \vee q$.

Ex. 4. Symbolize the disjunction. "We stop inflation or we increase wages."

Sol. Letting p stand for "we stop inflation" and q stand for "We increase wages." We can symbolize the disjunction by $p \vee q$.

Another construction that is important in logic is that of negating a given statement.

Definition 27.07 (*c*).

The ***negation*** *of a statement is obtained by inserting the word "not" in the statement or by prefixing it with such phrases as "it is not true that" or "it is not the case that." The symbol ~ is used to denote the negation of a statement.*

Ex. 5. If p stands for the statement, "It is cold today," translate the statement $\sim p$ into English.

Sol. The statement $\sim p$ can be written, "It is not cold today," or "It is not true that it is cold today."

Remark. Caution is required while inserting the word "not" at some proper place in the given statement to obtain the negation of the statement. Thus, the negation of the statement "All squares are rectangles" is

"Not all squares are rectangles." …(1)

This is not equivalent to "All squares are not rectangles."

However, (1) is equivalent to "There exists a square which is not a rectangle."

Ex. 6. Write the negation of the statement, p : Kolkata is a big city.

Sol. The negationof p is

$\sim p$: Kolkata is not a big city.

or $\sim p$: It is not the case that Kolkata is a big city.

or $\sim p$: It is false that Kolkata is a big city.

Note. We observe that, all the three statements have been symbolized by $\sim p$ (negation of statement p) because they have the same meaning in English language.

Ex. 7. Write the negation of the following statements.

(*i*) p : I went to station yesterday.

(*ii*) q : 3 + 5 = 8.

(*iii*) r : All natural number are integers.

Sol. (*i*) $\sim p$: I did not go to station yesterday **or** It is not the case that I went to station yesterday or It is false that I went to station yesterday.

(*ii*) $\sim q$: $3 + 5 \neq 8$ **or** It is not the case that $3 + 5 =$ **8 or** It is false that $3 + 5 = 8$.

(*iii*) $\sim r$: Not all natural numbers are integers **or** There exists a natural number which is not an integer **or** It is not the case that all natural numbers are integers **or** It is false that all natural numbers are integers.

27.08. Negation of compound statements

1. Negation of conjunction

The negation of a conjunction implies the negation of at least one of the two components. Therefore,

The negation of a conjunction $p \wedge q$ is the disjunction of the negation of p and the negation of q, that is,

$$\sim (p \wedge q) = \sim p \vee \sim q$$

Ex. 8. Write the negation of each of the following conjunctions.

(*i*) New Delhi is in India and Lahore is in Pakistan.

(*ii*) 3 + 4 = 7 and 6 < 9.

Sol. (*i*) Let p : New Delhi is in India and q : Lahore is in Pakistan.

Then, the conjunction in (*a*) is given by $p \wedge q$.

Now, $\sim p$: New Delhi is not in India, and

$\sim q$: Lahore is not in Pakistan

$\therefore \sim(p \wedge q) = \sim p \vee \sim q$, that is, New Delhi is not in India or Lahore is not in Pakistan.

(*ii*) Let p : $3 + 4 = 7$ and q : $6 < 9$. Then $\sim p$: $3 + 4 \neq 7$, $\sim q$: $6 \nless 9$.

$\therefore \sim(p \wedge q) = 3 + 4 \neq 7$ or $6 \nless 9$.

2. Negation of disjunction

A disjunction $p \vee q$ means that either p or q or both exist. Therefore, the negation of the disjunction would mean the negation of both p and q simultaneously. Therefore,

$$\sim (p \vee q) = \sim p \wedge \sim q$$

Ex. 9. Write the negation of each of the following statements.

(*i*) Shekhar is in class 6 or Poonam is in class 7.

(*ii*) 5 is greater than 3 or 4 is less than 8.

Sol. (*i*) Let p : Shekhar is in class 6 and q : Poonam is in class 7.

Then, $\sim (p \vee q)$: Shekhar is not in class 6 and Poonam is not in class 7.

(*ii*) Let p : 5 is greater than 3, and q : 4 is less than 8.

Then, $\sim (p \vee q)$ = 5 is not greater than 3 and 4 is not less than 8.

3. Negation of a negation

It is obvious that the negation of negation of a statement is the statement itself. Thus, $\sim(\sim p) = p$.

For example, if p : Crows are black, then $\sim p$: Crows are not black.

$\therefore \sim(\sim p)$: It is not the case that crows are not black **or** It is false that the crows are not black **or** Crows are black.

Ex. 10. Consider the two statements:

***p* : Raj Kapoor is alive.**

***q* : Raj Kapoor lives in Delhi.**

Write the following statements in symbolic form.

(*i*) Raj Kapoor is alive and he lives in Delhi.

(*ii*) Either Raj Kapoor is alive or he lives in Delhi.

(*iii*) Raj Kapoor is neither alive, nor does he live in Delhi.

(*iv*) It is not true that Raj Kapoor is alive and he lives in Delhi.

Sol. (*i*) $p \wedge q$

(*ii*) $p \vee q$

(*iii*) $\sim p \wedge \sim q$

(*iv*) $\sim(p \wedge q)$.

The student should make careful note of the use of parentheses to indicate what items are to be regarded as a unit. Thus, in the solution to (*d*) of Ex. 10, $\sim(p \wedge q)$ means that the entire statement $p \wedge q$ is negated. It is important to distinguish $\sim p \wedge q$ from $\sim(p \wedge q)$.

Ex. 11. Let *p* be the statement, "Anish likes Kiran," and let *q* be the statement, "Kiran likes Anish." Write in words.

(*i*) $\sim(p \wedge q)$ (*ii*) $\sim p \vee \sim q$ (*iii*) $\sim p \wedge \sim q$.

Sol. (*i*) It is not true that Anish likes Kiran and Kiran likes Anish. That is, it is not true that Anish and Kiran like each other.

(*ii*) Either Anish does not like Kiran or Kiran does not like Anish.

(*iii*) Anish does not like Kiran and Kiran does not like Anish. That is, Anish and Kiran dislike each other.

Ex. 12. Write the negation of each of the following statements.

(*i*) My grade is an A or a B.

(*ii*) It is not true that 3 is less than 4.

(*iii*) All pets are mammals.

(*iv*) Some students failed to qualify the entrance test.

Sol. (*i*) My grade is not an A and it is not a B.

(*ii*) 3 less than 4.

(*iii*) Some pets are not mammals.

(*iv*) No students failed to qualify the entrance test.

Remark. It may be noted that, in general, the negation of the statement

"All ..... are ..... " is "Some ..... are not ....." and the negation of the statement

"Some ..... are..... " is "No ..... are ....."

EXERCISE 27 (*c*)

1. Write the given statement in symbolic form using the letter in parentheses to represent the corresponding component.

(*i*) This is April (p) and income tax returns must be filed (q).

(*ii*) Accountancy is a required subject for Chartered Accountants (m) but not for engineers (n).

(*iii*) Mukesh Patel is a teacher (t) or a lawyer (u).

(*iv*) Jack went up the hill (c) and Jill went up the hill (d).

(*v*) I plan to take science (a) or commerce (c) in class 11.

(*vi*) I will not drive to Jaipur ($\sim d$) but I shall go by train (t) or by plane (p).

2. Let p be "Shruti can type," and let q be "Shruti takes shorthand." Write the following statements in symbolic form:

(*i*) Shruti can type and take shorthand.

(*ii*) Shruti can type but she does not take shorthand.

(*iii*) Shruti can neither type nor take shorthand.

(*iv*) It is not true that Shruti can type and take shorthand.

3. Use p : Ramesh is rich; q : Pradeep is poor. Think of "poor" as "not rich", and write each of these statements in symbolic form.

(*i*) Ramesh is poor and Pradeep is rich. (*ii*) Pradeep and Ramesh are both rich.

(*iii*) Neither Ramesh nor Pradeep is rich. (*iv*) Ramesh is not rich and Pradeep is poor.

(*v*) It is not true that Ramesh and Pradeep both are rich.

(*vi*) Either Ramesh is poor or Pradeep is poor.

(*vii*) Either Ramesh or Pradeep is rich.

4. Use p : I like this school; q : I like Mr. Saxena. Express each of the following statements in words.

(*i*) $p \wedge q$ (*ii*) $\sim q$ (*iii*) $\sim p$ (*iv*) $(\sim p) \wedge (\sim q)$

(*v*) $(\sim p) \wedge q$ (*vi*) $p \vee q$ (*vii*) $\sim (p \wedge q)$ (*viii*) $\sim [(\sim p) \wedge q]$

5. Give the negation of each of the following statements.

(*i*) Either he is bald or he is tall.

(*ii*) Nobody does not like Madhuri.

(*iii*) It is not true that the set of prime numbers is finite.

(*iv*) All circles are round.

(*v*) Some students passed this course.

ANSWERS

1. (*i*) $p \wedge q$ (*ii*) $m \wedge \sim n$ (*iii*) $t \vee u$ (*iv*) $c \wedge d$

(*v*) $a \vee c$ (*vi*) $\sim d \wedge (t \vee p)$

2. (*i*) $p \wedge q$ (*ii*) $p \wedge \sim q$ (*iii*) $\sim p \wedge \sim q$ (*iv*) $\sim (p \wedge q)$

3. (*i*) $(\sim p) \wedge (\sim q)$ (*ii*) $(\sim q) \wedge p$ (*iii*) $(\sim p) \wedge q$. (*iv*) $(\sim p) \wedge q$

(*v*) $\sim [p \wedge (\sim q)]$ (*vi*) $(\sim p) \vee q$ (*vii*) $p \vee (\sim q)$

4. (*i*) I like this school and I like Mr. Saxena.

(*ii*) I do not like Mr. Saxena.

(*iii*) I do not like this school.

(*iv*) I do not like this school and I do not like Mr. Saxena.

(*v*) I do not like this school and I like Mr. Saxena.

(*vi*) I like this school or I like Mr. Saxena.

(*vii*) It is not true that I like this school and I like Mr. Saxena.

(*viii*) It is not true that I do not like this school and I like Mr. Saxena.

5. (*i*) He is not bald and he is not tall.

(*ii*) Somebody does not like Madhuri.

(*iii*) The set of prime numbers is finite.

(*iv*) Some circles are not round.

(*v*) No student passed this course.

27.9. Truth values of statements

We regard T (for true) and F (for false) as the possible **truth values** of a statement. Thus, the statement, "Jawahar Lal Nehru was the first prime minister of India," has the truth value T, the statement, "Chennai is the capital of India," has the truth value F.

One of the principal problems in logic is that of determining the truth value of a compound statement when the truth values of its components are known. In order to attempt this problem, we must first assign appropriate truth values to such compounds as $p \wedge q$, $p \vee q$ and $\sim p$. Although, we have introduced the symbols $\wedge$, $\vee$ and $\sim$ to the student, their definitions remain incomplete. We shall complete their definitions by assigning appropriate truth values to statements involving these symbols.

27.10. Truth and falsity of a conjunction

Suppose you are offered a position in a company that requires that:

p : The applicant be at least 21 years of age.

q : The applicant should be an M.B.A.

It is evident that to be eligible for this position you must meet both requirements, that is, both p and q must be true. Thus, it seems desirable to say that for a conjunction to be true, both components must be true. Otherwise, the conjunction is false.

If we have a conjunction with two components, p and q, we have four possible pairs of truth values for these statements, namely:

1. p true, q true 2. p true, q false 3. p false, q true 4. p false, q false

As in the preceding example, it seems reasonable to assign the value T to the statement $p \wedge q$ only when both components are true.

27.11. Truth tables

A truth table is a convenient way of summarising the truth values of a statement $S(p, q, r, \ldots)$ and its components $p, q, r, \ldots$A truth table consists of rows and columns. The initial columns are filled with the possible truth values of the components and the last column is filled with the truth values of the compound statement S. The letter "***T***" is used to indicate the truth value of a true statement. The letter "***F***" is used to indicate the truth value of a false statement.

Remark. If a compound statement is made up of only one simple statement like '$\sim p$', then the truth table will consist of $2^1 = 2$ rows and if a compound statement is made up of two simple statements (like $p \wedge q$), then the truth table will consist of $2^2 (= 4)$ rows.

Ex. 13. Construct the truth table for $p \wedge q$.

Sol. First we list all the possible combinations of true and false for p and q as shown in Table 1. There being 2 components p and q in the compound statement $p \wedge q$, there would be $2^2 = 4$ rows.

Table 1

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Table 2

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Since we have agreed that $p \wedge q$ is true only when both p and q are true, the complete truth table is as given in Table 2.

Illustrations

(*i*) Let us consider the conjunction $5 + 3 = 8$ and $7 - 1 = 4$. Here the component $5 + 3 = 8$ is true but the component $7 - 1 = 4$ is false. Hence, the conjunction is false.

(*ii*) The statement 'Kusum is healthy and she is fair' is true only if both the conditions are satisfied, *i.e.*, (*i*) Kusum must be healthy (*ii*) Kusum must be fair. If (either or both) of these conditions is not satisfied, the conjunction is a false statement.

(*iii*) '3 is an odd number and 4 is an even number' is a true statement, since both the components of this conjunction are true statements.

Remark. The reader needs to be cautioned that other English words such as "but", "nevertheless", "still", "however", and so on are sometimes used in place of the connective "and", Thus, the statement, "Jyoti is pretty but not very smart," is a conjunction. On the other hand, the statement, "Jyoti and Madhu are sisters," is not a conjunction.

27.12. Truth and falsity of a disjunction

The connective 'or' can possibly be used in two ways which are referred to as the **exclusive** use and the **inclusive** use. In the exclusive use, the word ***'or'*** means **one or the other but not both.** The exclusive use of *or* may be illustrated by the sentence. "Either stay here or go home." Clearly, a person can't do both. Similarly, the sentence, "A whole number is either odd *or* even" also illustrates the exclusive use of the connective '*or*'. A whole number can't be both. The inclusive use of the word **'*or*' implies one or the other or both.** This is illustrated by the following sentences.

1. "It will rain today *or* it will rain tomorrow." Here it is also possible that it may rain on both the days.
2. "I shall be in my room *or* I shall be reading the newspaper." Clearly I could be both in my room or reading the newspaper.
3. "A certain whole number is an even number or it is a multiple of 5." Here, also a number like 10 could be both; of course the number 4 which is an even number but not a multiple of 5 also fulfils the stated condition. Likewise, a number 15 which is not even, but is a multiple of 5 also fulfils the stated condition.

In mathematics, the double usage of term is avoided. **Mathematicians have decided to use "*or*" only in the inclusive sense**. It is this meaning of "*or*" that we have always to keep in mind. The implication of this use of word "*or*" is that a ***compound statement containing the connective 'or' is a true statement if either of its components is true or if both of the components are true.*** Another way to put this is to say that such a compound statement is true if ***at least one component is true***.

(The advertiser who writes, "The candidate must know English or Hindi," certainly would not reject a candidate if he knew both the languages).

If the letters p and q represent two simple statements, then the truth table below shows how the truth values of the simple statements are used to assign a truth value to the ***'or'*** statement.

It should be noted that the expression $p \vee q$ (p or q) is true for all cases except when both p and q are false.

Table 3

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex. 14. Assign a truth value to each of the following statements:

(*i*) $4 < 4 \vee 4 < 5$

(*ii*) $3 + 5 = 8 \vee 0 > 1$

(*iii*) $5 \times 3 = 16 \vee 8 + 7 = 18$

(*iv*) $1 > 0 \vee 3 \times 8 - 1 = 23$

Sol. The truth value of first statement is true since one of its components, *viz*., $4 < 5$ is true.

The truth value of the second statement is true since one of its components, *viz*., $3 + 5 = 8$ is true.

The truth value of the third statement is false since both of its components are false.

The truth value of the fourth statement is true since both the components of the statement are true.

Note. In the definition of $A \cup B$ as $\{x \mid x \in A \text{ or } x \in B\}$, we use the word '*or*' as the keyword and here it means $x \in A$ or $x \in B$, or both. In disjunction also the word '*or*' is used in the same sense and so the two ideas of union and disjunction are related. This is the reason why the symbols $\vee$ and $\cup$ are similar.

27.13. Truth table for a negation (~ p)

Table 4

p	$\sim p$
T	F
F	T

In accordance with the ordinary usage, a statement and its negation have ***opposite*** truth value. The truth table of p and $\sim p$ is given in Table 4. The truth table expresses the fact that, "If p is true, then $\sim p$ is false, and if p is false, then $\sim p$ is true."

Remark. Since $\sim p$ consists of only one simple statement p, so there are 2^1 (= 2) rows in the truth table.

27.14. We now give some examples to illustrate how we can construct truth tables for various compound statements

Ex. 15. Construct the truth table for the statement $\sim p \vee q$.

Sol. We proceed by the following steps.

Step 1. Write in the first two columns the four possible pairs of truth values for the statements p and q.

Table 5 (*a*)

p	q	$\sim p$	$\sim p \vee q$
T	T		
T	F		
F	T		
F	F		

Step 1

Table 5 (*b*)

p	q	$\sim p$	$\sim p \vee q$
T	T	F	
T	F	F	
F	T	T	
F	F	T	

Step 2

Step 2. Using column 1 as a reference, we negate the statement p.

Step 3. 'Using columns 3 and 2 and recalling that a disjunction is false only when both components (3 and 2) are false, we enter the values for $\sim p \vee q$ in the column labelled 4. We note that the statement $\sim p \vee q$ is false only in row 2, because in this case both components ($\sim p$, q) are false. The completed truth table appears in Table 5 (*c*).

Table 5 (*c*)

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T
1	2	3	4

Illustration

Let p be "I lie," and let q be "I would tell you." When will the statement, "I do not lie or I would tell you," be false ?

Sol. The statement under consideration can be symbolized as $\sim p \vee q$. From table 5 (*c*), we see that $\sim p \vee q$ is false when p is true and q is false. Hence, the given statement is false when I lie and I would not tell you.

Ex. 16. Construct a truth table for the statement $p \wedge (\sim q)$.

Sol. First construct a table with the appropriate headings as follows:

Table 6 (*a*)

p	q	$\sim q$	$p \wedge \sim q$
T	T		
T	F		
F	T		
F	F		

Table 6 (*b*)

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	
T	F	T	
F	T	F	
F	F	T	
1	2	3	4

Now fill in the column headed "$\sim q$" by writing the negation of the values given under q in the second column [Table 6 (*b*)]. Now recalling that a conjunction is true only when both components

(1) and (3) are true, enter the values for $p \wedge \sim q$ in column (4). The table now appears as follows [Table 6 (*c*)].

Table 6 (*c*)

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Ex. 17. Construct the truth table for the statement $\sim(p \wedge \sim q)$.

Sol.

Step 1. Write in the first two columns the four possible pairs of truth values for the statements p and q.

Step 2. Write the values of $\sim q$ in column (3).

Step 3. Recalling that a conjunction is true only when both its components are true, fill in the values for $p \wedge \sim q$ in column (4). Now negate the values of $p \wedge \sim q$ and enter the values for $\sim(p \wedge \sim q)$ in column (5).

Table 7

p	q	$\sim q$	$(p \wedge \sim q)$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T
1	2	3	4	5

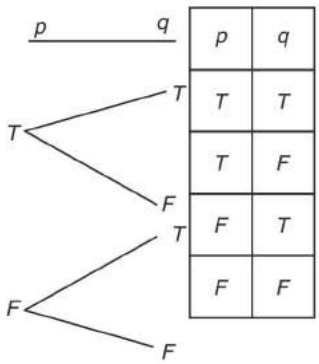
27.15. Truth values for more complicated sentences

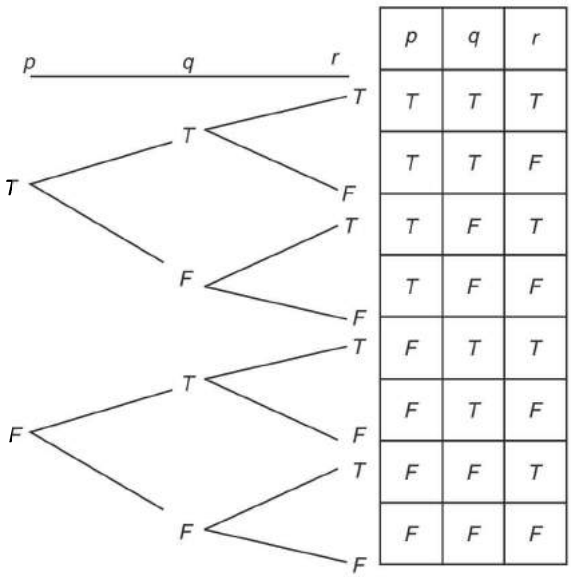
The five basic truth tables can be used to construct truth tables for more complicated sentences than considered so far. This is done by assigning all possible combinations of truth values for components of the given sentences and by successively computing truth values for parts of the sentences.

Ex. 18. Write the truth table for $(p \vee q) \wedge r$.

Number of rows in a truth table.

We have to account for all possible combinations of truth values for p, q and r. Tree diagrams are useful in identifying the various cases in a logical statement. When there are *two* component statements, p and q, there are 2^2, or 4 cases, and therefore, 4 lines in the truth table; when there are three components, p, q and r, there are 2^3, or 8 cases, and therefore, 8 lines in the truth table.





In general, if there are n component statements, there are 2^n combinations of possible truth values and, thus 2^n lines in the truth table.

Sol. Step 1. Write in the first three columns the possible truth values for the statements p, q and r. In column 1, we enter four T's and four F's, in column 2 we enter two T's two F's and so on; in column 3 we enter one T, one F, one T and so on. (See Table 8)

Table 8

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F
1	2	3	4	5

Step 2. In column 4, enter the values for $p \vee q$, recalling that a disjunction is false only when both components are false.

Step 3. In column (5), using columns 2 and 4, write the values for $(p \vee q) \wedge r$, recalling that a conjunction is true only when both components are true.

Remark. The truth table shows that $(p \vee q) \wedge r$ is true when r is true and at least one of p and q is true.

EXERCISE 27 (*d*)

1. Let p, q, r and s represent simple statements. Assume that p is false, q is true, r is false, and s is true. Determine the truth value of each statement expressed below:

(*i*) $q \wedge r$ (*ii*) $r \vee p$ (*iii*) $p \wedge s$ (*iv*) $p \vee s$
(*v*) $\sim q$ (*vi*) $q \vee s$ (*vii*) $\sim r$
(*viii*) $s \wedge q$ (*ix*) $r \wedge p$

2. Let a, b, c and d represent simple statements. Assume that $a \wedge d$ is true, $b \wedge c$ is false, and $\sim c$ is false.

(*i*) What is the truth value of a? (*ii*) What is the truth value of d?
(*iii*) What is truth value of c? (*iv*) What is the truth value of b?

3. Assume that two given statements p and q are both true and indicate whether or not you would expect each of the following statements to be true :

(*i*) $p \wedge q$ (*ii*) $p \vee q$ (*iii*) $p \vee (\sim q)$ (*iv*) $(\sim p) \vee (\sim q)$

Construct truth tables for:

4. $(\sim p) \wedge q$ **5.** $(\sim p) \wedge (\sim q)$ **6.** $\sim(p \wedge q)$ **7.** $p \vee (\sim q)$
8. $\sim[p \vee (\sim q)]$ **9.** $\sim (\sim p \wedge \sim q)$ **10.** $(p \wedge q) \vee (\sim p \wedge q)$
11. $p \wedge (q \vee r)$ [**Hint.** See Solved Ex. 18] **12.** $(\sim p \wedge \sim q) \vee (p \wedge \sim q)$
13. $(p \vee q) \vee (r \wedge \sim q)$

14. Let p be "Ananya is beautiful," and let q be "Ananya is 165 centimetres tall."

(*i*) Under what conditions is the statement, "Ananya is beautiful and 165 centimetres tall." true?
(*ii*) Under what conditions is the statement, "Ananya is beautiful and 165 centimetres tall," false?
(*iii*) Under what conditions is the statement, "Ananya is beautiful or 165 centimetres tall," true?
(*iv*) Under what conditions is the statement, "Ananya is beautiful or 165 centimetres tall," false?

ANSWERS

1. (*i*) F (*ii*) F (*iii*) F (*iv*) T (*v*) F
(*vi*) T (*vii*) T (*viii*) T (*ix*) F

2. (*i*) T (*ii*) T (*iii*) T (*iv*) F

3. All are true except (*iv*).

4.

p	q	$(\sim p)$	$(\sim p)\wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

5.

p	q	$\sim p$	$\sim q$	$(\sim p)\wedge(\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

6.

p	q	$p\wedge q$	$\sim(p\wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

7.

p	q	$\sim q$	$p\vee(\sim q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

8.

p	q	$\sim q$	$p\vee(\sim q)$	$\sim[p\vee(\sim q)]$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

9.

p	q	$\sim p$	$\sim q$	$\sim p\wedge\sim q$	$\sim(\sim p\wedge\sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

10.

p	q	$\sim p$	$p\wedge q$	$\sim p\wedge q$	$(p\wedge q)\vee(\sim p\wedge q)$
T	T	F	T	F	T
T	F	F	F	F	F
F	T	T	F	T	T
F	F	T	F	F	F

11.

p	q	r	$q\vee r$	$p\wedge(q\vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

12.

p	q	$\sim p$	$\sim q$	$\sim p\wedge\sim q$	$p\wedge\sim q$	$(\sim p\wedge\sim q)\vee(p\wedge\sim q)$
T	T	F	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	F	F	F
F	F	T	T	T	F	T

13.

p	q	r	$p\vee q$	$\sim q$	$r\wedge\sim q$	$(p\vee q)\vee(r\wedge\sim q)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	F	F

14. (*i*) Only if Ananya is both beautiful and 165 cm tall.
(*ii*) If Ananya is not beautiful and/or not 165 cm tall.
(*iii*) If Ananya is beautiful and/or 165 cm tall.
(*iv*) Only if Ananya is neither beautiful nor 165 cm tall.

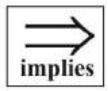
27.16. Implications

(A) Conditional sentences. (If ..... then)

We frequently come across statements like the following:

1. If it rains, then there will be no picnic.
2. If I have no home work, then I will go to the movie.
3. If *ABCD* is a parallelogram, then $AB = CD$ and $AD = BC$.
4. If *ABC* is a triangle, then $AB + BC > AC$.
5. If x names a prime number greater than 2, then x names an odd number.
6. If $5 + 4 = 9$, then $9 - 4 = 5$.

In each of the above statements two simple sentences are connected by ***If ....., then*** to form a new composite sentence. A compound statement using the connective *If ....., then* is called a **conditional**. The connective *If …, then* in the conditional may be replaced by the connective ***implies***. Thus, statements 1 and 4 could also be stated as under:



"It rains" *implies* "the picnic will be cancelled" "*ABC* is a triangle" *implies* "$AB + BC > AC$".

For this reason the statements in the *If ....., then* form are also referred to as **implications**.

The logical symbol for a conditional or an implication is $\Rightarrow$.

If we use the variables p and q to represent any two statements, then any conditional can be symbolised as simply $p \Rightarrow q$, which is read either as "If p, then q" or as "p implies q". The symbol $\Rightarrow$ is also called the implication symbol. For example,

(1) $x = 3 \Rightarrow x^2 = 9$ (2) $A \subseteq B \Rightarrow x \in A, x \in B$.

It may be noted that the symbolic representation of the conditional involves three symbols — the variable (p) at the tail of the arrow, and the variable (q) at the head of the arrow. The two statements that make up a conditional are its components. The component that is expressed by the "if" clause (the variable 'p' at tail) is the **antecedent** and the component that is expressed by the 'then' clause (the variable 'q' at the head) is the **consequent**. For example:

In the conditional "If $5 + 4 = 9$, then $9 - 4 = 5$," the statement $5 + 4 = 9$ is the *antecedent* and the statement $9 - 4 = 5$ is the *consequent*.

27.17. Truth table for a conditional

Think of the following situations:

A husband says to his wife, "If I get a promotion in my job, I shall take you on a foreign tour". The question is when is the husband good or bad.

Let p = the husband gets a promotion and q = The husband takes his wife on a foreign tour.

- If he gets the promotion and takes his wife on foreign tour, he is a good man (p is true, q is true, thus $p \Rightarrow q$ is true).
- If he gets promotion but does not take his wife on foreign tour, then he is a bad man (p is true, q is false, thus $p \Rightarrow q$ *is false*).
- If he does not get the promotion but takes his wife on foreign tour then he is a good man (p is false, q is true, thus $p \Rightarrow q$ is true)
- If he does not get the promotion and does not take his wife on a foreign tour, he remains a good man (p is false, q is false, thus $p \Rightarrow q$ is true)

Suppose that your friend promises you, "If the weather is nice (p), then I will go to the movie (q)." Under what conditions does your friend break his promise, that is, when is the statement, "If the weather is nice, then I will go to the movie," false? It is obvious that your friend lies only when the weather is nice and he does not go to the movie.

Table 9

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The above discussion suggests that $p \Rightarrow q$ is false when p is true and q is false, otherwise, $p \Rightarrow q$ is true. That is exactly, the way in which the truth table for $p \Rightarrow q$ is designed (see Table 9). The truth table expresses the fact that the conditional statement ***"if p, then q" is false only when p is true and q is false, otherwise it is true.***

Remark. Probably, most people agree with the first and the second rows. The third and fourth possibilities in which the antecedent (p) is false, are less familiar to us and do not readily appeal to a layman's common sense. As the possibilities do not agree with our intuition, we find it hard to accept that we should assign truth values to conditionals where the antecedent is false. In logic it is essential that every statement be either true or false. There can be none which are neither true nor false. This is the reason why conditionals with false antecedents cannot be ignored. Even though it is not clear what happens if p is false, the mathematicians have made the *definition that whenever p is false then the conditional $p \Rightarrow q$ is true*.

Ex. 19. Give the truth value of each of the following statements.

(*i*) If 4 + 5 = 9, then 9 – 5 = 4. **(*ii*) If 4 + 5 = 9, then 9 – 5 = 0.**

(*iii*) If 4 + 5 = 21, then 21 + 9 = 30. **(*iv*) If 4 + 5 = 21, then 21 = 9.**

Sol. All of these statements are of the form, $p \Rightarrow q$, where p is the antecedent and q is the consequent.

(i) Because p is true and q is true, the statement $p \Rightarrow q$ is true.

(ii) Because p is true and q is false, the statement $p \Rightarrow q$ is false.

(iii) Because p is false and q is true, the statement $p \Rightarrow q$ is true.

(iv) Because p is false and q is false, the statement $p \Rightarrow q$ is true.

Ex. 20. Find all possible numbers x such that the following statement is true.

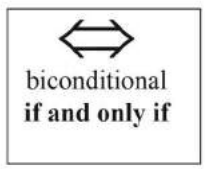
If 4 + 9 = 49, then x + 2 = 5

Sol. Let p be "4 + 9 = 49," and let q be "x + 2 = 5" Because p is a false statement, the statement $p \Rightarrow q$ is true regardless of the truth value of q. Thus, for all numbers k, the statement, "If 4 + 9 = 49, then x + 2 = 5," is true.

Ex. 21. Let p be "x is a fruit," and let q be "x is ripe." Under what conditions is the statement $p \Rightarrow q$ false?

Sol. The statement $p \Rightarrow q$ is false only when p is true and q is false, hence the given statement is false if x is a fruit that is not ripe.

(B) The biconditional (The connective "If and only If")



In certain statements, the conditional is used twice, with the antecedent and the consequent of the first conditional reversed in the second. For example, the statement, "If Shweta is an woman, then Shweta is a female and if Shweta is a female, then Shweta is an woman," uses the conditional twice in this manner. It is for this reason that such statements are called ***biconditionals*** and are symbolized as $p \Leftrightarrow q$. Because the conditional $p \Rightarrow q$ is false when p is true and q is false, and $q \Rightarrow p$ is false when q is true and p is false, the biconditional should be false when p is true and q is false, or when q is true and p is false. The truth table defining this new connective appears in Table 10. The statement $p \Leftrightarrow q$ is usually read, ***"p if and only if q"*** (abbreviated sometimes as iff). The truth table expresses the fact that **the biconditional $p \Leftrightarrow q$ is true only when p and q have the same truth value.**

Table 10

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Illustrations

1. Very often, we come across the use of phrase 'if and only if', *i.e.*, the biconditional in Mathematics. For example, let us consider the compound statement, **"Two triangles are equiangular if and only if their corresponding sides are proportional."**

From the above statement, we can write

(*i*) "If two triangles are equiangular, then their corresponding sides are proportional," which is an implication and **conversely.**

(*ii*) "If corresponding sides of two triangles are proportional, then they are equiangular," which is also an implication.

2. Consider the statement, 'Savita loves Anil, if and only if she accepts the invitation ($p \Leftrightarrow q$)'. This is the same thing as

(*i*) If Savita loves Anil, then she accepts the invitation ($p \Rightarrow q$).

(*ii*) If Savita accepts the invitation, then she loves Anil ($q \Rightarrow p$).

3. $5x + 2 = 12$, iff $x = 2$ ($p \Leftrightarrow q$) . This is the same thing as

(*i*) If $5x + 2 = 12$, then $x = 2$ ($p \Rightarrow q$). (*ii*) If $x = 2$, then $5x + 2 = 12$ ($q \Rightarrow p$).

27.18. Truth value of $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

As we have been doing so far, we proceed by steps.

Note that the statement $(p \Rightarrow q) \wedge (q \Rightarrow p)$ has two components $(p \Rightarrow q)$ and $(q \Rightarrow p)$.

Step 1. We write in the first two columns the four pairs of truth values for the statement p and q.

Table 11

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T
1	2	3	4	5

Step 2. Now recalling that statement $p \Rightarrow q$ is false only when p is true and q is false, we fill in the values in column 3 for the component $p \Rightarrow q$ assigning the value F to the second row and T's to the rest of the rows.

Step 3. Similarly, we enter the values in column 4 for the component $(q \Rightarrow p)$ recalling that the statement $q \Rightarrow p$ is false only when q is true and p is false. We assign the value F to the third row in column 4 and T to the rest of the lines.

Step 4. Recalling that a conjunction of two statements is true only when both components are true, we fill in the values in column 5 for the conjunction of the two statements $(p \Rightarrow q)$ and $(q \Rightarrow p)$.

Remark. We find from tables 10 and 11 that statements $[(p \Rightarrow q) \wedge (q \Rightarrow p)]$ and $(p \Leftrightarrow q)$ have identical truth tables, and hence they are equivalent; that is, $[(p \Rightarrow q) \wedge (q \Rightarrow p)] \leftrightarrow (p \Leftrightarrow q)$.

Definition. ***If p and q represent any statements, the biconditional $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$.***
Also, $p \Leftrightarrow q$ is true only when p and q are both true or both false.

For example,

The biconditional $(5 + 3 = 8) \Leftrightarrow (11 - 7 = 4)$ is true, since both the statements are true.

The biconditional (Men have wings) $\Leftrightarrow$ (Birds can fly) is false, since one statement is true and one is false.

The biconditional $(3 - 4 = 0) \Leftrightarrow (5 \times 7 = 10)$ is true, since both statements are false.

Note. We observe that the truth values of $p \Rightarrow q$ and $q \Rightarrow p$ do not necessarily agree. However, $p \Leftrightarrow q$ and $q \Leftrightarrow p$ always agree in the values.

EXERCISE 27 (*e*)

1. Write each sentence in the "If ..... then" form.

(*i*) Roses are vegetables if carrots are flowers.

(*ii*) All ducks are birds.

(*iii*) Vertical angles are equal.

(*iv*) Freezing water expands.

(*v*) A set with no elements is called the empty set.

(*vi*) A racer wins the race only if he runs fast.

(*vii*) Any two parallel lines are coplanar.

2. Let p be "I will marry her," and let q be "she is beautiful." Translate into symbolic form.

(*i*) If she is beautiful, then I will marry her.

(*ii*) If I will marry her, then she is beautiful.

(*iii*) She is beautiful if and only if I will marry her.

(*iv*) If she is not beautiful, then I will not marry her.

(*v*) If I will not marry her, then she is not beautiful.

(*vi*) If she is beautiful, then I will not marry her.

3. Determine whether p, q and 'If p, then q' are true or false in each case given below:

	p	q
(*i*)	3 is a prime number	3 is an even number
(*ii*)	$5 < 7$	5 is odd
(*iii*)	$3 > 2$	$2 \times 7 = 14$
(*iv*)	$1 > 5$	2 is even
(*v*)	$5 \times 3 = 16$	$2 + 7 = 6$
(*vi*)	$3\,(5 \div 6) < 1$	$8 - 3 \div 6 > 9$

4. Write T before each true statement and write F before each false statement. Then give the truth value of the conditional expressed.

(*i*) If Asia is a continent, then Delhi is in Japan.

(*ii*) If monkeys climb trees, then 6 is divisible by 2.

(*iii*) If oxygen is a gas, then gold is a compound.

(*iv*) Water is dry $\Rightarrow$ snow is hot.

(*v*) Snow is cold $\Rightarrow$ water is wet.

(*vi*) If $3 = 5$, then 7 is a prime number.

(*vii*) $5 \times 6 - 4 = 21 \Rightarrow 2\,(5 \div 15 + 3) = \dfrac{20}{3}$.

(*viii*) If a triangle is a rectangle, then a circle is a rhombus.

(*ix*) If 51 is the product of + 17 and –3, then lions can fly in the air.

(*x*) If $\sqrt{5}$ is an integer, then 3 is an integer.

ANSWERS

1. (*i*) If carrots are flowers, then roses are vegetables.

(*ii*) If the creature is a duck, then it is a bird.

(*iii*) If two angles are vertical angles, then they are equal.

(*iv*) If water is freezing, then it expands.

(*v*) If a set has no element, then it is called the empty set.

(*vi*) Only if he runs fast, then a racer wins the race.

(*vii*) If two lines are parallel, then they are coplanar.

2. (*a*) $q \Rightarrow p$ (*b*) $p \Rightarrow q$ (*c*) $q \Leftrightarrow p$ (*d*) $\sim q \Rightarrow \sim p$ (*e*) $\sim p \Rightarrow \sim q$
(*f*) $q \Rightarrow \sim p$

3.

	p	q	$p \Rightarrow q$
(*i*)	*T*	*F*	*F*
(*ii*)	*T*	*T*	*T*
(*iii*)	*T*	*T*	*T*
(*iv*)	*F*	*T*	*T*
(*v*)	*F*	*F*	*T*
(*vi*)	*F*	*F*	*T*

4. (*i*) *F* (*ii*) *T* (*iii*) *F* (*iv*) *T* (*v*) *T*
(*vi*) *T* (*vii*) *T* (*viii*) *T* (*ix*) *T* (*x*) *T*

27.19. Converse, Inverse and Contrapositive

The composite sentence that is used most in mathematics is the conditional. Now we will investigate three close relatives of the conditional, namely, the *converse*, the *inverse* and the *contrapositive*.

The converse

Examine the following sentences:

1. If 15 is an odd number, then 3 is a factor of 15.
2. If 3 is a factor of 15, then 15 is an odd number.

In these sentences conditional (2) is obtained by interchanging the antecedent and the consequent of conditional (1) Conditional (2) is *converse* of conditional (1).

Definition. ***The converse*** *of a given conditional is a new conditional formed by interchanging the antecedent and the consequent of the given conditional.*

Thus, $q \Rightarrow p$ represents the converse of the conditional $p \Rightarrow q$. It is not difficult to obtain the truth table for $q \Rightarrow p$ from that of $p \Rightarrow q$ if we remember that in the converse the consequent has become the antecedent and the antecedent has become the consequent and that a conditional is false only when the antecedent is true and the consequent is false. The truth table for $p \Rightarrow q$ and its converse $q \Rightarrow p$ is shown.

Table 12

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	*T*	*T*	*T*
T	*F*	*F*	*T*
F	*T*	*T*	*F*
F	*F*	*T*	*T*

It may be seen on comparing the truth values in the column headed $p \Rightarrow q$, line for line, with truth values in the column headed $q \Rightarrow p$ that a conditional and its converse do not always agree in truth value. The truth value of the converse has always got to be investigated on its own merit.

Consider the following examples.

1. The conditional, 'If birds have wings then trees have wings', is false because its antecedent is true and consequent is false. Its converse.

'If trees have wings, then birds have wings' is true because it is a conditional whose antecedent is false and consequent is true.

The table also shows that *the converse of a true conditional is true when both the antecedent and the consequent of the given conditional are true and also when both the antecedent and the consequent are false.*

2. The conditional 'If 3 is a factor of 15, then 15 is an odd number' is the true converse of true conditional, 'If 15 is an odd number, then 3 is a factor of 15'.

27.20. The inverse

This conditional is another relative of the conditional. We will use the idea of negation to help us establish this concept. Let us examine the following statements:

1. If 5 is a prime number, then there is life on the moon.

2. If 5 is not a prime number, then there is no life on the moon.

Here in conditional (2), the 'if-part' is the denial of the 'if-part' of the conditional (1). The 'then part' (2) is the denial of the 'then-part' of conditional (1). Conditional (2) is the inverse of conditional (1).

Definition. ***The inverse of a given conditional is a new conditional whose antecedent is the negation of the original antecedent and whose consequent is the negation of the original consequent.*** **That is, if the given conditional is $p \Rightarrow q$, the inverse of this conditional is $\sim p \Rightarrow \sim q$ read "not p implies not q", or "if not p, then not q".**

Ex. 22. State the inverse of the conditional given below:

"If it rains, then I shall not go out".

Sol. By definition, the inverse of a given conditional is formed by negating the antecedent and consequent of the given conditional. The negation of the antecedent is '*It does not rain*'. The negation of the consequent is *I shall go out*. Therefore, the inverse of the given conditional is as follows:

"If it does not rain, then I shall go out". We now construct the truth table of the inverse. This can be done as follows:

Table 13

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim p \Rightarrow \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Comparing the truth values in the column headed $p \Rightarrow q$, line for line, with the truth values in the column headed $\sim p \Rightarrow \sim q$, we conclude that the inverse of a conditional is not always true when the conditional is true. The truth value of the inverse has got to be investigated on its own merits.

27.21. The Contrapositive

The next relative of the conditional which we shall now consider is the ***contrapositive***. Let us examine the following sentences.

1. If 15 is an odd number, then 3 is a factor of 15.
2. If 3 is not a factor of 15, then 15 is not an odd number.

If we compare the components of the two conditionals, we find that the 'if-part' of conditional (2) is the denial of the 'then-part' of conditional (1) and the 'then-part' of conditional (2) is the denial of the 'if-part' of conditional (1). In other words, we can say that conditional (2) is obtained from conditional (1) by interchanging the antecedent and the consequent of (1) and negating each component. Statement (2) is the *contrapositive* of statement (1).

Definition. ***A contrapositive of a given conditional is a new conditional whose antecedent is the negation of the original consequent and whose consequent is the negation of the original antecedent.*** That is, if the given conditional is $p \Rightarrow q$, the contrapositive of this conditional is $\sim q \Rightarrow \sim p$, read "not q implies not p." or "if not q, then not p". In other words, the contrapositive is the inverse of the converse of a given conditional, or contrapositive is converse of the inverse.

Ex. 23. State the contrapositive of the conditional given below:

"If it rains, then I shall not go out"

Sol. We form the negation of the antecedent, which is 'It does not rain'. We form the negation of the consequent, which is 'I shall go out'. The contrapositive is obtained on interchanging these negations. It is given below.

"If I shall go out, then it does not rain."

If $2 + 3 \neq 7$, then $x \not\geq 2$ is the contrapositive of the conditional, if $x > 2$, then $2 + 3 = 7$.

It is easily seen that the inverse of a conditional is the contrapositive of the converse of the conditional. In other words $p \Rightarrow q$ is the same as $\sim q \Rightarrow \sim p$. (See Table 14) ***[ISC]***

The truth table for the contrapositive is constructed by keeping in mind how the truth value of any statement compares with the truth value of its negation. It is given below:

Table 14

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

An examination of the table reveals that a *conditional* $p \Rightarrow q$ and *its contrapositive*, $\sim q \Rightarrow \sim p$ *always agree in truth value*. That is, when one is true, the other is true. When one is false, the other is false. Such statements that always agree in truth value are **logically equivalent**.

Similarly, *the converse* $q \Rightarrow p$ *and the inverse* $\sim p \Rightarrow \sim q$ *of a conditional are also logically equivalent since the inverse is the contrapositive of the converse. This may further be verified by comparing the truth values of the two from the truth table.* (Table 15)

Summary

Given any simple statement of implication, we may obtain the following:

Statement	: $\boldsymbol{p \Rightarrow q}$	**If *p*, then *q***
Converse	: $q \Rightarrow p$	If q, then p.
Inverse	: $(\sim p) \Rightarrow (\sim q)$	If not p then not q.
Contrapositive	: $(\sim q) \Rightarrow (\sim p)$	If not q, then not p.

Consider the following examples.

(*i*)	**Statement**	**: If it is raining, then I stay at home.**
	Converse	: If I stay at home, then it is raining.
	Inverse	: If it is not raining, then I do not stay at home.
	Contrapositive	: If I do not stay at home, then it is not raining.
(*ii*)	**Statement**	: **If $\Delta ABC \cong \Delta XYZ$, then $\Delta ABC \sim \Delta XYZ$**
	Converse	: If $\Delta ABC \sim \Delta XYZ$, then $\Delta ABC \cong \Delta XYZ$
	Inverse	: If ΔABC is not congruent to ΔXYZ, then ΔABC is not similar to ΔXYZ.
	Contrapositive	: If ΔABC is not similar to ΔXYZ, then ΔABC is not congruent to ΔXYZ.
(*iii*)	**Statement**	**: If $x + 3 = 8$, then $x = 5$**
	Converse	: If $x = 5$, then $x + 3 = 8$
	Inverse	: If $x + 3 \neq 8$, then $x \neq 5$
	Contrapositive	: If $x \neq 5$, then $x + 3 \neq 8$.

Lastly, here is the truth table for the four implication we have been looking at.

Table 15

		Conditional	*Converse*			*Inverse*	*Contrapositive*
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

same

same

Remarks.

1. The result $p \Leftrightarrow q$ is true iff both $p \Rightarrow q$ and $q \Rightarrow p$ are true finds application in mathematics, when to establish the validity of a result, you have to prove that both a statement and its *converse* are true.

2. The contrapositive of a statement is used in proving theorems in which a direct proof is difficult, but the proof of the contrapositive is easy. The following example illustrates this idea.

'Prove that if n^2 is odd, then n is odd.'

Let p be "n^2 is odd," and let q be "n is odd." We have to prove that if p is true, q is always true. Because the statement $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$, it is sufficient to prove the equivalent statement, "if $\sim q$ is true, then $\sim p$ is true." To this end, we assume that $\sim q$ is true, that is, we assume that n is even. If n is even, then $n = 2k$, where k is an integer. Then $n^2 = (2k)^2 = 4k^2 = 2\,(2k^2)$, which is also even. Hence, when $\sim q$ is true, $\sim p$ is always true. We have thus proved that "if n^2 is odd, then n is odd."

3. **Necessary and sufficient conditions.** Frequently, in mathematics the words necessary and sufficient are used in conditional statements. To say that p is sufficient for q means that when p happens (is true), q will also happen (is also true). Hence, "p is sufficient for q" is equivalent to "If p, then q, *i.e.*, $p \Rightarrow q$."

Similarly, the sentence q is necessary for p means that if q does not happen, neither will p. That is, $\sim q \Rightarrow \sim p$. The statement $\sim q \Rightarrow \sim p$ is equivalent to $p \Rightarrow q$, so the sentence "q is necessary for p" is equivalent to "If p, then q," that is, $p \Rightarrow q$.

Illustrations

A necessary condition that a given whole number be exactly divisible by 9 is that it be divisible by 3. It is not a sufficient condition, for 6, 15, 21, are each divisible by 3, but not by 9. A sufficient condition that a given whole number be exactly divisible by 9 is that it be divisible by 18. This is not a necessary condition since numbers such as, 27, 45, 63 are each divisible by 9, but not by 18. A necessary and sufficient condition that a given whole number be divisible by 9 is that sum of its digits should be divisible by 9.

EXERCISE 27 (*f*)

1. Write the converse, inverse and contrapositive of the following statements.
(*i*) If you do not drink your milk, you will not be strong.
(*ii*) If you drink milk, you will be strong.
(*iii*) You will be strong only if you drink your milk.

2. Write the converse of each of the following statements. In which cases is the converse true?
(*i*) If an integer is even, then its square is divisible by 4.
(*ii*) If it is raining, then there are clouds in the sky.
(*iii*) In order to get this job, I must be a graduate.
(*iv*) If Mr. Saxena is elected to office, then all our problems are over.

3. Consider the statements:
p : You will work hard q : You will become wealthy.
Translate each of the symbolic statements into an English sentence.
(*i*) $p \Rightarrow q$ (*ii*) $q \Rightarrow p$ (*iii*) $(\sim p) \Rightarrow (\sim q)$ (*iv*) $(\sim q) \Rightarrow (\sim p)$

4. Compare the following statements:
(*i*) $p \Rightarrow q$ (*ii*) If p, then q
(*iii*) p is a sufficient condition for q.
(*iv*) q is a necessary condition for p.
(*v*) p, only if q.

5. Construct truth tables for each of the following:
(*i*) $(p \Rightarrow q) \wedge (q \Rightarrow p)$ (*ii*) $q \Rightarrow [(\sim p) \vee q]$ (*iii*) $[(\sim p) \wedge q] \Rightarrow (p \vee q)$

6. Write the converse, inverse and contrapositive for the statement $(\sim p) \Rightarrow q$.

7. Write the inverse of the converse of $p \Rightarrow q$.

8. Write the converse of the inverse of $p \Rightarrow q$.

9. Write the contrapositive of the inverse of $p \Rightarrow q$.

10. Write the converse of the contrapositive of $p \Rightarrow q$.

11. Write the contrapositive of the contrapositive of $p \Rightarrow q$.

12. Does completing each of the problems 6 through 10 result in a conditional? What is the relationship of each resulting condition to the original conditional $p \Rightarrow q$?

13. If p and q are any two propositions then prepare the truth table for $p \Rightarrow q$, $\sim q \Rightarrow \sim p$ and show that the above statements are equivalent.
Hence, or otherwise determine which of the following two arguments is valid?
(*i*) Given : If you work hard, then you pass the course.
Given : You did not work hard.
Conclusion : You did not pass the course.
(*ii*) Given : If you work hard, then you pass the course.
Given : You did not pass the course.
Conclusion : You did not work hard.
[**Hint.** See Table 16]

Table 16

		Conditional	*Converse*			*Inverse*	*Contrapositive*
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

same

same

If "you work hard" is denoted by p and "you pass the course" by q, then

(*i*) $p \Rightarrow q$ and $\sim p \Rightarrow \sim q$ are not the same. Therefore, if $p \Rightarrow q$, then $\sim p \Rightarrow \sim q$ is not valid.

(*ii*) $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are the same. Therefore, if $p \Rightarrow q$, then $\sim q \Rightarrow \sim p$ is valid.]

ANSWERS

1. (*i*) Converse : If you are not strong, then you do not drink your milk.
Inverse : If you drink your milk, then you are strong.
Contrapositive : If you are strong, then you drink your milk.
(*ii*) Converse : If you are strong, then you drink your milk.
Inverse : If you do not drink your milk, then you are not strong.
Contrapositive : If you are not strong, then you do not drink your milk.
(*iii*) Converse : If you drink your milk, then you are strong.
Inverse : If you are strong, then you do not drink your milk.
Contrapositive : If you do not drink your milk, then you are not strong.

2. (*i*) If the square of an integer is divisible by four, then the integer is even. True
(*ii*) If there are clouds in the sky, then it is raining. False
(*iii*) If I am a graduate, then I can get this job. False
(*iv*) If all our problems are over, then Mr. Saxena is elected to office. False

3. (*i*) If you work hard, then you will become wealthy.
(*ii*) If you will become wealthy, then you will work hard.
(*iii*) If you will not work hard, then you will not become wealthy.
(*iv*) If you will not become wealthy, then you will not work hard.

4. All five statements are equivalents, *i.e.*, all are different ways of saying. "If p, then q."

5. (*i*)

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(*ii*)

p	q	$\sim p$	$\sim p \vee q$	$q \Rightarrow [(\sim p \vee q)]$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(*iii*)

p	q	$\sim p$	$(\sim p) \wedge q$	$p \vee q$	$(\sim p \wedge q) \Rightarrow p \vee q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	T

[Recall that $p \Rightarrow q$ is false only when p is true and q is false]

6. Converse $q \Rightarrow (\sim p)$, inverse $p \Rightarrow (\sim q)$, contrapositive $(\sim q) \Rightarrow p$

7. $\sim q \Rightarrow \sim p$ **8.** $\sim q \Rightarrow \sim p$ **9.** $q \Rightarrow p$ **10.** $\sim p \Rightarrow \sim q$ **11.** $p \Rightarrow q$

12. Yes, contrapositive, contrapositive, converse, inverse, original condition.

27.22. Validating statements

Examining the **validity** of a statement means ***to check when a particular statement is true and when it is not true.*** This depends upon which of the special connectives *'and', 'or'* and which of the implications *'if and only if', 'if-then'* and which of the quantifiers *'for every', 'there exists'*, appear in the given statement.

p	q	$p \wedge q$
T	***T***	***T***
T	F	F
F	T	F
F	F	F

General rules for checking whether a statement is true or not.

Rule 1: *'For statements with connective'AND'*

Recall the truth table for $p \wedge q$. Given two statements p and q, we find out whether '*p and q'* is true as under:

Step 1. Show that the statement p is true

Step 2. Show that the statement q is true

Conclusion : 'p and q is true'

Ex. 24. Given two statements

***p* : 12 is a multiple of 3, *q* : 12 is a multiple of 4.**

Write the compound statement with the connective 'and' and check its validity.

Sol. The compound statement is *'12 is a multiple of 3 and 4'*. We know that both p and q are true statements. So the compound statement *'p and q'* is also true, *i.e.*, it is a valid statement.

Ex. 25. If *p* and *q* are two statements given by

***p* : 20 is a multiple of 4, *q* : 20 is a multiple of 7.**

Write the compound statement '*p* and *q*' and check its validity.

p	q	$p \vee q$
T	T	T
T	***F***	***T***
F	***T***	***T***
F	F	F

Sol. The compound statement is *'20 is a multiple of 4 and 7.'*

Since 20 is a multiple of 4, but it is not a multiple of 7, therefore, p is true but q is false. Hence, the compound statement *'p and q'* is false, *i.e.*, it is not valid.

Rule 2: *For statements with connective 'OR'*

Recall the truth table for $p \vee q$. Given two statements p and q, we find out whether 'p or q' is true as under:

By assuming that p is false, show that q must be true

or

By assuming that q is false, show that p must be true.

Ex. 26. Given two statements.

***p* : 20 is a multiple of 4, *q* : 20 is a multiple of 7.**

Write the compound statement '*p* or *q*' and check its validity.

Sol. The compound statement is '*20 is a multiple of 4 or 7*'. Here, if we assume that p is false, *i.e.*, 20 is not a multiple of 4, then, clearly, q is true. Therefore, the compound statement 'p or q' is true, *i.e.*, it is valid.

Aliter. Also, we could have proceeded as under:

If we assume that q is false, *i.e.,* 20 is not a multiple of 7, then, clearly, p is true. Therefore, the compound statement 'p or q' is true, *i.e.*, it is valid.

Ex. 27. Check the validity of the following statement:

'Square of an integer is positive or negative'.

Sol. The given statement is a disjunction whose components are

p : Square of an integer is positive.

q : Square of an integer is negative.

Assume that q is false, *i.e.*, square of an integer is not negative, then it follows that p is true. Therefore, the given compound statement is true.

Rule 3: *For statements with implication 'IF THEN'*

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Recall the true table for *'If p and then q'*. Given two statements p and q, we find out whether '$p \Rightarrow q$' as under:

Case 1 : *Direct method* : By assuming that p is true, prove that q must be true.

Case 2 : *Contrapositive method* : By assuming q is false, prove that p must be false.

Rule 4 : *For statements with the implication 'IF AND ONLY IF'*

Recall the following truth table.

p	q	$p \Rightarrow q$ If p then q	$q \Rightarrow p$ If q then p	$p \Leftrightarrow q$ (p if and only if q)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Given two statements p and q, we find out whether the statement 'if and only if' is true as under:

Step 1. If p is true, then q is true.

Step 2. If q is true, then p is true.

Ex. 28. Check whether the following statement is true or not:

'If x and y are odd integers, then xy is an odd integer.'

Sol. The given statement is of the form 'If p, then q'.

First method : (*Direct method*). Let p be true. Then,

p is true $\Rightarrow x$ and y are odd integers

$\Rightarrow x = 2m + 1, y = 2n + 1$ for some integers m, n

$\Rightarrow xy = (2m + 1)(2n + 1)$

$\Rightarrow xy = 2(2mn + m + n) + 1$

$\Rightarrow xy$ is an odd integer

$\Rightarrow q$ is true

Thus, p is true $\Rightarrow q$ is true

Hence, '*If p, then q*' is a true statement.

Second method (*Contrapositive method*). Let q be not true. Then,

q is not true $\Rightarrow xy$ is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow p$ is not true.

Thus, q is false $\Rightarrow p$ is false

Hence, 'If p, then q' is a true statement.

Ex. 29. By contrapositive, check the validity of the following statement:

If a, b are integers such that ab is odd, then both a and b are odd.

Sol. Let p and q be two statements given by

p : ab is an odd integer

q : Both a and b are odd integers

We have to check whether $p \Rightarrow q$ is true by contrapositive method, *i.e.*, we have to check the validity of $\sim q \Rightarrow \sim p$.

Let q be not true, *i.e.*, $\sim q$ is true.

$\Rightarrow$ It is false that both a and b are odd integers.

$\Rightarrow$ At least one of a and b is an even integer.

Let $a = 2n$ for some integer n. Then

$ab = 2nb$ for some integer n

$\Rightarrow ab$ is an even integer

$\Rightarrow ab$ is not an odd integer

$\Rightarrow \sim p$ is true

Thus, $\sim q$ is true $\Rightarrow \sim p$ is true.

Hence, the given statement is true.

27.23. Contradiction

In establishing the validity of a statement by the method of contradiction, we make an assumption and then after making certain operations arrive at a result which *contradicts* our supposition.

Ex. 30. Verify by the method of contradiction that $\sqrt{3}$ is irrational.

Sol. Let p be the statement given by

p : $\sqrt{3}$ is irrational

If possible, let p be not true, *i.e.*, let p be false. Then,

p is false $\Rightarrow$ $\sqrt{3}$ is rational

$\Rightarrow \sqrt{3} = \frac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow \quad 3 = \frac{a^2}{b^2} \Rightarrow a^2 = 3b^2$

$\Rightarrow 3$ divides a^2

$\Rightarrow 3$ divides a

$\Rightarrow a = 3c$, for some integer c

$\Rightarrow a^2 = 9c^2 \Rightarrow 3b^2 = 9c^2$

$\Rightarrow b^2 = 3c^2 \Rightarrow 3$ divides b^2

$\Rightarrow 3$ divides b

$\Rightarrow 3$ is a common factor of both a and b.

This contradicts our assumption that a and b have no common factor. So, our assumption that $\sqrt{3}$ is rational is wrong.

Hence, the statement $\sqrt{3}$ is irrational' is true.

27.24. Invalidity of statements by counter examples

In order to show that a given statement is false, we may give an example of a situation where the statement is not valid. Such an example is called a counter example.

Ex. 31. By giving a counter example, show that the following statement is false.

'If *n* is an even integer, then *n* is prime'.

Sol. We know that 8 is an even number which is not prime. Similarly, 4, 6, 10, 12 etc., are even integers which are not primes.

Hence, the given statement is false.

EXERCISE 27 (*g*)

1. Show that the statement
p : 'If x is a real number such that $x^3 + 4x = 0$, then $x = 0$' is true by
(*i*) Direct method, (*ii*) Method of contradiction (*iii*) Method of contrapositive

2. Show that the statement 'For any real numbers a and b, $a^2 = b^2$ implies that $a = b$' is not true by giving counter examples.

3. Show that the following statement is true by the method of contrapositive.
p : If x is an integer and x^2 is even, then x is also even.

4. Given below are two statements:
p : 30 is a multiple of 5.
q : 30 is a multiple of 9.
Write the compound statement, connecting these two statements with 'and' and 'or'. In both cases, check the validity of the compound statement.

5. Verify by the method of contradiction that $\sqrt{7}$ is irrational.

ANSWERS AND HINTS

1. (*i*) *Direct method* : $x^3 + 4x = 0 \Rightarrow x(x^2 + 4) = 0$. But $x^2 + 4, x \in R \neq 0$, therefore, $x = 0$.

(*ii*) *Method of contradiction:*

Let $x \neq 0$, and let $x = p, p \in R$ and p is a root of $x^3 + 4x = 0$

$\Rightarrow p^3 + 4p = 0 \Rightarrow p(p^2 + 4) = 0$. But $p \neq 0$ (assumption),

also $p^2 + 4 \neq 0 \Rightarrow p(p^2 + 4)$, *i.e.*, $p^3 + 4p \neq 0$

This contradicts the given statement. So our assumption that $x \neq 0$ is wrong. Hence, $x = 0$.

(*iii*) *Method of contrapositive*

Let $x = 0$ be not true. Let $x = p \neq 0$.

Then, $p^3 + 4p = 0$, p being the root of $x^2 + 4 = 0$

$\Rightarrow p(p^2 + 4) = 0$. Now, $p \neq 0$. Also, $p^2 + 4 \neq 0$

$\Rightarrow p(p^2 + 4) \neq 0 \therefore x = 0$ is the root of $x^3 + 4x = 0$

2. Let $a = 1, b = -1$, then $a^2 = b^2 = 1$ but $a \neq b$.

3. The statement is of the form 'If p, then q'. Let q be false, *i.e.*, let x be not even

$\Rightarrow x = 2n + 1$, where n is an integer.

$\Rightarrow x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1 = 1$ more than an even number

$\Rightarrow x^2$ is odd

i.e., If q is not true, then p is not true is proved.

Hence, the given statement is true.

4. Compound statement with 'AND' is 30 is a multiple of 5 and 9. This is a false statement because p and q both are not true. Compound statement with 'OR' is 30 is a multiple of 5 or is a multiple of 9. This is a true statement since one of the components, *viz.*, p is true.

5. Type solved Ex. 30.

CHAPTER SUMMARY

1. A **statement** is a sentence which is either true or false, but not both.
e.g. 'Taj Mahal is located in Agra' and '1 is a prime number'.

2. If p denotes a statement, then $\sim p$ denotes negation of p.

3. A **compound statement** is a statement made up of two or more component statements. They are made by

(*i*) **Connectives**: '*AND*', '*OR*'

(*ii*) **Quantifiers**: 'There exists' $\exists$ and 'for every $\forall$

(*iii*) **Implications:** If p then q $(p \Rightarrow q)$ If and only if $(p \Leftrightarrow q)$. It means $p \Rightarrow q$ and $q \Rightarrow p$.
e.g. A triangle is equiangular if and only if it is equiangular is an implication with the two component statements

p : A triangle is equilateral

q : A triangle is equiangular.

4. (*i*) The **contrapositive** of the statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.

(*ii*) The **converse** of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

(*iii*) The **inverse** of the statement $p \Rightarrow q$ is the statement $\sim p \Rightarrow \sim q$.

5. Contradiction. In this method to prove that p is true, we start the assumption that negation p is true.

6. A statement is said to be **valid** or **invalid** according as it is true or false.

7. Validating statements means checking a statement whether it is true or false.

8. For ready reference we give below the symbols used for logical concepts and the five basic truth tables for compound statements. The letters p and q represent simple statements.

Operation	*How read*	*Symbolism*
Conjunction	p and q	$p \wedge q$
Disjunction	p or q	$p \vee q$
Conditional	If p, then q	$p \Rightarrow q$
Equivalence or Biconditional	p if and only if q	$p \Leftrightarrow q$
Negation	not p	p' or $\sim p$

Negation

p	$\sim p$
T	F
F	T

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

CHAPTER TEST

1. For each of the following compound statements, first identify the connective words and then break it into component statements.
(*i*) All rational numbers are real and all real numbers are not complex.
(*ii*) Square of an integer is positive or negative.
(*iii*) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

2. Identify the quantifier in the following statements and write the negation of the statements.
(*i*) There exists a number which is equal to its square.
(*ii*) For every real numbers x, x is less than $x + 1$.
(*iii*) There exists a capital for every state in India.

3. For any statement 'p' prove that $\sim(\sim p) \equiv p$.

4. Write the converse, contradiction and contrapositive of the statement
'If $x + 3 = 9$, then $x = 6$.'

5. For any statements, p and q, prove that $p \Rightarrow q \equiv (\sim p \vee q)$.

6. Write the following implications $(p \Rightarrow q)$ in the form $(\sim p \vee q)$ and write its negation.
'*If* ΔABC *is isosceles then the base angles A and B are equal.*'

ANSWERS

1. (*i*) The connective is 'AND'. Its component statements are:
p : All rational numbers are real.
q : All real numbers are not complex.
(*ii*) The connective word is 'OR'. Its component statements are:
p : Square of an integer is positive.
q : Square of an integer is negative.
(*iii*) The connective is 'AND'. The component statements are:
p : $x = 2$ is a root of the equation $3x^2 - x - 10 = 0$
q : $x = 3$ is a root of the equation $3x^2 - x - 10 = 0$

2. (*i*) The quantifier for the given statement is '*there exists*'. It is true.
Negation or the given statement is $\sim p$: There does not exist a number which is equal to its square.

(*ii*) The quantifier for the given statement is '*For every*'. It is true.
Negation of the given statement is
$\sim p$: For every real number x, x is not less than $x + 1$.

(*iii*) The quantifier for the given statement is '*there exists*'. It is true. Negation of this statement is
$\sim p$: There exists a state in India which does not have a capital.

3. Prepare the truth table as shown.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Since the columns of p and $\sim(\sim p)$ coincide, so $\sim(\sim p) \equiv p$.

4. Given statement is $x + 3 = 9 \Rightarrow x = 6$.

(*i*) Its converse is $x = 6 \Rightarrow x + 3 = 9$, *i.e.*, if $x = 6$, then $x + 3 = 9$.

(*ii*) Its contradiction is $x = 6 \Rightarrow x + 3 \neq 9$, *i.e.*, if $x = 6$, then $x + 3 \neq 9$.

(*iii*) Its contrapositive is $x \neq 6 \Rightarrow x + 3 \neq 9$, *i.e.*, if $x \neq 6$, then $x + 3 \neq 9$.

5. Prepare the truth table.

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

In this table, the columns headed $p \Rightarrow q$ and $\sim p \vee q$ coincide. Hence, $(p \Rightarrow q) \equiv (\sim p \vee q)$.

6. Let p : ΔABC is isosceles
q : The base angles A and B are equal.
Then, $(\sim p \vee q)$: Either ΔABC is not isosceles or the base angles A and B are equal. The negation of the given statement is $(p \wedge \sim q)$, given by ΔABC is isosceles, and the base angles A and B *are not equal.*

CHAPTER 28

STATISTICS
(Continued from Chapter 20)

28.01. Mean of the combined distributions

When two sets of scores have been combined into a single distribution, then the mean of the combined distribution is the weighted *mean of the means of the components, the weights being the total frequencies in those components.*

In other words, $$\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

where $\bar{x}$ is the mean of the combined distribution, $\bar{x}_1$ and $\bar{x}_2$ are the means of the component distributions, n_1 and n_2 are the total frequencies of the component distributions.

Proof. $\frac{\Sigma x}{n_1} = \bar{x}_1$, $\frac{\Sigma y}{n_2} = \bar{x}_2$

$\therefore$ $\Sigma x = n_1 \bar{x}_1$, $\Sigma y = n_2 \bar{x}_2$

Now, by definition,

$$\bar{x} = \frac{\Sigma x + \Sigma y}{n_1 + n_2} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

This formula may be extended to any number of distributions. Thus, if the combined distribution consists of three component parts, then $\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + n_3 \bar{x}_3}{n_1 + n_2 + n_3}$.

Ex. 1. There are 60 students in a class of which 25 are girls. The average weight of 25 girls is 40 kg and that of 35 boys is 53 kg. Find the mean weight in kg of the entire class.

Sol. Let $\bar{x}_1$, $\bar{x}_2$ denote the average weight of girls and boys respectively. Then $\bar{x}_1$ = 40, $\bar{x}_2$ = 53, n_1 = 25, n_2 = 35.

$\therefore$ Average weight of the class $\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$

$$= \frac{25 \times 40 + 35 \times 53}{25 + 35} = \frac{1000 + 1855}{60} = \frac{2855}{60} = \frac{571}{12} = \textbf{47.583 kg.}$$

Ex. 2. The mean monthly salary paid to all employees of a company was ₹ 5,000. The mean annual salary to male and female employees was ₹ 5200 and ₹ 4200 respectively. Determine the percentage of males and females employed by the company.

Sol. Here, $\bar{x} = 5000$, $\bar{x}_1 = 5200$, $\bar{x}_2 = 4200$. Let $n_1 + n_2 = 100$

Then $n_2 = 100 - n_1$ where n_1 denotes males and n_2 denotes females.

$\therefore$ Using the formula $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$, we have $5000 = \dfrac{n_1 \times 5200 + (100 - n_1)\,4200}{100}$

$\Rightarrow$ $500000 = 5200n_1 + 420000 - 4200n_1$,

$80000 = 1000n_1$

$\Rightarrow$ $n_1 = 80$

$\therefore$ $n_1 = 80$, $n_2 = 100 - n_1 = 20$. Hence, number of males and females in the company are 80% and 20% respectively.

28.02. Combined standard deviation

You have learnt that we can find the combined mean $\bar{x}$ of two series of sizes n_1 and n_2 and arithmetic means $\bar{x}_1$ and $\bar{x}_2$ respectively by using the formula

$$\bar{x} = \frac{n_1\,\bar{x}_1 + n_2\,\bar{x}_2}{n_1 + n_2}$$

Let the standard deviation of these series be σ_1 and σ_2 respectively. Then their combined standard deviation σ or σ_{12} is given by the formula

$$\sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}} = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where $d_1 = (\bar{x}_1 - \bar{x})$ and $d_2 = (\bar{x}_2 - \bar{x})$ and $\bar{x}$ is the combined mean.

Ex. 3. The means of two samples of sizes 500 and 600 were respectively 186 and 175. The corresponding standard deviations were respectively 9 and 10. The variable studied was height in centimetres. Obtain the mean and variance of the combined sample.

Sol. Here, $n_1 = 500$, $\bar{x}_1 = 186$, $\sigma_1 = 9$

$n_2 = 600$, $\bar{x}_2 = 175$, $\sigma_2 = 10$

$\therefore$ Combined mean $(\bar{x}) = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \dfrac{500 \times 186 + 600 \times 175}{500 + 600} = \dfrac{198000}{1100}$ **= 180 cm**

Combined s.d. $\sigma_{12} = \sqrt{\dfrac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_1^2 + d_2^2)}{n_1 + n_2}}$

where $d_1 = \bar{x} - \bar{x}_1 = 180 - 186 = -6$, $d_2 = \bar{x} - \bar{x}_2 = 180 - 175 = 5$

$\therefore$ $\sigma_{12} = \sqrt{\dfrac{500\,(9^2 + (-6)^2) + 600\,(10^2 + 5^2)}{500 + 600}} = \sqrt{\dfrac{500 \times 117 + 600 \times 125}{1100}}$

$= \sqrt{\dfrac{58500 + 75000}{1100}} = \sqrt{\dfrac{133500}{1100}} = \sqrt{\dfrac{1335}{11}} = \sqrt{121.36}$

$\therefore$ Variance of the combined sample = σ_{12}^2 = **121.36.**

Ex. 4. The mean and standard deviations of monthly salary (in ₹) are given below for two factories A and B.

Factory	No. of employees	Average monthly salary	Standard deviation of monthly salary
A	100	2500	200
B	120	2800	250

(*i*) Find the pooled mean and pooled standard deviation.

(*ii*) Which factory has better consistent salary structure?

Sol. (*i*) Pooled $(\bar{x}) = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \dfrac{100 \times 2500 + 120 \times 2800}{100 + 120}$ **= 2663.63.**

(*ii*) Pooled Standard deviation $(\sigma_{12}) = \sqrt{\dfrac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$

where $d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$.

Here, $d_1 = 2500 - 2663.63 = -163.63$, $d_2 = 2800 - 2663.63 = 136.37$

$$\therefore \quad \sigma_{12} = \sqrt{\frac{100\,(200^2 + (-163.63)^2) + 120\,(250^2 + (136.37)^2)}{100 + 120}}$$

$$= \sqrt{\frac{100\,(40000 + 26774.7769) + 120\,(62500 + 18596.7769)}{220}}$$

$$= \sqrt{\frac{100 \times 66774.7769 + 120 \times 81096.7769}{220}} = \sqrt{\frac{6677477.69 + 9731613.228}{220}}$$

$$= \sqrt{\frac{16409090.92}{220}} = \sqrt{74586.776909} = \sqrt{74586.78} = \mathbf{273.106}$$

(*ii*) Coeff. of variation for factory $A = \dfrac{\sigma_1}{\bar{x}_1} \times 100 = \dfrac{200}{2500} \times 100 = 8\%$

Coeff. of variation for factory $B = \dfrac{\sigma_2}{\bar{x}_2} \times 100 = \dfrac{250}{2800} \times 100 = 8.9\%$

Since coeff. of variation is less for factory A, it is more consistent.

EXERCISE 28 (*a*)

1. One set of 100 observations has the mean 15 and another set of 150 observations has the mean 16. Find the mean of 250 observations by combining the two sets of given observations.

2. The mean age of 40 students is 16 years and the mean age of another group of 60 students is 20 years. Find out the mean age of all the 100 students combined together.

3. The mean of marks obtained in an examination by a group of 100 students is found to be 49.46. The mean of the marks obtained in the same examination by another group of 200 students was 52.32. Find the mean of the marks obtained by both the groups of students taken together.

4. The number of students in section XA and XB are 30 and 35 respectively. The mean scores of students in the mathematics test are as follows:

XA	XB	XA and XB combined
70	?	62

Find the mean score of XB.

5. Two samples of sizes 50 and 100 are given. The mean of these samples respectively are 56 and 50. Find the mean of size 150 by combining. **[*ISC BH 2004*]**

6. The mean and standard deviation of the marks obtained by the groups of students, consisting of 50 each, are given below. Calculate the mean and standard deviation of the marks obtained by all the 100 students.

Groups	*Mean*	*Standard deviation*
1	60	8
2	55	7

7. The mean and standard deviation of distribution of 100 and 150 items are 50, 5 and 40, 6 respectively. Find the mean and standard deviation of all the 250 items taken together.

ANSWERS

1. 15.6 **2.** 18.4 **3.** 51.37 **4.** 55.14 **5.** 52 **6.** $\bar{X}_{12} = 57.5, \sigma_{12} = 7.92$

7. $\bar{X}_{12} = 44, \sigma_{12} = 7.46$

28.03. Introduction

In chapter 20, you learnt how to find arithmetic mean for simple and frequency distributions-both ungrouped and grouped and also method of finding median for simple distribution.

In this chapter, we will tell you how to find median and other partition values for frequency distributions. You will also learn how to find mode graphically.

28.04. Median in a discrete distribution (Simple distribution)

If a set of numbers is given, they can be arranged in order of magnitudes, preferably the highest score at the top, then the middle point of the array gives the position of the median and the score at that point is called the ***median*** or the ***median score***. There will be an equal number of items above and below the median.

For a set of observations arranged in order of size, the median is the value 50% of the way through the distribution. Described often as the middle value, the median is an average that is unaffected by extreme values.

For instance, if seven boys of different heights are made to stand in a row, the tallest first, the next tallest next, and so on, the *median height* is of the *fourth* boy from either end. If there is an even number of boys, say eight, it would be natural to take as median the height midway between that of the fourth and that of the fifth boy, *i.e.,* the position given by $\frac{4+5}{2}$ or 4.5th item from either end.

Median for an ungrouped simple distribution

For ungrouped data consisting of n observations in order of size.

$$\textbf{Median } (M_d) = \textbf{Size of the } \frac{n+1}{2}\textbf{th observation}$$

Ex. 5. A student secured the following marks in seven subjects: 50, 53, 61, 49, 45, 63, 48. Find the median score.

Sol. Arranging the marks in ascending order, we have 45, 48, 49, 50, 53, 61, 63. Since the series contains seven items, $n = 7$, $\frac{n+1}{2} = \frac{7+1}{2} = 4$

$\therefore$ M_d = Size of 4th item = **50 marks.**

Ex. 6. Find the median of the following set of numbers:

10, 75, 3, 81, 18, 27, 4, 48, 12, 47, 9, 15

Sol. Arranging the numbers in ascending order, we have

3, 4, 9, 10, 12, 15, ↓, 18, 27, 47, 48, 75, 81

Here, $n = 12$ and $\frac{n+1}{2} = \frac{12+1}{2} = 6.5$

In this case, 6.5th item lies midway between 6th and 7th items, as shown by the arrow above.

M_d = Size of 6.5th item = average of 15 and 18 = $\frac{15+18}{2} = \mathbf{16.5.}$

Ex. 7. The weights (in kilogram) of 15 students are as follows:

31, 35, 27, 29, 32, 43, 37, 41, 34, 28, 36, 44, 45, 42, 30

Find the median. If the weight 44 kg is replaced by 46 kg and 27 by 25 kg, find the new median.

Sol. Writing the given weights in increasing order, we have

27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 41, 42, 43, 44, 45

Here $n = 15 \Rightarrow \frac{n+1}{2} = \frac{15+1}{2} = 8$ $\therefore$ Median = size of 8th term = **35 kg**

When 44 kg is replaced by 46 kg and 27 kg by 25 kg, the new array is

25, 28, 29, 30, 31, 32, 34, 35, 36, 37, 41, 42, 43, 45, 46

$\therefore$ New median = 8th term = **35 kg.**

Ex. 8. The following table represents the marks obtained by a batch of 12 students in certain class tests in Statistics and Physics.

S.No.	1	2	3	4	5	6	7	8	9	10	11	12
Marks (Statistics)	53	54	32	30	60	46	28	25	48	72	33	65
Marks (Physics)	55	41	48	49	27	24	23	20	28	60	43	67

Indicate in which subject the level of achievement is higher?

Sol. The level of achievement is higher in that subject for which median marks are greater. Arranging the marks in the two subjects in ascending order, we have

S. No.	1	2	3	4	5	6	7	8	9	10	11	12
Marks (Statistics)	25	28	30	32	33	46	48	53	54	60	65	72
Marks (Physics)	20	23	24	27	28	41	43	48	49	55	60	67

Since the number of students is 12 (even), therefore, the marks of the middle most student are the mean of the marks of the 6th and 7th student.

Median marks in Statistics = $\frac{46+48}{2} = \mathbf{47}$

Median marks in Physics = $\frac{41+43}{2} = \mathbf{42}$

Since the median marks in statistics are greater than the median marks in physics, therefore, the level of achievement of the students is higher in statistics.

28.05. Median in an ungrouped frequency distribution

Method. Step 1. *Arrange the data in ascending or descending order of magnitude.*
Step 2. *Find out cumulative frequencies (c.f.).*

Step 3. *Find* $\frac{n}{2}$ *if the distribution is continuous and* $\frac{n+1}{2}$ *if it is discrete.*

Note. *When the given distribution does not reveal its nature, i.e., whether it is discrete or continuous, then you may solve the problem by taking* $\frac{n+1}{2}$ *or* $\frac{n}{2}$. *Most of the authors generally take* $\frac{n+1}{2}$.

Step 4. *Look at the cumulative frequency columns for the c.f. which is* $\geq \frac{n+1}{2}$ *or* $\frac{n}{2}$ *as the case may be. The corresponding value of the variable gives the median.*

Ex. 9. Locate the median from the following data:

Size of shoes	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
Frequency	10	18	22	25	40	15	10	8	7

Sol.

Size of shoes	*f*	*c.f.*	*Size of shoes*	*f*	*c.f.*
4.0	10	10	6.5	15	130
4.5	18	28	7.0	10	140
5.0	22	50	7.5	8	148
5.5	25	75	8.0	7	155
6.0	40	115			

Median = size of $\frac{n+1}{2}$ th item = size of $\left(\frac{155+1}{2}\right) = \frac{156}{2} = 78$th item.

This lies, under the *c.f.* column, in 115. Size of the shoes corresponding to this is 6.0. Hence, median size of the shoes = **6.**

Ex. 10. Find the median of the following distribution.

x	8	5	6	10	9	4	7
f	6	4	5	8	9	6	4

Sol. First arrange the value of x in ascending order and then find cumulative frequencies.

x	f	$c.f.$
4	6	6
5	4	10
6	5	15
7	4	19
8	6	25
9	9	34
10	8	42
Total	42	

$$\frac{n+1}{2} = \frac{42+1}{2} = \frac{43}{2} = 21.5$$

$\therefore$ The median is the size of the 21.5th item, *i.e.*, the average of the 21st and 22nd items. The value of each item from 20th to 25th is 8.

$\therefore$ Median = Mean of 21st and 22nd items = $\frac{8+8}{2} = 8$

28.06. Median in a grouped distribution

Let us consider the following distribution:

Marks (*x*)	*Frequency* (*f*)	*Frequency density* (*f.d.*)	(*c.f.*)
1 – 10	0	0	0
11 – 20	30	3	30
21 – 30	60	6	90
31 – 40	220	22	310
41 – 50	540	54	850
51 – 60	490	49	1340
61 – 70	310	31	1650
71 – 80	180	18	1830
81 – 90	110	11	1940
91 – 100	60	6	$n = 2000$

We draw the histogram as shown in Fig. 28.01.

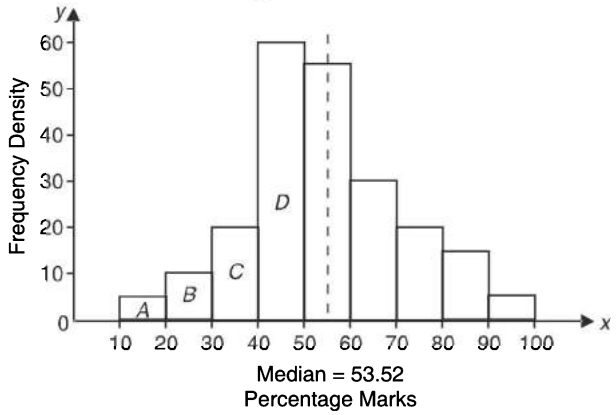


Fig. 28.01

The sum of the areas of the rectangles *A, B, C* and *D* is equal to the frequency 0 + 30 + 60 + 220 + 540, or 850. Now since $\frac{n}{2} = 1000$, therefore, the remaining frequency difference of 150 we have to get from the rectangle next to the rectangle *D*. We apply the method of interpolation as shown here:

The frequency of 490 is represented by the width 60.5 – 50.5, *i.e.,* 10.

$\therefore$ The frequency of 150 is represented by the width $\frac{10}{490} \times 150$ or 3.06. Hence, the median is equal to 50.5 + 3.06 or 53.56. In the figure the dotted line is the ordinate that divides the total area of the histogram into two parts of equal area.

It should be noted that it is not necessary to draw the histogram, though the histogram helps a good deal to understand why the position of the median is given by $\frac{n}{2}$. Even without the histogram the median can be estimated by adding one more column of cumulative frequency *(c.f.)* as shown below:

Since $\frac{n}{2} = 1000$, which is more that 850 but less than 1340 (see cumulative frequency column), therefore, it is clear that the median lies in the class 51 – 60, which we shall call as the median class. For the cumulative frequency of 850 the value of the variable is the exact upper limit 50.5 of the class 41 – 50. The value of x for the difference of 1000 – 850 or 150 can be estimated by the method of interpolation as shown below:

The frequency of 490 corresponds to the class interval of 10 *i.e.,* (60.6 – 50.5).

$\therefore$ The frequency of 150 corresponds to the class interval of $\frac{10}{490} \times 150$ or 3.06.

Hence, the median is equal to 50.5 + 3.06 or 53.56.

The above method can be concisely put in the form of a formula which is given below:

$$\textbf{Median } (M_d) = L + \frac{i}{f}\left(\frac{n}{2} - c\right)$$

where n is total number of frequencies, i.e., $n = \sum f_i$.

L *is the lower limit of the class in which median lies.*
i *is the class interval of the median class.*
f *is the frequency of the class in which the median lies.*
c *is the cumulative frequency of the class just lower than the median class.*

If we apply this formula to the example given above, we get

$$\frac{n}{2} = \frac{2000}{2} = 1000, \qquad c = 850,$$

$$i = 10, f = 490, \qquad L = 50.5$$

$\therefore$ $$M_d = 50.5 + \frac{10}{490}(1000 - 850)$$

$$= 50.5 + 3.06 = 53.56$$

Ex. 11. The marks (out of 75) obtained by 60 students in a certain examination are given below. Find the median marks.

Marks	15 – 20	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55	55 – 60	60 – 65
No. of Students	4	5	11	6	5	8	9	6	4	2

Sol.

Marks	*Number of students (f)*	*c.f.*
15 – 20	4	4
20 – 25	5	9
25 – 30	11	20
30 – 35	6	26
35 – 40	5	31
40 – 45	8	39
45 – 50	9	48
50 – 55	6	54
55 – 60	4	58
60 – 65	2	60

Here, $\frac{n}{2} = \frac{60}{2} = 30$. The cumulative frequency just greater than 30 is 31 and the corresponding class is 35 – 40. Therefore, the median class is 35 – 40.

$$\text{Median} = L + \frac{i}{f}\left(\frac{n}{2} - c\right) = 35 + \frac{5}{5}\ (30 - 26) = \textbf{39 marks.}$$

Ex. 12. The following table gives the population of males in different age groups.

Age group (in years)	**5 – 14**	**15 – 24**	**25 – 34**	**35 – 44**	**45 – 54**	**55 – 64**	**65 – 74**
No. of males (in lakhs)	**447**	**307**	**279**	**220**	**157**	**91**	**39**

Calculate their median age.

Sol. First it is necessary to convert the given inclusive distribution into the exclusive distribution.

Age group (in years)	*Class boundaries*	*No. of males (in lakhs)*	*c.f.*
5 – 14	4.5 – 14.5	447	447
15 – 24	14.5 – 24.5	307	754
25 – 34	24.5 – 34.5	279	1033
35 – 44	34.5 – 44.5	220	1253
45 – 54	44.5 – 54.5	157	1410
55 – 64	54.5 – 64.5	91	1501
65 – 74	64.5 – 74.5	39	1540

Here, $\frac{n}{2} = \frac{1540}{2} = 770$. The *c.f.* just greater than 770 is 1033 and the corresponding class is 24.5 – 34.5 which is the median class. Using the formula.

$$\text{Median} = L + \frac{i}{f}\left(\frac{n}{2} - c\right), \text{ we have } M_d = 24.5 + \frac{10}{279}\ (770 - 754) = \textbf{25.07 years.}$$

Ex. 13. The following table shows age distribution of persons in a particular region:

Age (years)	**No. of persons ('000)**	**Age (years)**	**No. of persons ('000)**
Below 10	**2**	**Below 50**	**14**
Below 20	**5**	**Below 60**	**15**
Below 30	**9**	**Below 70**	**15.5**
Below 40	**12**	**70 and above**	**15.6**

Find the median age.

Sol. The frequencies are given as cumulative frequencies. First, we convert them into absolute frequencies.

Age groups	*No. of persons ('000)*	*c.f.*
0 – 10	2	2
10 – 20	3	5
20 – 30	4	9
30 – 40	3	12
40 – 50	2	14
50 – 60	1	15
60 – 70	0.5	15.5
70 and above	0.1	15.6

$$\frac{n}{2} = \frac{15.6}{2} = 7.8$$

Median lies in the 20 – 30 age group

$$\therefore \quad M_d = L + \frac{i}{f}\left(\frac{n}{2} - c\right)$$

$$= 20 + \frac{10}{4}(7.8 - 5)$$

$$= 20 + \frac{10}{4} \times 2.8$$

$$= 20 + 7 = \mathbf{27\ years}$$

Ex. 14. Compute median from the following data of a continuous distribution:

Mid Values	**115**	**125**	**135**	**145**	**155**	**165**	**175**	**185**	**195**
Frequency	**6**	**25**	**48**	**72**	**116**	**60**	**38**	**22**	**3**

Sol. Here, we are given the mid values of the class intervals of a continous distribution. The difference between the two mid values is 10. Therefore, we subtract $\frac{1}{2}$ (10) = 5 from each to get the lower limit and add 5 to get the upper limit of each class interval. The given distribution is then re-written as under :

Class Interval	*Frequency (f)*	*c.f.*
110 – 120	6	6
120 – 130	25	31
130 – 140	48	79
140 – 150	72	151
150 – 160	116	267
160 – 170	60	327
170 – 180	38	365
180 – 190	22	387
190 – 200	3	390
Total	390	

$$\frac{n}{2} = \frac{390}{2} = 195$$

which lies in the class 150 — 160.

$$\therefore \quad \text{Median} = L + \frac{i}{f}\left(\frac{n}{2} - c\right)$$

$$= 150 + \frac{10}{116}(195 - 151)$$

$$= 150 + \frac{440}{116} = 150 + 3.79$$

$$= \mathbf{153.79}$$

Ex. 15. Following is the distribution of marks, obtained by 50 students in a certain subject:

Marks more than	**0**	**10**	**20**	**30**	**40**	**50**
No. of students	**50**	**46**	**40**	**20**	**10**	**3**

Calculate the median marks.

Sol. First, we re-write the distribution after converting the given cumulative frequencies into absolute frequencies.

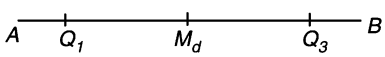
Marks more than	*No. of Students*	*Class interval*	*No. of Students (f)*	*c.f.*
0	50	0 – 10	4 = (50 – 46)	4
10	46	10 – 20	6 = (46 – 40)	10
20	40	20 – 30	20 = (40 – 20)	30 ←
30	20	30 – 40	10 = (20 – 10)	40
40	10	40 – 50	7 = (10 – 3)	47
50	3	50 and above	3	50

Here $\frac{n}{2} = \frac{50}{2} = 25$ $\therefore$ Median lies in the class 20 – 30.

$\therefore$ Median $= L + \frac{i}{f}\left(\frac{n}{2} - c\right) = 20 + \frac{10}{20}(25 - 10) = 20 + \frac{1}{2}(15) = 20 + 7.5 =$ **27.5 marks.**

28.07. Further partition values — Quartiles, deciles, percentiles, etc.

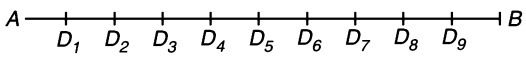
Just as medians divide distributions into two equal subgroups, quartiles divide into four, quintiles into five, sextiles into six, deciles into ten and percentiles (sometimes called centiles) into one hundred. Remember that medians, quartiles, quintiles, sextiles, deciles, and percentiles are points of division. Do you notice that the median (sometimes called Q_2) is the 5th decile (D_5) or the 50th percentile (P_{50}) ($M_d = Q_2 = D_5 = P_{50}$). Also, D_1 (first decile) is the tenth percentile (P_{10}), the lower quartile Q_1 is the 25th percentile ($Q = P_{25}$) and the upper quartile Q_3 is the 75th percentile ($Q_3 = P_{75}$). Quarters should not be confused with quartiles.



(Also called Q_2)

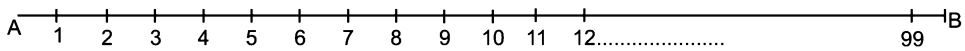
Median and Quartiles

(Distribution divided into 4 equal parts)



Deciles

(Distribution divided into ten equal parts)



Percentiles

(Distribution divided into 100 equal parts)

For data arranged in order of size:

- **the lower quartile Q_1** is the value 25% of the way through the distribution.
- **the upper quartile Q_3** is the value 75% of the way through the distribution.
- **the first decile D_1** is the value 10% of the way through the distribution.
- **the nth percentile, P_n** is the value n% of the way through the distribution.

28.08. Interquartile range

The difference between the upper and the lower quartile values is called interquartile range. The upper quartile is denoted by Q_3 and the lower quartile by Q_1.

Thus interquartile range = upper quartile – lower quartile = $Q_3 - Q_1$

Quartile deviation = Semi-interquartile range = $\frac{Q_3 - Q_1}{2}$.

Similarly, the difference $D_9 - D_1$ between the upper decile and the lower decile is called the **interdecile range**, and the difference $P_{90} - P_{10}$ between the upper percentile and the lower percentile is called **interpercentile range**.

We can calculate quartiles with the help of the following formulae:

$$Q_1 = L + \frac{i}{f}\left(\frac{n}{4} - c\right), \quad Q_3 = L + \frac{i}{f}\left(\frac{3n}{4} - c\right)$$

For deciles we use $\frac{n}{10}$ for the first decile, $\frac{2n}{10}$ for the second decile, $\frac{3n}{10}$ for the third decile, and so on. For percentiles we use $\frac{n}{100}$ for the first percentile, $\frac{2n}{100}$ for the second percentile, $\frac{3n}{100}$ for the third percentile, and so on.

Note 1. If the variable in the grouped frequency distribution is discrete, then in all the above formulae $(n + 1)$ will be used instead of n.

Note 2. The median is not necessarily midway between Q_1 and Q_3, although this will be so for a symmetrical distribution.

The following table will help you in understanding the method of computation of different fractiles.

Measure	*Ungrouped data (Simple and frequency distribution)*	*Grouped data*
Q_1	$\frac{n+1}{4}$ th item	$\frac{n}{4}$ th item; $Q_1 = L + \frac{i}{f}\left(\frac{n}{4} - c\right)$
Q_3	$\frac{3(n+1)}{4}$ th item	$\frac{3n}{4}$ th item; $Q_3 = L + \frac{i}{f}\left(\frac{3n}{4} - c\right)$
D_1	$\frac{n+1}{10}$ th item	$\frac{n}{10}$ th item; $= D_1 = L + \frac{i}{f}\left(\frac{n}{10} - c\right)$
D_7	$\frac{7(n+1)}{10}$ th item	$\frac{7n}{10}$ th item; $= D_7 = L + \frac{i}{f}\left(\frac{7n}{10} - c\right)$
P_1	$\frac{n+1}{100}$ th item	$\frac{n}{100}$ th item; $= P_1 = L + \frac{i}{f}\left(\frac{n}{100} - c\right)$
P_{47}	$\frac{47(n+1)}{100}$ th item	$\frac{47n}{100}$ th item; $= P_{47} = L + \frac{i}{f}\left(\frac{47n}{100} - c\right)$

28.09. Solved examples

Type I. Individual Series (Simple Distribution)

Ex. 16. Find the median, Q_1, Q_3 and the interquartile range for the following distribution:

5, 3, 6, 3, 13, 9, 8, 24, 19, 20, 18

Sol. Arranging the given set of data in ascending order, we have

3 3 (5) 6 8 (9) 13 18 (19) 20 24

Q_1 M_d Q_3

$\frac{n+1}{2} = \frac{11+1}{2} = 6$ $\quad\therefore\quad M_d$ = size of the 6th term = 9

$\frac{n+1}{4} = \frac{11+1}{4} = 3$ $\quad\quad Q_1$ = size of the 3rd term = 5

$\frac{3(n+1)}{4} = \frac{3\times(11+1)}{4} = 9$ $\quad\quad Q_3$ = size of the 9th term = 19

Ex. 17. Calculate median, quartiles, 8th decile and sixty-first percentile from the following data of the marks obtained by 10 students in an examination.

22, 26, 14, 30, 18, 11, 35, 41, 12, 32

Sol. Arranging the given data in ascending order, we have

11, 12, 14, 18, 22, 26, 30, 32, 35, 41

$\frac{n+1}{2} = \frac{10+1}{2} = 5.5$

$\therefore$ Med. = size of 5.5th item

$$= \frac{1}{2}\text{ (5th item + 6th item)} = \frac{22+26}{2} = \frac{48}{2} = \textbf{24 marks}$$

$$\frac{n+1}{4} = \frac{10+1}{4} = 2.75, \qquad \frac{3(n+1)}{4} = 3 \times 2.75 = 8.25$$

$\therefore$ Q_1 = size of 2.75th item = 2nd item + 0.75 (3rd item – 2nd item)

= 12 + 0.75 (14 – 12) = 12 + 1.5 = **13.5 marks**

Q_3 = size of 8.25th item = 8th item + 0.25 (9th item – 8th item)

= 32 + 0.25 (35 – 32) = 32 + 0.25 × 3 = 32 + 0.75 = **32.75 marks**

$$\frac{8(n+1)}{10} = \frac{8(10+1)}{10} = \frac{88}{10} = 8.8, \qquad \therefore \quad D_8 = \text{size of 8.8th item}$$

= 8th item + 0.8 (9th item – 8th item) = 32 + 0.8 (35 – 32)

= 32 + 0.8 × 3 = 32 + 2.4 = **34.4 marks**

$$\frac{61(n+1)}{100} = \frac{61(10+1)}{100} = \frac{61\times 11}{100} = \frac{671}{100} = 6.71 \qquad \therefore \ P_{61} = \text{size of 6.71th item}$$

= 6th item + 0.71 (7th item – 6th item)

= 26 + 0.71 (30 – 26) = 26 + 2.84 = **28.84 marks.**

Type II. Discrete Series (Ungrouped frequency distributions)

Ex. 18. Locate median, upper and lower quartiles from the following data.

Size of shoes	Frequency	Size of shoes	Frequency
4.0	10	6.5	15
4.5	18	7.0	10
5.0	22	7.5	8
5.5	25	8.0	7
6.0	40		

Sol. Prepare the cumulative frequency table.

Size of shoes	*f*	*c.f.*
4.0	10	10
4.5	18	28
5.0	22	50
5.5	25	75
6.0	40	115
6.5	15	130
7.0	10	140
7.5	8	148
8.0	7	155

$$\frac{n+1}{2} = \frac{155+1}{2} = 78$$

$\therefore$ M_d = size of 78th item = 6

$\therefore$ Median size of shoes = **6**

$$\frac{n+1}{4} = \frac{155+1}{4} = 39, \frac{3(n+1)}{4} = 3 \times 39 = 117$$

$\therefore$ Lower Quartile (Q_1) = size of 39th item = **5**

Upper Quartile (Q_3) = size of 117th item = **6.5**

Type III. Grouped frequency distributions

Ex. 19. Find lower quartile, median, upper quartile, seventh decile, sixtieth percentile for the following frequency distribution.

Wages (₹)	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of persons	1	3	11	21	43	32	9

Sol.

Wages (₹)	*f*	*c.f.*
10 – 20	1	1
20 – 30	3	4
30 – 40	11	15
40 – 50	21	36
50 – 60	43	79
60 – 70	32	111
70 – 80	9	120

$\frac{n}{4} = \frac{120}{4} = 30$ $\therefore$ Q_1 lies in the class 40 – 50.

$\therefore \quad Q_1 = L + \frac{i}{f}\left(\frac{n}{4} - c\right) = 40 + \frac{10}{21}(30 - 15)$

$= 40 + \frac{150}{21} = 40 + 7.14 = 47.14$

$\frac{n}{2} = \frac{120}{2} = 60$ $\therefore$ M_d lies in the class 50 – 60.

$\therefore \quad M_d = L + \frac{i}{f}\left(\frac{n}{2} - c\right) = 50 + \frac{10}{43}(60 - 36)$

$= 50 + \frac{240}{43} = 50 + 5.58 = \mathbf{55.58}$

$\frac{3n}{4} = \frac{3 \times 120}{4} = 90 = Q_3$ lies in the class 60 – 70.

$\therefore \quad Q_3 = L + \frac{i}{f}\left(\frac{3n}{4} - c\right) = 60 + \frac{10}{32}(90 - 79) = 60 + \frac{110}{32} = 60 + 3.44 = \mathbf{63.44.}$

$\frac{7n}{10} = \frac{7 \times 120}{10} = 84$ $\therefore$ D_7 lies in the class 60 – 70.

$\therefore \quad D_7 = L + \frac{i}{f}\left(\frac{7n}{10} - c\right) = 60 + \frac{10}{32}(84 - 79) = 60 + \frac{50}{32} = 60 + 1.56 = \mathbf{61.56.}$

$\frac{60n}{100} = \frac{60 \times 120}{100} = 72$ $\therefore$ P_{60} lies in the class 50 – 60.

$\therefore \quad P_{60} = L + \frac{i}{f}\left(\frac{60n}{100} - c\right) = 50 + \frac{10}{43}(72 - 36) = 50 + \frac{360}{43} = 50 + 8.37 = \mathbf{58.37.}$

Ex. 20. Find the values of D_8 and P_{75} from the following continuous frequency distribution.

Class	**10 – 14**	**15 – 19**	**20 – 24**	**25 – 29**	**30 – 34**	**35 – 39**
Frequency	**3**	**7**	**16**	**12**	**9**	**5**

Sol. First change the given distribution into exclusive.

Class boundaries	*Frequency f*	*c.f.*
9.5 – 14.5	3	3
14.5 – 19.5	7	10
19.5 – 24.5	16	26
24.5 – 29.5	12	38
29.5 – 34.5	9	47
34.5 – 39.5	5	52

$\frac{8n}{10} = \frac{8 \times 52}{10} = \frac{416}{10} = 41.6$

$\therefore$ D_8 lies in the class 29.5 – 34.5.

$\therefore \quad D_8 = L + \frac{i}{f}\left(\frac{8n}{10} - c\right)$

$= 29.5 + \frac{5}{9}(41.6 - 38)$

$= 29.5 + \frac{5 \times 3.6}{9} = 29.5 + 2 = \mathbf{31.5.}$

$\frac{75n}{100} = \frac{75 \times 52}{100} = \frac{3900}{100} = 39$

$\therefore$ P_{75} lies in the 29.5 – 34.5 class.

$\therefore \quad P_{75} = L + \frac{i}{f}\left(\frac{75n}{100} - c\right) = 29.5 + \frac{5}{9}(39 - 38) = 29.5 + \frac{5}{9} = 29.5 + 0.56 = \mathbf{30.06.}$

Ex. 21. For the following data compute the values of upper and lower quartiles, median, D_6, P_{20} and O_3.

Marks	No. of students	Marks	No. of students
Below 10	5	40 – 50	90
10 – 20	25	50 – 60	40
20 – 30	40	60 – 70	20
30 – 40	70	Above 70	10

Sol.

Marks	*f*	*c.f.*
Below 10	5	5
10 – 20	25	30
20 – 30	40	70
30 – 40	70	140
40 – 50	90	230
50 – 60	40	270
60 – 70	20	290
Above 70	10	300
	$n = 300$	

$$\frac{n}{4} = \frac{300}{4} = 75, \frac{3n}{4} = 3 \times 75 = 225$$

$\therefore$ Q_1 lies in the class 30 – 40.

$$\boldsymbol{Q_1} = L + \frac{i}{f}\left(\frac{n}{4} - c\right) = 30 + \frac{10}{70}(75 - 70) = 30 + 0.714 = \mathbf{30.714}$$

Q_3 lies in the class 40 – 50.

$$\boldsymbol{Q_3} = L + \frac{i}{f}\left(\frac{3n}{4} - c\right) = 40 + \frac{10}{90}(225 - 140)$$

$$= 40 + \frac{85}{9} = 40 + 9.44 = \mathbf{49.44} \text{ or } 49 \text{ approx.}$$

This means that there are 75% students (225) whose marks are 49 and below and 25% students (75) whose marks are above 49.

$$\frac{n}{2} = \frac{300}{2} = 150 \quad \therefore \quad \mathrm{M}_d \text{ lies in the class } 40 - 50.$$

$$\therefore \quad \mathbf{M}_d = L + \frac{i}{f}\left(\frac{n}{2} - c\right) = 40 + \frac{10}{90}(150 - 140) = 40 + \frac{10}{90} = 40 + 1.11 = \mathbf{41.11} \text{ or } 41 \text{ approx.}$$

This means that there are 50% students (150) whose marks are 41 and below and 50% students whose marks are above 41.

$$\frac{6n}{10} = \frac{6 \times 300}{10} = 180 \quad \therefore \quad D_6 \text{ lies in the class } 40 - 50.$$

$$\therefore \quad \boldsymbol{D_6} = L + \frac{i}{f}\left(\frac{6n}{10} - c\right) = 40 + \frac{10}{90}(180 - 140) = 40 + 4.44 = \mathbf{44.44} \quad \text{or} \quad 44 \text{ approx.}$$

This means that there are 60% students (180) whose marks are below 44 and 40% students (12) whose marks are above 44.

$$\frac{20n}{100} = \frac{20 \times 300}{100} = 60 \quad \therefore \quad P_{20} \text{ lies in the class } 20 - 30.$$

$$\therefore \quad \boldsymbol{P_{20}} = L + \frac{i}{f}\left(\frac{20n}{100} - c\right) = 20 + \frac{10}{40}(60 - 30) = 20 + 7.5 = \mathbf{27.5} \text{ or } 28 \text{ approx.}$$

This means that there are 20% students (60) whose marks are below 27.5 and 80% students (240) whose marks are above 27.5.

$$\frac{3n}{8} = \frac{3 \times 300}{8} = 112.5 \quad \therefore \quad O_3 \text{ lies in the class } 30 - 40.$$

$$\therefore \quad \boldsymbol{O_3} = L + \frac{i}{f}\left(\frac{3n}{8} - c\right) = 30 + \frac{10}{70}(112.5 - 70) = 30 + 6.07 = \mathbf{36.07} \text{ or } \mathbf{36} \text{ approx.}$$

Ex. 22. The table shows the number of candidates whose marks are in the ranges indicated. The marks are all whole numbers and the maximum is 49.

Marks	No. of candidates	Marks	No. of candidates
0 – 5	1	25 – 35	28
5 – 10	3	30 – 35	20
10 – 15	6	35 – 40	13
15 – 20	9	40 – 45	3
20 – 25	15	45 – 50	2

Estimate the mean, the median and the 30th percentile.

Sol. We re-write the frequency distribution as shown below:

Marks	*Actual Limits*	x	f	*c.f.*	$d = x - 27$	$t = \frac{d}{s}$	ft
0 – 5	–0.5 – 4.5	2	1	1	–25	–5	–5
5 – 10	4.5 – 9.5	7	3	4	–20	–4	–12
10 – 15	9.5 – 14.5	12	6	10	–15	–3	–18
15 – 20	14.5 – 19.5	17	9	19	–10	–2	–18
20 – 25	19.5 – 24.5	22	15	34←	–5	–1	–15
25 – 30	24.5 – 29.5	27	28	62←	0	0	0
30 – 35	29.5 – 34.5	32	20	82	5	1	20
35 – 40	34.5 – 39.5	37	13	95	10	2	26
40 – 45	39.5 – 44.5	42	3	98	15	3	9
45 – 50	44.5 – 49.5	47	2	100	20	4	8
			100				–5

(*i*) Let the assumed mean A be equal to 27.

$$\text{Mean} = A + i\frac{\Sigma ft}{\Sigma f} = 27.5 + 5 \cdot \frac{-5}{100} = 27 - 0.25 = 26.75 = \textbf{27 marks,}$$

correct to the nearest integer.

(*ii*) The position of the median is given by $\frac{n}{2}$ or $\frac{100}{2}$ or 50,

which lies in the class interval 24.5 – 29.5.

$$M_d = L + \frac{i}{f}\left(\frac{n}{2} - c\right) = 24.5 + \frac{5}{28}(50 - 34) = 24.5 + 2.9 = \textbf{27.4} = \textbf{27 marks,}$$

correct to the nearest integer.

(*iii*) The position of the 30th percentile is given by $\frac{30n}{100}$ or $\frac{30 \times 100}{100}$ or 30,

which lies in the class interval 19.5 – 24.5.

$$\therefore \quad P_{30} = L + \frac{i}{f}\left(\frac{30n}{100} - c\right) = 19.5 + \frac{5}{15}(30 - 19) = 19.5 + 3.7 = 23.2 = \textbf{23 marks,}$$

correct to the nearest integer.

EXERCISE 28 (*b*)

1. Find the median of the following sets of data:

(*i*) 2, 3, 5, 7, 9 (*ii*) 4, 8, 12, 16, 20, 23, 28, 32

(*iii*) 60, 33, 63, 61, 44, 48, 51 (*iv*) 13, 22, 25, 8, 11, 19, 17, 31, 16, 10

2. Find the median of the following data:
41, 43, 127, 99, 61, 92, 71, 58, 57. If 58 is replaced by 85, what will be the new median?

3. Calculate the median of the following incomes.

Wages in ₹	20	21	22	23	24	25	26	27	28
No. of workers	8	10	11	16	20	25	15	9	6

4. Compute the median of the following distributions:

f	2	3	4	5	6	7
x	3	8	10	12	16	14

5. Marks obtained by 38 students are given below. Calculate the median marks:

Marks	20	90	50	70	60	95
No. of students	4	5	8	10	6	5

6. Find whether the following statements are true or false:

(*i*) The median of a frequency distribution is the most commonly occurring value.

(*ii*) The median of a discrete ungrouped frequency distribution containing a number of items is the value of the middle item, the data being arranged in ascending or descending order.

7. In a school examination it is decided that exactly half the pupils will pass. Name the measure of central tendency that is used.

8. (1, 2, 3, 6, 8) is a set of five positive integers whose mean is 4 and median is 3. Write down two other sets of five positive integers, each having the same mean and median as this set.

Find the median from the following distributions:

9.

Marks	10 – 25	25 – 40	40 – 55	55 – 70	70 – 85	85 – 100
Frequency	6	20	44	26	3	1

10.

Class interval	11 – 15	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45	46 – 50
Frequency	7	10	13	26	35	22	11	6

11.

Value	*Frequency*	*Value*	*Frequency*
Less than 10	4	Less than 50	96
Less than 20	16	Less than 60	112
Less than 30	40	Less than 70	120
Less than 40	76	Less than 80	125

12.

Size	*Frequency*	*Size*	*Frequency*
More than 50	0	More than 20	123
More than 40	40	More than 10	165
More than 30	98		

13. Find the median and first and third quartile for the following data:
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

14. Compute Q_1, Q_3, D_3, D_6 and D_8 for the following data:
14, 7, 13, 12, 13, 17, 8, 10, 6, 15, 18, 21, 20

15. Following are the scores of 12 students in a class test of 30 marks:
18, 20, 9, 15, 21, 26, 14, 13, 27, 22, 16, 28
Find D_7 and P_{33}.

16. Compute Q_1, Q_3, D_6 and P_{45} for the following data:

x	18	19	20	21	22	23	24	25	26	27
f	15	18	25	27	40	25	19	16	8	7

17. Following are the different sizes and number of shoes in a shoe shop. Calculate median, first quartile, third quartile, 6th decile and 80th percentile.

Size of shoes	*No. of shoes*	*Size of shoes*	*No. of shoes*
4.5	4	8	40
5	8	8.5	20
5.5	12	9	15
6	15	9.5	24
6.5	20	10	12
7	35	10.5	5
7.5	50	11	3

18. Find out first quartile, third quartile and first decile.

Size of item	*f*	*Size of item*	*f*
0 – 10	2	40 – 50	35
10 – 20	18	50 – 60	20
20 – 30	30	60 – 70	6
30 – 40	45	70 – 80	3

19. Calculate the median, 3rd decile and 20th percentile for the following data:

x	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25
f	7	18	25	30	20

20. Find the interquartile range, quartile deviation for the following data.

Age in years	20	30	40	50	60	70	80
No. of members	3	61	132	153	140	51	3

[**Hint.** Inter-quartile range = $Q_3 - Q_1$, Q.D. = $\frac{Q_3 - Q_1}{2}$]

21. Find the interquartile range, semi-interquartile range, and coefficient of quartile deviation from the following frequency distribution.

Marks	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
No. of students	60	45	120	25	90	80	120	60

[**Hint.** Coeff of $Q.D.$ = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$]

ANSWERS

1. (*i*) 5 (*ii*) 18 (*iii*) 51 (*iv*) 16.5 **2.** 61, 71 **3.** ₹ 24 **4.** 12

5. 70 **6.** (*i*) False (*ii*) True **7.** Median

8. {1, 2, 3, 4, 10}, {1, 2, 3, 5, 9} **9.** 48.18 marks **10.** 31.79

11. 36.25 **12.** 32.7 **13.** $M_d = 12$, $Q_1 = 6$, $Q_3 = 18$

14. $Q_1 = 9$, $Q_3 = 17.5$, $D_3 = 11$, $D_6 = 14.5$, $D_8 = 19$

15. $D_7 = 24$, $P_{33} = 15.5$. **16.** $Q_1 = 20$, $Q_3 = 24$, $D_6 = 22$, $P_{45} = 22$

17. Med = 7.5 size, $Q_1 = 7$ size, $Q_3 = 8.5$ size, $D_6 = 8$, $P_{80} = 9$.

18. $Q_1 = 26.58$, $Q_3 = 46.93$, $D_1 = 17.7$ **19.** $M_d = 15$, $D_3 = 11$, $P_{20} = 8.6$

20. 20 years; Q.D. = 10 years, coefft of Q.D. = 0.2

21. 38.75, 19.375, 0.365

28.10. Mode

The word mode is derived from the French word 'La mode' meaning the fashion. Mode in statistical language is that value which occurs most often in a series, that is a value which is most typical. If garment manufacturers say that short collars are now in fashion, the statement implies that maximum number of people nowadays wear short collar shirts. If we say the mode is size No. 7 in shoes, it means that in a given data maximum number of people wear size no. 7. Thus, *mode is that value of observation which occurs the greatest number of times or with the greatest frequency.*

1. **The value of the variable which occurs most frequently in a distribution is called mode.**
— *Kenney and Keeping*
2. **"The mode of the distribution is the value at the point around which the items tend to be most heavily concentrated. It may be regarded as the most typical of a series of values.**
— *Croxton and Cowden*
3. **"Mode is the value which has the greatest frequency in its immediate neighbourhood."**
— *A.M. Tuttle*

28.11. Computation of mode

(*i*) Let us consider the following distribution:

Variable x	*Frequency* f
5	15
6	25
7	20
8	10

Here since 25 is the maximum frequency, the corresponding value of x, namely, 6 is the mode.

(*ii*) Suppose 10 students score the following marks in a certain subject.

40, 62, 78, 90, 45, 90, 65, 79, 90, 98

Here 90 repeats the maximum, *i.e.,* three times, therefore the mode is 90. In certain cases, mode may not exist or there may be more than one mode.

Thus,

(*i*) 50, 48, 40, 74, 63 (No mode)

(*ii*) 64, 67, 64, 70, 64, 70, 68, 70, 65, 70 is bimodal, the two modes being 64 and 70.

A series having one mode is called **unimodal,** having two modes is called **bimodal,** having three modes — **trimodal,** having more than three modes — **multimodal.**

Now, we will discuss various methods of computing mode and illustrate the same with the help of solved examples.

Case I. Individual series (Simple distribution)

Method 1. By Inspection

Ex. 23. Find the mode of the following data:

7, 9, 8, 7, 9, 6, 9, 10, 4, 3, 5, 9, 9

Sol.

Arrange the given data in ascending (or descending) order.

3, 4, 5, 6, 7, 7, 8, 9, 9, 9, 9, 9, 10

On inspection, we find that 9 occurs the maximum number of times. So 9 is the mode.

Method 2. By Converting into Discrete Series

The given data are converted into discrete series to identify quickly the most frequently repeated value.

Ex. 24. The given data relate to the weekly expenditure of 40 students of the hostelers of a certain institute. Find the modal expenditure.

(Expenditure in rupees)

79	**75**	**76**	**79**	**78**	**81**	**84**	**80**	**81**	**77**	**80**	**75**	**74**	**81**
82	**83**	**79**	**83**	**76**	**79**	**78**	**77**	**76**	**75**	**78**	**79**	**80**	**82**
84	**77**	**79**	**78**	**77**	**76**	**80**	**80**	**82**	**78**	**74**	**79**		

Sol.

Converting the given data to discrete series, we have

Weekly expenditure (₹)	74	75	76	77	78	79	80	81	82	83	84
Number of students (f)	2	3	4	4	5	7	5	3	3	2	2

The maximum frequency is 7 corresponding to which the expenditure is ₹ 79. Hence, the modal expenditure is ₹ 79.

Case II. Discrete series

In discrete series the mode can be determined by inspection

Ex. 25. Find out modal income of the series:

Daily income (₹)	**80**	**85**	**90**	**95**	**100**	**105**	**110**
Number of men	**20**	**32**	**40**	**65**	**48**	**28**	**16**

Sol. By inspection, we find that 95 occurs the maximum number of times. Hence, the modal income is ₹ 95.

Case III. Continuous frequency distribution

In case of continuous frequency distribution, we locate the modal class by inspection if it is obvious and by grouping method as done in case of discrete series discussed above if it is not very clear. The value of mode is then determined by applying the following formula. The symbol M_0 or Z is used to denote mode.

$$M_0 \text{ or } Z = L + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i$$

where L = lower limit of the modal class

f_m = frequency of the modal class

f_{m-1} = frequency of the class preceding the modal class

f_{m+1} = frequency of the class succeeding the modal class

i = the width of the modal class

Note 1. In the above formula it is assumed that the classes at continuous. If the classes are discontinuous, then first make them continuous.

Note 2. While applying this formula, it is necessary that the class intervals are of uniform width throughout. If they are unequal they should first be made equal on the assumption that the frequencies are equally distributed throughout the class otherwise we will obtain misleading results.

Illustration:

Class	0 – 10	10 – 20	20 – 40	40 – 50	50 – 70	70 – 80
Frequency	5	15	40	32	28	5

The amended series will be

Class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Frequency	5	15	20	20	32	14	14	5

The modal class is 40 — 50.

Ex. 26. Estimate the mode for the following frequency distribution:

Age (Years)	Frequency	Age (years)	Frequency
20–25	50	40–45	150
25–30	70	45–50	120
30–35	100	50–55	70
35–40	180	55–60	59

Sol. Here the modal class is 35 – 40 which corresponds to the maximum frequency of 180. The actual lower limit of the modal class is 34.5 and the class interval is 5.

$$\therefore \quad \text{Mode} = L + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i = 34.5 + \frac{180 - 100}{360 - 100 - 150} \times 5$$

$$= 34.5 + \frac{80}{110} \times 5 = 34.5 + 3.6 = 38.1 \text{ years.}$$

Ex. 27. Find the mode for the following table of test scores.

Marks	No. of students
Below 10	5
Below 20	18
Below 30	39
Below 40	76
Below 50	107
Below 60	131
Below 70	134

Sol. The above is a cumulative frequency table and this must be changed to an absolute frequency table before calculation.

Marks	*Actual limits*	*Number of students*
0 – 9	–0.5 – 9.5	5
10 – 19	9.5 – 19.5	13 (=18 – 5)
20 – 29	19.5 – 29.5	21 (= 39 – 18), etc.
30 – 39	29.5 – 39.5	37
40 – 49	39.5 – 49.5	31
50 – 59	49.5 – 59.5	24
60 – 69	59.5 – 69.5	3

The maximum frequency 37 occurs in the group (29.5 — 39.5).

$$\therefore M_0 = L + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i = 29.5 + \frac{37 - 21}{74 - 21 - 31} \times 10 = 29.5 + \frac{16}{22} \times 10$$

= 36.77 marks.

Unequal Class Intervals

Ex. 28. Compute the mode from the following data:

Class	Frequency	Class	Frequency
0 – 3	4	18 – 20	24
3 – 6	8	20 – 24	14
6 – 10	10	24 – 25	16
10 – 12	14	25 – 28	11
12 – 15	16	28 – 30	10
15 – 18	20	30 – 36	6

Sol. First we make the class intervals equal by combining two or more classes.

Class	0 – 6	6 – 12	12 – 18	18 – 24	24 – 30	30 – 36
Frequency	12	24	36	38	37	6

The mode lies in the class 18 – 24. Then

$$M_0 = L + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i = 18 + \frac{38 - 36}{2 \times 38 - 36 - 37} \times 6$$

$$= 18 + \frac{2}{3} \times 6 = 18 + 4 = 22$$

If the mode is ill-defined then mode may be obtained by using the formula

Mode = 3 Median – 2 Mean

EXERCISE 28 (*c*)

1. Find the mode of the following data:

(*i*) 3, 4, 7, 11, 4, 3, 4, 5, 6, 4, 1, 4, 2, 4, 4

(*ii*) Size of shoes: 4, 4.5, 5, 4.5, 5.5, 5, 6, 4.5, 4, 4.5

(*iii*) Wages (₹): 100, 120, 100, 120, 130, 120, 120, 130, 120, 100

(*iv*) Runs in an innings : 18, 32, 0, 40, 60, 69, 33, 69, 35, 11, 20

2. Find the mode from the following data in questions 2 to 7.

Marks	10	12	15	22	25	35	45	50	60
Number of students	4	6	10	14	20	19	10	6	3

3.

Size	8	10	12	14	16	18	20	22	24
Frequency	3	5	1	7	8	8	4	2	3

4.

Class interval	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	10	45	12	3

5.

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Number of students	8	16	28	60	38

6.

Marks	*Frequency*	*Marks*	*Frequency*
1 – 5	7	26 – 30	18
6 – 10	10	31 – 35	10
11 – 15	16	36 – 40	5
16 – 20	32	41 – 45	1
21 – 25	24		

[**Hint.** First change into continuous distribution]

7.

Mid-value	60	64	68	72	76
Frequency	25	8	10	6	4

8. (*i*) The mode of the following frequency distribution is 48.6. Find the unknown frequency.

Class	20 – 25	25 – 40	40 – 55	55 – 70	70 – 85
Frequency	6	20	44	?	3

(*ii*) If the frequency of the class 70 – 85 is 13 instead of 3, then what difference will it make?

[**Hint.** (*i*) Since $M_0 = 48.6$, the modal class is 40 — 55. Let the unknown frequency be f_2. Then applying the formula $M_0 = L + \dfrac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i$, we have $48.6 = 40 + \dfrac{44 - 20}{2 \times 44 - 20 - f_2} \times 15$.

(*ii*) If the frequency of the class 70 — 85 is 13 instead of 3, then also the mode will remain unaffected because (*i*) mode does not depend on all values (*ii*) It is not affected by extreme values. But if there was a change in the value of either of f_{m-1}, f_m and f_{m+1}, then the mode would have also changed.]

9. Find the mean, median and mode of the following:

(*i*) The data 17, 32, 35, 33, 15, 21, 41, 32, 11, 18, 20, 22, 11, 15, 35, 23, 38, 12.

(*ii*)

Size of item	11	12	13	14	15	16	17
Frequency	8	13	25	37	23	15	9

10. The mode of the following distribution is 240. Find out the missing frequency:

Size	*Frequency*	*Size*	*Frequency*
0 – 100	140	300 – 400	?
100 – 200	230	400 – 500	150
200 – 300	270	500 – 600	140

11. Find the median, mode, third quartile, 5th decile and 65th percentile from the following:

Value of the item	1	2	3	4	5	6	7	8	9	10
Frequency	25	30	38	45	46	58	32	34	25	22

12. Find median, mean and modal marks and also determine the limits between which 80% of the students have secured marks.

Marks	*No. of students*	*Marks*	*No. of students*
0 – 10	15	50 – 60	80
10 – 20	25	60 – 70	70
20 – 30	52	70 – 80	10
30 – 40	56	80 – 90	5
40 – 50	78	90 – 100	2

[**Hint.** The limits of 80% students are P_{90} and P_{10}.]

ANSWERS

1. (*i*) 4 (*ii*) 4.5 (*iii*) ₹ 120 (*iv*) 69 **2.** 25 marks **3.** 16 **4.** 15.15 **5.** 35.93 marks
6. 18.84 marks **7.** 60.38 **8.** (*i*) $f_2 = 26.14$, *i.e.,* 26
9. (*i*) $\bar{x} = 23.94$, $M_d = 21.5$, M_O is ill-defined (*ii*) $\bar{x} = 14.04$, $M_d = 14$, $M_O = 14$
10. 210 **11.** $M_d = 5$, $M_O = 5$, $Q_3 = 7$, $D_5 = M_d = 5$, $P_{65} = 6$
12. $M_d = 46.22$, $\bar{x} = 44.62$, $M_O = 51.67$, $P_{90} = 66.94$, $P_{10} = 19.72$

LOCATING MODE GRAPHICALLY

28.12. Procedure for graphing location of mode

Step 1. *Draw a histogram of the data, for convenience only three rectangles may be drawn, i.e., rectangles for the modal, pre-modal and post-model classes.*

Step 2. *Draw two lines diagonally inside the modal class rectangle, starting from each upper corner of the modal rectangle to the upper corner of the adjacent rectangle.*

Step 3. *From the point of intersection of both these lines draw a perpendicular to the x-axis; the point where this perpendicular meets the x-axis is the value of the mode.*

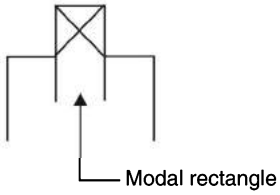


Fig. 28.02

Note. Mode can also be determined by drawing a frequency polygon. In such a case, a perpendicular is drawn from the apex of the frequency polygon and the point where the perpendicular touches the x-axis, gives the modal value. Mode cannot be determined graphically if two or more classes have the same highest frequency. For example, for the data given below, mode cannot be determined graphically.

Size of shop	3 – 6	6 – 9	9 – 12	12 – 15	15 – 18	18 – 21
No. of persons	8	13	15	18	18	8

Ex. 29. Find the mode of the distribution given below, by drawing a histogram.

Classes	**0 – 20**	**20 – 40**	**40 – 60**	**60 – 80**
Frequency	**15**	**6**	**18**	**10**

Sol. We draw the histogram by taking 1cm = 20 units on horizontal axis and 1cm = 5 units on vertical axis. Now, inside the highest rectangle, we join A to D and B to C. AD and BC meet at M. From M we draw the perpendicular MN to the horizontal axis meeting it at N. We estimate ON approximately equal to 52 units.

Hence the required mode = **52.**

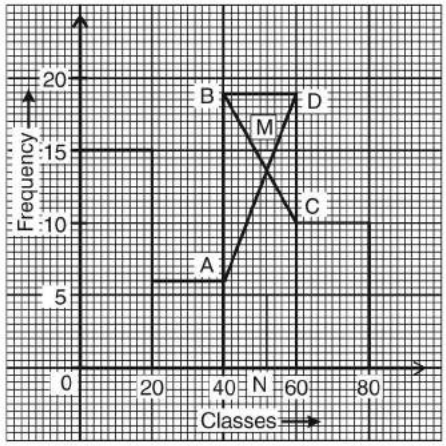


Fig. 28.03

Ex. 30. Construct a histogram for the following distribution. Determine the mode graphically.

x	1 – 9	9 – 17	17 – 25	25 – 33	33 – 41	41 – 49	49 – 57	57 – 65
f	20	30	26	16	10	8	4	4

Verify the result with the help of interpolation.

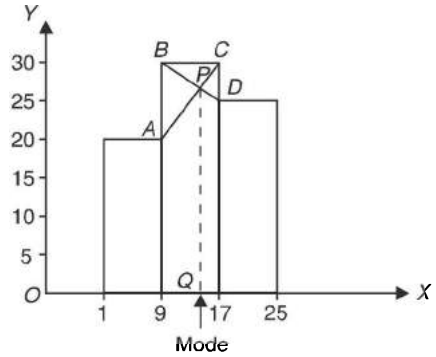


Fig. 28.04

Sol. By inspection the modal class is 9 – 17. We draw three rectangles of the histogram of the given distribution : of modal class 9 – 17, of pre-modal class 1 – 9 and of post-modal class 17 – 25. Draw lines AC and BD diagonally as shown and perpendicular PQ from their point of intersection P to the x-axis. Read off the value of Q on the x-axis. It gives the mode.

Verification: Applying the formula $M_0 = L + \dfrac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i$, we get

$$M_0 = 9 + \frac{30 - 20}{2 \times 30 - 20 - 26} \times 8 = 9 + \frac{10}{14} \times 8$$

$$= 9 + 5.71 = \mathbf{14.71}$$

EXERCISE 28 (*d*)

Find the mode of the given distribution by drawing a histogram.

1.

Class Inveral	0 – 10	10 – 20	20 – 30	30 – 40
Frequency	10	45	12	3

2.

Marks	1 – 5	6 – 10	11 – 15	16 – 20	21 – 25
f	7	10	16	32	24

3.

Marks	26 – 30	31 – 35	36 – 40	41 – 45
f	18	10	5	1

4.

Mid-value	60	64	68	72	76
f	25	8	10	66	4

5.

Class	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	5	11	19	21	16	10	8	6	3	1

6. The daily profits in rupees of 100 shops are distributed as under:

Profits per shop (₹)	0 – 100	100 – 200	200 – 300	300 – 400	400 – 500	500 – 600
No. of shops	12	18	27	20	17	7

Draw a histogram of the data and find the modal value. Verify by direct calculation.

7. Determine the value of mode of the following distribution graphically and verify the result.

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
No. of students	5	12	14	10	8	6

8. (*i*) In an asymmetrical distribution mean is 58 and median is 61. Calculate mode.

(*ii*) If mode in a tolerably asymmetrical distribution is 12 and median is 16, what would be the most probable mean? **[*ISC*]**

(*iii*) Find the median if mean is 40 and mode is 36.

[**Hint.** Apply the formula Mode = 3 Median – 2 Mean]

ANSWERS

1. 15.15 **2.** 18.8 **3.** 28.96 **4.** 60.38 **5.** 32.8

6. ₹ 256.25 **7.** 23.33 marks **8.** (*i*) 67 (*ii*) 18 (*iii*) 38.7

CHAPTER SUMMARY

1. Combined mean

$\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$, where $\bar{x}$ is the mean of the combined distribution $\bar{x}_1$ and $\bar{x}_2$ are the means of the component distribution, n_1 and n_2 are the total frequencies of the component distributions.

2. Combined standard deviation

$$\sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}} = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where $d_1 = (\bar{x}_1 - \bar{x})$ and $d_2 = (\bar{x}_2 - \bar{x})$ and $\bar{x}$ is the combined mean.

3. Median:

(*a*) In a discrete distribution (simple distribution)

Median (M_d) = size of $\dfrac{n+1}{2}$ th term when *n* is odd

= Average of $\dfrac{n}{2}$th and $\left(\dfrac{n}{2}+1\right)$th term where *n* stands for the number of items in the set, arranged in ascending or descending order.

Remark : We can simply find $\dfrac{n+1}{2}$ and proceed as in solved Ex. 2.

(*b*) For an ungrouped frequency distribution

Step 1. *Arrange the data in ascending or descending order of magnitude.*

Step 2. *Find out cumulative frequencies (c.f).*

Step 3. *Find* $n = \Sigma fi$

Step 4. *Look at the c.f. column for the c.f. which is* $\geq \dfrac{n+1}{2}$.

The corresponding value of the variable gives the median.

(*c*) For grouped frequency distribution

$$\text{Median } (M_d) = L + \frac{i}{f}\left(\frac{n}{2} - c\right)$$

where $\dfrac{n}{2}$ is the median number.

c is the cumulative frequency of the class just lower than the median class.
i is the class interval of the median class.
f is the frequency of the median class.
L is the actual lower limit of the median class.

4. Quartiles, Deviles and Percentiles

Measure	Discrete series and ungrouped frequency distribution	Grouped frequency distribution
Q_1	size of $\frac{n+1}{4}$ th item	$L+\frac{i}{f}\left(\frac{n}{4}-c\right)$
Q_3	size of $\frac{3(n+1)}{4}$ th item	$L+\frac{i}{f}\left(\frac{3n}{4}-c\right)$
D_1	size of $\frac{n+1}{10}$ th item	$L+\frac{i}{f}\left(\frac{n}{10}-c\right)$
D_7	size of $\frac{7(n+1)}{10}$ th item	$L+\frac{i}{f}\left(\frac{7n}{10}-c\right)$
P_1	size of $\frac{n+1}{100}$ th item	$L+\frac{i}{f}\left(\frac{n}{100}-c\right)$
P_{47}	size of $\frac{47(n+1)}{100}$ th item	$L+\frac{i}{f}\left(\frac{47n}{100}-c\right)$
Remark : In case of ungrouped distributions, ungrouped frequency distributions are found.		

5. Mode (M_0 or Z) $= L+\frac{f_m-f_{m-1}}{2f_m-f_{m-1}-f_{m+1}}\times i$

where L = lower limit of the modal class
f_m = frequency of the modal class
f_{m-1} = frequency of the class preceding the modal class
f_{m+1} = frequency of the class succeeding the modal class
i = width of the modal class

6. Relation between Mean, Median and Mode

Mode = 3 Median – 2 Mean.

CHAPTER TEST

1. The means of two sets of sizes 40 and 60 respectively are 15 and 16 and the standard deviations are 3 and 4. Obtain the mean and standard deviation of the composite set of 100 items when the two sets are pooled together.

2. Find the median of the following values:

7 cm, 9 cm, 10 cm, 12 cm, 15 cm, 18 cm, 20 cm.

3. The marks obtained by 12 students out of 50 are as under:

25, 24, 23, 32, 40, 27, 30, 25, 20, 15, 16, 45

Find the median marks.

4. Compute Q_3, D_6 and P_{70} for the following data:

28, 17, 12, 25, 26, 19, 13, 27, 21, 16

5. Obtain the median for the following data and describe the information conveyed by it.

Number of students absent	5	6	7	8	9	10	11	12	13	15	18	20
Number of days	1	5	11	14	16	13	10	70	4	1	1	1

6. Calculate the median for the following distribution:

x	1	2	6	9	11	8	5	4
f	5	7	9	11	13	15	17	19

7. Calculate the median, first quartile and third decile for the following data:

Weekly income (in ₹)	58	59	60	61	62	63	64	65	66
No. of workers	2	3	6	15	10	5	4	3	2

8. Draw an ogive for the following distribution:

Duration (in sec)	0 – 30	30 – 60	60 – 90	90 – 120	120 – 150	150 – 180	180 – 210
No. of cells	25	75	50	25	150	50	25

Hence, find Q_1, Q_3, D_9, P_4 and P_{40} from it.

9. Calculate the median, Q_3, D_7 and P_{70} for the following distribution:

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	3	10	17	7	6	4	2	1

10. Graphically find the mode of the following distribution:

Height (in cm)	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
No. of students	4	3	8	11	6	2

ANSWERS

1. $\bar{x}_{12} = 15.6, \sigma_{12} = 3.67$ **2.** 12 cm **3.** 25 marks

4. $Q_3 = 26.5, D_6 = 23, P_{70} = 25.5$

5. 12, this shows that for first half the number of days, 12 or less than 12 students remained absent while on the remaining half of the number of days. 12 or more than 12 students remained absent.

6. 8 **7.** $M_d = 61, Q_1 = 61, D_3 = 61$

8. $Q_1 = 60, Q_3 = 145, D_9 = 171, P_4 = 19.2, P_{40} = 102$

9. $M_d = 49.5$ marks, $Q_3 = 40.83$ marks, $D_7 = 37.14$ marks, $P_{70} = D_7 = 37.14$ marks.

10. 64

CHAPTER 29

CORRELATION ANALYSIS

29.01. Correlation defined

Definition of Correlation. *When two variables are so related that a change in one is accompanied by a change in the other in such a way that (i) an increase in one is accompanied by an increase or decrease in the other, or (ii) decrease in one by a decrease or increase in the other, and the greater the magnitude of the change in one, the greater the magnitude of the change in the other, then the variables are said to be correlated.*

For example,

(*i*) an increase in the intensity of cold results in greater sale of woollen clothes.

(*ii*) an increase in the price of a commodity results in a decrease in its demand.

(*iii*) an increase in the heights of children is accompanied by an increase in their weights.

(*iv*) a decrease in the price of a commodity is accompanied by an increase in its demand.

29.02. Positive and negative correlation

Whether correlation is positive or negative would depend upon the direction in which the variables are moving. If both the variables move in the same direction, *i.e.*, when an increase or decrease in one corresponds to an increase or decrease respectively in the other, then the correlation between the two variables is said to be **positive or direct.**

If both the variables move in opposite directions, *i.e.*, when an increase in one corresponds to a decrease in the other or a decrease in one corresponds to an increase in the other, then the correlation between the two variables is said to be **negative or inverse.**

Illustrations:

Positive correlation

Height (cm)	160	162	163	166	170	175
Weight (kg)	58	60	61	65	68	70

Negative correlation

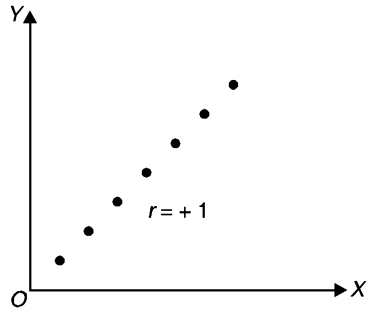
Price (₹ per unit)	20	18	15	14	12	10
Demand (units)	80	100	120	150	180	200

29.03. Degree of correlation

Correlation may be perfect or imperfect. When the changes in the corresponding values of two variables are proportional, directly or inversely, the correlation between them is said to be perfect. It is perfect positive if the increase (or decrease) in the values of one variable is accompanied by a proportional increase (or decrease) in the values of a second variable, *e.g.*, the correlation between the circumferences of circles and their radii is perfect positive. If the reverse is the case, *i.e.*, if the increase (or decrease) in one is accompanied by a proportional decrease (or increase) in the other variable, the correlation between the two variables is said to be perfect negative, *e.g.*, if a rectangle has constant area, the correlation between the lengths of its sides is perfect negative.

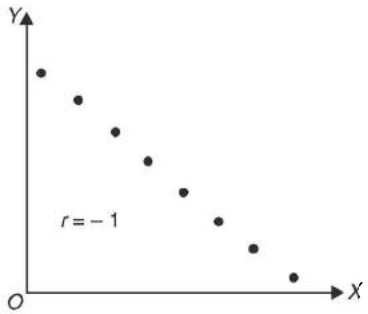
Such perfect (positive or negative) correlations are met with only in exact sciences like Mathematics, Physics, Chemistry, etc., but not in social and economic phenomenon. In such a phenomenon, the changes in one variable are not generally proportional to the changes in the other. In this case, the correlation between them, if it exists, is said to be imperfect positive or negative depending upon its nature. Imperfect correlation again may be high, moderate or low. The degree of imperfect correlation lies between perfect correlation and no correlation. Thus, we may have high positive correlation, *e.g.*, between incomes and standard of living or we may have negative correlation, *e.g.*, between supply and price of commodity.

Similarly, we have situation where correlation may be moderate (or low), negative or positive.



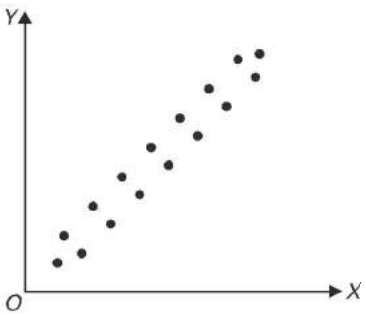
Perfect positive correlation

All the plotted points lie on a st. line rising from the lower left hand corner to the upper right hand corner.



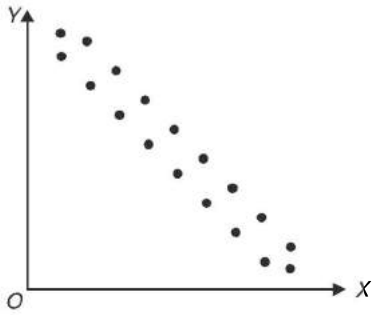
Perfect negative correlation

All the plotted points lie on a st. line falling from the upper left hand corner to the lower right hand corner.



High degree positive correlation

The plotted points (x_i, y_i), $i = 1, 2, 3, \ldots, n$ fall in a narrow band and the points are rising from lower left hand corner to the upper right hand corner.



High degree negative correlation

If the plotted points (x_i, y_i), $i = 1, 2, 3, \ldots, n$ fall in a narrow band from upper left hand corner to the lower left hand corner.

Fig. 29.01

29.04. Methods of studying correlation

The various methods to determine whether two variables are correlated or not are:

(*i*) ***Scatter diagram method***

(*ii*) ***Karl Pearson's coefficient of correlation***

(*iii*) ***Rank method*** **(Spearman's and Kendall's coefficient)**

Out of the above, only the Karl Pearson's and Spearman's methods are in the syllabus now.

29.05. Scatter diagram

Scatter diagram is a graphic device for finding **correlation between two variables. One variable normally, an indpendent** variable or time is plotted on the horizontal axis. This is also called a predicting variable. The other variable known as the dependent variable or one to be predicted is shown on the vertical axis. The movements of the pairs of these variables shown by dots on the graph reveal whether they move in the same or the opposite direction.

If the points form a band of some width, it will indicate imperfect correlation between the two variables. The direction of the band indicates the nature of correlation. If the band slopes upward, it indicates positive correlation and if it slopes downward then it indicates negative correlation. The width of the band gives an idea of the degree of correlation. The narrower the band the greater is the degree of correlation.

When the points do not form a band, *i.e.*, they are scattered in all directions it indicates that there is no correlation between the variables.

In the case of perfect correlation, the points will be on a straight line.

The method is mainly used when we are interested in finding out whether there is correlation and only in getting a rough idea about its nature and degree. It does not give us any measure of correlation. The following diagrams illustrate the various cases:

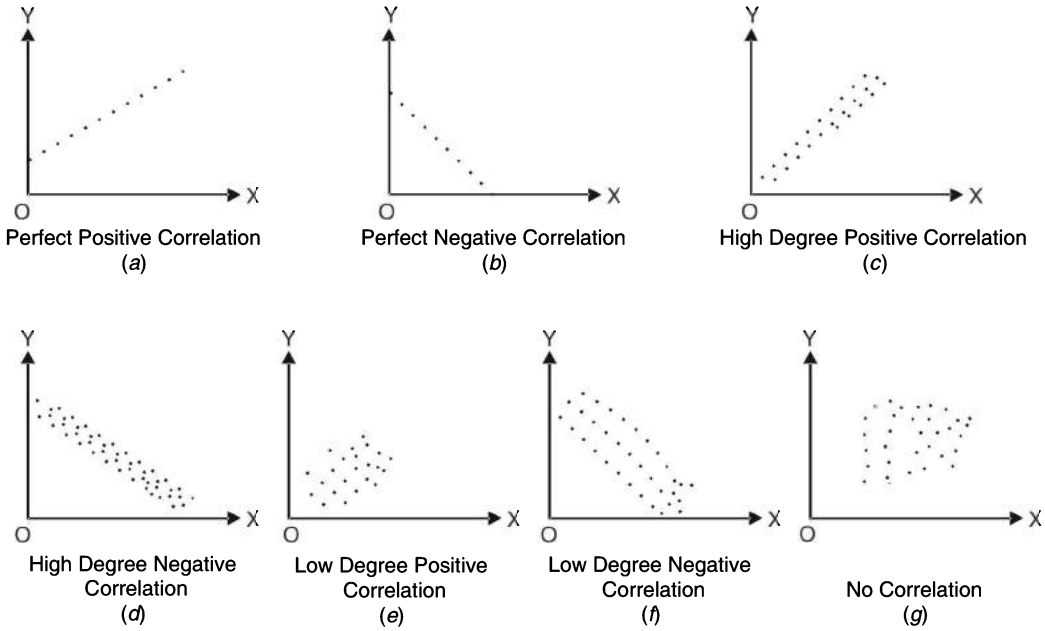


Fig. 29.02

29.06. The line of best fit

Most often there is not a straight line which passes through all the points but we can still draw the straight line which comes closest to finding all the points. We can estimate the position of the line by eye. This line shows the general trend of the relationship between the two sets of data. It may or may not pass through any of the data points.

A line that is drawn to pass as close as possible to all the plotted points on a scatter graph is called the line of best fit.

The closer the points are to this line of best fit the higher the correlation. The gradient of the line is not important except that a vertical or horizontal line of best fit means that the variables are not connected. Here are some typical examples and their interpretation:

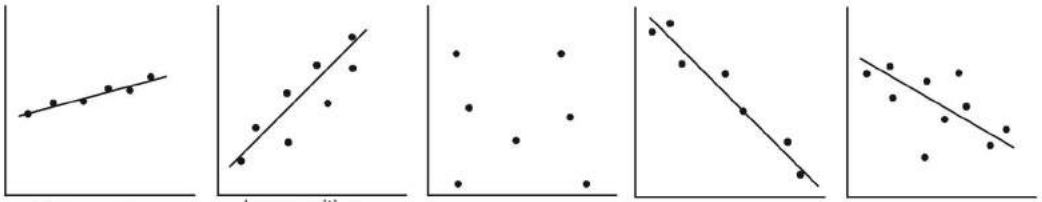


Fig. 29.03

Note. The method to find equation of line of best fit, also called line of regression, will be discussed in class 12.

Ex. 1. Construct a straight line which approximates the following data, *i.e.*, the line of best fit.

X	1	3	4	6	8	9	11	14
Y	1	2	4	4	5	7	8	9

Sol. Plot the points (1, 1), (3, 2), (4, 4), (6, 4), (8, 5), (9, 7), (11, 8) and (14, 9) on a rectangular coordinate system as shown in Fig. 29.04

A straight line approximating the data is drawn freehand in the figure.

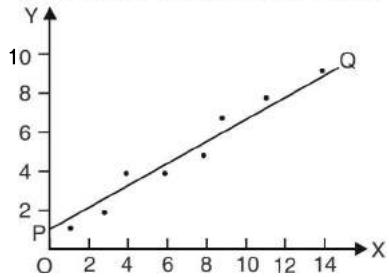


Fig. 29.04

29.07. Karl Pearson's coefficient of correlation

A mathematical method for measuring the degree of correlation between two variables X and Y was suggested by **Karl Pearson**. It is known as Pearsonian coefficient of correlation and is denoted by the symbol r or $\rho\,(X, Y)$*. The formula for finding the correlation coefficient is based on the concept of *co-variance* which we define below :

Co-Variance

Definition. If the variable X takes the values of $x_1, x_2, x_3, \ldots x_n$ and another variable Y takes the values $y_1, y_2, y_3, \ldots y_n$ then the covariance between the two variables x and y is written as cov (x, y) and is defined as

$$\text{Cov}\,(X, Y) = \frac{(x_1 - \bar{X})(y_1 - \bar{Y}) + (x_2 - \bar{X})(y_2 - \bar{Y}) + \ldots + (x_n - \bar{X})(y_n - \bar{Y})}{n}$$

where $\bar{X}$ and $\bar{Y}$ denote the arithmetic means of the two series, *i.e.*,

$$\bar{X} = \frac{x_1 + x_2 + \ldots + x_n}{n} \text{ and } \bar{Y} = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

i.e., $\text{Cov}\,(X, Y) = \frac{1}{n}\sum (x - \bar{X})(y - \bar{Y})$ or written more simply, $\text{Cov}\,(X, Y) = \frac{1}{n}\Sigma d_x d_y$

where $d_x = x_i - \bar{x},\ d_y = y_i - \bar{y}$ **(Formula 1)**

$$\textbf{Cov}\,(X, Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{\Sigma d_x d_y}{n} \textbf{ where } d_x = x_i - \bar{x}, d_y = y_i - \bar{y}$$

* The symbol ρ (pronounced 'ro') is a letter of the Greek alphabet corresponding to the English r.

If we divide the covariance by the product of the individual standard deviations, the quotient so obtained is called the correlation coefficient. As it was suggested by **Karl Pearson,** so it is called Karl Pearson's coefficient of correlation. Thus,

First Formula

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y} = \frac{\text{Cov}(X, Y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} = \frac{\frac{\Sigma d_x d_y}{n}}{\sigma_x \sigma_y} = \frac{\Sigma d_x d_y}{n . \sigma_x . \sigma_y}$$

Since $\sigma_x = \sqrt{\frac{\Sigma (d_x)^2}{n}}$ and $\sigma_y = \sqrt{\frac{\Sigma (d_y)^2}{n}}$, therefore,

$$\rho(X, Y) \text{ or } r = \frac{\frac{\Sigma d_x d_y}{n}}{\sqrt{\frac{\Sigma (d_x)^2}{n}} \cdot \sqrt{\frac{\Sigma (d_y)^2}{n}}} = \frac{\Sigma d_x d_y}{\sqrt{(\Sigma d_x^2)\,(\Sigma d_y^2)}}$$

$$\rho(X, Y) \text{ or } r = \frac{\Sigma d_x d_y}{n . \sigma_x . \sigma_y} = \frac{\Sigma d_x d_y}{\sqrt{(\Sigma d_x^2) . (\Sigma d_y^2)}}$$

where $d_x = x_i - \bar{x}, d_y = y_i - \bar{y}$ *are the deviations taken from the actual mean.*

Since the above formula is based on $\frac{\Sigma d_x d_y}{n}$, *i.e.*, product of the deviations of the observed values of X and Y from their means, this method is also called **product moment method.**

29.08. Interpretation of correlation coefficient

(*i*) The coefficient of correlation shall always be between –1 and +1.

(*ii*) When r is +1, there is perfect correlation between the variables.

(*iii*) When r is –1, there is perfect negative correlation between the variables.

(*iv*) When r is between 0.7 to 0.999, there is a high degree of correlation between the variables. The correlation shall be positive if the sign of r is plus (+) and negative if the sign of r is minus (–).

(*v*) When r is between 0.5 to 0.699, there is a moderate degree of correlation between the variables.

(*vi*) When r is less than 0.5 there is a low degree of correlation between the variables.

(*vii*) When r is zero, there is no correlation between the variables.

Ex. 2. Given $r = 0.5$, $\Sigma xy = 120$, $\sigma_y = 8$ and $\Sigma x^2 = 90$, find the number of items, (x and y are deviations from arithmetic average).

Sol. We have, $r = \frac{\Sigma xy}{n \sigma_x \sigma_y}$ or $r^2 = \frac{(\Sigma xy)^2}{n^2 \sigma_x^2 \sigma_y^2}$

$$\therefore \quad 0.25 = \frac{(120)^2}{n^2 \times \left(\frac{90}{n}\right) \times 64} \qquad \left[\because \sigma_x^2 = \frac{\Sigma x^2}{n} = \frac{90}{n}\right]$$

$$= \frac{14400}{n \times 90 \times 64} = \frac{5}{2n} \quad \Rightarrow \quad 0.25 \times 2n = 5 \quad \Rightarrow \quad n = 10.$$

Ex. 3. The following table gives the test scores and sales by nine salesmen during last one year in a certain firm:

Test scores	**14**	**19**	**24**	**21**	**26**	**22**	**15**	**20**	**19**
Sales (in 000' ₹)	**31**	**36**	**48**	**37**	**50**	**45**	**33**	**41**	**39**

Compute the Karl Pearson's Coefficient of Correlation and interpret the result. (*ISC 1993*)

Sol.

x	y	$d_x = x - 20$	$d_y = y - 40$	$d_x d_y$	d_x^2	d_y^2
14	31	–6	–9	54	36	81
19	36	–1	–4	4	1	16
24	48	4	8	32	16	64
21	37	1	–3	–3	1	9
26	50	6	10	60	36	100
22	45	2	5	10	4	25
15	33	–5	–7	35	25	49
20	41	0	1	0	0	1
19	39	–1	–1	1	1	1
$\Sigma x = 180$	$\Sigma y = 360$			$\Sigma d_x d_y = 193$	$\Sigma d_x^2 = 120$	$\Sigma d_y^2 = 346$

$$\bar{x} = \frac{180}{9} = 20; \quad \bar{y} = \frac{360}{9} = 40$$

$$r = \frac{\Sigma d_x d_y}{\sqrt{\Sigma d_x^2 \cdot \Sigma d_y^2}} = \frac{193}{\sqrt{120 \times 346}} = \frac{193}{203.75} = 0.94, \text{ approx.}$$

which shows a very high relationship.

Ex. 4. Compute the correlation coefficient between the corresponding values of X and Y in the following table:

X	2	4	5	6	8	11
Y	18	12	10	8	7	5

(ISC 2007 Type, 1992)

Sol.

X	Y	$X - \bar{X} = d_x$	d_x^2	$Y - \bar{Y} = d_y$	d_y^2	$d_x d_y$
2	18	–4	16	8	64	–32
4	12	–2	4	2	4	–4
5	10	–1	1	0	0	0
6	8	0	0	–2	4	0
8	7	2	4	–3	9	–6
11	5	5	25	–5	25	–25
36	60		50		106	–67

$$\bar{X} = \frac{36}{6} = 6, \quad \bar{Y} = \frac{60}{6} = 10$$

$$r = \frac{\Sigma d_x d_y}{\sqrt{\Sigma d_x^2 \times \Sigma d_y^2}} = -\frac{67}{\sqrt{50 \times 106}} = -\frac{67}{\sqrt{5300}} = \frac{-67}{72.80} = -0.92$$

Hence, the variables X and Y are highly *negatively correlated.*

29.09. Second formula for ρ(X, Y) (without using deviations from means, i.e., by using directly the values x_i and y_i)

We know that

$$\Sigma (x_i - \bar{x})^2 = \Sigma (x_i^2 - 2x_i.\bar{x} + \bar{x}^2) = \Sigma x_i^2 - 2\bar{x}.\Sigma x_i + n.\bar{x}^2$$

$$= \Sigma x_i^2 - 2\Sigma \frac{x_i}{n}.\Sigma x_i + n.\frac{(\Sigma x_i)^2}{n^2} = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

Similarly, $\Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \dfrac{(\Sigma y_i)^2}{n}$

$$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma(x_i y_i - x_i\bar{y} - y_i\bar{x} + \bar{x}\,\bar{y}) = \Sigma x_i y_i - \bar{y}\Sigma x_i - \bar{x}\Sigma y_i + n.\bar{x}\,\bar{y}$$

$$= \Sigma x_i y_i - \bar{y}(n\bar{x}) - \bar{x}\,(n\bar{y}) + n\bar{x}\,\bar{y} = \Sigma x_i y_i - n\bar{x}\,\bar{y}$$

$$= \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

$$\rho(X, Y) = \frac{\sum_{i=1}^{n} \Sigma (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

$$= \frac{\Sigma x_i y_i - \dfrac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n}}\sqrt{\Sigma y_i^2 - \dfrac{(\Sigma y_i)^2}{n}}}$$

or simply,

$$\rho\,(X, Y) \text{ or } r = \frac{\Sigma xy - \dfrac{\Sigma x\,.\,\Sigma y}{n}}{\sqrt{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}\,.\,\sqrt{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}} \text{ or } \frac{n\Sigma xy - \Sigma x\,.\,\Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\,.\,\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

where 'x' and 'y' stand for the values of items in X and Y series.

Ex. 5. Calculate the coefficient of correlation between x and y for the following data.

x	1	2	3	4	5	6	7	8	9	10
y	3	10	5	1	2	9	4	8	7	6

Sol.

x	y	x^2	y^2	xy
1	3	1	9	3
2	10	4	100	20
3	5	9	25	15
4	1	16	1	4
5	2	25	4	10
6	9	36	81	54
7	4	49	16	28
8	8	64	64	64
9	7	81	49	63
10	6	100	36	60
55	55	385	385	321

Since the given values are small we can use the formula

$$r = \frac{\Sigma xy - \dfrac{\Sigma x\,.\,\Sigma y}{n}}{\sqrt{\left\{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right\}}\sqrt{\left\{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right\}}}$$

$$= \frac{321 - \dfrac{55 \times 55}{10}}{\sqrt{\left\{385 - \dfrac{(55)^2}{10}\right\}\left\{385 - \dfrac{(55)^2}{10}\right\}}}$$

$$= \frac{321 - 302.5}{\sqrt{(385 - 302.5)\,(385 - 302.5)}}$$

$$= \frac{18.5}{\sqrt{82.5 \times 82.5}} = \frac{18.5}{82.5} = \mathbf{0.224.}$$

Note. You may also use the alternative form of the formula given above.

Ex. 6. In order to find the correlation coefficient between two variables x and y from 12 pairs of observations, the following calculations were made:

$\Sigma x = 30$, $\Sigma y = 5$, $\Sigma x^2 = 670$, $\Sigma y^2 = 285$, $\Sigma xy = 334$. On subsequent verification it was found that the pair ($x = 11$, $y = 4$) was copied wrongly, the correct values being ($x = 10$, $y = 14$). Find the correct value of correlation coefficient.

Sol. Corrected Σx = given Σx – incorrect value + correct value = $30 - 11 + 10 = 29$

Similarly, corrected $\Sigma y = 5 - 4 + 14 = 15$, corrected $\Sigma x^2 = 670 - (11)^2 + (10)^2 = 649$,

corrected $\Sigma y^2 = 285 - (4)^2 + (14)^2 = 465$, corrected $\Sigma xy = 334 - 11 \times 4 + 10 \times 14 = 430$

The correct value of correlation coefficient is given by

$$r = \frac{\Sigma xy - \frac{\Sigma x.\,\Sigma y}{n}}{\sqrt{\left[\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right]}} = \frac{430 - \frac{29 \times 15}{12}}{\sqrt{\left[649 - \frac{(29)^2}{12}\right]\left[465 - \frac{(15)^2}{12}\right]}}$$

$$= \frac{393.75}{\sqrt{578.92 \times 446.25}} = \frac{393.75}{508.27} = \mathbf{0.7747.}$$

29.10. Third formula for $\rho(X, Y)$. (When the deviations are taken from an assumed mean)

If the values of x_i's and y_i's are large or involve fractions, then neither of the two formulas given so far is convenient to use. In such cases, the computation of the correlation coefficient can be simplified by considering the deviations u_i and v_i of the values of x_i and y_i from assumed means a and b respectively for the variates x and y.

If $u_i = \frac{x_i - a}{h}, v_i = \frac{y_i - b}{k}$, *i.e.*, $x_i = hu_i + a$, $y_i = kv_i + b$, then on substituting in the formula

$$\rho(X, Y) = \frac{\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{n}}{\sqrt{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}} \cdot \sqrt{\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}}}$$ and simplifying, we obtain the following:

$$\rho(X, Y) = \frac{\sum_{i=1}^{n} u_i v_i - \frac{1}{n}\left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} v_i\right)}{\sqrt{\sum_{i=1}^{n} u_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} u_i\right)^2} \sqrt{\sum_{i=1}^{n} v_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} v_i\right)^2}}$$

Using short notations u and v in place of $\sum_{i=1}^{n} u_i$ and $\sum_{i=1}^{n} v_i$ respectively, we may write this formula in a simpler and convenient form as under:

$$\boldsymbol{r = \frac{\Sigma uv - \frac{(\Sigma u)(\Sigma v)}{n}}{\sqrt{\Sigma u^2 - \frac{(\Sigma u)^2}{n}} \sqrt{\Sigma v^2 - \frac{(\Sigma v)^2}{n}}}} \quad \textbf{or} \quad \boldsymbol{\frac{n\Sigma uv - (\Sigma u)(\Sigma v)}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}.}$$

Where $u = X - A$ or $\frac{X - A}{h}$, $v = Y - B$ or $\frac{Y - B}{k}$, A and B being the assumed means and n = number of pairs of observation. **This formula is used when actual means are fractions.**

Note. In some books you will find the notation d_x and d_y for u and v respectively. Using this notation, the above formula can be written as under

$$\rho(X, Y) \text{ or } r(X, Y) = \frac{\Sigma d_x d_y - \frac{\Sigma d_x . \Sigma d_y}{n}}{\sqrt{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{n}} \cdot \sqrt{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{n}}}$$

In this book we have used the notation d_x and d_y respectively for the deviation from $\bar{x}$ and $\bar{y}$, *i.e.*, actual means.

Useful tips:

1. Find the arithmetic means right in the beginning to know whether the means are whole numbers or fractions so that you may apply the formula accordingly.

2. If the given values are small, then apply formula (II) involving direct values only.

Ex. 7. Calculate Karl Pearson's correlation coefficient between the marks in English and Hindi obtained by 10 students.

Marks in English	**10**	**25**	**13**	**25**	**22**	**11**	**12**	**25**	**21**	**20**
Marks in Hindi	**12**	**22**	**16**	**15**	**18**	**18**	**17**	**23**	**24**	**17**

Sol. $\bar{x} = \frac{184}{10} = 18.40,\ \bar{y} = \frac{182}{10} = 18.2$

x	y	$u = x - 18$	u^2	$v = y - 18$	v^2	uv
10	12	−8	64	−6	36	48
25	22	7	49	4	16	28
13	16	−5	25	−2	4	10
25	15	7	49	−3	9	−21
22	18	4	16	0	0	0
11	18	−7	49	0	0	0
12	17	−6	36	−1	1	6
25	23	7	49	5	25	35
21	24	3	9	6	36	18
20	17	2	4	−1	1	−2
$\Sigma x = 184$	$\Sigma y = 182$	$\Sigma u = 4$	$\Sigma u^2 = 350$	$\Sigma v = 2$	$\Sigma v^2 = 128$	$\Sigma uv = 122$

We take 18 as the assumed mean for both the series

$$r = \frac{\Sigma uv - \frac{\Sigma u\, \Sigma v}{n}}{\sqrt{\left\{\Sigma u^2 - \frac{(\Sigma u)^2}{n}\right\}\left\{\Sigma v^2 - \frac{(\Sigma v)^2}{n}\right\}}} = \frac{122 - \frac{4 \times 2}{10}}{\sqrt{\left\{350 - \frac{(4)^2}{10}\right\}\left\{128 - \frac{(2)^2}{10}\right\}}}$$

$$= \frac{122 - 0.8}{\sqrt{348.4 \times 127.6}} = \frac{121.2}{\sqrt{44455.84}} = \frac{121.2}{210.85} = \mathbf{0.57} \text{ (to 2 dp)}$$

Note. Answers to problems or correlation may be written up to 2 decimal places.

Ex. 8. Given the following pairs of values of the variables X and Y:

X	2	4	6	8	10	12	14	16
Y	16	14	12	10	8	6	4	2

Find the Karl Pearson's coefficient of correlation, make a scatter diagram and interpret the result.

Sol. Calculate $\rho(X, Y)$ yourself. **Ans.** –1.

To draw the scatter diagram, plot the points (2, 16), (4, 14), (6, 12), (8, 10), (10, 8), (12, 6), (14, 4) and (16, 2). Fig. 29.05 shows the required scatter diagram.

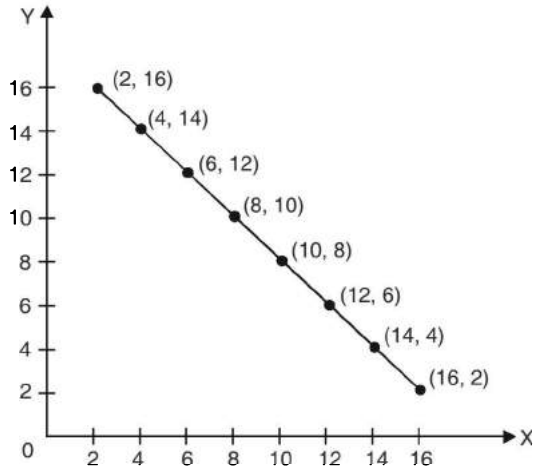


Fig. 29.05

The points in the scatter diagram proceed in a line from top to the bottom which indicates that X and Y are in perfect negative correlation.

Ex. 9. (*i*) Calculate the value of the correlation coefficient for the following data:

(1, 13), (2, 23), (3, 33), (4, 43), (5, 53), (6, 63), (7, 73), (8, 83), (9, 93), (10, 103), (11, 10.5), (12, 11), (13, 11.5), (14, 12), (15, 12.5), (16, 13), (17, 13.5), (18, 14), (19, 14.5), (20, 15).

(*ii*) Draw the scatter diagram.

(*iii*) Comment on the result.

Sol. (*i*) Let the assumed mean for the first variate X be 10 and that for the second variate be 13. Here, $n = 20$.

X	Y	$u = X - 10$	$v = Y - 13$	u^2	v^2	uv
1	13	–9	0	81	0	0
2	23	–8	10	64	100	–80
3	33	–7	20	49	400	–140
4	43	–6	30	36	900	–180
5	53	–5	40	25	1600	–200
6	63	–4	50	16	2500	–200
7	73	–3	60	9	3600	–180
8	83	–2	70	4	4900	–140
9	93	–1	80	1	6400	–80

10	103	0	90	0	8100	0
11	10.5	1	−2.5	1	6.25	−2.5
12	11	2	−2	4	4.00	−4
13	11.5	3	−1.5	9	2.25	−4.5
14	12	4	−1	16	1.00	−4
15	12.5	5	−0.5	25	0.25	−2.5
16	13	6	0	36	0.00	0
17	13.5	7	0.5	49	0.25	3.5
18	14	8	1	64	1.00	8
19	14.5	9	1.5	81	2.25	13.5
20	15	10	2	100	4.00	20
Total		$\Sigma u = 10$	$\Sigma v = 447.5$	$\Sigma u^2 = 670$	$\Sigma v^2 =$ 28521.25	$\Sigma vu =$ − 1172.5

$$\rho(X, Y) = \frac{\Sigma uv - \frac{1}{n}(\Sigma u)(\Sigma v)}{\sqrt{\left(\Sigma u^2 - \frac{(\Sigma u)^2}{n}\right)}\cdot\sqrt{\left(\Sigma v^2 - \frac{(\Sigma v)^2}{n}\right)}}$$

$$= \frac{-1172.5 - \frac{1}{20}(10)(447.5)}{\sqrt{\left(670 - \frac{(10)^2}{20}\right)}\sqrt{\left(28521.25 - \frac{(447.5)^2}{20}\right)}}$$

$$= -\frac{1396.25}{\sqrt{665}\sqrt{18508.438}} = -\frac{1396.25}{25.788 \times 136.046} = -\frac{1396.25}{3508.3542} = \mathbf{-0.39798}$$

(*ii*) Plotting the points (1, 13), (2, 23),.....(20,15) we obtain the scatter diagram as shown in Fig. 29.06.

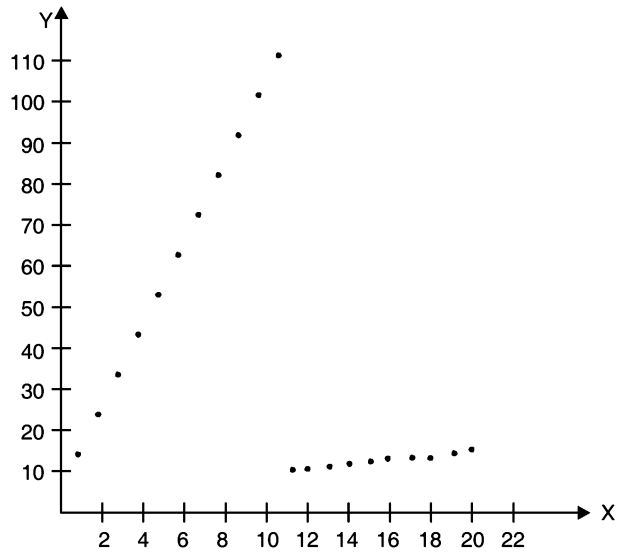


Fig. 29.06

Clearly, the scatter diagram consists of two straight lines.

(*iii*) With the scatter diagram we would conclude that there is no relationship between the two variables.

Ex. 10. The data below gives the values of (X, Y), for 16 persons where X is the age in years and Y the weights in kilograms:
(25, 55), (30, 62), (30, 61), (30, 63), (35, 67), (40, 70), (45, 71), (45, 72), (45, 73), (50, 70), (55, 67), (55, 65), (55, 68), (60, 62), (60, 61), (65, 55).
(*i*) Calculate the value of the correlation coefficient.
(*ii*) Comment on the results.

Sol. (*i*) Let the assumed mean for x-variate be 45 and assumed mean for y-variate be 64. Here, $n = 16$. Prepare the table and calculate the correlation coefficient yourself. You will obtain $r(X, Y) = 0.03$ (nearly).

(*ii*) To draw the scatter diagram, plot the points whose coordinates are the given pairs of values of the two variates. Fig. 29.07 shows the required scatter diagram. It shows the points lying close to a curve which first increases and then decreases.

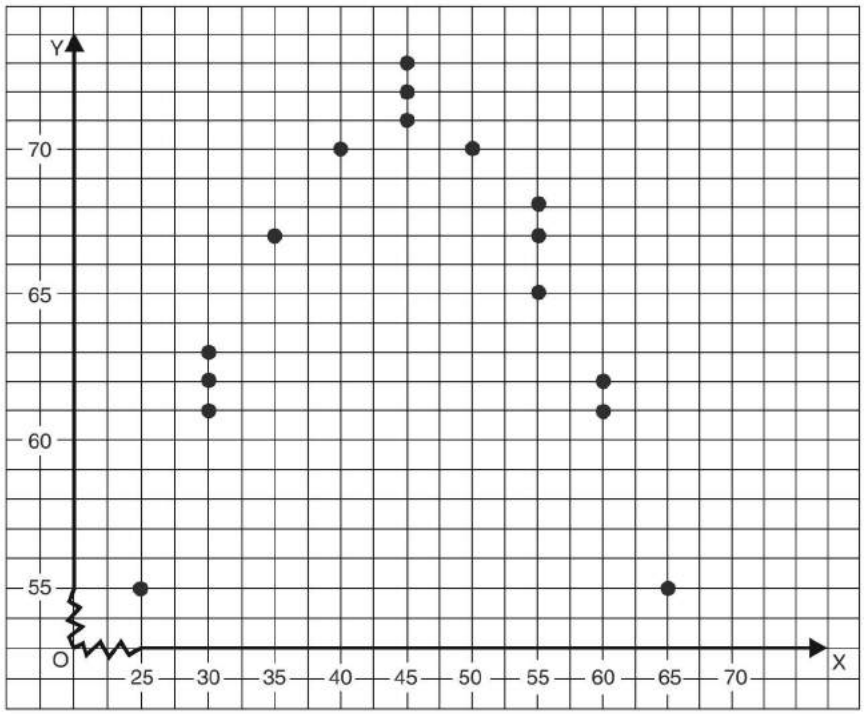


Fig. 29.07

(*iii*) The relationship between the variables is not disclosed by the coefficient of correlation as it is not linear.

EXERCISE 29(*a*)

Type 1. (*On Scatter diagram*)

1. A physicist is experimenting with the resistance in a circuit she is using. She measures and records the resulting current.

Resistance (ohms)	5	10	15	20	25	30	50
Current (amps)	10	4.9	3.2	2.4	1.9	1.7	1.0

(*i*) Draw a scatter graph of her results.
(*ii*) Estimate the current for a resistance of 40 ohms.
(*iii*) Estimate the resistance for a current of 7.5 amps.

2. In a small survey the heights of eight boys were measured and their shoe sizes were recorded.

Height (cm)	172	182	164	190	167	169	175	185
Shoe size	$8\frac{1}{2}$	$10\frac{1}{2}$	7	13	$8\frac{1}{2}$	8	10	12

Draw a scatter graphs and use it to find out whether there is a relationship between these sets of data.

Type 2. (*Based on first formula :* $r = \dfrac{\Sigma d_x d_y}{\sqrt{(\Sigma d_x^2)(\Sigma d_y^2)}}$, *where* $d_x = x - \bar{x}$, $d_y = y - \bar{y}$)

Calculate Karl Pearson's coefficient of correlation between the values of X and Y for the following data. Comment on the value of r.

3.

X	1	2	3	4	5
Y	7	6	5	4	3

4.

X	1	2	3	4	5	6	7	8	9
Y	12	11	13	15	14	17	16	19	18

5.

	X series	*Y series*
Number of pairs of observation	15	15
Arithmetic mean	25	18
Standard deviation	3.01	3.03
Sum of the squares of deviation from the mean	136	138
Sum of the product of the deviations of x and y series from their respective means	122	

6. Calculate the Pearson's coefficient of correlation between the ages of husband and wife.

Age of husband	35	34	40	43	56	20	38
Age of wife	32	30	31	32	53	20	33

7. Given $r = 0.8$, $\Sigma xy = 60$, $\sigma_y = 2.5$ and $\Sigma x^2 = 90$, find the number of items. x and y are deviations from their respective mean.

Type 3. (*Using direct values*) *when they are small numbers.*

Formula : $r = \dfrac{\Sigma xy - \dfrac{\Sigma x . \Sigma y}{n}}{\sqrt{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}\sqrt{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}}$ or $\dfrac{n\Sigma xy - \Sigma x . \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}.\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$, ***where x and y denote the values of items in x and y series.***

Calculate Karl Pearson's coefficient of correlation between the values of x and y for the following data.

8. (1, 2), (2, 4), (3, 8), (4, 7), (5, 10), (6, 5), (7, 14), (8, 16), (9, 2), (10, 20)

9.

X	–3	–2	–1	0	1	2	3
Y	9	4	1	0	1	4	9

10. $n = 50,\ \Sigma x = 75,\ \Sigma y = 80,\ \Sigma x^2 = 150,\ \Sigma y^2 = 140,\ \Sigma xy = 120.$

11. $n = 10,\ \Sigma x = 55,\ \Sigma y = 40,\ \Sigma x^2 = 385,\ \Sigma y^2 = 192$ and $\Sigma(x + y)^2 - 947.$

Type 4. (***Using deviations from assumed means, when $\bar{x}$ and $\bar{y}$ are not integers.***)

$$\textbf{Formula : } r = \frac{\Sigma uv - \frac{(\Sigma u).(\Sigma v)}{n}}{\sqrt{\Sigma u^2 - \frac{(\Sigma u)^2}{n}}\sqrt{\Sigma v^2 - \frac{(\Sigma v)^2}{n}}} \text{ or } \frac{n\Sigma uv - (\Sigma u).(\Sigma v)}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}.\sqrt{n\Sigma v^2 - (\Sigma v)^2}},$$

where $u = X - A$ or $\frac{X - A}{h}$, $v = Y - B$ or $\frac{Y - B}{k}$, A and B being assumed means.

12.

X	16	18	21	20	22	26	27	15
Y	22	25	24	26	25	30	33	14

(***ISC BM 2003***)

13.

X	1	2	4	5	7	8	10
Y	2	6	8	10	14	16	20

14. Calculate Karl Pearson's correlation coefficient between the marks in English and Hindi obtained by 10 students.

Marks in English	10	25	13	25	22	11	12	25	21	20
Marks in Hindi	12	22	16	15	18	18	17	23	24	17

15. Show that the coefficient of correlation ρ between two variables x and y is given by $\rho = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_y\sigma_x}$ where σ_x^2, σ_y^2 and σ_{x-y}^2 are the variances of x, y and $x - y$ respectively.

16. A computer expert while calculating correlation coefficient between X and Y from 25 pairs of observations obtained the following results:

$n = 25,\ \Sigma X = 125,\ \Sigma X^2 = 650,\ \Sigma Y = 100,\ \Sigma Y^2 = 460,\ \Sigma XY = 508$

It was, however, later discovered at the time of checking that he had copied down two pairs as $\begin{vmatrix} X & Y \\ 6 & 14 \\ 8 & 6 \end{vmatrix}$ while the correct values were $\begin{vmatrix} X & Y \\ 8 & 12 \\ 6 & 8 \end{vmatrix}$. Obtain the correct value of correlation coefficient. (***NM Olymp 1998***)

17. A computer obtained the data: $n = 30,\ \Sigma x = 120,\ \Sigma y = 90,\ \Sigma x^2 = 600,\ \Sigma y^2 = 250$ and $\Sigma xy = 356$. Later on, it was found that pairs $\begin{vmatrix} x & 8 & 12 \\ y & 10 & 7 \end{vmatrix}$ are wrong while the correct values are $\begin{vmatrix} x & 8 & 10 \\ y & 12 & 8 \end{vmatrix}$. Find the correct values of $\rho\,(X, Y)$. (***NM Olymp 1996***)

18. Show that Pearson's coefficient of correlation lies between -1 and $+1$, *i.e.*, $-1 \le r \le 1$ or $|r| \le 1$.

ANSWERS

1. (*ii*) 1.3 amps (*iii*) 6.5 ohms

2. The points are approximately along a straight line suggesting that the shoe size of a boy is related to his height.

3. –1, perfect negative correlation **4.** 0.933, high positive correlation

5. 0.89 **6.** 0.937 **7.** $n = 10$ **8.** 0.61 **9.** 0

10. 0 **11.** –0.6812 **12.** 0.89 **13.** 1, positive and perfect correlation

14. 0.575 **16.** 0.667 **17.** 0.0504

HINTS TO SELECTED QUESTIONS

11. For applying the formula, we first need to calculate the value of Σxy.

$\Sigma(x+y)^2 = 947 \Rightarrow \Sigma(x^2 + 2xy + y^2) = 947 \Rightarrow \Sigma x^2 + 2\,\Sigma xy + \Sigma y^2 = 947$

$\Rightarrow \quad 385 + 2\,\Sigma xy + 192 = 947 \Rightarrow \Sigma xy = 185.$

15. $\sigma^2_{x-y} = \frac{1}{n}\Sigma[(x-y)-(\bar{x}-\bar{y})]^2 = \frac{1}{n}\Sigma[(x-\bar{x})-(y-\bar{y})]^2$

$= \frac{1}{n}\Sigma[(x-\bar{x})^2 + (y-\bar{y})^2 - 2(x-\bar{x})(y-\bar{y})]$

$= \frac{1}{n}\Sigma(x-\bar{x})^2 + \frac{1}{n}\Sigma(y-\bar{y})^2 - 2.\frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y})$

$= \quad \sigma_x^2 + \sigma_y^2 - 2\rho.\sigma_x\sigma_y \quad \Rightarrow \quad \rho = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

16. See Solved Ex. 6

Corrected ΣX = Given ΣX – (sum of the incorrect values) + (sum of the correct values)

$= 125 - (6 + 8) + (8 + 6) = 125$

Corrected $\Sigma X^2 = 650 - (6^2 + 8^2) + (8^2 + 6^2) = 650$, Similarly, find out corrected ΣY and ΣY^2

Corrected $\Sigma XY = 508 - (6 \times 14 + 8 \times 6) + (8 \times 12 + 6 \times 8) = 520$

Now apply formula for $r(X, Y)$.

18. Let x and y be the deviations of X and Y series from their means and σ_x, σ_y be their standard deviations. Then, we have

$$\Sigma\left(\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right)^2 = \Sigma\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{2xy}{\sigma_x\sigma_y}\right)$$

$$= \Sigma\left(\frac{x^2}{\sigma_x^2}\right) + \Sigma\left(\frac{y^2}{\sigma_y^2}\right) + \Sigma\left(\frac{2xy}{\sigma_x\sigma_y}\right) = \frac{\Sigma x^2}{\sigma_x^2} + \frac{\Sigma y^2}{\sigma_y^2} + \frac{2\Sigma xy}{\sigma_x\sigma_y} \qquad ...(1)$$

Now $\sigma_x^2 = \dfrac{\Sigma x^2}{n} \quad \therefore \quad \dfrac{\Sigma x^2}{\sigma_x^2} = n.$ Similarly, $\dfrac{\Sigma y^2}{\sigma_y^2} = n$

Also $r = \dfrac{\Sigma xy}{n\sigma_x.\sigma_y}$ (Art. 29.07, First formula on Page 6) $\Rightarrow \quad \dfrac{2\Sigma xy}{\sigma_x\sigma_y} = 2nr$

$\therefore \ \Sigma\left(\dfrac{x}{\sigma_x} + \dfrac{y}{\sigma_y}\right)^2 = n + n + 2nr = 2n(1 + r)$ [From (1)]

But the square of a real number cannot be negative, therefore $\Sigma\left(\dfrac{x}{\sigma_x} + \dfrac{y}{\sigma_y}\right)^2 \geq 0$

$\Rightarrow \quad 2n(1+r) \geq 0 \quad \Rightarrow \quad 1+r \geq 0 \Rightarrow r \geq -1 \Rightarrow -1 \leq r$ ...(2)

Similarly, by expanding $\sum\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right)^2$ it can be shown that this expression is equal to $2n(1-r)$ and since $\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right)^2 \geq 0$, therefore, $1-r \geq 0$, *i.e.*, $r \leq 1$ . ...(3)

Combining the results (2) and (3), we get $-1 \leq r \leq +1$ or $|r| \leq 1$.

For alternative proof, see appendix at the end of the chapter.

29.11. Spearman's rank correlation coefficient

Sometimes such problems are faced that it is possible to arrange the various items of a series in serial order but the quantitative measurement of their values is difficult ; for example, it is possible for a class teacher to arrange his students in ascending or descending order of intelligence, even though intelligence cannot be measured quantitatively. No doubt, the quantitative study about the intelligence of students can be made by holding an examination and assigning them marks, but this method can never be said to be infallible. There are many such attributes which are incapable of quantitative measurements, for example, honesty, character, morality, etc.

In such cases it is possible to rank the individual in some order. The most intelligent individual may be given rank 1, next rank 2 and so on.

The correlation coefficient between two series of ranks is called 'Rank Correlation Coefficient'. The formula for coefficient of rank correlation as given by Edward Spearman is

$$R = 1 - \frac{6\sum D^2}{n(n^2-1)} \quad \text{or} \quad R = 1 - \frac{6\sum D^2}{n^3 - n}$$

where D is the difference between the corresponding ranks of the two series and n is the number of individuals in each series.

Note 1. Instead of assigning ranks 1, 2, 3, … from highest to lowest, we can also assign these ranks from the lowest to highest, *i.e.*, rank 1 to the least intelligent, rank 2 to the next more intelligent, next rank 3, and so on.

Note 2. Remember that the algebraic sum of the rank differences is always 0, *i.e.*, $\sum D$ is always = 0. If it is not so, then some mistake has been committed at the time of assigning ranks.

Note 3. The interpretations of the values of R are the same as given on Page 7 in Art. 29.08.

29.12. Solved examples

We may come across two types of problems:

(*i*) *When ranks are given* (*ii*) *When ranks are not given.*

Type 1. ***When ranks are given***

Working rule

Step 1. *Compute D, the difference of the ranks.*

Step 2. *Compute D^2 and get the sum ΣD^2.*

Step 3. *Substitute the values in the formula.*

Ex. 11. Following are the ranks obtained by 10 students in two subjects, Statistics and Mathematics. To what extent the knowledge of the students in the two subjects is related?

Statistics	1	2	3	4	5	6	7	8	9	10
Mathematics	2	4	1	5	3	9	7	10	6	8

Sol.

Rank of Statistics (x)	*Rank of Mathematics* (y)	$D = (x - y)$	D^2
1	2	−1	1
2	4	−2	4
3	1	+2	4
4	5	−1	1
5	3	+2	4
6	9	−3	9
7	7	0	0
8	10	−2	4
9	6	+3	9
10	8	+2	4
			$\Sigma D^2 = 40$

$$r = 1 - \frac{6\,\Sigma D^2}{n\,(n^2 - 1)}$$

$$= 1 - \frac{6 \times 40}{10\,(10^2 - 1)}$$

$$= 1 - \frac{240}{990} = 1 - 0.24$$

$$= +\ \mathbf{0.76}$$

Caution. When the ranks are already given as in the above example, do not commit the mistake of assigning new ranks.

Type 2. ***When ranks are not given.***

When no ranks are given, but actual data are given, then we should assign ranks. We can give ranks by taking the highest as 1 or the lowest value as 1, next to the highest (lowest) as 2 and follow the same procedure for both the variables.

Ex. 12. The marks obtained by the students in Physics and in Mathematics are as follows:

Marks in Physics	35	23	47	17	10	43	9	6	28
Marks in Mathematics	30	33	45	23	8	49	12	4	31

Compute their ranks in the two subjects and coefficient of correlation of ranks. (Interpret the result.) ***(ISC 2002, CBSE)***

Sol.

x	y	*Ranks in* x ; R_1	*Ranks in* y ; R_2	$D = R_1 - R_2$	D^2
35	30	3	5	−2	4
23	33	5	3	2	4
47	45	1	2	−1	1
17	23	6	6	0	0
10	8	7	8	−1	1
43	49	2	1	1	1
9	12	8	7	1	1
6	4	9	9	0	0
28	31	4	4	0	0
					$\Sigma D^2 = 12$

$$r = 1 - \frac{6\,\Sigma D^2}{n\,(n^2-1)} = 1 - \frac{6\times 12}{9\,(81-1)} = 1 - \frac{72}{720} = 1 - \frac{1}{10} = \frac{9}{10} = \mathbf{0.9.}$$

The high value of r indicates a very high relationship. This means that the students who are good in Physics are good in Mathematics also and vice-versa.

Ex. 13. Ten competitors in a beauty contest are ranked by three judges in the following order.

I Judge	**1**	**5**	**4**	**8**	**9**	**6**	**10**	**7**	**3**	**2**
II Judge	**4**	**8**	**7**	**6**	**5**	**9**	**10**	**3**	**2**	**1**
III Judge	**6**	**7**	**8**	**1**	**5**	**10**	**9**	**2**	**3**	**4**

Use rank correlation coefficient to discuss which pair of judges have the nearest approach to common tastes in beauty.

Sol.

First Judge (R_1)	*Second Judge* (R_2)	*Third Judge* (R_3)	$D_{12} = R_1 - R_2$	$D_{13} = R_1 - R_3$	$D_{23} = R_2 - R_3$	D_{12}^2	D_{13}^2	D_{23}^2
1	4	6	− 3	− 5	− 2	9	25	4
5	8	7	− 3	− 2	1	9	4	1
4	7	8	− 3	− 4	− 1	9	16	1
8	6	1	2	7	5	4	49	25
9	5	5	4	4	0	16	16	0
6	9	10	− 3	− 4	− 1	9	16	1
10	10	9	0	1	1	0	1	1
7	3	2	4	5	1	16	25	1
3	2	3	1	0	− 1	1	0	1
2	1	4	1	− 2	− 3	1	4	9
						$\Sigma D_{12}^2 = 74$	$\Sigma D_{13}^2 = 156$	$\Sigma D_{23}^2 = 44$

$$r_{12} = 1 - \frac{6\,\Sigma D_{12}^2}{n\,(n^2-1)} = 1 - \frac{6\times 74}{10\times 99} = \mathbf{0.55}$$

$$r_{13} = 1 - \frac{6\,\Sigma D_{13}^2}{n\,(n^2-1)} = 1 - \frac{6\times 156}{10\times 99} = \mathbf{0.05}$$

$$r_{23} = 1 - \frac{6\,\Sigma D_{23}^2}{n\,(n^2-1)} = 1 - \frac{6\times 44}{10\times 99} = \mathbf{0.73.}$$

Since r_{23} is maximum, we conclude that the pair of second and third judges has the nearest approach to beauty.

Ex. 14. The coefficient of rank correlation of marks obtained by 10 students in English and Economics was found to be 0.5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the correct coefficient of rank correlation. **(*ISC 2009 Type*)**

Sol. $r = 1 - \dfrac{6\,\Sigma D^2}{n\,(n^2-1)}$. Substituting the given values, we get

$$0.5 = 1 - \frac{6\Sigma D^2}{10(100-1)} \Rightarrow \frac{6\Sigma D^2}{990} = 1 - 0.5 = 0.5 \Rightarrow 6\Sigma D^2 = 0.5 \times 990$$

$$\Rightarrow \quad \Sigma D^2 = \frac{0.5 \times 990}{6} = 82.5 \therefore \text{ corrected value of } \Sigma D^2 = 82.5 - 3^2 + 7^2 = 122.5.$$

$$\therefore \quad \text{The correct value of } r = 1 - \frac{6 \times 122.5}{10(100-1)} = 1 - \frac{735}{990} = 1 - 0.74 = \mathbf{0.26 \text{ approx.}}$$

Ex. 15. Find out rank correlation from the following data:

S.No.	1	2	3	4	5	6	7	8	9	10
Rank differences	–2	–4	–1	+3	+2	0	–2	+3	+3	–2

Sol. First we find the unknown rank difference by using the fact that $\Sigma D = 0$. This gives the value of the unknown rank difference = + 11 – 9 = 2.

Now, we have

S.No.	1	2	3	4	5	6	7	8	9	10
D	–2	–4	–1	3	2	0	–2	3	3	–2
D^2	4	16	1	9	4	0	4	9	9	4

$N = 10$
$\Sigma D = 0$
$\Sigma D^2 = 60$

$$\therefore \quad r = 1 - \frac{6\Sigma D^2}{n(n^2-1)} = 1 - \frac{6 \times 60}{10(10^2-1)} = 1 - \frac{360}{990} = \frac{630}{990} = \frac{7}{11} = 0.636 = \mathbf{0.64} \text{ (to 2 dp)}$$

29.13. Correlation for tied ranks

If in either or both series two or more individuals (or items) have the same score (*i.e.*, tie rank) then the Spearman's rank correlation coefficient formula, $R = 1 - \frac{6\Sigma D^2}{n(n^2-1)}$ fails to give the correlation coefficient and a correction or modification in the formula becomes necessary because this formula is based on the supposition that ranks of various items are different and that no rank is given to more than one item.

The problem is solved by assigning a common rank to each of the individuals who are in a tie. This common rank is the average of the ranks of these individuals.

For example, we have the series

90 85 80 80 80 80 78 72 69 69 55 54

Rank 1 and rank 2 are assigned to two values, 90 and 85 and the next value 80 appears four times then the rank $\frac{3+4+5+6}{4} = 4.5$ will be assigned to each of the value 80 at the four places and the next lower value than 80, *viz.*, 78 would be assigned the rank 7 because 6 ranks have already been assigned. Now, we see that the value 69 is repeated twice at rank 9 and 10, then the common rank assigned to each value would be $\frac{9+10}{2} = 9.5$ and the next value 55 will have the rank 11 and the value 54, the rank 12. Thus, we have

Series (X)	90	85	80	80	80	80	78	72	69	69	55	54
Rank assigned	1	2	4.5	4.5	4.5	4.5	7	8	9.5	9.5	11	12

Note. We could have started by assiging ranks from lowest value, *i.e.*, rank 1 to 54, rank 2 to 55, then rank $\frac{3+4}{2} = 3.5$ to each of the values 69, rank 5 to 72, rank 6 to 78, then rank $\frac{7+8+9+10}{4} = 8.5$ to each of the values 80, rank 11 to 85 and lastly rank 12 to 90. Then, the ranks would have been as under:

Series (X)	90	85	80	80	80	80	78	72	69	69	55	54
Rank assigned	12	11	8.5	8.5	8.5	8.5	6	5	3.5	3.5	2	1

Correction factor : After assigning common rank to items with repeated values, an adjustment or correction factor is added to the Spearman's rank correlation formula.

If in a series there are m items whose ranks are common, then for correction $\frac{1}{12}(m^3 - m)$ is added for each repeating value in both the series. The modified formula, then is given by

$$R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3) + \ldots\right]}{n(n^2 - 1)}$$

where m_1, m_2, m_3, ... are the numbers of times a value is repeated.

Ex. 16. The figures below give the numbers of passenger-carrying vehicles under the awaiting repair, and the number of persons killed in train accidents in Great Britain in the years 1938–48. Find the correlation between these figures and comment on the result. The method of ranks may be used. ***(HSC)***

Sol.

Year	x *Vehicles under repair (1000's)*	y *No. of Persons*	*Rank of x*	*Rank of y*	D	D^2
1938	2.7	30	7	9	– 2	4
1939	2.5	27	10	10	0	0
1940	2.7	50	7	6	1	1
1941	2.5	76	10	2	8	64
1942	2.5	37	10	7	3	9
1943	2.7	14	7	11	– 4	16
1944	3.5	34	5	8	– 3	9
1945	4.9	75	3	3	0	0
1946	5.4	60	1.5	5	– 3.5	12.25
1947	5.4	121	1.5	1	0.5	0.25
1948	3.8	74	4.0	4.0	0	0
						$\Sigma D^2 = 115.5$

In x series 5.4 occurs twice. The rank of 5.4 is $\frac{1+2}{2} = 1.5$. The next lower value 4.9 is assigned as rank as 3. Similarly, 2.7 occurs thrice. The rank of 2.7 is $\frac{6+7+8}{3} = 7$, and next 2.5 occurs thrice. The rank of 2.5 is $\frac{9+10+11}{3} = 10$.

Owing to these ranks the coefficients will have to be corrected by adding $\frac{1}{12}(m^3 - m)$ to the value of ΣD^2. In respect of the x series, this addition will be

$$\frac{1}{12}(2^3 - 2) + \frac{1}{12}(3^3 - 3) + \frac{1}{12}(3^3 - 3) = \frac{6 + 24 + 24}{12} = 4.5 \qquad \text{(Art. 29.13)}$$

as 5.4 occurs twice, 2.7 occurs thrice and 2.5 occurs thrice.

If there had been repeated values in y-series also, we would have added the correction for them also.

$$\therefore \quad r = 1 - \frac{6[(\Sigma D^2) + 4.5]}{n^3 - n} = 1 - \frac{6[115.5 + 4.5]}{11^3 - 11}$$

$$= 1 - \frac{6 \times 120}{1331 - 11} = 1 - \frac{720}{1320} = 1 - 0.545 = 0.455 = \mathbf{0.46} \text{ (to 2dp)}$$

Since the value 0.455 or r lies between 0.40 and 0.70, therefore, it signifies substantial or marked relationship. It means that ordinarily, the higher the number of passenger-carrying vehicles under repair, the higher is the number of persons killed in train accidents in Britain. But it does not necessarily follow that one is the cause of the other. There can be many other causes, for example, wear and tear, etc.

Ex. 17. Find out the rank correlation coefficient between the heights of fathers and sons from the following data:

Height of fathers in inches	**65**	**66**	**67**	**67**	**68**	**69**	**70**	**72**
Height of sons in inches	**67**	**68**	**65**	**68**	**72**	**72**	**69**	**71**

Sol.

x	y	*Rank of* x (R_1)	*Rank of* y (R_2)	$D = R_1 - R_2$	D^2
65	67	8	7	1	1
66	68	7	5.5	1.5	2.25
67	65	5.5	8	–2.5	6.25
67	68	5.5	5.5	0	0
68	72	4	1.5	2.5	6.25
69	72	3	1.5	1.5	2.25
70	69	2	4	–2	4
72	71	1	3	–2	4
					$\Sigma D^2 = 26$

In the x-series, 67 occurs twice and its rank is $\frac{5+6}{2} = 5.5$

In the y-series, 72 occurs twice and its rank is $\frac{1+2}{2} = 1.5$

68 occurs twice and its rank is $\frac{5+6}{2} = 5.5$

The correction factor for x-series $= \frac{m^3 - m}{12} = \frac{2^3 - 2}{12} = 0.5$

The correction factor for y-series $= \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2) = 0.5 + 0.5 = 1$

$\therefore$ Total correction factor = 0.5 + 1 = 1.5

$$\therefore \quad R = 1 - \frac{6[\Sigma D^2 + 1.5]}{n(n^2-1)} = 1 - \frac{6(26+1.5)}{8(8^2-1)} = 1 - \frac{6 \times 27.5}{504}$$

$$= \frac{504 - 165}{504} = \frac{339}{504} = 0.672619 = 0.673 = \mathbf{0.67} \text{ (to 2 dp)}$$

Ex. 18. Find the Spearman's rank coefficient of correlation from the following data:

X	48	33	40	9	16	16	65	25	16	57
Y	13	13	24	6	15	4	20	9	6	19

Sol. We will solve this question by assigning ranks from lowest to highest, *i.e.*, rank 1 to the smallest value, rank 2 to the next value and so on.

Series X		*Series Y*		*Rank Differences*	*Squares*
X	*Rank - X* (R_1)	Y	*Rank - Y* (R_2)	$D = R_1 - R_2$	D^2
48	8	13	5.5	+ 2.5	6.25
33	6	13	5.5	+ 0.5	0.25
40	7	24	10	– 3	9
9	1	6	2.5	– 1.5	2.25
16	3	15	7	– 4	16
16	3	4	1	+ 2	4
65	10	20	9	+ 1	1
25	5	9	4	+ 1	1
16	3	6	2.5	+ 0.5	0.25
57	9	19	8	+ 1	1
$n = 10$		$n = 10$		$\Sigma D = 0$	$\Sigma D^2 = 41$

16 is repeated 3 times in series X and so $\frac{1}{12}(3^3 - 3)$ will be added to ΣD^2. In series Y, 13 is repeated 2 times and 6 is also repeated 2 times. So the factor $\frac{1}{12}(2^3 - 2)$ will be added twice to ΣD^2.

$$\therefore \quad R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \ldots\right]}{n(n^2-1)}$$

$$= 1 - \frac{6\left[41 + \frac{1}{12}(3^3 - 3) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2)\right]}{10(10^2 - 1)}$$

$$= 1 - \frac{6[41 + 2 + 0.5 + 0.5]}{1000 - 10} = 1 - \frac{6 \times 44}{990} = \frac{990 - 264}{990}$$

$$= \frac{726}{990} = 0.733 = \mathbf{0.73} \text{ (to 2 dp)}$$

EXERCISE 29 (*b*)

Find Rank Correlation Coefficient by Spearman's formula in the following questions.

Type 1. [Based on the formula $R = 1 - \frac{6\sum D^2}{n(n^2-1)}$]

1. The marks obtained by nine students in Physics and Mathematics are given below:

Physics	48	60	72	62	56	40	39	52	30
Mathematics	62	78	65	70	38	54	60	32	31

Calculate Spearman's coefficient correlation and interpret the result.

2. In a skating competition the judges gave the five competitors the following marks:

Competitors	*A*	*B*	*C*	*D*	*E*
Ist Judge	5.7	5.8	5.9	5.6	5.5
2nd Judge	5.6	5.7	6.0	5.5	5.8

Calculate a coefficient of rank correlation. **(*ISC*)**

3. The marks in history and mathematics of twelve students in a public examination are given in the table below:

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
History	69	36	39	71	67	76	40	20	85	65	55	34
Mathematics	33	52	71	25	79	22	83	81	24	35	46	64

Calculate a coefficient of correlation by ranks. What deduction can be made from the result? **(*ISC*)**

4. The marks given to five competitors by three different judges were as follows:

Competitors	*A*	*B*	*C*	*D*	*E*
Judge X	10	9	4	3	1
Judge Y	8	4	9	3	6
Judge Z	6	8	10	2	4

The result was decided by the average mark of the two judges whose marks showed the best correlation. Calculate:

(*i*) the coefficient of correlation by ranks for each pair of judges;

(*ii*) the final order of the competitors. **(*ISC*)**

5. The coefficient of rank correlation between the marks in Statistics and Mathematics obtained by a certain group of students is $\frac{2}{3}$ and the sum of the squares of the differences in ranks is 55. Find the number of students in the group.

Type 2. (Using correction factor)

6. The final positions of twelve clubs in a football league and the average attendances at their home matches were as follows:

Club	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Position	1	2	3	4	5	6	7	8	9	10	11	12
Attendance (thousands)	27	30	18	25	32	12	19	11	32	12	12	15

Calculate a coefficient of correlation by ranks and comment on your result.

What other factors do you think might affect the number of spectators apart from the positions of the clubs in the league? ***(ISC)***

7. The competitors in a beauty contest were awarded marks out of 20 by three judges with the following results:

Competitors	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Judge X	2	11	11	18	6	5	8	16	13	15
Judge Y	6	11	16	9	14	20	4	3	13	17
Judge Z	13	9	18	16	3	5	6	20	11	15

Determine the rank correlation coefficient in order to decide which two judges have the nearest approach to common taste in beauty. State the final order of the competitors if the result of the competition was decided by the average mark of the two judges having the nearest approach to the common taste in beauty.

8. (*i*) Ten recruits were subjected to a selection test to ascertain their suitability for a certain course of training. At the end of the training, they were given a proficiency test.
The marks, secured by recruits in the selection test x and proficiency test (y) are given below:

Recruits	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Test (x)	10	15	12	17	13	16	24	14	22	20
Test (y)	30	42	45	46	33	34	40	35	39	45

Calculate a coefficient of correlation by rank and comment on your result.

(ISC 2009 Type)

(*ii*) Describe the correlation you would expect to find between:
(*a*) the ages and weights of children under six year old,
(*b*) the width of a river at different points and the distance of these points from the sea,
(*c*) the amount of oxygen in the air at a place and its height above sea level.

(ISC)

Note. Use Spearman's formula in (*i*).

9. The following table gives the record of the heights of eight athletes and the measurements of their long jump and high jump to the nearest cm:

	Height	*Long Jump*	*High Jump*
A	158	324	175
B	165	365	185
C	162	380	180
D	170	400	184
E	175	350	200
F	163	350	172
G	178	425	188
H	164	375	180

Calculate the coefficient of rank correlation between height and long jump and between height and high jump. Comment on the result. ***(ISC)***

ANSWERS

1. $r = 0.667$. This indicates a moderate positive relationship between marks in Physics and Mathematics.
2. $r = 0.4$
3. $r = -0.776$. There is a negative correlation of a high to very high order. This means that a very good student of history is a very bad student of mathematics and *viceversa*.
4. Correlation between the marks of judges *X, Y* and *Z, i.e.*, $r_{xy} = 0.3$, $r_{xz} = 0.5$, $r_{yz} = 0.7$. So the best correlation was between the marks of the judges *Y* and *Z*; the final order of the competitors are *C*, *A*, *B*, *E*, and *D*.
5. 10
6. $r = 0.48$. There is a marked relationship between the positions of the clubs in the league and the number of spectators; it is not a high relationship. Club1attained the 9th position although the maximum number of spectators witnessed the match.

   Many other factors such as weather, non-availability of conveyance, etc., might affect the number of spectators apart from the positions of the clubs in the league.
7. $r_{xy} = -0.19$, $r_{yz} = -0.29$, $r_{zx} = 0.652$. Therefore, the judges *X* and *Z* have the nearest approach to common taste in beauty. The final order of the competitors is *H, D, J, C, I, B, A, G, F, E*.
8. (*i*) $r = 0.41$, which shows that there is a marked or substantial relationship.

   (*ii*) (*a*) Direct, (*b*) Inverse, (*c*) No correlation because the amount of oxygen at a place depends not only on height but also on various other factors like pollution, etc.
9. Spearman's Coefficient of correlation between height and long jumps is 0.52 and between height and high jump is 0.86. Thus, there is a marked relationship between height and long jump and very high relationship between height and high jump.

APPENDIX

Alternative proof of the property that Pearsons' correlation coefficient *r* lies between –1 and + 1.

The above property can be easily established by using the following inequality known as **Cauchy-Schwartz inequality**.

If u is a variable which takes n values $u_1, u_2, \ldots, u_n$ and v is another variable which takes n values $v_1, v_2, \ldots, v_n$, then

$$(u_1v_1 + u_2v_2 + \ldots + u_nv_n)^2 \le (u_1^2 + u_2^2 + \ldots u_n^2)\,(v_1^2 + v_2^2 + \ldots v_n^2)$$

i.e., $$(\Sigma uv)^2 \le (\Sigma u^2)(\Sigma v^2)$$

This inequality becomes an equality, *i.e.*, $(\Sigma uv)^2 = (\Sigma u^2)(\Sigma v^2)$ if $\frac{u_1}{v_1} = \frac{u_2}{v_2} = \ldots = \frac{u_n}{v_n}$.

We can use this inequality in proving the property of r that $-1 \le r \le 1$.

We have

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2)\,(\Sigma y^2)}} \text{ where } x = X - \bar{X},\ y = Y - \bar{Y}.$$

$$\Rightarrow \quad r^2 = \frac{(\Sigma xy)^2}{(\Sigma x^2)(\Sigma y^2)} \qquad \ldots (1) \quad \text{[squaring both sides]}$$

As per the Cauchy-Schwartz inequality,

$$(\Sigma xy)^2 \le (\Sigma x^2)(\Sigma y^2)$$

$$\Rightarrow \qquad \frac{(\Sigma xy)^2}{(\Sigma x^2)(\Sigma y^2)} \leq 1 \qquad \ldots(2)$$

(1) and (2) $\Rightarrow r^2 \leq 1$, *i.e.*, $-1 \leq r \leq +1$.

Note. Since $r^2 \leq 1 \Rightarrow -1 \leq r \leq 1$ and $-1 \leq r \leq 1 \Rightarrow r^2 \leq 1$, the two statements can together be expressed as $r^2 \leq 1 \Leftrightarrow -1 \leq r \leq 1$.

CHAPTER SUMMARY

1. Formulae for evaluating Karzl Pearson's coefft. of correlation

(*i*) $r = \dfrac{\Sigma dxdy}{\sqrt{\Sigma dx^2 \cdot \Sigma dy^2}}$, where $d_x = x_i - \bar{x}$, $dy = y_i - \bar{y}$ are deviation taken from the actual mean.

(*ii*) $r = \dfrac{\Sigma xy - \dfrac{\Sigma x \cdot \Sigma y}{n}}{\sqrt{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}} \cdot \sqrt{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}}$, where x and y are the values of items in x and y series.

(*iii*) $r = \dfrac{\Sigma uv - \dfrac{(\Sigma u)(\Sigma v)}{n}}{\sqrt{\Sigma u^2 - \dfrac{(\Sigma u)^2}{n}} \cdot \sqrt{\Sigma v^2 - \dfrac{(\Sigma v)^2}{n}}}$, where $u = X - A$ or $\dfrac{x - A}{h}$

$v = y - B$ or $\dfrac{y - B}{k}$,

A and B being the assumed means and n = number of pairs of observation. This formula is used when actual means are fractions.

2. Interpretation of correlation coefficient

(*i*) It always lies between – 1 and + 1.

(*ii*) $r = 1$ indicates perfect correlation, $r = -1$ perfect negative correlation, $r = 0$ is the range 0.7 to 0.999 high degree of correlation and 0.5 to 0.699 a moderate degree of correlation, less than 0.5 low degree of correlation and $r = 0$ no correlation between the variables.

3. Spearman's rank correlation coefficient

(*i*) $R = 1 - \dfrac{6\Sigma D^2}{n(n^2 - 1)}$ where D is the difference between the corresponding ranks of the two series and $\dfrac{n}{1}$ is the number of individuals in each series.

(*ii*) **Using Correlation factor:**

$$R = \frac{1 - 6\left[\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3) + \ldots\right]}{m(n^2 - 1)}$$

where $m_1, m_2, m_3, \ldots$ are the number of times a value is repeated.

CHAPTER TEST

1. Find the coefficient of correlation from the following pairs of observations:

(1, 3), (2, 2), (3, 5), (4, 4), (5, 6)

2. Find the Karl Pearson's coefficient of correlation between x and y for the following data:

x	16	18	21	20	22	26	27	15
y	22	25	24	26	25	30	33	14

(*ISC 2013*)

3. From the following data, calculate the Karl Pearson's coefficient of correlation, it being given that $\bar{y} = 8$.

x	6	2	10	4	8
y	?	11	5	8	7

4. Calculate Karl Pearson's coefficient between the values of x and y if $\Sigma x = 18$, $\Sigma x^2 = 90$, $n = 10$, $\Sigma y = 25$, $\Sigma y^2 = 120$, $\Sigma xy = 65$.

5. A psychologist selected a random sample of 22 students. He grouped them in 11 pairs so that the students in each pair have nearly equal scores in an intelligence test. In each pair, one student was taught by method A and the other by method B and examined after the course. The marks obtained by them after the course are as follows:

Pairs	1	2	3	4	5	6	7	8	9	10	11
Method A	24	29	19	14	30	19	27	30	20	28	11
Method B	37	35	16	26	23	27	19	20	16	11	21

Calculate Spearman's Rank correlation. **(*ISC 2014*)**

6. In a contest the competitors were awarded marks out of 20 by two judges. The scores of the 10 competitors are given below. Calculate spearman's rank correlation. **(*ISC 2015*)**

Pairs	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Judge A	2	11	11	18	6	5	8	16	13	15
Judge B	6	11	16	9	14	20	4	3	13	17

ANSWERS

1. 0.8 **2.** 0.894 **3.** – 0.92

4. 0.35 **5.** – 0.0296 **6.** – 0.194

HINTS TO SELECTED QUESTIONS

3. First find y by using $\frac{? + 11 + 5 + 8 + 7}{5} = 8$.

CHAPTER 30

INDEX NUMBERS AND MOVING AVERAGES

30.01. Index numbers – Introduction and meaning

Index numbers indicate the state of a certain variable or a group of variables such as industrial production, wholesale prices, consumer prices of commodities, exports-imports etc., at any given time or place in comparison with the state of the same variable or variables at some standard time or place. They are widely used in almost all areas of economic activities.

Index numbers were first constructed in the year 1760 as against the year 1500 mainly for measuring the effect of changes in prices.

The payments of dearness allowance to employees and periodic increase announced therein is directly linked to the rise in cost of living which is measured with the help of index numbers.

The various definitions of index numbers given by eminent economists and authors will help you understand better the concept of Index numbers.

1. *An index number is a statistical measure designed to show changes in a variable or a group of related variables with respect to time, geographic location or other characteristics such as income, production etc.* **– Spiegal**
2. *Index number is a single ratio (usually in percentage) which measures the combined (i.e., average) change of several variables between two different times, places or situations.* **– Tultle**
3. *Index numbers are used to measure the changes in some quality which we cannot observe directly.* **– Bowley**
4. *Index numbers are devices for measuring differences in the magnitude of a group of related variables.* **– Croxton & Cowden**
5. *In its simplest form an index number is nothing more than a relative number or a 'relative' which expresses the relationship between two figures, where one of the figures is used as a base.* **– Morris Hamburg**

From the above definitions, it is clear that 'Index' numbers are specialised averages which measure the extent to which changes take place in a group of related variables and also the effect of these changes.

30.02. Characteristics of index numbers

1. *Index numbers are specialised averages.* Thus, while normally, averages can be used to compare only those variables which are in the same units, index numbers help in comparing the changes in variables which are in different units.

For example, while constructing a consumer price index the various items taken into account, such as (*i*) food (wheat, sugar, pulses, etc.), (*ii*) cloth, (*iii*) lighting, (*iv*) house rent and (*v*) other miscellaneous items are in different units.

2. *Index numbers are expressed in percentages.* But the percentage size is never used.

3. *Index numbers measure changes not capable of direct measurement.*

4. *Index numbers are comparative in nature.* They compare changes taking place over a period of time.

30.03. Uses of index numbers

Index numbers are today one of the most important statistical tools and are widely used in a number of economic activities especially in measuring relative changes. Some of the uses are as follows:

1. *They measure the relative change.* Index numbers help in getting better idea of changes in the level of prices, production, employment etc., and many of the policies are guided by index numbers.

For example, the rate of D.A. and any increase therein from time to time, depends upon the cost of living index.

2. They reveal trends and tendencies.

3. They help in framing suitable policies.

4. They help in making comparisons by reducing the changes to percentages which are easily comparable.

5. They act as indicators of inflationary or deflationary tendencies. They act as a sort of economic barometers, indicating the state of affairs in various sectors of economy.

30.04. Classification of index numbers

There are various types of index numbers depending on the economic activity they measure.

1. *Price index.* It is an index number which compares the prices for a group of commodities at a certain time or at a place with prices of a base period and are used to measure the value of money. We have wholesale price index numbers and retail price index numbers.

2. *Quantity index.* Quantity index numbers study the changes in the value of goods produced or consumed such as industrial production, agricultural production, import, export, etc.

3. *Value index.* These index numbers compare the total value of a certain period with the total value of the base period. The total value is equal to the price of each, multiplied by the quantity. Thus, we have index numbers of profits, sales, inventories etc.

4. *Special purpose.* In this book, we shall discuss only the first two, *i.e.*, price and quantity index numbers.

30.05. Problems in the construction of index numbers

The construction of index numbers involves the following major problems. We shall study these, mainly, with reference to price index numbers.

1. *Purpose or object.* The purpose for which an index number is constructed must be clearly defined. No index number can serve for all purposes.

For example, one consumer price index number cannot be suitable for indicating the changes in consumer prices in different localities and for different classes of people. Similarly, one index number of industrial production cannot be suitable for different localities and different sectors of industry. Thus, it is essential to decide in advance the purpose of constructing the Index numbers. The selection of items, their number, qualities etc., would depend on this.

For example, if we want to construct the Index Numbers for economically weaker classes, then there is no point in including the luxury goods.

2. *Selection of items.* The number of components of a system is generally very large. It is, therefore, not possible to consider all of them in the construction of an index number. Only the important ones need to be considered. Thus, for a price index number, we need not consider the price of all commodities consumed. Instead, we select only a few relevant and important items which are widely used by the class of persons for whom the index number is to be constructed.

3. *Quotations.* Obtaining regular quotations (*i.e.,* statement of values) of selected items such as prices, production figures etc., also poses a problem. These may be obtained from standard trade journals, government publications, production centres etc. Care should be taken to ensure that the quotations relate to the same quality of the item. Quotations received from different sources are averaged to give the final quotation of that time which is to be used in the construction of the index number.

4. *Choice of base.* The standard time or place which is used to make comparisons is called the *base*. Every index number must have a base. We cannot say whether the price level has increased or decreased unless we compare the price level of the current year with the price level of the previous year. The base period should be selected with care and should be recent and normal. A normal year is one which is free from economic and natural disturbances, such as, failures of rains, earthquakes, strikes, war etc. The year to be selected as base year should be normal year and a recent one.

The base may be of the following type:

(*a*) *Fixed base.* The prices of a particular year, selected as a base period are taken to be 100. The changes in the prices of subsequent years are shown as the percentages of the base year.

(*b*) *Average base.* Sometimes it is difficult to select a fixed base. In such a situation, the average of several years in considered better to reduce the influence of abnormalities.

(*c*) *Chain base.* In fixed base method the base year once selected, remains fixed and all index numbers are based on the same base year. In this method, instead of having a fixed base year, a system of chain base is used. It is the previous year that is taken as the base for the current year and the change is calculated as a percentage of that year. For instance, for 1998, the base year is 1997; for 1997, the base year is 1996 ; for 1996 the base year is 1995, and so on. Since no normal year is there, long period comparison cannot be made.

5. *Choice of an average.* Since index numbers are specialised averages, a decision has to be made as to which particular average, *i.e.*, arithmetic mean, median, mode, geometric mean or harmonic mean should be used for constituting the index. Median, mode and harmonic mean are almost never used. For all practical purposes, the choice lies between the arithmetic mean and the geometric mean. The arithmetic mean is more commonly used, since it is easier to calculate and understand. However, index number based on this suffers from all disadvantages that an arithmetic mean has, and therefore, geometric mean is considered best as an average of relatives, in the construction of index numbers. In the construction of index numbers, we are concerned with the ratios or relative changes for which geometric mean is more suited.

6. *Weightage.* An index number is the average of relatives corresponding to various items under consideration. A simple average (arithmetic or geometric) is used when the times are considered to be a great importance. An index number so obtained is called a *simple index number*. Sometimes, there arise situations, when the different commodities cannot be given equal importance or weightage. Thus, when the price of rice is doubled and the price of sugar is halved, the people will suffer more because of the increase in the price of rice rather than the increase in the price of sugar because consumption of sugar is not so essential or important as that of rice. The importance is highlighted or stressed by assigning proper weightage to the values of items in proportion to their relative importance.

There are two methods of weighting:

(*a*) Implicit weighting, (*b*) Explicit weighting.

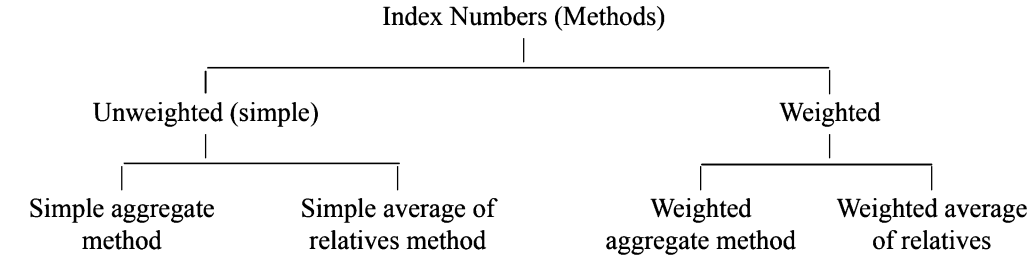
Implicit weights are not expressly laid down. These are implied by the nature of commodities selected. Thus, an important commodity may be included four times. Explicit weights are numbers assigned to the various items proportional to their relative degree of importance.

7. *Choice of formulae.* A large number of formulae have been advised for constructing the index. The statistician has to choose a formula which is most appropriate. Different formulas usually produce different results when applied to the same price and quantity data. The choice of the formula would depend upon the data available and the nature, purpose and scope of the enquiry.

30.06. Construction of index numbers

The various methods of constructing index numbers can be grouped under two heads:

(*a*) Unweighted indices, and (*b*) Weighted indices. In the unweighted indices weights are not expressly assigned whereas in the weighted indices weights are assigned to the various items. The different methods available are depicted in the following diagram.



1. Simple aggregate method

This is the simplest method of constructing the index numbers.

Before taking up the weighted method of constructing index numbers, we will explain the term 'Price Relative'.

30.07. Price relative

A price relative or simply a relative is the ratio of the price of a certain commodity in a given period to its price in an earlier fixed period called the *base period* or reference period expressed as a percentage.

$$\therefore \quad \text{Price relative} = \frac{\text{current price}}{\text{base price}} \times 100$$

or $$\textbf{Price relative} = \frac{p_1}{p_0} \times 100$$

where p_0 and p_1 denote the commodity price during the base period and given period respectively.

For example, let sugar be sold at ₹ 10 per kg in the year 1998 and ₹ 15 per kg in the year 2000, then

Relative price in 2000 with 1998 as base year $= \frac{15}{10} \times 100 = 150$. It means that if the price was ₹ 100 in the year 1998, then it was ₹ 150 in the year 2000.

30.08. Unweighted methods of constructing index numbers

There are two methods of constructing unweighted index numbers:

(1) *Simple aggregate method*

(2) *Simple average of price relative method.*

30.09. Simple aggregate method

This is the simplest method of constructing index numbers and is obtained as under.

Consider the following table of prices of commodities consumed in a working class family in 1949 and 1950:

	1949 Price	*1950 Price*
Bread and Flour	₹ 4 per kg	₹ 5 per kg
Meat	₹ 24 per kg	₹ 30 per kg
Potatoes	₹ 2 per kg	₹ 3 per kg
Tea	₹ 36 per kg	₹ 36 per kg
Sugar	₹ 5 per kg	₹ 6 per kg
Butter	₹ 18 per kg	₹ 27 per kg
Margarine	₹ 12 per kg	₹ 15 per kg
Eggs	₹ 30 per dozen	₹ 45 per dozen
Total	₹ 131	₹ 167

If the total prices in 1949 is represented by 100, then the total of prices in 1950 will be represented by $\frac{167}{131} \times 100$ or 127.5 which is called the simple aggregate price index for 1950, taking 1949 as the base year.

Procedure

Step 1. *Add the prices of the different commodities of the current year.*

Step 2. *Add the prices of these commodities of the base year.*

Step 3. *Divide the total obtained in step 1 by the total obtained in step 2 and multiply the quotient by 100.*

The result gives the required price index number.

Symbolically : $$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

where P_{01} denotes the price index number for the current year with reference to the base year.

Σp_1 denotes the total of current year prices of all commodities under consideration,

Σp_0 denotes the total of base year prices of the same commodities as considered for Σp_1.

Type 1.

Ex. 1. A small industrial concern used three raw materials *A*, *B* and *C* in its manufacturing process. The prices of the materials were as shown below:

Commodity	Prices in ₹ in the year 1995	Prices in ₹ in the year 2005
A	4	5
B	60	57
C	36	42

Using 1995 as the base year, calculate a simple aggregate price index for 2005. ***(ISC 2008)***

Sol. First, we construct the following table.

Commodity	*Base price (₹) in the year 1995* p_0	*Current price (₹) in the year 2005* p_1
A	4	5
B	60	57
C	36	42
	$\Sigma p_0 = 100$	$\Sigma p_1 = 104$

Hence, required price index by simple aggregate method is $p_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{104}{100} \times 100 = \mathbf{104}$.

Ex. 2. Compute a price index for the following by simple aggregate method:

Commodity	*A*	*B*	*C*	*D*	*E*	*F*
Price in 1986 (₹)	20	30	10	25	40	50
Price in 1991 (₹)	25	30	15	35	45	55

Sol. $\Sigma p_0 = 20 + 30 + 10 + 25 + 40 + 50 = 175$

$\Sigma p_1 = 25 + 30 + 15 + 35 + 45 + 55 = 205.$

$\therefore$ Simple aggregate index $P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{205}{175} \times 100 = \mathbf{117.143.}$

This means that the prices have increased in the year 1991 to the extent of 17 % when compared to the prices of 1986.

Ex. 3. For the data given below, compute the index numbers for various years by taking 1997 as the base year:

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Price of a commodity	4	5	6	7	8	10	9	11	12

Sol. Here we have to calculate the index number for each year taking 1997 as the base year. This is done in the following table.

Year	*Price of Commodity*	*Index number* $\left(\frac{\Sigma p_1}{\Sigma p_0}\right) \times 100$ *Taking 1997 as base year*
1992	4	$\frac{4}{10} \times 100 = 40$
1993	5	$\frac{5}{10} \times 100 = 50$
1994	6	$\frac{6}{10} \times 100 = 60$
1995	7	$\frac{7}{10} \times 100 = 70$
1996	8	$\frac{8}{10} \times 100 = 80$
1997	10	$\frac{10}{10} \times 100 = 100$
1998	9	$\frac{9}{10} \times 100 = 90$
1999	11	$\frac{11}{10} \times 100 = 110$
2000	12	$\frac{12}{10} \times 100 = 120$

Ex. 4. The index number for the year 2005 taking 2000 as base year for the following data was found to be 125. Find the missing entries in the data if $\Sigma p_0 = 360$.

Commodities	A	B	C	D	E	F
Prices in 2000 (in ₹)	40	60	20	50	x	110
Prices in 2005 (in ₹)	55	70	40	y	100	115

Sol. First we tabulate the given data in the following form: ***(Type ISC 2014)***

Commodities	*Price in 2000 (in ₹)* p_0	*Price in 2005 (in ₹)* p_1
A	40	55
B	60	70
C	20	40
D	50	y
E	x	100
F	110	115
	$\Sigma p_0 = 280 + x$	$\Sigma p_1 = 380 + y$

Given:

(*i*) $\Sigma p_0 = 360 \Rightarrow 280 + x = 360 \Rightarrow x = \mathbf{80}$.

(*ii*) Index number = 125 $\quad \therefore$ Using $p_{01} = \dfrac{\Sigma p_1}{\Sigma p_0} \times 100$, $125 = \dfrac{380 + y}{360} \times 100 \quad \Rightarrow y = \mathbf{70}$.

Type 2. When an average over several years is used as base.

Ex. 5. For the data given in Ex. 3, compute index numbers for various years by taking 1992 to 1994 as base period.

Sol. Here the base period is not a single year. It is given to be the period from 1992 to 1994. So we take the average of prices in 1992, 1993 and 1994 as the base price.

$$\text{Average} = \frac{4 + 5 + 6}{3} = 5$$

Index numbers taking 1992 to 1994 as the base period

Year	*Price of Commodity*	*Index Numbers* $\left(\dfrac{p_1}{p_0} \times 100\right)$
1992	4	$\frac{4}{5} \times 100 = 80$
1993	5	$\frac{5}{5} \times 100 = 100$
1994	6	$\frac{6}{5} \times 100 = 120$
1995	7	$\frac{7}{5} \times 100 = 140$
1996	8	$\frac{8}{5} \times 100 = 160$
1997	10	$\frac{10}{5} \times 100 = 200$
1998	9	$\frac{9}{5} \times 100 = 180$
1999	11	$\frac{11}{5} \times 100 = 220$
2000	12	$\frac{12}{5} \times 100 = 240$

Note two points in the above example.

(1) The base price has been taken as the average of the prices in 1992, 1993, 1994.

(2) Being a single commodity, the index number for each year is the same as price relative, *i.e.*, $\frac{p_1}{p_0} \times 100$.

30.10. Simple average of price relative method

In this method, the price relative of each commodity is calculated separately and then averaged. The average may be determined by using any measure of central value, *i.e.*, A.M. or G.M. or H.M. or Median or Mode, but in this book we will use only A.M. to find the average of price relative.

If n is the number of items (commodities), then

$$P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right)}{n}$$

Ex. 6. Calculate the price index number by simple average of relative method for the data of Ex. 2.

Sol. Procedure :

Step 1. *Compute the price relatives* $\left(\frac{p_1}{p_0} \times 100\right)$ *of all the commodities under consideration.*

Step 2. *Find the average of these price relatives by using A.M.*

Commodity	*Price in 1986* (p_0)	*Price in 1991* (p_1)	*Price relative* $\frac{p_1}{p_0} \times 100$
A	20	25	125
B	30	30	100
C	10	15	150
D	25	35	140
E	40	45	112.5
F	50	55	110
	175	205	737.5

Using A.M., $P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right)}{n} = \frac{737.5}{6} = \mathbf{122.92}$

This shows that as compared to 1986, there is an increase in the prices of commodities included in the index to the extent of about 23 %.

Remark. Due to the two different methods used, the indexes obtained are different. In Ex. 2, the index number was obtained as 117.143.

Ex. 7. From the following data, compute price index by using simple average of price relatives:

Commodity and unit	Price in 1989 (in ₹)	Price in 1990 (in ₹)
Butter (kg)	20.00	21.00
Cheese (kg)	15.00	14.00
Milk (lt)	3.00	3.00
Bread (1)	2.80	2.80
Eggs (doz)	6.00	8.00
Ghee (1tin)	250.00	260.00

(ISC 2004)

Sol. First, we tabulate the given data in the following form:

Commodity and unit	*Price in 1989 (in ₹) (p_0)*	*Price in 1990 (in ₹) (p_1)*	*Price relatives* $\frac{p_1}{p_0}\times 100$
Butter (kg)	20.00	21.00	$\frac{21}{20}\times 100 = 105$
Cheese (kg)	15.00	14.00	$\frac{14}{15}\times 100 = 93.33$
Milk (lt)	3.00	3.00	$\frac{3}{3}\times 100 = 100$
Bread (l)	2.80	2.80	$\frac{2.80}{2.80}\times 100 = 100$
Eggs (doz)	6.00	8.00	$\frac{8}{6}\times 100 = 133.33$
Ghee (1 tin)	250.00	260.00	$\frac{260}{250}\times 100 = 104$
			$\Sigma \frac{p_1}{p_0}\times 100 = 635.66$

$$\therefore \quad P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0}\times 100\right)}{n} = \frac{635.66}{6} = \mathbf{105.94}.$$

Hence, the price index for 1990 taking 1989 as the base year is 105.94.

Ex. 8. Using simple average of price relatives method, the price index for 2001, taking 1991 as base year, was found to be 127. If $\Sigma p_0 = 263$, find x and y from the following data:

Commodities	*A*	*B*	*C*	*D*	*E*	*F*
Prices in 1991 (in ₹)	80	70	50	*x*	18	25
Prices in 2001 (in ₹)	100	87.50	61	22	*y*	32.50

Sol. $\Sigma p_0 = 80+70+50+x+18+25 = 243+x = 263 \text{(given)} \Rightarrow \boldsymbol{x = 20}$.

Then, we construct the following table.

Commodities	*Price in 1991 (in ₹)* (p_0)	*Price in 2001 (in ₹)* (p_1)	*Price relative* $\left(\frac{p_1}{p_0}\times 100\right)$
A	80	100	$\frac{100}{80}\times 100 = 125$
B	70	87.50	$\frac{87.50}{70}\times 100 = 125$
C	50	61	$\frac{61}{50}\times 100 = 122$
D	20	22	$\frac{22}{20}\times 100 = 110$
E	18	*y*	$\frac{y}{18}\times 100 = \frac{50y}{9}$
F	25	32.50	$\frac{32.50}{25}\times 100 = 130$
			$\Sigma\left(\frac{p_1}{p_0}\times 100\right) = 612+\frac{50y}{9}$

$$\therefore \quad P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0}\times 100\right)}{n} = \frac{612+\frac{50y}{9}}{6} = 127 \text{ (given)} \Rightarrow \boldsymbol{y = 27.}$$

EXERCISE 30 (*a*)

Using the simple aggregate method, calculate the price index number from the following data:

1.

Commodities	*A*	*B*	*C*	*D*	*E*
1993 prices (in ₹)	50	40	10	5	2
1995 prices (in ₹)	80	60	20	10	6

2.

Commodities	*A*	*B*	*C*	*D*	*E*	*F*
1990 prices (in ₹)	40	60	20	50	80	100
1998 prices (in ₹)	50	60	30	70	90	110

3.

Commodities	*A*	*B*	*C*	*D*
Price in 1997	90	40	90	30
Price in 1998	95	60	110	35

4. Using 2005 as base year, the index numbers for the price of a commodity in 2006 and 2007 are 118 and 125. Calculate the index numbers for 2005 and 2007 if 2006 is taken as the base year.

(Type ISC 1994)

Compute a price index for the following by using price relative method.

5.

Commodities	*A*	*B*	*C*	*D*	*E*
Price in 1991 (in ₹)	20	40	60	80	100
Price in 1992 (in ₹)	70	45	70	90	105

6.

Material	*Cement*	*Timber*	*Steel*	*Bricks*
Price in 1969 (in ₹)	5	9.5	35	12
Price in 1970 (in ₹)	8	14.3	42	24

7. The index number for the following data for the year 2008, taking 2004 as base year was found to be 116. The simple aggregate method was used for calculation. Find the numerical value of x and y if the sum of the prices in the year 2008 is ₹ 203. ***(ISC 2010)***

Commodity	*Price (in ₹) in the year 2004*	*Price (in ₹) in the year 2008*
A	20	25
B	10	30
C	30	15
D	25	45
E	x	35
F	50	y

8. Construct index numbers by the simple average of relative method for 1990 and 1991 with 1989 as the base year.

Commodity *Price (in ₹) per unit*	*A*	*B*	*C*	*D*	*E*
1989	100	40	30	10	20
1990	120	45	35	12	22
1991	150	60	45	15	23

9. Construct the index number for 1991 taking 1990 as the base year from the following data by simple average of price relative method.

Commodity	*A*	*B*	*C*	*D*	*E*
Price in 1990 (₹)	100	80	160	220	40
Price in 1991 (₹)	140	120	180	240	40

(ISC 1996)

10. Construct index number from the following data for 1991 and 1992 taking 1990 as base by using the method of simple average of price relatives:

Group	Price in 1990 (in ₹)	Price in 1991 (in ₹)	Price in 1992 (in ₹)
A	20.00	24.00	22.00
B	1.25	1.50	1.00
C	5.00	8.00	8.00
D	2.00	2.25	2.12

11. The following data relate to the price of rice in different years.

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Price (in ₹)	6	7	7	8	10	14	12	13	14	15

Find out price relatives

(*i*) taking 1988 as base ; (*ii*) 1992 as base;

(*iii*) Taking average of 1988, 1989 and 1990 as base.

12. Compute a price index for the following by (*i*) simple aggregate and (*ii*) average of price relative method.

Commodity	*A*	*B*	*C*	*D*	*E*	*F*
Price in 1994 (₹)	20	30	10	25	40	50
Price in 1999 (₹)	25	30	15	35	45	55

13. Construct an index for 1998 taking 1997 as base by Average of Relatives.

Commodity	*A*	*B*	*C*	*D*	*E*
Price in 1997	5	4	8	11	2
Price in 1998	7	6	9	12	2

14. Construct the consumer price index number for 1990 taking 1989 as the base year and using simple average of price relative method for the following data:

Commodities	*Price in 1989*	*Price in 1990*
Butter	20	21
Cheese	16	12
Milk	3	3
Eggs	2.80	2.80

(*ISC 2003*)

ANSWERS

1. 164.5 **2.** 117.143 **3.** 120 **5.** 159.33 **6.** 157.625
7. $x = ₹\,40$, $y = ₹\,53$ **8.** 1990 → 115.83, 1991 → 143.0 **9.** 122.32
10. 128.13, 114
11. (*i*) 100, 116.67, 116.67, 133.33, 166.67, 233.33, 200, 216.67, 233.3, 250
(*ii*) 60, 70, 70, 80, 100, 140, 120, 130, 140, 150
(*iii*) 89.96, 104.95, 104.95, 119.94, 149.93, 209.90, 179.91, 194.90, 209.90, 224.89
12. (*a*) 117.143 (*b*) 122.92 **13.** 122.32 **14.** 95

HINTS TO SELECTED QUESTIONS

4. Let p_1, p_2, p_3 be the prices in 2005, 2006, 2007. Then $\frac{p_2}{p_1}\times 100 = 118$ and $\frac{p_3}{p_1}\times 100 = 125$.

Price index of 2005 with 2006 as base $= \frac{p_1}{p_2}\times 100 = \frac{100}{118}\times 100 = 84.75$.

Price index of 2007 with 2006 as base $= \frac{p_3}{p_2}\times 100 = \left(\frac{p_3}{p_1}\right)\left(\frac{p_1}{p_2}\right)\times 100 = \frac{125}{100}\times\frac{84.75}{100}\times 100 =$ **105.93.**

7. $\Sigma p_0 = 20 + 10 + 30 + 25 + x + 50 = 135 + x$; $\Sigma p_1 = 25 + 30 + 15 + 45 + 35 + y = 150 + y$

$\Rightarrow \quad 203 = 150 + y \qquad \Rightarrow \quad y = ₹\,53$

Simple Aggregate Index $= \frac{\Sigma p_1}{\Sigma p_0}\times 100 \qquad \therefore \quad 116 = \frac{203\times 100}{135 + x}$.

11. Average of 1988, 1989 and 1990 prices $= \frac{6+7+7}{3} = 6.67$

For table as under:

Year	*Price in ₹*	*Price relatives*		
		1988 (base)	*1992 (base)*	*Average of 1988 1989, 1990 (base)*
1988	6	100	$\frac{6}{10}\times 100 = 60$	$\frac{6}{6.67}\times 100 = 89.96$
1989	7	$\frac{7}{6}\times 100 = 116.67$	$\frac{7}{10}\times 100 = 70$	$\frac{7}{6.67}\times 100 = 104.95$
.....	.....	.....	.....	.....
1995				

30.11. Use of weights

The index numbers we have studied before are unweighted and they assign equal importance to all the items included in the index. The purpose of weighting is to make the index numbers more representative and to give more importance to them.

In order to use proper weights, we must answer the questions:

1. *By what do we 'weight'?*

We can 'weight' by whatever seems appropriate to bring out the economic importance of the commodities involved. The weights can be production figures, consumption figures, sales figures, etc.

2. *What type of weights do we use?*

The type of weight to be used depends upon the method of construction of index number that we are using. In this case of aggregate method, we have to use **quantity weights**. A *quantity weight* denoted by q means the amount of commodity produced, distributed or consumed in some period. In this case of price relatives, we have to use **value weights**. A *value weight*, combines price with quantity produced, sold, distributed or consumed.

3. *From which time period do we take the weights?*

As for the period from which we have to take the weights, a number of choices are available. We can use base year quantities, or current year quantities, or average or base and current year or some such combination of these methods.

30.12. Weighted index numbers

Weighted index numbers are of two types. They are:

1. Weighted aggregate index numbers;
2. Weighted average of price relatives.

Here, we will confine our discussion to weighted average of price relatives only.

30.13. Weighted aggregate method

If along with the base prices p_0 and current prices p_1 of a number of items, the weights w of each items are given, then the index number based on weighted aggregates is given by

$$P_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$$

Sometimes, the quantities in the base year are taken as weights. In such cases the weighted index number is constructed by using the formula

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100.$$

Ex. 9. Calculate the index number for the year 2005 with 2000 as the base year by weighted aggregate method from the following data:

Commodity	Price in ₹		Weights
	year 2000	year 2005	
A	140	180	10
B	400	550	7
C	100	250	6
D	125	150	8
E	200	300	4

(*ISC 2007*)

Sol.

Commodity	*Price (in ₹) in the year 2000* p_0	*Price (in ₹) in the year 2005* p_1	*Weights* w	p_1w	p_0w
A	140	180	10	1800	1400
B	400	550	7	3850	2800
C	100	250	6	1500	600
D	125	150	8	1200	1000
E	200	300	4	1200	800
				Σp_1w = 9550	Σp_0w = 6600

$$\therefore\ P_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100 = \frac{9550}{6600} \times 100 = \mathbf{144.69.}$$

Hence, the index number for the year 2005, with 2000 as the base year, is 144.69.

Ex. 10. A family buys quantities of rice, wheat, fish, potato and coal whose prices are given below, in the ratio 3 : 1 : 3 : 2 : 2. Find the weighted aggregate price index.

Commodity	**Rice**	**Wheat**	**Fish**	**Potato**	**Coal**
Base Price (₹)	**30**	**22**	**54**	**20**	**15**
Current Price (₹)	**35**	**25**	**64**	**25**	**18**

Sol. We construct the following table.

Commodity	*Base Price (₹)* p_0	*Current price (₹)* p_1	*Weight* w	p_0w	p_1w
Rice	30	35	3	90	105
Wheat	22	25	1	22	25
Fish	54	64	3	162	192
Potato	20	25	2	40	50
Coal	15	18	2	30	36
Total				Σp_0w = 344	Σp_1w = 408

Hence, the required index number using weighted aggregates is

$$P_{01} = \frac{\Sigma\, p_1 w}{\Sigma\, p_0 w} \times 100 = \frac{408}{344} \times 100 = \mathbf{118.60}.$$

Ex. 11. Construct index number for price for the year 2007 with 2005 as the base year from the following data by taking quantities in the base year as weights:

Commodity	**2005**		**2007**	
	Price	**Quantity**	**Price**	**Quantity**
A	**2**	**8**	**4**	**6**
B	**5**	**10**	**6**	**5**
C	**4**	**14**	**5**	**10**
D	**2**	**19**	**2**	**3**

Sol.

Commodity	2005		2007			
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)	p_1q_0	p_0q_0
A	2	8	4	6	32	16
B	5	10	6	5	60	50
C	4	14	5	10	70	56
D	2	19	2	3	38	38
					Σp_1q_0 = 200	Σp_0q_0 = 160

$$\therefore \quad P_{01} = \frac{\Sigma\, p_1q_0}{\Sigma\, p_0q_0} \times 100 = \frac{200}{160} \times 100 = \mathbf{125}.$$

30.14. Weighted average of price relative method

In this method, we make use of price relatives. When the base and current prices of a number of items along with weights or quantities are given, their weighted average of price relatives is given by

$$P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right) \times w}{\Sigma w} = \frac{\Sigma(\text{price relative} \times \text{weights})}{\Sigma(\text{weight})}$$

$$= \frac{\Sigma wx}{\Sigma w}, \text{ if we denote } \Sigma\left(\frac{p_1}{p_0} \times 100\right) \text{ by } x.$$

If p_0 and q_0 denote respectively the price and quantity of a commodity in the base year, then its weight (w) is given by p_0q_0. So, we can write the above formula as

$$P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right) \times p_0q_0}{\Sigma\, p_0q_0}$$

Note. Compute base year values p_0q_0 for each commodity and consider them as weights, if weights are not given.

Ex. 12. The quotations for four different commodities for the years 2000 and 2005 are given below. Calculate the index number for 2005, with 2000 as the base year, by using the weighted average of price relatives method.

Commodity	Weight	Price (in ₹) in the Year 2000	Price (in ₹) in the Year 2005
A	5	2.00	4.50
B	7	2.50	3.20
C	6	3.00	4.50
D	2	1.00	1.80

(Type ISC 2011, ISC 2006)

Sol.

Commodity	*Weight* (w)	*Price (in ₹) in the year 2000* p_0	*Price (in ₹) in the year 2005* p_1	*Price relatives* $x = \frac{p_1}{p_0} \times 100$	wx
A	5	2.00	4.50	$\frac{4.50}{2.00} \times 100 = 225$	1125
B	7	2.50	3.20	$\frac{3.20}{2.50} \times 100 = 128$	896
C	6	3.00	4.50	$\frac{4.50}{3.00} \times 100 = 150$	900
D	2	1.00	1.80	$\frac{1.80}{1.00} \times 100 = 180$	360
	$\Sigma w = 20$				$\Sigma wx = 3281$

$$\therefore \quad P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right) \times w}{\Sigma w} \quad \Rightarrow \quad P_{01} = \frac{3281}{20} = \mathbf{164.05}.$$

Ex. 13. Taking 1975 as the year, with an index number 100, calculate an index number for 1979, based on weighted average of price relatives derived from the table given below:

Commodity	***A***	***B***	***C***	***D***
Weight	**30**	**15**	**25**	**30**
Price per unit in 1975	**20**	**10**	**5**	**40**
Price per unit in 1979	**24**	**20**	**30**	**40**

(ISC 2012, 2009 Type)

The weights are now changed so that the weight for *A* is 40 and *C* is 10 and the total weight is 100. If the value of the index no. in 1979 with the changed weight is 182, calculate the weights applied to *B* and *D*. ***(ISC 2002, 1980)***

Sol. (i)

Commodity	*Price per unit*		*Price relatives for 1979* $x = \frac{p_1}{p_0} \times 100$	*Weights* w	$w.x$
	p_0 *1975*	p_1 *1979*			
A	20	24	120	30	3600
B	10	20	200	15	3000
C	5	30	600	25	15000
D	40	40	100	30	3000
				$\Sigma w = 100$	$\Sigma wx = 24600$

Index number for 1979 $= \frac{\Sigma wx}{\Sigma w} = \frac{24600}{100} = \mathbf{246}$.

(*ii*)

Commodity	p_0	p_1	x	w	$w \cdot x$
A	20	24	120	40	4800
B	10	20	200	x	$200x$
C	5	30	600	10	6000
D	40	40	100	$50 - x$	$5000 - 100x$
					$15800 + 100x$

$$182 = \frac{15800 + 100x}{100} \Rightarrow x = 24 \text{ or } 50 - x = 26$$

Hence, the weights for *B* and *D* are **24** and **26** respectively.

Ex. 14. Compute price index for the following data by applying weighted average of price relative method.

Item	Price in 1991 (₹)	Price in 1992 (₹)	Quantity in 1991
Wheat	**2.00**	**2.50**	**40 kg**
Sugar	**3.00**	**3.25**	**20 kg**
Milk	**1.50**	**1.75**	**10 litre**

Sol.

Items	*Price in 1991* p_0 *(in ₹)*	*Price in 1992* p_1 *(in ₹)*	*Quantity in 1991* (q_0)	$w = p_0 q_0$	$x = \frac{p_1}{p_0} \times 100$	wx
Wheat	2.00	2.50	40	80	125	10000
Sugar	3.00	3.25	20	60	108.33	6499.8
Milk	1.50	1.75	10	15	116.67	1750.05
			Total	$\Sigma w =$ 155		$\Sigma wx =$ 18249.85

$$P_{01} = \frac{\Sigma wx}{\Sigma w} = \frac{18249.85}{155} = \mathbf{117.74.}$$

Ex. 15. Calculate weighted index number for 2001 from the following data:

Item	*A*	*B*	*C*
Quantity (units)	**20**	**15**	**10**
Price in 2000 (in ₹)	**200**	**100**	**20**
Price in 2001 (in ₹)	**320**	**120**	**28**

(ISC 2003)

Sol. We compute the weighted index number by using weighted arithmatic mean.

Item	*Quantity* q_0	*Price in 2000 (in ₹)* p_0	*Price in 2001(in ₹)* p_1	$w = p_0q_0$	*Price relatives* $x = \frac{p_1}{p_0} \times 100$	wx
A	20	200	320	4000	$\frac{320}{200} \times 100 = 160$	640000
B	15	100	120	1500	$\frac{120}{100} \times 100 = 120$	180000
C	10	20	28	200	$\frac{28}{20} \times 100 = 140$	28000
				$\Sigma w = \Sigma p_0q_0 = 5700$		$\Sigma wx = 848000$

$$\therefore \quad P_{01} = \frac{\Sigma wx}{\Sigma w} = \frac{848000}{5700} = \mathbf{148.77}.$$

Ex. 16. The price relatives and weights of a set of commodities are given below:

Commodity	***A***	***B***	***C***	***D***
Price relative	**125**	**120**	**127**	**119**
Weight	x	$2x$	y	$y + 3$

If the sum of weights is 40 and the index for the set is 122, find the numerical values of x and y. ***(ISC 1988)***

Sol. We first tabulate the given data in the following form.

Commodity	*Weight* w	*Price relative*	$w \times$ *Price relative*
A	x	125	$125x$
B	$2x$	120	$240x$
C	y	127	$127y$
D	$y + 3$	119	$119y + 357$
Total	$\Sigma w = 3x + 2y + 3$		Σ(price relative $\times w$) $= 365x + 246y + 357$

Given: $\Sigma w = 40 \Rightarrow 3x + 2y + 3 = 40 \Rightarrow 3x + 2y = 37$ ... (*i*)

$$P_{01} = \frac{\Sigma \text{ (Price relatives} \times \text{weights)}}{\Sigma \text{ (weights)}} \Rightarrow 122 = \frac{365x + 246y + 357}{40} \Rightarrow 365x + 246y = 4523 \quad ...(ii)$$

Solving (*i*) and (*ii*), we get $x = 7$ and $y = 8$.

Ex. 17. An index of retail prices is the mean of the five other price index numbers which are weighted as follows:

A	***B***	***C***	***D***	***E***
Food	**Rent**	**Household goods**	**Fuel and lights**	**Miscellaneous**
348	**88**	**106**	**65**	**393**

The original index of retail prices are established at 100. If the following percentage increases in the various indices have occurred since that time, find the index of retail prices now.

***A* = 55%; *B* = 180%; *C* = 60%; *D* = 45%; *E* = 90%.**

(ISC 1991)

Sol. Given : The base year prices for commodities *A, B, C, D* and *E* are 100 each and the price relatives $\left(x = \frac{p_1}{p_0} \times 100\right)$ are respectively 155, 280, 160, 145, 190.

We form the following table.

Commodity	*Weights* (*w*)	*Price relatives* $x = \frac{p_1}{p_0} \times 100$	*wx*
Food	348	155	53940
Rent	88	280	24640
Household goods	106	160	16960
Fuel and light	65	145	9425
Miscellaneous	393	190	74670
	$\Sigma w = 1000$		$\Sigma wx = 179635$

$\therefore$ Required index $= \frac{\Sigma wx}{\Sigma w} = \frac{179635}{1000} =$ **179.635**.

EXERCISE 30 (*b*)

1. Explain briefly, what is meant by a "weighted average".

Calculate a cost of living index from the following table of prices and weights.

	Weight	*Price index*
Food	35	108.5
Rent	9	102.6
Clothes	10	97.0
Fuel	7	100.9
Miscellaneous	39	103.7

(SC, ISC 1990)

2. Taking 1975 as the base year with an index number 100, calculate an index number for 1985 based on weighted average of price relatives derived from the table given below:

Commodity	*A*	*B*	*C*	*D*
Weight	20	30	10	40
Price per unit in 1975	10	20	5	40
Price per unit in 1985	30	35	10	80

(ISC 1987)

3. Calculate the index number for the year 1979 with 1970 as base from the following data using weighted average of price relatives.

Commodity	*Weights*	*Price in* ₹	
		1970	*1979*
A	22	2.50	6.20
B	48	3.30	4.40
C	17	6.25	12.75
D	13	0.65	0.90

(ISC 2005)

4. Construct a composite index number, as a weighted mean from the following data:

Index number	122	145	101	98	137	116
Weight	7	2	4	1	6	5

(*SC*)

5. Construct a composite index number from the following index numbers and weights:

Index numbers	127	142	186	172	115
Weight	5	4	3	6	8

(*ISC*)

6. A small industrial concern used three raw materials A, B and C in its manufacturing process. The price, in £ per kg, of these materials are shown below:

	1957	*1967*
A	4	5
B	60	57
C	36	42

Using 1957 as the base year, calculate for 1967.

(*i*) a simple aggregate price index.

(*ii*) price relatives for the three materials and hence a simple average of relatives index. Does either index suffer from any disadvantage? If the number of kg's of A, B and C used per year are 30, 5 and 10 respectively, calculate a weighted aggregate price index for 1967 using 1957 as the base year. **(*ISC*)**

7. A manufacturer uses 4 raw materials A, B, C, D in the production of a certain commodity. Masses of raw materials used in manufacturing are in the ratio 2 : 3 : 4 : 1. The prices, in ₹, of the materials per kilogram in the years 1978, 1980 are given in the following table:

	A	*B*	*C*	*D*
1978	8	12	6	18
1980	9.50	13	7.50	20

Calculate the index number for the total cost of the raw materials used for the manufacture of the commodity in 1980, using 1978 as the base year.

If the commodity is sold for ₹ 5.75 in 1978, calculate the selling price in 1980, on the assumption that selling prices are directly proportional to the cost of raw material. **(*ISC 1984*)**

8. The table shows the averages prices of coffee, sugar and milk in 1979 and 1980, and the weights used to calculate the cost of making a cup of coffee.

	Cost in 1979 (per kg) ₹ (p_0)	*Cost in 1980 (per kg) ₹ (p_1)*	*Weights (w)*
Sugar	3	7	3
Milk	3	3.50	4
Coffee	90	120	2

Calculate, correct to one decimal place, the index number for the cost of a cup of coffee in 1980 using:

(*i*) weighted price relatives, (*ii*) weighted aggregates

taking the index number for 1979 as 100 in each case. **(*ISC*)**

9. An enquiry into the budget of the middle class families in a city in England gave the following information:

Expenses on	*Food* 35%	*Rent* 15%	*Clothing* 20%	*Fuel* 10%	*Misc.* 20%
Prices (1928)	£ 150	£ 30	£ 75	£25	£40
Prices (1929)	£ 145	£ 30	£ 65	£ 23	£ 45

What changes in cost of living figures of 1928 as compared with that of 1929 are seen?

10. Calculate the cost of living index number from the following group data:

Group	*Weights*	*Group Index No.*
Food	47	247
Fuel and Lighting	7	293
Clothing	8	289
House rent	13	100
Miscellaneous	14	236

(ISC 1993)

11. The following commodities have the given price indices relative to a base of 100. The weights are also given.

	Relative index	*Weight*
Butter	181	4
Bread	116	12
Tea	110	3
Bacon	152	7

Calculate the new index for this set of commodities. ***(ISC)***

12. Calculate an index number for the second year, taking the first year as base, taking into account the prices of the four commodities (in ₹ per kg) and the weights given hereunder:

	A	*B*	*C*	*D*
I year	30	28	36	28
II year	42	35	45	42
Weight	24	14	6	25

(ISC)

13. Construct the consumer price index number for 1988 on basis of 1998 from the following data:

Commodity	*A*	*B*	*C*	*D*	*E*
Weights	40	25	5	20	10
Price (₹ per unit) 1988	16.00	40.00	0.50	5.12	2.00
Price (₹ per unit) 1998	20.00	60.00	0.50	6.25	1.50

(ISC 2012)

14. Calculate the index number for the year 2006 with 1996 as the base year by the weighted average of price relatives method from the following data.

Commodity	*A*	*B*	*C*	*D*	*E*
Weight	40	25	5	20	10
Price (₹ per unit) year 1996	32.00	80.00	1.00	10.24	4.00
Price (₹ per unit) year 2006	40.00	120.00	1.00	15.36	3.00

(ISC 2009)

15. Calculate the cost of living index for the following data:

Commodity	*Weight*	$x = \frac{p_1}{p_0} \times 100$
A	22	248
B	48	133.30
C	17	204
D	13	138.50

(ISC 1997)

16. Find the consumer price index number for 1991 on the base of 1990 from the following data, using the method of weighted relatives:

Item	*Quantity*	*Price in 1990* (₹)	*Price in 1991* (₹)
A	20 units	200	320
B	14 units	400	420
C	15 units	100	120
D	18 units	40	60
E	10 units	20	28

(ISC 1999)

17. From the following data compose price index by applying weighted average of price relatives method using arithmetic means:

Commodity	p_0 (₹)	q_0	p_1 (₹)
Sugar	9.00	20 kg	15.00
Flour	5.00	40 kg	10.00
Milk	6.00	10 lit.	15.00

(ISC 1999)

18. The following table shows the prices per unit in 1980 and 1984 with weights of the commodities *A*, *B*, *C*, *D* ;

Commodity	*Weights*	*Price in units in 1980*	*Price in unit in 1984*
A	20	25	30
B	25	20	30
C	15	50	70
D	40	5	10

Taking 1980 as base year with an index number 100, calculate the index number of 1984 based on weighted average of price relatives. ***(ISC 2000, ISC 2011)***

19. The price quotations of four different commodities for 2001 and 2009 are as given below. Calculate the index number for 2009 with 2001 as the base year by using weighted average of price relative method.

Commodity	*Weight*	*Price (in ₹)*	
		2009	*2001*
A	10	9.00	4.00
B	49	4.40	5.00
C	36	9.00	6.00
D	4	3.60	2.00

(ISC 2011)

ANSWERS

1. 104.4 **2.** 212.5 **3.** 171.24 **4.** 122
5. 143
6. (*i*) A simple aggregate price index is £ 104 ;
(*ii*) A simple average of relatives index is £ $112\frac{2}{9}$. Both suffer from the disadvantage that no weights are used in either of the methods. A weighted aggregate price index for 1967 using 1957 as the base year is £ $119\frac{22}{27}$.
7. 114.9 ; ₹ 6.61 **8.** (*i*) 159.3 (*ii*) 136.8
9. Index number for 1929 is equal to 97.86. It means that the living in 1929 was cheaper than the living in 1928.
10. 231.2 approx. **11.** 135 **12.** 139.3 **13.** 124.41
14. 130 **15.** 171.229 **16.** 137.27 **17.** 193.18
18. 162.5 **19.** 128.10

HINTS TO SELECTED QUESTIONS

4. Composite index number *is the average of the index numbers for different groups of variables*. Here, you may proceed as under:

$$\frac{\sum(\text{Index No.} \times \text{Weight})}{\sum \text{Weight}}.$$

8. For (*ii*) use $100 \times \frac{\sum p_1 w}{\sum p_0 w}$

10.

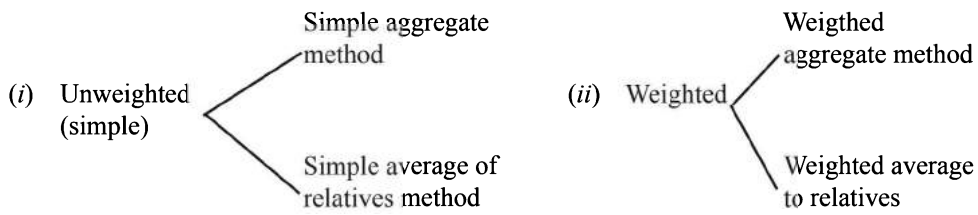
Group	*Weights (w)*	*Group Index No. (x)*	*w.x.*
Food	47	247	11609
Fuel and Lighting	7	293	2051
Clothing	8	289	2312
House rent	13	100	1300
Miscellaneous	14	236	3304
	$\Sigma w = 89$		$\Sigma w \cdot x = 20576$

$\therefore$ Cost of living index $= \frac{\sum w \cdot x}{\sum w} = \frac{20576}{89} = 231.2$ approx.

16. Type Solved Ex. 15 **17.** Type Solved Ex. 15 **18.** Type Solved Ex. 12 **19.** Type Solved Ex. 12

CHAPTER SUMMARY

1. **Index numbers** are specialised averages that measure changes non capable of direct measurement. These are comparative in nature and measure changes taking place over a period of time.
2. **Constructing Index numbers.** The various methods for constructing index numbers.



3. Formulae

(*i*) Price relative $= \frac{p_1}{p_0} \times 100 = \frac{\text{current price}}{\text{base price}} \times 100$

(*ii*) Simple aggregate method: $P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$, where p_{01} denotes the price index number for the current year with reference to the base year. Σp_1 denotes the total of current year prices of all commodities under consideration. Σp_0 denotes the total of base year prices of the same commodities as considered for Σp_1.

3. Simple average of price relative method

$$P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right)}{n}$$

Where n dentoes the number of commodities.

4. Weighted aggregate method:

$$P_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$$

- Sometimes, the quantities in the base year are taken as weights. In such cases the weighted index number is constructed by using the formula

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$ (Solved Ex. 11)

5. Weighted average of price relative method

$$\boldsymbol{P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right) \times w}{\Sigma w} = \frac{\Sigma(\text{Price relatives} \times \text{weight})}{\Sigma(\text{weights})}}$$

$$\boldsymbol{= \frac{\Sigma wx}{\Sigma w}, \text{ denoting } \Sigma\left(\frac{p_1}{p_0} \times 100\right) \text{ by } x.}$$

If p_0 and q_0 denote respectively the price and quantity of a commodity in the base year, then its weight $(w) = p_0 q_0$ and so the above formula can be written as

$$P_{01} = \frac{\Sigma\left(\frac{p_1}{p_0} \times 100\right) \times p_0 q_0}{\Sigma p_0 q_0}$$

CHAPTER TEST

1. Construct the consumer price index number for 1990, taking 1989 as the base year and using simple average of price relative method for the following data:

Commodities	*Price in 1989*	*Price in 1990*
Butter	20	21
Cheese	16	12
Milk	3	3
Egg	2.80	2.80

(ISC 2003)

2. A small industrial concern used three raw materials A, B and C in its manufacturing process, the prices of the materials was as shown below:

Commodity	*Price (in ₹) in the year 1995*	*Price (in ₹) in the year 2005*
A	4	5
B	60	57
C	36	42

Using 1995 as the base year, calculate a simple aggregate price index for 2005. **(*ISC 2008*)**

3. Find the consumer price index number for the year 2010 as the base year by using method of weighted aggregates.

Commodity	*A*	*B*	*C*	*D*	*E*
year 2000 price (in ₹) per unit	16	40	0.50	5.12	2.00
Year 2010 price (in ₹) per unit	20	60	0.50	6.25	1.50
Weights	40	25	5.00	20.00	10.00

(*ISC 2012*)

4. The price of six different commodities for years 2009 and year 2011 are as follows:

Commodities	*A*	*B*	*C*	*D*	*E*	*F*
Price in 2009 (₹)	35	80	25	30	80	x
Price in 2011 (₹)	50	y	45	70	120	105

The index number for the year 2011 taking 2009 as the base year for the above data was calculated to be 125. Find the values of x and y if the total price in 2009 is ₹ 360. **(*ISC 2014*)**

ANSWERS AND HINTS

1. 95. Type Solved Ex. 6, 7.

2. 104, Solved Ex. 1, Ch. 30.

3. 138.39.

4. x = ₹ 110, y = ₹ 60. Type solved Ex. 4 on page chp 30-9.

CHAPTER 31

MOVING AVERAGE

31.01. Introductory remarks

Consider the following situations:

1. The planners in the Government may like to estimate the likely population of the country in the next 5 to 10 years so that necessary planning may be done in advance to fulfil the needs of the increased population in food, housing, clothing, electricity, fuel, etc.
2. A multinational would like to know in advance deciding to go in for any new venture about the likely demand of its product before and the profit it can earn.

For making such estimates, and making forecasts, it would be necessary to have information about the performance of the variables in the past.

For this, we collect statistical data at successive intervals of time. Such data are called **Time Series**. These may be series relating to prices, production of commodities-agricultural or industrial, national income and foreign exchange reserves, etc. Thus, **Time series** is *a record arranged chronologically, of the values of a variate* (*i.e., commodity*) *during a particular period, taken at successive intervals of time*.

The following is an example of time series:

Population of India (in millions) from 1901 to 1971

Year	1901	1911	1921	1931	1941	1951	1961	1971
Population (*in millions*)	238.4	252.1	254.3	279.0	318.7	361.1	439.2	547.4

Thus a time series is given by a bivariate distribution, one of the two variables being time and the other value of the phenomenon measured at different intervals of time.

When the values of the variate are plotted against time, on a graph paper, and the points so obtained are joined by straight line segments, we obtain a graphical record of the behaviour of the variate. Graph of a time series so obtained is called a **historigram** of the time series. The historigram gives a rough idea about the nature of changes in the values of the variate with time.

Illustration:

The following figures give the production (in 1000 quintals) of a sugar factory.

Year	1941	42	43	44	45	46	47	48	49	50	51	52	53	54	55
Production in quintals	80	85	90	93	86	94	99	92	88	90	93	96	100	93	105

When represented graphically, this time series gives the following historigram.

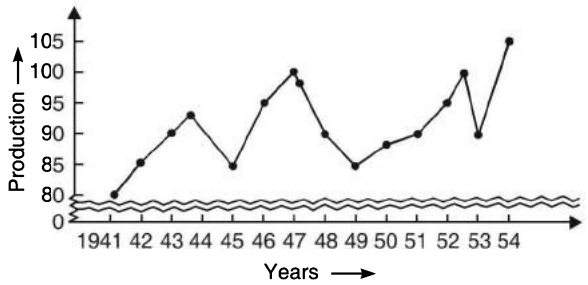


Fig. 31.01

This historigram shows that on the whole the production is steadily increasing. Even though there are ups and downs over shorter durations, a steady rising of the production figures is clearly seen. The variation in the values of the variate (*i.e.*, production figures) indicated by the historigram, show fluctuations. These fluctuations are of many different kinds. However, they distinctly reveal a steady rise in the production figures, superimposed by increases and decreases, with shorter intervals. The fluctuations are the combined effect of various causes, which sometimes induce sharp rise and sharp fall in the production figures, though the steady influence is one which indicates continuous rise. The influences or causes which operate over a long time, need to be separated from those which operate over a short time. The long term influences help, when identified, to estimate the values beyond the period for which they are recorded. Knowledge about the short time influences then enables to make adjustments in the estimated values. Segregating the various types of fluctuations in the time series is known as the *analysis of the time series of the production figures*. The main problem of *analysis of the time series* is to separate the long time and the short time fluctuations.

31.02. Variations or fluctuations or oscillations

The four basic types are:

1. Secular Trend or Long term Trend
2. Seasonal Variations
3. Cyclical Variations
4. Irregular, erratic or random fluctuations.

1. Secular or long-time trend

The general tendency of the time series data to increase or decrease or stagnate during a long period of time is called the secular trend, also known as longterm trend. Although, some up and down movements may occur over short intervals of time, yet the tendency to grow or decline persists and is clearly visible. For example, in India, population, price of consumer goods, agricultural production etc., had a general tendency to increase with time, while mortality rate, value of the rupee, etc., had a tendency to decrease.

2. Seasonal variations

These variations may occur due to weather and other natural forces or social conditions. Umbrellas are sold more in rainy season, refrigerators and air-conditioners in summers and woollen clothes in winters. Demand for certain commodities increase in festivals and in marriage season.

3. Cyclic variations

Many industrial or financial sectors experience repeated booms and slumps over regular periods of time. Since these periods are

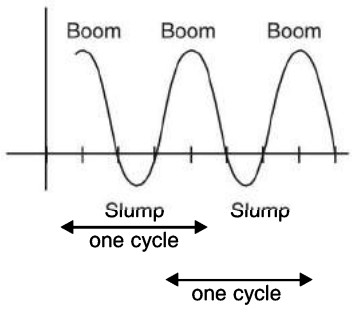


Fig. 31.02

short (but larger than a year) they do not affect the trend. Such periodic changes are known as *Cyclic changes*. When isolated and plotted against time they show movements from peaks to troughs and back at regular intervals of time. Thus, the periodic movement is in the form of cycles.

31.03. Moving averages

In this method, fluctuations due to the cyclical changes are eliminated by averaging the values of the variate for a specified number of successive years. The number of years over which the values are averaged, depends upon the average length of cycles found in the series. The time-interval covering the number of years is called the *period of the cycles*.

Illustrations. The following illustrations will help to strengthen the concept of moving averages further.

1. The following graph called a historigram or a time series shows how a given value changes over time.

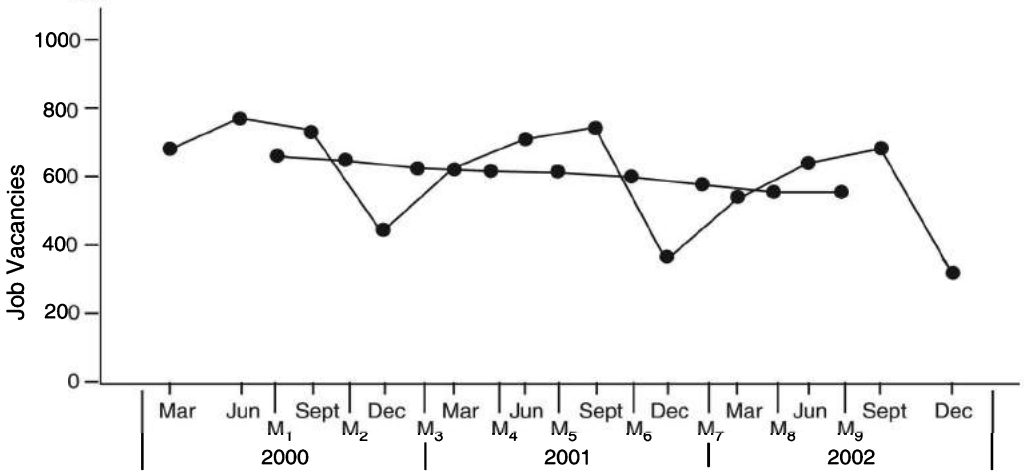


Fig. 31.03

It is difficult from the graph to see whether the number of job vacancies has risen or fallen over the three years. There is too much variation within each year.

We can account for this variation by working out a **moving average.** This means taking the average of consecutive pieces of data.

Here is the table of results for the above graph.

Year	March	June	September	December
2000	682	785	742	423
2001	622	722	752	385
2002	550	639	661	376

There are four values for each year, so we look at groups of four. The first moving average, M_1, is the mean of the first four values.

$$M_1 = \frac{682 + 785 + 742 + 423}{4} = 658$$

The next moving average, M_2, is found by 'moving up one place'.
It is the mean of the values from June 2000 to March 2001.

If we consider groups of four we are finding a **four point moving average.**

$$M_2 = \frac{785 + 742 + 423 + 622}{4} = 643$$

The 4-point moving average for the whole data set is:

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
658	643	627	630	620	602	582	559	557

We can plot the 4-point moving average on the same graph as our original data. Each average is plotted at the mid-point of the values from which it was generated.

It is easy to infer now that the number of job vacancies fell between the beginning of 2000 and the end of 2002.

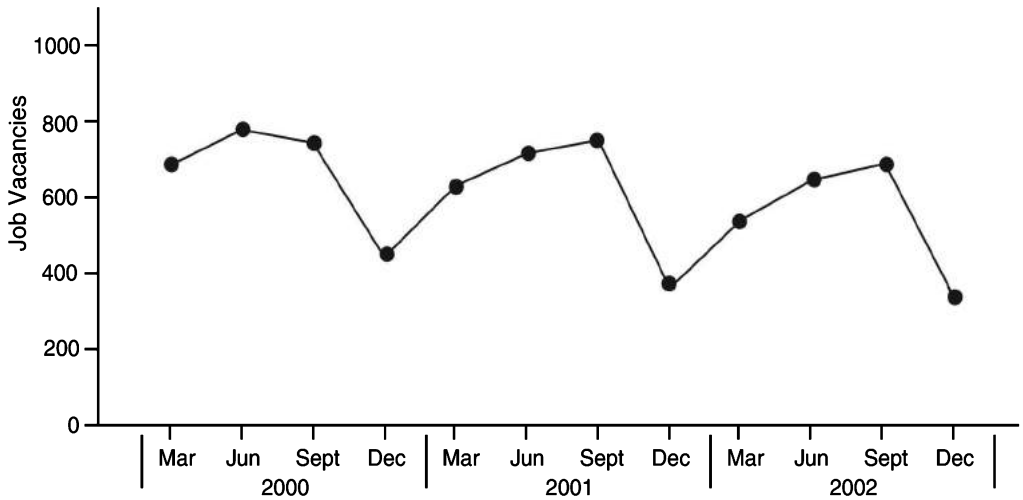


Fig. 31.04

2. Take the case of a car dealer who keeps a record of how many cars were sold in each month of a year.

Months	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Cars	10	23	36	17	16	25	44	8	15	28	47	13

We can graph this set of data (Fig. 31.05).

The resulting line graph shows a normal variation of business for the dealer, but does not reveal the general trend of business. That it is improving, declining or remaining the same.

We can show the general trend by first calculating the mean for each four month span, month on month, as below:

Mean for Jan, Feb, Mar and April (M_1) $(10 + 23 + 36 + 17) \div 4 = 21.5$

Mean for Feb, Mar, Apr and May (M_2) $(23 + 36 + 17 + 16) \div 4 = 23$

Mean for Mar, Apr, May and Jun (M_3) $(36 + 17 + 16 + 25) \div 4 = 23.5$

Mean for Apr, May, Jun and Jul (M_4) $(17 + 16 + 25 + 44) \div 4 = 25.5$

Mean for May, Jun, Jul and Aug (M_5) $(16 + 25 + 44 + 8) \div 4 = 23.25$

Mean for Jun, Jul, Aug and Sep (M_6) $(25 + 44 + 8 + 15) \div 4 = 23$

Mean for Jul, Aug, Sep and Oct (M_7) $(44 + 8 + 15 + 28) \div 4 = 23.75$

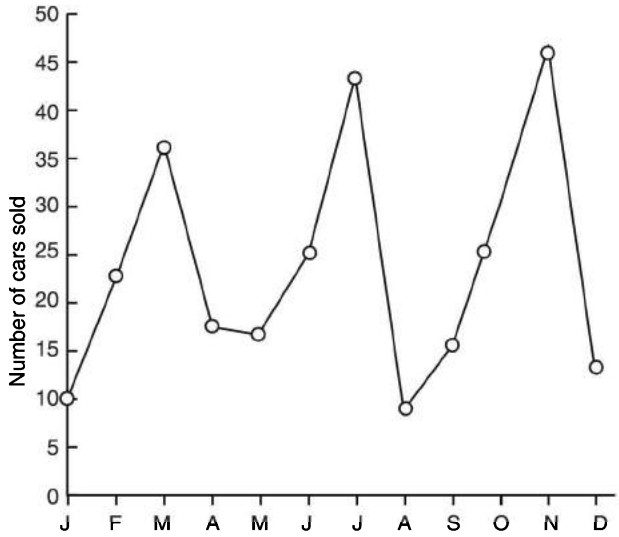


Fig. 31.05

Mean for Aug, Sep, Oct and Nov (M_8) $(8 + 15 + 28 + 47) \div 4 = 24.5$

Mean for Sep, Oct, Nov and Dec (M_9) $(15 + 28 + 47 + 13) \div 4 = 25.75$

We then plot, on the first graph, each mean value at the mid-point of the corresponding four-month span. This produces a much smoother graph, which, in this case, shows a slight upward trend. In other words, business is improving.

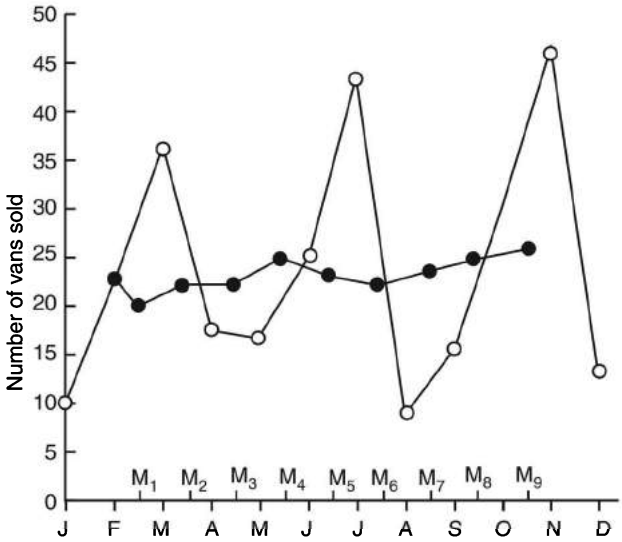


Fig. 31.06

In this example, we used an interval of four months but there is nothing special about this interval. It could well have been five or six months, except that we would then have needed data for more months to give sufficient mean values to show a trend. The number of months, weeks or even years used for moving averages depends on the likely variations of the data. We would not expect to use less than three or more than twelve items of data at a time.

31.04. General method for finding moving averages

A **moving average** gives a clear indication of the **trend** of a set of data. It smoothes out, for example, seasonal differences, monthly variations or daily differences.

Suppose the period of the cycles for a given time series is 5 years. Then the values for the first 5 years are added and their arithmetic mean is taken. This mean is written against the mid-year, *i.e.*, the 3rd year. Leaving the value of the first year, mean of the values of the next 5 years—from 2nd to 6th year—is taken. This mean is written against the mid-year of this group, *i.e.*, the 4th year.

Then we take the next group of 5 years and proceed in the same manner as for the first two groups of years, until the mean of the values of the last 5 years is written down against their mid-year. In this way, we get a new series *viz.*, the series of the arithmetic means of groups of 5 successive years corresponding to all the years except the first two and the last two. This is due to the fact that the mean of the first group of 5 successive years is written against the 3rd, the mean of the 2nd group of successive years is written against the 4th, ..., the mean of last group of 5 successive years is written against the last but second year. The values of the new series are known as *moving averages*. Since, in the present case, they are arithmetic means taken over 5-year periods, they are called 5 *year moving averages*.

We obtain 3-year, 7-year, 9-year, ..., moving averages in a similar manner. The number of successive years over which average would have to be taken in these cases will be respectively 3, 7, 9, ..., etc.

If the period of a time series is even, say 6 years, then we find the moving averages for that period, say 6-yearly moving averages, exactly as in the above case for 5-yearly moving average. But in this case we write the averages in the rows between that of the 3rd and the 4th years, the 4th and the 5th years, and so on. Then we take the arithmetic means of successive pairs of averages. The results are then written down against the central year of each pair, *i.e.*, against the 4th, 5th, ..., year respectively. The new averages are called *centred 6-year moving averages*.

In either case, *i.e.*, when the period of a time series is odd or even, the final series of moving averages represents the trend.

The moving averages smooth out the cyclical movements in the given time series. The moving averages are plotted against the corresponding years and the successive points thus obtained are joined by straight line segments. The resulting polygonal graph indicates the trend of the given series.

If the values for the years 1, 2, 3, 4, ... are $a, b, c, d, \ldots$ then the formula for calculating 3-yearly moving average is $\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}, \ldots$ and for 5 yearly moving average is $\frac{a+b+c+d+e}{5}, \frac{b+c+d+e+f}{5}, \frac{c+d+e+f+g}{5}, \ldots$

31.05. Procedure for calculating the moving averages

Case 1. When period of moving average is an odd number.

(A) For 3-yearly moving averages

Year	*Values*	*3-yearly total*	*3-yearly moving average*
I	a		
II	b	$A_1 = a+b+c$	$\frac{1}{3}A_1$
III	c	$A_2 = b+c+d$	$\frac{1}{3}A_2$
IV	d	$A_3 = c+d+e$	$\frac{1}{3}A_3$

Method: Step 1. *Add the values of the first 3 years and place the total against the middle year (year II).*

Step 2. *Leave the first year's value and add up the values of the next 3 years (II, III, IV) and place the total against the middle year, i.e., year III.*

Step 3. *Continue this process until the last year's value is taken for calculating*

V	e	$A_4 = d + e + f$	$\frac{1}{3}A_4$
VI	f	$A_5 = e + f + g$	$\frac{1}{3}A_5$
VII	g	$A_6 = f + g + h$	$\frac{1}{3}A_6$
VIII	h		

moving average.

Step 4. *On dividing the 3-yearly totals by 3, you will obtain the 3-yearly moving average.*

Step 5. *On plotting the points* $\left(II, \frac{1}{3}A_1\right), \left(III, \frac{1}{3}A_2\right), \left(IV, \frac{1}{3}A_3\right), \ldots$ *on a graph paper and joining them, you will obtain the trend-line.*

Ex. 1. Calculate the 3-yearly moving averages of the data given below:

Year	**1980**	**1981**	**1982**	**1983**	**1984**	**1985**	**1986**	**1987**	**1988**	**1989**
Sales (Millions of rupees)	**3**	**4**	**8**	**6**	**7**	**11**	**9**	**10**	**14**	**12**

Sol.

Year	*Sales, (Millions of rupees)*	*3-yearly totals*	*3-yearly moving average (trends)*
1980	3		
1981	4	15	5
1982	8	18	6
1983	6	21	7
1984	7	24	8
1985	11	27	9
1986	9	30	10
1987	10	33	11
1988	14	36	12
1989	12		

Explanation

1. Find the 3-year moving totals. The total for 1980, 1981, 1982 is 3 + 4 + 8 = 15. Place it against the middle year 1981. The total for 1981, 1982, 1983 is 18. Place it against the middle year 1982. Continue likewise. We will not have a total for 1980 and for 1989.

2. Divide the moving totals by 3. This gives us the 3-yearly moving average column.

3. **Graph.** Plot the actual and trend values (*i.e.*, moving averages), and draw the graph.

Note. The graph shows that the sales have a regular 3-year cycle. For example, from a peak at 1982 to the next peak 1985, we have 3-years. Then the next peak comes 3 years later at 1988.

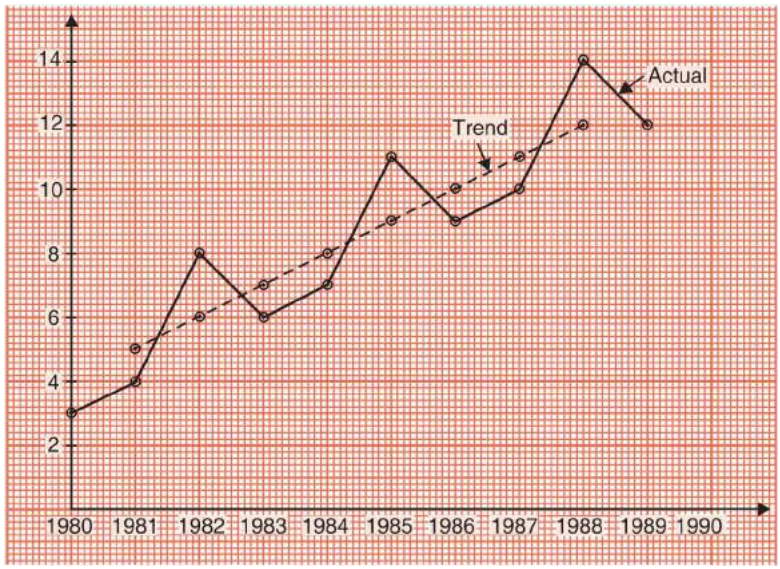


Fig. 31.07

When the 3-year moving averages are plotted on the graph as in Fig. 31.07 they fall on a straight line, and the cyclical fluctuations have been smoothed out. The straight line is the trend line we seek. The hypothetical data have been adjusted so that the trend line is a straight line; however, in general, we shall have a smooth curve.

Ex. 2. Consider the following data:

Dates is the month of April	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
Number of units sold	**2**	**5**	**0**	**12**	**13**	**25**	**45**	**13**	**31**	**18**	**11**	**2**	**3**	**1**

Calculate 3-day moving averages and display these and the original figures on the same graph. ***(ISC 2010)***

Sol.

Dates in the month of April	**Number of units sold**	**3-day moving total**	**3-day moving average**
12	2	—	—
13	5	7	2.33
14	0	17	5.66
15	12	25	8.33
16	13	50	16.66
17	25	83	27.66
18	45	83	27.66
19	13	89	29.66
20	31	62	20.66
21	18	60	20
22	11	31	10.33
23	2	16	5.33
24	3	6	2
25	1	—	—

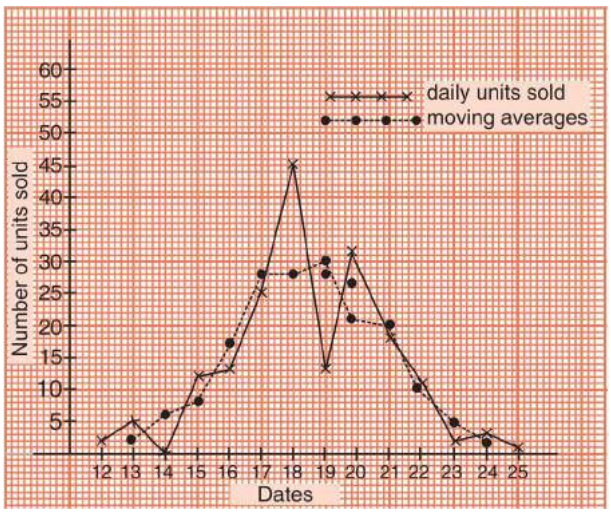


Fig. 31.08

(B) For 5-yearly moving averages

Year	*Values*	*5-yearly total*	*5-yearly moving average*
I	a	—	—
II	b	—	—
III	c	$a+b+c+d+e=A_1$	$\frac{1}{5}A_1$
IV	d	$b+c+d+e+f=A_2$	$\frac{1}{5}A_2$
V	e	$c+d+e+f+g=A_3$	$\frac{1}{5}A_3$
VI	f	$d+e+f+g+h=A_4$	$\frac{1}{5}A_4$
VII	g	—	—
VIII	h	—	—

Method 1. Add the values of the first five years and write the total against the middle year (year III). Now, leave value of Ist year, and add the values of the years II, III, IV, V, VI and write the total against the middle year (year IV) and continue likewise.

2. These totals are divided by 5 and the moving averages are written in the next column.

3. Graph. The trend line is obtained by plotting the points $\left(\text{III}, \frac{1}{5}A_1\right)$, $\left(\text{IV}, \frac{1}{5}A_2\right), \left(\text{V}, \frac{1}{5}A_3\right), \left(\text{VI}, \frac{1}{5}A_4\right)$.

Ex. 3. Calculate the five-yearly moving averages of a number of students studying in a commercial college from the following figures:

Year	**1981**	**1982**	**1983**	**1984**	**1985**
No. of students	**332**	**317**	**357**	**392**	**402**
Year	**1986**	**1987**	**1988**	**1989**	**1990**
No. of students	**405**	**410**	**427**	**405**	**438**

Draw a graph to represent the moving averages. ***(ISC 2012, Specimen Paper)***

Sol. Computation of Five yearly Moving Average

Year (1)	*No. of Students (2)*	*5 Yearly Moving Totals (3)*	*5 Yearly Moving Average (4)*
1981	332	—	—
1982	317	—	—
1983	357	1800	360.0
1984	392	1873	374.6 or 375
1985	402	1966	393.2 or 393
1986	405	2036	407.2 or 407
1987	410	2049	409.8 or 410
1988	427	2085	417.0
1989	405	—	—
1990	438	—	—

The graph is drawn below:

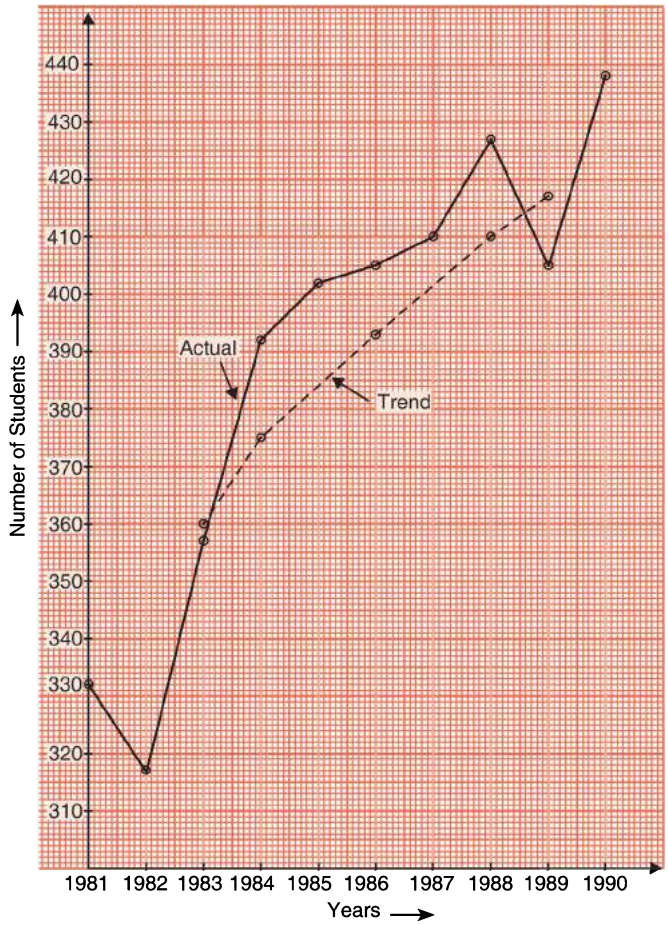


Fig. 31.09

Ex. 4. Take a five yearly period of moving average and determine short term oscillations from the following data:

Year	*Production (in '000 units)*	*Year*	*Production (in '000 units)*
1985	**14**	**1991**	**20**
1986	**17**	**1992**	**24**
1987	**22**	**1993**	**25**
1988	**28**	**1994**	**29**
1989	**26**	**1995**	**30**
1990	**18**	**1996**	**23**

Sol. Determination of short term oscillations taking 5-yearly moving average:

Year (1)	*Production (in '000 units)* (2)	*5-yearly moving totals col* (3)	*5-yearly moving* average (3) ÷ 5 (4)	*Short term oscillations col* (2) – *col* (4)
1985	14	—	—	—
1986	17	—	—	—
1987	22	107	21.4	+ 0.6
1988	28	111	22.2	+ 5.8

1989	26	114	22.8	+ 3.2
1990	18	116	23.2	− 5.2
1991	20	113	22.6	− 2.6
1992	24	116	23.2	+ 0.8
1993	25	128	25.6	− 0.6
1994	29	131	26.2	+ 2.8
1995	30	—	—	—
1996	23	—	—	—

Ex. 5. The average number, in lakhs of working days lost in strikes during each year of the period (1981–90) was as under:

1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
1.5	1.8	1.9	2.2	2.6	3.7	2.2	6.4	3.6	5.4

Calculate the 3-yearly moving average and draw the moving average graph. ***(ISC 2005)***

Sol.

Year	*Number of days (in lakhs)*	*3-yearly moving total*	*3-yearly moving average*
1981	1.5	—	—
1982	1.8	5.2	1.7
1983	1.9	5.9	1.967 or 2.0
1984	2.2	6.7	2.2
1985	2.6	8.5	2.8
1986	3.7	8.5	2.8
1987	2.2	12.3	4.1
1988	6.4	12.2	4.066 or 4.1
1989	3.6	15.4	5.1
1990	5.4	—	—

The required graph is shown below:

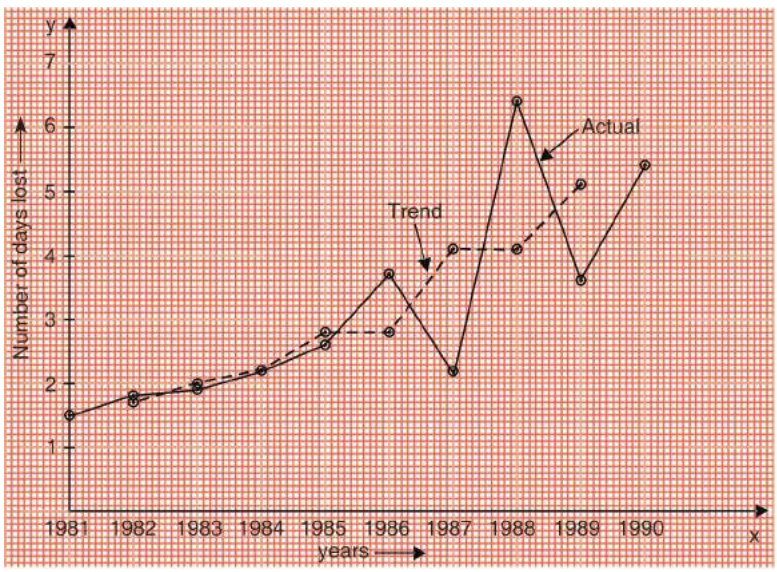


Fig. 31.10

Ex. 6. The table shows the number of students on the roll of a school on the first day of each term.

Term	1981	1982	1983	1984	1985	1986
January	252	261	288	299	312	312
April	284	258	312	322	306	324
September	254	261	312	274	306	284

Draw the graph of these figures. Calculate a set of moving averages using the most number of observations. Give reasons for your choice. Plot these moving averages on the same graph paper and comment on the general trend. ***(ISC 1989)***

Sol. Since there are three terms so we calculate 3-term moving average.

Year	*Term*	*Number of students*	*3-term moving total*	*moving average*
1981	Jan.	252	—	—
	Apr.	284	790	263.3
	Sep.	254	799	266.3
1982	Jan.	261	773	257.7
	Apr.	258	780	260
	Sep.	261	807	269
1983	Jan.	288	861	287
	Apr.	312	912	304
	Sep.	312	923	307.7
1984	Jan.	299	933	311
	Apr.	322	895	298.3
	Sep.	274	908	302.7
1985	Jan.	312	892	297.3
	Apr.	306	924	308
	Sep.	306	924	308
1986	Jan.	312	942	314
	Apr.	324	920	306.7
	Sep.	284	—	—

The required graph is shown below:

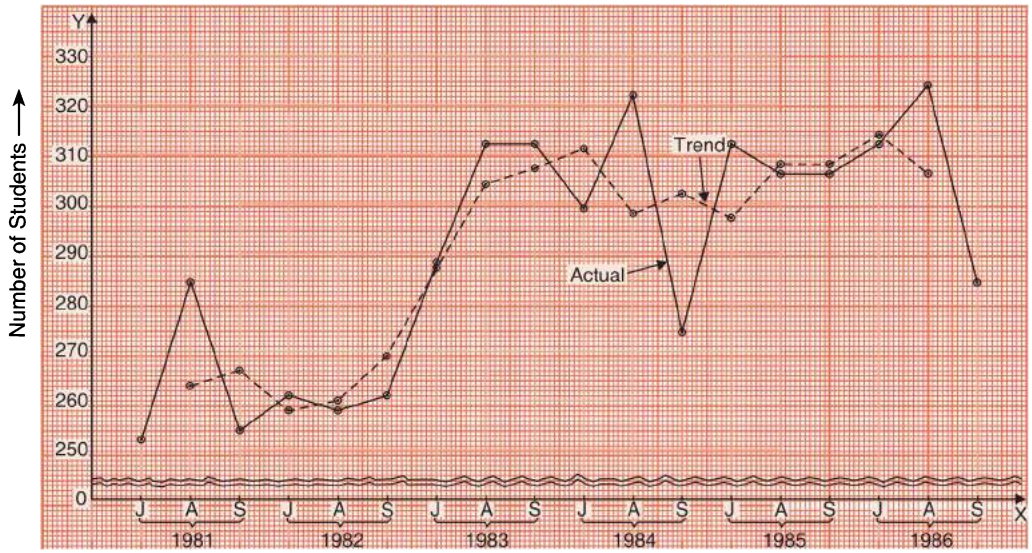


Fig. 31.11

Here, we see a steadily increasing long term trend though these are some seasonal variations.

Case 2. When period of moving average is an even number.

For 4-yearly moving averages

Year	*Value*	*4-yearly moving total*	*4-yearly moving average*	*4-yearly moving average centred*
I	a			
II	b			
	←	$a+b+c+d=A_1$	$\frac{1}{4}A_1=M_1$	
III	c		←	$\frac{1}{2}(M_1+M_2)=B_1$
	←	$b+c+d+e=A_2$	$\frac{1}{4}A_2=M_2$	
IV	d		←	$\frac{1}{2}(M_2+M_3)=B_2$
	←	$c+d+e+f=A_3$	$\frac{1}{4}A_3=M_3$	
V	e		←	$\frac{1}{2}(M_3+M_4)=B_3$
	←	$d+e+f+g=A_4$	$\frac{1}{4}A_4=M_4$	
VI	f			
VII	g			

Method 1. *Add the values of the first four years and place the total* (A_1) *in between the 2nd and 3rd years.*

2. *Leave the first year value and add the values of the next four years and place the total* (A_2) *in between the 3rd and 4th years.*

Continue this process until the last year is taken into account.

3. *Divide 4-yearly moving totals* (A_1, A_2, A_3, ...,) *by 4. This will give 4-yearly moving averages* (M_1, M_2, M_3, ...).

4. *Add first two moving averages and divide by 2 to get the moving average centred* (B_1). *Place it against 3rd year. Now, add next two moving averages* M_2 *and* M_3 *and divide by* 2 *to get the next moving average centred* B_2. *Place it against the 4th year.*

Continue this until the last moving average has been included.

5. Graph. *Plot the points* (III, B_1), (IV, B_2), (V, B_3) *and join them to get the trend line.*

Ex. 7. The following table shows the average monthly production of coal in millions of tonnes for the years 1987-1996.

Year	*Average monthly production of coal (in million tonnes)*	*Year*	*Average monthly production of coal (in million tonnes)*
1987	**50.0**	**1992**	**38.1**
1988	**36.5**	**1993**	**32.6**
1989	**43.0**	**1994**	**41.7**
1990	**44.5**	**1995**	**41.1**
1991	**38.9**	**1996**	**33.8**

Determine: (*i*) 4-yearly moving average figures.

(*ii*) 4-yearly centered moving average.

Sol. 4-yearly moving average and 4-yearly centred average

Year	*Average monthly production of coal (in million tonnes)*	*4-yearly moving total*	*4-yearly Moving average*	*4-yearly moving average centred*
1987	50.0			—
1988	36.5			—
	←	174.0	43.5	
1989	43.0		←	42.10
	←	162.9	40.7	
1990	44.5		←	40.90
	←	164.5	41.1	
1991	38.9		←	39.80
	←	154.1	38.5	
1992	38.1		←	38.15
	←	151.3	37.8	
1993	32.6		←	38.10
	←	153.5	38.4	
1994	41.7		←	37.85
	←	149.2	37.3	
1995	41.1			
1996	33.8			

Ex. 8. The profit of a soft drink firm (in thousands of rupees) during each month of the year is as given below.

Months	**Profit (in thousands of Rupees)**
January	3.6
February	4.3
March	4.3
April	3.4
May	4.4
June	5.4
July	3.4
August	2.4
September	3.4
October	1.8
November	0.8
December	1.2

Calculate the 4-monthly moving averages and plot these and the original data on a graph sheet. ***(ISC 2011)***

Sol.

Months	**Profit (in thousands of rupees)**	**4-monthly moving total**	**4-monthly moving average**	**4-monthly moving average centred**
Jan.	3.6			
Feb.	4.3			
	→	15.6	3.9	
March	4.3		→	4.0
	→	16.4	4.1	

April	3.4			4.2375 → 4.2
	→	17.5	4.375	→
May	4.4			4.2625 → 4.3
	→	16.6	4.15	→
June	5.4			4.025 → 4.0
	→	15.6	3.9	→
July	3.4			3.775 → 3.8
	→	14.6	3.65	→
Aug.	2.4			3.2
	→	11.0	2.75	→
Sep.	3.4			2.425 → 2.4
	→	8.4	2.1	→
Oct.	1.8			1.95 → 2.00
	→	7.2	1.8	→
Nov.	0.8			
Dec.	1.2			

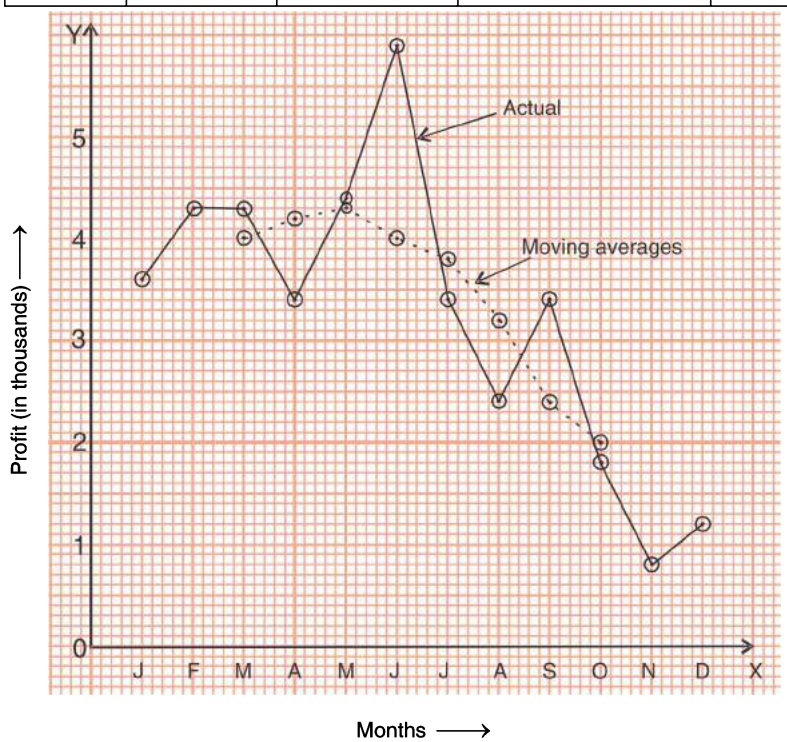


Fig. 31.12

The required graph is shown as above: The general comment is that the profits go on increasing from January to May and then start decreasing from June to Dec.

Ex. 9. The table below gives details of the electricity generated in million kilowatt hours in each quarter for the years 2002 to 2004.

Year	Quarter			
	1st	2nd	3rd	4th
2002	8	7	6	9
2003	10	7	7	10
2004	11	7	8	10

Calculate the 4-quarterly moving average and show these moving averages on a graph.

(Type ISC 2014, ISC 2006)

Sol.

Year	*Quarter*	*Elec. generated million kilowatt*	*4-quarterly total*	*4-quarterly moving average*	*4-quarterly moving average centred*
2002	1st	8			
	2nd	7	30	7.5	7.75
	3rd	6	32	8	8
	4th	9	32	8	8.125
2003	1st	10	33	8.25	8.375
	2nd	7	34	8.5	8.625
	3rd	7	35	8.75	8.75
	4th	10	35	8.75	8.875
2004	1st	11	36	9	9
	2nd	7	36	9	
	3rd	8			
	4th	10			

The first 4-quarterly moving average $= \frac{1}{4}(8+7+6+9) = \frac{1}{4}(30) = 7.5$

The second 4-quarterly moving average $= \frac{1}{4}(7+6+9+10) = \frac{1}{4}(32) = 8$

The third 4-quarterly moving average $= \frac{1}{4}(6+9+10+7) = \frac{1}{4}(32) = 8$

The first moving average centred $= \frac{7.5+8}{2} = 7.75$

The second moving average centred $= \frac{8+8}{2} = 8$

The corresponding historigram and the 4-quarterly moving averages are shown in the graph drawn below.

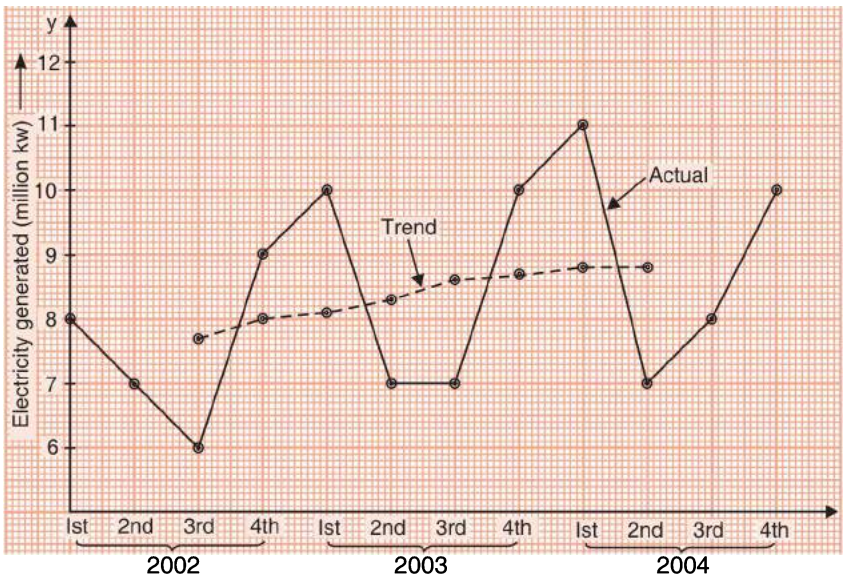


Fig. 31.13

Ex. 10. Coded monthly sales figures of a particular brand of T.V. for 18 months commencing January 1, 2005 are as follows:

Year	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
2005	18	16	23	27	28	19	31	29	35	27	28	24
2006	24	28	29	30	29	22						

Calculate 6-monthly moving averages and display these and the original figures on the same graph using the same axes for both. ***(ISC 2008)***

Sol. Calculation of 6-monthly moving average.

Year (a)	*Month* (b)	*Monthly sale* (c)	*6-monthly moving total* ($D = \Sigma c$)	*6-monthly moving average* $\left(d = \frac{1}{6}\Sigma c\right)$	*6-monthly centred moving average*
2005	Jan.	18			
	Feb.	16			
	Mar.	23	131	21.833	
	Apr.	27	144	24.000	22.92
	May	28	157	26.167	25.08
	June	19	169	28.167	27.17
	July	31	169	28.167	28.17
	Aug.	29	169	28.167	28.17
	Sept.	35	174	29.000	28.52
	Oct.	27	167	27.833	22.42
	Nov.	28	166	27.667	22.75
	Dec.	24	166	27.667	22.17
2006	Jan.	24	163	27.167	26.92
	Feb.	28	164	27.333	27.25
	Mar.	29	162	27.000	27.17
	Apr.	30			
	May	29			
	June	22			

First 6-monthly moving average = $\frac{18+16+23+27+28+19}{6} = 21.833$

Second 6-monthly moving average = $\frac{16+23+27+28+19+31}{6} = 24.000,$ etc.

First 6-monthly centred moving average = $\frac{21.833+24.00}{2} = 22.92,$ etc.



Fig. 31.14

EXERCISE 31

1. This table shows the number of students in a school getting at least a grade C in mathematics for the years 1994 to 2001.

(*i*) Represent this data as a time series.

1994	1995	1996	1997	1998	1999	2000	2001
97	118	115	117	121	125	111	125

(*ii*) Calculate the 3-point moving average and plot it on the same graph.

(*iii*) Are the school's maths results improving?

(*iv*) Explain why this is not a good way to work out whether the school's results are improving.

2. The profits of a soft drink firm in thousands of litres during each month of a year were:

January	February	March	April	May	June
1.2	0.8	1.4	1.6	2.0	2.0
July	August	September	October	November	December
3.6	4.8	3.4	1.8	0.7	1.2

Calculate 3-monthly moving averages and illustrate graphically. ***(ISC 1992 type)***

3. The number of traffic offences committed in a certain city over a period of 3 years is given in the following table:

	Jan.–March	*April–June*	*July–Sept.*	*Oct.–Dec.*
1968	74	56	48	69
1969	83	52	49	81
1970	94	60	48	79

Calculate 4-quarterly moving averages and illustrate these and original figures on one graph using the same axis for both. Comment briefly on a local politician's claim that traffic offences were on the increase.

4. Find the 4-quarterly moving averages in the following table which gives the quarterly index numbers of coal production (for the years 1936 – 1938). Also plot on the same graph the quarterly index numbers as well as the 4-quarterly moving average. Comment on the nature of the general trend. ***(ISC 2006, 1993 Type)***

	Quarters			
Year	1	2	3	4
1936	93.3	81.7	81.5	89.1
1937	93.8	92.3	86.5	93.7
1938	97.6	82.3	79.0	89.3

5. The annual incomes of a firm were recorded every quarter for 4 years. The results are shown in this table.

	1999	2000	2001	2002
1st quarter	₹ 18,00,000	₹ 20,00,000	₹ 21,00,000	₹ 22,50,000
2nd quarter	₹ 14,50,000	₹ 17,80,000	₹ 19,50,000	₹ 21,00,000
3rd quarter	₹ 13,50,000	₹ 15,00,000	₹ 18,00,000	₹ 19,80,000
4th quarter	₹ 19,00,000	₹ 18,30,000	₹ 19,20,000	₹ 20,50,000

(*i*) Work out the 4-point moving average for the data.

(*ii*) Plot the original data and the moving average on the same graph.

(*iii*) Comment on how the firm's incomes have changed over the 4-years.

6. The following table shows the daily sales of milk at a local corner shop for a month.

Sun	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
12	8	6	9	4	11	15
11	7	7	6	3	15	14
14	9	7	7	5	12	15
11	12	8	7	4	14	19

Make a table showing the moving average using a 7-day span, and draw a graph to show the trend of milk sales over the month.

7. The following table gives the monthly expenditure on a motor car for a period of two years.

Year	*January*	*February*	*March*	*April*	*May*	*June*
1961	£ 18.2	7.4	9.4	10.6	11.3	9.2
1962	£ 11.5	11.0	6.9	14.1	9.0	8.3
Year	*July*	*August*	*September*	*October*	*November*	*December*
1961	£ 9.8	10.6	8.2	7.7	19.2	8.7
1962	£ 13.9	7.9	7.5	16.5	8.2	10.7

Calculate 12-monthly moving averages for the two years and display them and the original table on the same graph. ***(SC)***

8. A new film was shown at a theatre and ran for six weeks. The attendances are shown in the table.

	Mon	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
First week	243	268	407	384	348	489
Second week	445	501	623	621	527	684
Third week	602	625	800	763	728	800

Fourth week	800	800	800	800	800	800
Fifth week	721	785	800	800	800	800
Sixth week	647	664	683	642	608	726

(*i*) Plot a line graph from the above data.

(*ii*) Calculate the 6-day moving average for the data and plot this on the same graph.

(*iii*) Comment on the weekly attendance.

9. The table below given details of the electricity generated in million kilowatt hours for public supply in each quarter of the years 1952 to 1955.

	Quarter			
Year	1	2	3	4
1952	8.9	7.1	6.7	9.3
1953	10.1	7.5	7.1	10.5
1954	11.7	7.5	8.3	10.9
1955	12.5	8.3	9.5	11.7

Draw a graph illustrating these figures.

Calculating a set of moving averages using the most suitable number of observations; give reasons of your choice. On the same diagram as before draw a graph showing the moving averages.

(*ISC*)

10. The number of letters, in hundreds, posted in a certain city on each day of a fortnight was as follows:

	Sunday	*Monday*	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
First Week	35	70	36	59	62	60	71
Second Week	39	72	38	56	63	71	75

Calculate the 7-day moving averages and display these and the original figures graphically on the same diagram, using the same scale and axes.

What is the general trend? **(*ISC 2004*)**

11. In an influenza epidemic the numbers of cases diagnosed were:

Date (March)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Numbers	2	0	5	12	20	27	46	30	31	18	11	5	0	1

On what days do the mode and upper and lower quartiles occur?

Calculate 3-day moving averages and display them and the original figures on the same graph.

(*SC, ISC*)

12.

	Date rate per thousand Quarter ended			
Year	*March*	*June*	*September*	*December*
1953	13.9	10.3	8.1	10.6
1954	13.8	9.8	7.8	10.8
1955	14.2	10.1	7.8	10.0

Plot these figures on a graph.

Calculate the 4-quarterly moving averages and plot on the same graph. **(*SC*)**

13. Registered unemployed (hundreds)

	1957	*1958*
January	638	596
February	602	548
March	509	491
April	462	462
May	359	365
June	295	325
July	290	308
August	322	328
September	377	377
October	392	380
November	480	474
December	542	536
Average for year	439	432.5

Plot these monthly figures on a graph. Calculate the 12-monthly moving averages and plot these on the same graph. ***(SC)***

14. A Ballet Company gave a 6-weeks' season at a large hall capable of seating 4000 people and the attendances in hundreds, at the evening performances, are recorded in the following table.

Attendance, in hundreds, to nearest hundred

	Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
First week	12	13	20	19	17	24
Second week	22	25	31	31	26	34
Third week	30	31	40	38	36	40
Fourth week	40	40	40	40	40	40
Fifth week	38	39	40	40	40	40
Sixth week	32	33	34	32	30	36

Plot a graph of the above time-series and include on the same diagram the graph of 6-day moving averages.

Comment on the weekly cycle on attendances and state, with reasons, if you think, an extension of the season of eight weeks, would have been justified. ***(NU)***

15. Production of passenger cars, U.S.A. (tens of thousands)

Year	*Quarters*			
	I	*II*	*III*	*IV*
1927	26	36	24	11
1928	29	36	36	22
1929	40	52	43	17

Calculate the 4-quarterly moving averages and then draw the graphs of the given series and the moving averages. Briefly comment on the general trend. ***(IU)***

16. The aggregate number, in millions, of working days lost in strikes during each year of the period 1950-60 was

1950	'51	'52	'53	'54	'55	'56	'57	'58	'59	'60
1.4	1.7	1.8	2.2	2.5	3.8	2.1	8.4	3.5	5.3	3.0

Draw a graph to represent this information. Calculate the 3-yearly moving averages and draw the 3-yearly moving averages graph, using the same axes and scales. What is the main purpose in drawing moving average graph? Comment on whether the purpose is achieved in this case.

(*Type ISC 2005*)

17. The profits of a soft drink firm in thousands of rupees during each month of a year were

Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
1.2	0.8	1.4	1.6	2.0	2.4	3.6	4.8	3.4	1.8	0.8	1.2

Plot these on a graph.

Calculate 4-monthly moving averages and plot these on the same graph. Comment on the general trend.

(*ISC 2013, 2011, 2007 Type, 1992*)

18. Calculate 5-yearly moving averages for the following data of the commercial and industrial failures in a country from 1982 to 1997.

Year	*No. of failures*	*Year*	*No. of failures*
1982	23	1990	9
1983	26	1991	13
1984	28	1992	11
1985	32	1993	14
1988	12	1996	3
1989	10	1997	1

Display the actual and trend values on the same graph using the same axes for both.

(*ISC 1998*)

19. The table given below shows the daily attendance in thousands at a certain exhibition over a period of two weeks:

Week 1	52	48	64	68	52	70	72
Week 2	55	47	61	65	58	75	81

Calculate 7-day moving averages and illustrate these and original information on the same graph using the same scales.

(*ISC 2001*)

20. The profit of a soft-drink firm (in thousands of rupees) during each month of the year is as given below:

Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
3.6	4.3	4.3	3.4	4.4	5.4	3.4	2.4	3.4	1.8	0.8	1.2

Calculate the 4-monthly moving averages and plot these and the original data on a graph sheet.

(*ISC 2011*)

ANSWERS

1. The set of data has been smoothed out. The trend is increasing. The number is increasing, but might be because the school is getting bigger.

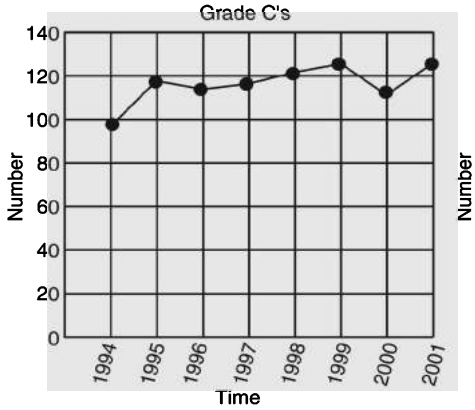


Fig. 31.15



Fig. 31.16

2. The 3-monthly averages are:

Feb. 1.13, March 1.27, Apr. 1.67, May 1.87, June 2.53, July 3.47, Aug. 3.93, Sept. 3.33, Oct. 1.97, Nov. 1.23.

3. 4-quarterly moving averages centred are 62.9, 63.5, 63.1, 64.8, 67.6, 70.9, 70.5.

The local politician's claim that the traffic offences were on the increase was correct.

4. 86.4, 86.525, 89.175, 90.425, 91.575, 92.525, 90, 88.125, 87.025.

5. 16,25,000; 16,75,000; 17,57,500; 17,95,000; 17,77,500, 18,02,500; 18,45,000; 19,20,000; 19,42,500; 19, 80,000; 20,17,500; 20,62,500; 20,95,000.

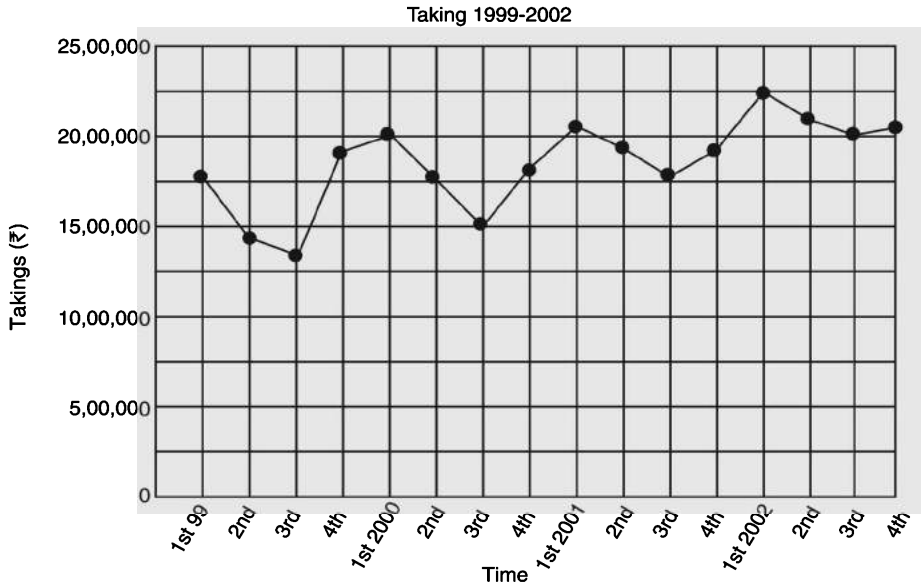
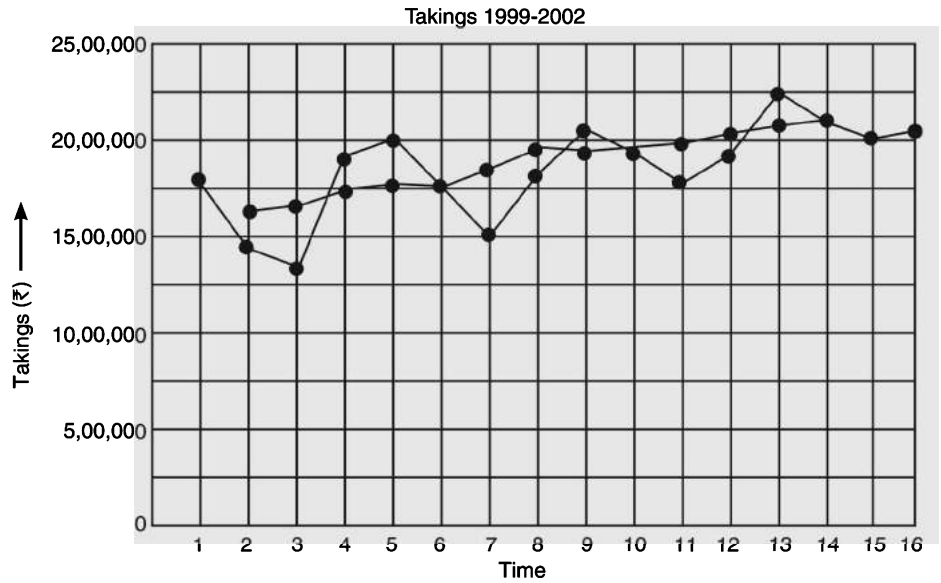


Fig. 31.17



The data have been smoothed out. It shows a steady increase in takings.

Fig. 31.18

6. Moving average: 9.3, 9.1, 9.0, 9.1, 8.7, 8.6, 9.1, 9.0, 9.4, 9.7, 9.7, 9.9, 10.1, 9.7, 9.9, 9.4, 9.9, 10.0, 10.0, 9.9, 10.1, 10.6.

7. 1961 July → £ 10.58, '61 Aug → 10.45, '61 Sept. → 10.49, '61 Oct → 10.53; '61 Nov. → 10.58, '61 Dec. → 10.45, '62 Jan. → 10.59, '62 Feb. → 10.64, '62 Mar. → 10.50, '62 Apr. → 10.83, '62 May → 10.74, '62 June → 10.37.

8. Moving average: 356, 390, 429, 465, 504, 534, 566, 593, 613, 643, 666, 700, 719, 752, 781, 781, 788, 800, 800, 786, 784, 784, 784, 784, 784, 772, 751, 732, 706, 674, 661 (gradually increases, peaks in 4th week, then slowly decreases).

9. We shall find 4-quarterly moving averages as it will eliminate the 12-monthly cycle and leave the general trend of the data.

1952 Quarter 3 → 8.15, 4 → 8.35; '53 Quarter 1 → 8.45, 2 → 8.65, 3 → 9.00, 4 → 9.20;

'54 Quarters 1 → 9.35, 2 → 9.55, 3 → 9.70, 4 → 9.90; '55 Quarter 1 → 10.15, 2 → 10.40.

10. First week — Wed. → 56.14, Thu. → 56.71, Fri. → 57.00, Sat. → 57.28

Second week — Sun. → 56.85, Mon. → 57.00, Tue. → 58.57, Wed. → 59.14.

The general trend is that the number of letters posted is on the increase every day, week after week.

11. 7th, 9th, 6th; 2.3, 5,6, 12.3, 19.6, 31.0, 34.3, 35.6, 26.3, 20.0, 11.3, 5.3, 2.0.

12. 10.7 for Sept. '53, 10.6 for Dec. '53, 10.5 for Mar. '54, 10.5 for June. '54, 10.6 for Sept. '54, 10.7 for Dec. '54, 10.7 for Mar. '55, 10.62 for June '55.

13. 437, 433, 430, 430, 430, 431, 433, 434, 435, 434, 433, 433.

14. 17.5, 19.2, 21.2, 23, 25, 26.5, 28.2, 29.5, 30.5, 32, 33.2, 34.8, 35.8, 37.5, 39, 39, 39.3, 40, 40, 39.7, 39.5, 39.5, 39.5, 39.5, 39.5, 38.5, 37.5, 36.5, 35.2, 33.5, 32.8. An extension of the season of eight weeks was justified as the attendence was still very satisfactory.

15. 24.63, 25.00, 26.50, 29.38, 32.13, 35.50, 38.38, 38.63. The general trend showed that the demand of cars was increasing year after year.

16. 1.63, 1.90, 2.17, 2.83, 2.80, 4.77, 4.67, 5.73, 3.93 million working days. The main purpose in drawing moving average graph is to find the general trend. The purpose is achieved in this case because the 3-yearly moving graph shows that the number of working days lost is on the increase.

17. Mar. 1.4, Apr. 1.7, May 2.1, June 2.8, July 3.4, Aug. 3.5, Sept. 3.1, Oct. 2.3. The general comment is that the profits go on increasing from Jan. to Aug. and then start decreasing from Sept. to Dec.

18. Following are the 5-yearly moving averages:

1984 → 25.8, 1985 → 23.6, 1986 → 20.8, 1987 → 17.2

1988 → 12.6, 1989 → 11.2, 1990 → 11.8, 1991 → 11.4

1992 → 11.8, 1993 → 11.8, 1994 → 9.8, 1995 → 7.8

19. 68 → 60.82, 52 → 61.29, 70 → 61.15, 72 → 60.71, 55 → 60.29, 47 → 61.14, 61 → 61.86, 65 → 63.15.

20. 3.9, 4.1, 4.375, 4.15, 3.9, 3.65, 2.75, 2.1, 1.8

HINTS TO SELECTED QUESTIONS

4. See Solved Ex. 8 **9.** See Solved Ex. 8 **12.** Type Solved Ex. 8
15. Type Solved Ex. 8 **16.** Same type as Solved Ex. 5

CHAPTER TEST

1. The following table gives the recorded monthly sales figures of a certain type of television for the 18-month period commencing 1st January 1989.

1989	Jan.	Feb.	March	April	May	June
	18	16	23	27	28	19
1989	July	Aug.	Sept.	Oct.	Nov.	Dec.
	31	29	35	27	28	24
1990	Jan.	Feb.	March	April	May	June
	24	28	29	30	29	22

Calculate the 6-monthly moving averages and display these and the original figures on the same graph using the same axes for both.

Comment briefly on the purpose of moving average graphs. ***(ISC 1996, 1994)***

2. The following table gives the numbers of failures of commercial industries in a country during the years 1975 to 1990.

Year	1975	1976	1977	1978	1979	
No. of failures	23	26	28	32	20	
Year	1980	1981	1982	1983	1984	
No. of failures	12	12	10	9	13	
Year	1985	1986	1987	1988	1989	1990
No. of failures	11	14	12	9	3	1

Draw a graph illustrating the figures.

Calculate the 4-yearly moving average and plot them on the same graph. ***(ISC 1997)***

3. The average number, in lakhs, of working days lost in strikes during each year of the period 1981-90 was

1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
1.5	1.8	1.9	2.2	2.6	2.7	2.2	6.4	3.6	5.4

Calculate the 3-yearly moving average and draw the moving average graph. ***(ISC 2005)***

4. The profit of a soft drink firm (in thousands of rupees) during each month of the year is as given below:

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Profits (in thousands of rupees)	3.6	4.3	4.3	3.4	4.4	5.4	3.4	2.4	3.4	1.8	0.8	1.2

Calculate the 4-monthly moving averages and plot these and the original data on a graph sheet. ***(ISC 2011)***

5. The quartely profits of a small scale industry (in thousands of rupees) is as follows:

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2012	39	47	20	56
2013	68	59	66	72
2014	88	60	60	67

Calculate 4-quarterly moving averages. Display these and the original figures graphically on the same graph sheet. ***(ISC 2015)***

6. The number of road accidents in the city due to rash driving over a period of 3 years, is given in the following table:

Year	Jan. – Mar.	April – June	July – Sept.	Oct. – Dec.
2010	70	60	45	72
2011	79	56	46	84
2012	90	64	45	82

Calculate four quarterly moving averages and illustrate them and original figures on one graph using the same axes for both. ***(ISC 2014)***

ANSWERS AND HINTS

1. Type Solved Ex. 10.

2. 26.88, 24.75, 21, 16.25, 12.13, 10.88, 10.88, 11.75, 12.13, 12, 10.5, 7.88.

3. 1.73, 1.97, 2.23, 2.83, 2.83, 4.10, 4.07, 5.13.

4. 4-monthly averages are 3.9, 4.1, 4.375, 4.15, 3.9, 3.65, 2.75, 2.1, 1.8. Moving average centred are 4.0, 4.2375, 4.2625, 4.025, 3.775, 3.2, 2.425, 1.95.

5. 40.5, 66.25, 68.75. Type solved Ex. 9, page ch 31–15.

6. 61.75, 64, 63, 63.25, 66.25, 69, 71, 70, 75, 70.25. Type solved Ex. 9 on page ch 31–15.

MODEL TEST PAPERS

PAPER 1

Time: 3 hours *Max. Marks: 100*

The Question Paper consists of three sections A, B and C.
Candidates are required to attempt all the questions from ***Section A*** *and all the questions* ***EITHER*** *from* ***Section B Or Section C.***

Section A: *Internal choice has been provided in three questions of four marks each and two questions of six marks each.*

Section B: *Internal choice has been provided in two questions of four marks each.*

Section C: *Internal choice has been provided in two questions of four marks each.*

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

Slide rule may be used.

SECTION A

Question 1: **[10 × 2 = 20]**

(*i*) If $X = \{2, 3, 5, 7, 9\}$ be the universal set, $A = \{3, 7\}$, $B = \{2, 5, 7, 9\}$, then prove that $(A \cup B)' = A' \cap B'$.

(*ii*) Find the radius of a circle in which a central angle of 45° intercepts an arc of 187 cm.

(*iii*) Evaluate: $\sin(270° - \theta) \cdot \sin(90° - \theta) - \cos(270° - \theta) \cdot \cos(90° + \theta)$

(*iv*) Convert the given complex number "$-i$" in the polar form.

(*v*) Evaluate: $\lim\limits_{x \to -5^-} \dfrac{x+5}{|x+5|}$.

(*vi*) If $f(x) = mx + c$ and $f(0) = 1 = f'(0)$, find $f(-2)$.

(*vii*) In a single throws of two dice, what is the probability of getting a total of atmost 9?

(*viii*) Prove that $\dfrac{\cos A}{1 - \sin A} = \tan\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right)$

(*ix*) For the quadratic equation $(k - 1)x^2 = kx - 1$, $k \neq 1$, find k so that the roots are numerically equal but opposite sign.

(*x*) How many numbers greater than 3,00,000 can be formed by using all the digits of the number 111223?

Question 2: Define greatest integer function. Write its domain and range. Draw the graph of it. **[4]**

Or

Let 'f' be a function defined by $f : x \to 5x^2 + 2$, $x \in R$.

(*i*) Express 'f' as a set of ordered pairs using set builder notation.

(*ii*) Is 'f' a one one function?

(*iii*) Find the image of 3 under *f*.

(*iv*) Find *x* such that $f(x) = 2$.

Question 3: Find the total number of ways of selecting five letters of the word 'INDEPENDENT'. **[4]**

Question 4: Use the principle of mathematical induction, to prove that **[4]**

$$3 \cdot 2^2 + 3^2 \cdot 2^3 + 3^3 \cdot 2^4 + \ldots\ldots\ldots + 3^n \cdot 2^{n+1} = \frac{12}{5}(6^n - 1),\ n \in N.$$

Question 5: Find the locus of a complex number $z = x + iy$, $x, y \in R$ satisfying the relation $|2z + 3i| \geq |2z + 5|$. Illustrate the locus in Argand plane. **[4]**

Question 6: If $\cot\ \alpha = \frac{1}{2}, \alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = \frac{-5}{3}, \beta \in \left(\frac{\pi}{2}, \pi\right)$, find $\tan(\alpha + \beta)$ and find the quadrant in which $\alpha + \beta$ terminates. **[4]**

Or

Prove that $\cos\frac{\pi}{9} \cdot \cos\frac{2\pi}{9} \cdot \cos\frac{3\pi}{9} \cdot \cos\frac{4\pi}{9} = \frac{1}{2^4}$

Question 7: Differentiate, $f(x) = \sqrt{2x+3}$, by using 1st principle. **[4]**

Or

Evaluate: $\lim\limits_{x \to \pi} \frac{1 - \sin\frac{x}{2}}{(\pi - x)^2}$.

Question 8: Find the equation of two straight lines drawn through the point (0, 1) on which the perpendiculars dropped from the point (2, 2) are each of unit length. **[4]**

Or

A variable line passes through the point (–1, 2) and cuts the coordinate axes in points *A* and *B*. If the point *P* divides the segemnt *AB* internally in the ratio 2 : 3, show that locus of *P* is $5xy = 4x - 3y$.

Question 9: In the binomial expansion of $(1 + x)^{43}$, the coefficients of the $(2r + 1)$th and $(r + 2)$th terms are equal. Find *r*. **[4]**

Question 10: Find the equations to the circles which touch the *x*-axis at a distance of 4 units from the origin and cut of an intercept of 6 units from the *y*-axis. **[4]**

Question 11: Prove that $\cos^3 x + \cos^3\left(\frac{2\pi}{3} + x\right) + \cos^3\left(\frac{2\pi}{3} - x\right) = \frac{3}{4}\cos 3x$ **[6]**

Or

Solve: $\tan\theta + \sec\theta = \sqrt{3}$, $-360° \leq \theta \leq 360°$

Question 12: If *x* is real, show that the value of $\frac{x^2 + 2x - 3}{x^2 + 2x + 4}$ cannot lie between $-\frac{4}{3}$ and 1. **[6]**

Question 13: The sum of three numbers in GP is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an AP. Find the numbers. **[6]**

Or

Find the sum of the series $1 + 5 + 14 + 30 + 55 + \ldots$ upto *n* terms.

Question 14: Estimate standard deviation for the following frequency distribution. **[6]**

Size	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14
Freq.	1	3	6	8	11	13	13	14	10	12	8	5	2

SECTION B

Question 15: **[3 × 2 = 6]**

(*i*) Construct the truth table for $[(\sim p) \wedge q] \Rightarrow (p \vee q)$

(*ii*) Find the equation of ellipse, the distance between foci is 8 units and the distance between the directrices is 18 units.

(*iii*) Write the converse and contrapositive of the following statement:

A positive integer is prime only if it has no divisors other that 1 and itself.

Question 16: The tangents from P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are mutually perpendicular, show that the locus of P is the circle $x^2 + y^2 = a^2 - b^2$ **[4]**

Or

Find the vertex, focus and directrix of the parabola $y^2 - 2y + 8x - 23 = 0$.

Question 17: Find the ratio in which the line joining the points (2, 5, 4) and (3, 5, 4) is divided by the *YZ* plane. Also find the coordinates of the point of the section. **[4]**

Or

Determine the point in *ZX* plane which is equidistant from points (1, –1, 0), (2, 1, 2), (3, 2, –1).

Question 18: The axis of a parabola is $3x + 4y + 6 = 0$, its vertex is (–2, 0) and latus rectum is 4 in length. Find its equation. **[6]**

SECTION C

Question 19: **[3 × 2 = 6]**

(*i*) Given $r = 0.8$, $\Sigma xy = 60$, $\sigma_y = 2.5$ and $\Sigma x^2 = 90$, find the number of items. x and y are deviation from their respective means.

(*ii*) In a class of 15 students, 4 students failed and those who passed had marks 38, 45, 63, 35, 81, 99, 78, 57, 92, 39, 48. Find the median marks of the class.

(*iii*) Construct the index numbers for 2014 taking 2010 as the base year from the following data by simple average of price relative method:

Commodities	*A*	*B*	*C*	*D*	*E*
Price in 2010 (in ₹)	100	80	160	220	40
Price in 2014 (in ₹)	140	120	180	240	40

Question 20: Calculate Karl Pearson's coefficient of correlation for the following data. **[4]**

Marks in Mathematics	15	18	21	24	27	30	36	39	42	48
Marks in Statistics	25	25	27	27	31	33	35	41	41	45

Or

Heights (in cm) of a sample of 12 fathers and their oldest sons are given below:

Heights of father	165	160	170	163	173	158	178	168	173	170	175	180
Heights of son	173	168	173	165	175	168	173	165	180	170	173	178

Find Spearman's rank correlation coefficient.

Question 21: Find the values of a and b from the following data: **[4]**

Marks	10–20	20–30	30–40	40–50	50–60	Total
No. of candidates	5	a	15	b	7	47

Given that mode = 37.

Or

A sample of 35 observations has mean 80 and standard 4. A second sample of 65 observations has mean 70 and standard deviation = 3. Find the combined mean and standard deviation.

Question 22: The following table gives the number of failures of commercial industries in a country during the year 1975 to 1990: **[6]**

Year	1975	1976	1977	1978	1979	1980	1981	1982
Number of failures	23	26	28	32	20	12	12	10
Year	1983	1984	1985	1986	1987	1988	1989	1990
Number of failures	9	13	11	14	12	9	3	1

Draw a graph illustrating these figures. Calculate the 4 yearly moving averages and plot them on the same graph.

ANSWERS WITH SOLUTION

SECTION A

Solution 1:

(*i*) $X = \{2, 3, 5, 7, 9\}, A = \{3, 7\}, B = \{2, 5, 7, 9\}$

$A \cup B = \{2, 3, 5, 7, 9\} \Rightarrow (A \cup B)' = \phi$

$A' = \{2, 5, 9\}, \; B' = \{3\} \Rightarrow A' \cap B' = \phi$

$\therefore \; (A \cup B)' = A' \cap B'$ Verified.

(*ii*) $\theta = 45° = \dfrac{\pi^c}{4}, \; p = \dfrac{S}{\theta^c} = \dfrac{187 \times 4 \times 7}{22} = 238 \text{ cm}$

(*iii*) $\sin(270° - \theta) \cdot \sin(90° - \theta) - \cos(270° - \theta) \cdot \cos(90° + \theta)$

$= (-)\cos\theta \cdot \cos\theta - (-)\sin\theta\,(-)\sin\theta$

$= -(\cos^2\theta + \sin^2\theta) = -1$

(*iv*) $-i = 0 + (-1)i = 1\left[\cos\left(\dfrac{-\pi}{2}\right) + i\sin\left(\dfrac{-\pi}{2}\right)\right]$

where $r = 1, \; \theta = \dfrac{\pi}{2}$

(*v*) $\lim\limits_{x \to -5^-} \dfrac{x+5}{|x+5|} = \lim\limits_{h \to 0} \dfrac{-5-h+5}{|-5-h+5|} = \lim\limits_{h \to 0} \dfrac{-h}{h} = -1$

(*vi*) $f(x) = mx + c \qquad f(0) = 1$

$f'(x) = m \qquad \Rightarrow \; c = 1$

$f'(0) = 1 \qquad \Rightarrow \; m = 1$

$\therefore \; f(x) = x + 1 \qquad \Rightarrow \; f(-2) = -2 + 1 = -1$

(*vii*) Required probability = 1 – P (sum of two numbers appearing > 9)

= 1 – P (sum of the two numbers appearing 10, 11, 12)

$= 1 - \dfrac{6}{36} = \dfrac{5}{6}$. [Favourable outcomes are (4, 6), (5, 5), (6, 4) (5, 6), (6, 5) (6, 6).

(*viii*) L.H.S. $= \dfrac{\cos A}{1-\sin A} = \dfrac{\cos^2\dfrac{A}{2}-\sin^2\dfrac{A}{2}}{\left(\cos\dfrac{A}{2}-\dfrac{\sin A}{2}\right)^2} = \dfrac{\cos\dfrac{A}{2}+\sin\dfrac{A}{2}}{\cos\dfrac{A}{2}-\sin\dfrac{A}{2}}$

$= \dfrac{1+\tan\dfrac{A}{2}}{1-\tan\dfrac{A}{2}} = \tan\left(\dfrac{\pi}{4}+\dfrac{\pi}{2}\right)$

(*ix*) $(k-1)x^2 - k\cdot x + 1 = 0$

$a = k-1,\ b = -k,\ c = 1$

$\because$ Roots are numerically equal but opposite sign $\Rightarrow\ b = 0$

$\Rightarrow$ $\boxed{k = 0}$

(*x*) Since the number is greater than 3,00,000 and the given number is 1,11,223.

Therefore, the numbers's formate will be 3 * * * * *

Hence, remaining 5 digits (1, 1, 1, 2, 2) will arrange among themselves in remaining five place in $\dfrac{5!}{3!\ 2!}$ ways $= 10$ ways

Solution 2:

$[x]$ is the greater integer $\le x,\ \forall\ x \in R$

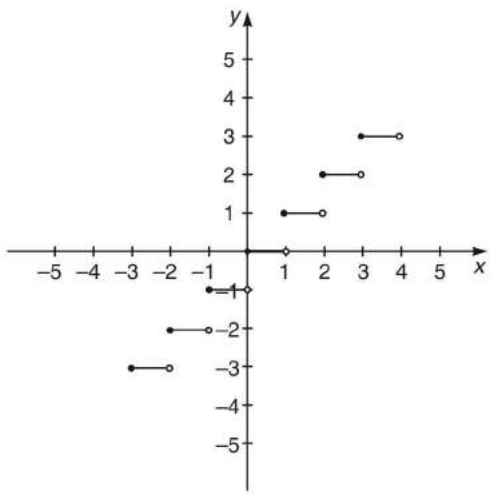
$[x] = x$, if x is an integer

= an integer immediately on the left of 'x', if x is not an integer

$\therefore\ D_f = R, R_f = Z$

For graph,

x: ...	$-3 \le x < -2$	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x < 2$	$2 \le x < 3$	$3 \le x < 4$	...
y: ...	–3	–2	–1	0	1	2	3	...



Or

(*i*) $f = \{(x, y) | y = 5x^2 + 2,\ x \in R\}$

(*ii*) $f(1) = f(-1) = 7 \Rightarrow$ Image 7 has more than one pre-images.

$\Rightarrow f$ is not one one.

(*iii*) $f(3) = 5 \times 9 + 2 = 47$

(*iv*) $f(x) = 2 \Rightarrow 5x^2 + 2 = 2 \Rightarrow x = 0.$

Solution 3:

In the word INDEPENDENT there are 3Ns (NNN), 3E's (EEE), 2Ds (DD) and 3 distinct letters I, P, T.

NNN EEE DD IPT

The following cases arise:

Case I: All 5 letters are different

Number of selection = $C(6, 5)\ C(6, 1) = 6$

Case II: 3 alike and 2 different

Number of selection = $C(2, 1)\ C(5, 2) = 20$

Case III: 3 alike and 2 alike

Number of selection = $C(2, 1)\ C(2, 1) = 4$

Case IV: 2 alike and 3 different

Number of selection = $C(3, 1)\ C(5, 3) = 30$

Case V: 2 alike, 2 alike and 1 different

Number of selection = $C(3, 2)\ C(4, 1) = 12$

Total number of selection = 6 + 20 + 4 + 30 + 12 = 72.

Solution 4:

$$P(n) : 3 \cdot 2^2 + 3^2 \cdot 2^3 + \ldots + 3^{n-1}\ 2^n + 3^n \cdot 2^{n+1} = \frac{12}{5}(6^n - 1)$$

$$P(1) = \frac{12}{5}(6-1) = 12 \text{ Verified}$$

Let $P(k)$ be true.

$$P(k): 3.2^2 + 3^2 2^3 + \ldots + 3^{k-1}\ 2^k + 3^k \cdot 2^{k+1} = \frac{12}{5}(6^k - 1) \quad \ldots (1)$$

$$P(k+1): [3.2^2 + 3^2 2^3 + \ldots + 3^k 2^{k+1}] + 3^{k+1}.2^{k+2}$$

$$= \frac{12}{5}(6^k - 1) + \frac{12}{5} 6^k \cdot 5 \qquad \text{[Using (1)]}$$

$$= \frac{12}{5}(6^{k+1} - 1) \quad \Rightarrow \quad P(k+1) \text{ is true.}$$

$\therefore$ $P(n)$ is true $\forall n \in N$.

Solution 5:

Let $z = x + iy,\ x, y \in R$

$|2z + 3i| \geq |2z + 5|$

$\Rightarrow |2x + i(2y + 3)| \geq |(2x + 5) + i2y|$

$\Rightarrow \sqrt{(2x)^2 + (2y+3)^2} \geq \sqrt{(2x+5)^2 + (2y)^2}$

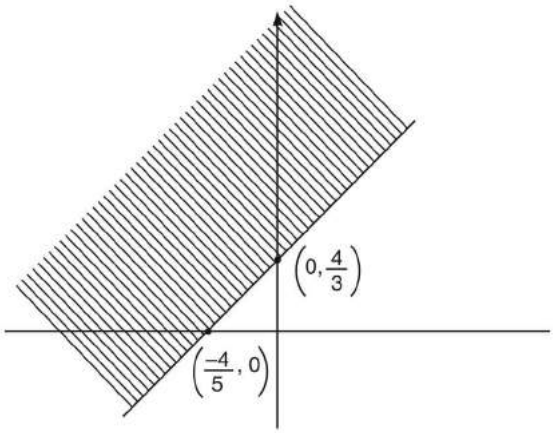
$\Rightarrow 4x^2 + 4y^2 + 9 + 12y \geq 4x^2 + 25 + 20x + 4y^2$

$\Rightarrow \quad 20x - 12y + 16 \le 0$

$\Rightarrow \quad 5x - 3y + 4 \le 0, \quad \Rightarrow \quad 5x - 3y \le -4$

$\Rightarrow \quad \dfrac{x}{-\frac{4}{5}} + \dfrac{y}{\frac{4}{3}} \le 1$

Which is the required locus.



Solution 6:

$\cot \alpha = \dfrac{1}{2} \quad \Rightarrow \quad \tan \alpha = 2$

$\sec \beta = \dfrac{-5}{3}$ and $\dfrac{\pi}{2} < \beta < \pi \quad \Rightarrow \quad \tan\beta = \dfrac{-4}{3}$

Using $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \dfrac{2 - \frac{4}{3}}{1 + 2 \cdot \frac{4}{3}} = \dfrac{2}{11}$

$\because \quad \pi < \alpha < \dfrac{3\pi}{2} \quad \Rightarrow \quad \dfrac{3\pi}{2} < \alpha + \beta < \dfrac{5\pi}{2}$

$\dfrac{\pi}{2} < \beta < \pi$, *i.e.*, 4th and 1st quadrant.

$\therefore$ $\tan(\alpha + \beta)$ is $> 0 \quad \Rightarrow \quad \alpha + \beta$ terminates in 1st quadrant.

Or

$$\cos\frac{\pi}{9} \cdot \cos\frac{2\pi}{9} \cdot \cos\frac{\pi}{3} \cdot \cos\frac{4\pi}{9} = \frac{1}{2} \frac{\left(2\sin\frac{\pi}{9} \cdot \cos\frac{\pi}{9}\right)}{2\sin\frac{\pi}{9}} \cdot \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$= \frac{1}{2.4\sin\frac{\pi}{9}} \left(2\sin\frac{2\pi}{9} \cdot \cos\frac{2\pi}{9}\right)\cos\frac{4\pi}{9}$$

$$= \frac{1}{2.8\sin\frac{\pi}{9}} \left(2\sin\frac{4\pi}{9} \cdot \cos\frac{4\pi}{9}\right) = \frac{1}{16\sin\frac{\pi}{9}} \sin\frac{8\pi}{9} = \frac{\sin\left(\pi - \frac{\pi}{9}\right)}{16\sin\frac{\pi}{9}} = \frac{\sin\frac{\pi}{9}}{16\sin\frac{\pi}{9}} = \frac{1}{16}$$

Solution 7:

$f(x) = \sqrt{2x+3}$

By 1st principle, $f'(x) = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{\sqrt{2(x+\delta x)+3} - \sqrt{2x+3}}{\delta x}$

$= \lim\limits_{\delta x \to 0} \dfrac{2x + 2\delta x + 3 - 2x - 3}{\delta x\left[\sqrt{2(x+\delta x)+3} + \sqrt{2x+3}\right]} = \lim\limits_{\delta x \to 0} \dfrac{2\delta x}{\delta x\left[\sqrt{2(x+\delta x)+3} + \sqrt{2x+3}\right]}$

$= \dfrac{1}{\sqrt{2x+3}}$ (By applying limit)

Or

$$\lim_{x \to \pi} \frac{1 - \sin\frac{x}{2}}{(\pi - x)^2}$$

Let $x - \pi = h$

$$= \lim_{h \to 0} \frac{1 - \cos\frac{h}{2}}{h^2} = \lim_{h \to 0} \frac{2\sin^2\frac{h}{4}}{h^2} = \frac{2}{16} \lim_{h \to 0} \frac{\sin\frac{h}{4}\sin\frac{h}{4}}{\frac{h}{4}\frac{h}{4}} = \frac{1}{8}$$

Solution 8:

Equation of line through (0, 1)

$y - 1 = mx \Rightarrow mx - y + 1 = 0$

Perpendicular distance from (2, 2) to $mx - y + 1 = 0$.

$\left|\dfrac{2m - 2 + 1}{\sqrt{m^2+1}}\right| = 1 \Rightarrow 4m^2 + 1 - 4m = m^2 + 1$

$\Rightarrow 3m^2 - 4m = 0 \Rightarrow m(3m - 4) = 0 \Rightarrow m = 0$ or $m = \dfrac{4}{3}$

Required equations are $y = 1$, $4x - 3y + 3 = 0$.

Or

Let required equation be $\dfrac{x}{-a} + \dfrac{y}{b} = 1$

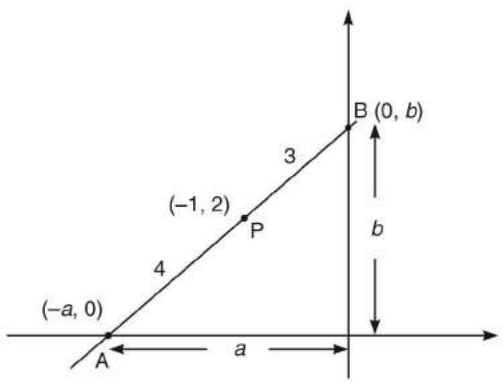
Passing through (–1, 2)

$+\dfrac{1}{a} + \dfrac{2}{b} = 1$

$P\left(\dfrac{-3a}{5}, \dfrac{2b}{5}\right) \approx (x, y) \Rightarrow \dfrac{3}{-5x} + \dfrac{4}{5y} = 1$

$\therefore\ a = \dfrac{-5x}{3}, b = \dfrac{5y}{2} \Rightarrow 5xy = 4x - 3y$

Which is the required locus.



Solution 9:

Given binomial expansion $(1 + x)^{43}$

$$T_{1+2r} = C(43, 2r) \cdot x^{2r}$$

$$T_{r+2} = T_{1+(1+r)} = C(43, r+1)x^{r+1}$$

Since the coefficients of $(2r + 1)$th and $(r + 2)$th terms are equal

$C(43, 2r) = C(43, r + 1)$ $\quad [\because C(n, a) = C(n, b) \Rightarrow a + b = n]$

$\Rightarrow \quad 43 = 2r + r + 1$

$\Rightarrow \quad r = 14.$

Solution 10:

Since the circle touching the x-axis at a distance of 4 units from the origin $\Rightarrow$ circle is passing through (4, 0).

Required equation of circle is $(x - h)^2 + (y - k)^2 = k^2$

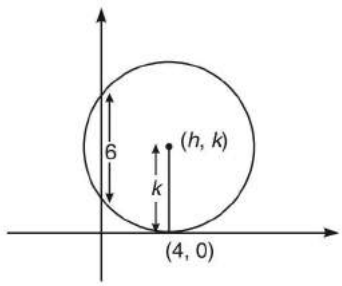
Satisfying by (4, 0),

$(4 - h)^2 + (0 - k)^2 = k^2$

$h^2 - 8h + 16 = 0 \Rightarrow h = 4$

Hence, equation of circle reduced to $x^2 + y^2 - 8x - 2ky = 0$

y-axis intercept, $2\sqrt{f^2 - c} = 6 \Rightarrow \sqrt{k^2} = 3 \Rightarrow k = \pm 3$



Hence, required equation of circle is $x^2 + y^2 - 8x \pm 6y = 0$.

Solution 11:

Let $a = \cos x$

$$b = \cos\left(\frac{2\pi}{3} + x\right)$$

$$c = \cos\left(\frac{2\pi}{3} - x\right)$$

$$a + b + c = \cos x + \left[\cos\left(\frac{2\pi}{3} - x\right) + \cos\left(\frac{2\pi}{3} + x\right)\right]$$

$$= \cos x + 2\cos\frac{2\pi}{3} \cdot \cos x = \cos x - \cos x = 0$$

$$\therefore \cos^3 x + \cos^3\left(\frac{2\pi}{3} + x\right) + \cos^3\left(\frac{2\pi}{3} - x\right)$$

$$= 3\cos x \times \left[2\cos\left(\frac{2\pi}{3} + x\right)\cos\left(\frac{2\pi}{3} - x\right)\right]$$

$$= 3\cos x \times \left[\cos^2 x - \sin^2\frac{2\pi}{3}\right] = 3\cos x \times \left[\cos^2 x - \frac{3}{4}\right]$$

$$= \frac{3}{4}\left[4\cos^3 x - 3\cos x\right] = \frac{3}{4}\cos 3x. \text{ Hence proved.}$$

Or

$$\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \sqrt{3}$$

$$\Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta = 1 \quad \Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$$

$\Rightarrow \quad \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6} = \cos\frac{\pi}{3} \quad \Rightarrow \quad \cos\left(\theta + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$

$\Rightarrow \quad \theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3},\ n \in I$

$\theta = 2n\pi - \frac{\pi}{2}$ or $\theta = 2n\pi + \frac{\pi}{6}$

$n = 0,\ \theta = -\frac{\pi}{2} \qquad \theta = \frac{\pi}{6}$

$n = 1,\ \theta = \frac{3\pi}{2} \qquad \theta = 2\pi + \frac{\pi}{6}$ (Neglected)

$n = -1,\ \theta = -\frac{-2\pi - \pi}{2}$ (Neglected) $\quad \theta = -2\pi + \frac{\pi}{6} = \frac{-11\pi}{6}$

Sol. $\left\{\frac{-11\pi}{6}, \frac{-\pi}{2}, \frac{\pi}{6}, \frac{3\pi}{2}\right\}$

Solution 12:

$$y = \frac{x^2 + 2x - 3}{x^2 + 2x + 4}$$

$\Rightarrow \quad x^2(y-1) + 2x(y-1) + (4y+3) = 0$

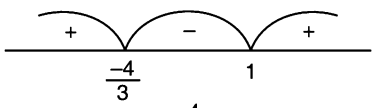
$\because \quad x$ is real $\Rightarrow \quad 4(y-1)^2 - 4(y-1)(4y+3) \geq 0$

$\Rightarrow \quad 4(y-1)(y-1-4y-3) \geq 0$

$\Rightarrow \quad 4(y-1)(-3y-4) \geq 0$

$\Rightarrow \quad (y-1)(3y+4) \leq 0$

$\Rightarrow \quad \frac{-4}{3} \leq y \leq 1$



$\Rightarrow$ y always lies between $-\frac{4}{3}$ and 1.

Solution 13:

Let the terms in GP be a, ar, ar^2.

According to question, $a(1 + r + r^2) = 56$ ... (*i*)

Again given that $a - 1$, $ar - 7$, $ar^2 - 21$ are in AP.

$\Rightarrow \quad 2(ar - 7) = a - 1 + ar^2 - 21$

$\Rightarrow \quad 2ar - 14 = a + ar^2 - 22$

$\Rightarrow \quad a + ar^2 = 8 + 2ar$ ... (*ii*)

From eqn. (*i*) and (*ii*), we have

$\Rightarrow \quad 8 + 3ar = 56 \qquad \Rightarrow \quad 3ar = 48$

$\Rightarrow \quad \boxed{ar = 16} \qquad \Rightarrow \quad a = \frac{16}{r}$ ... (*iii*)

From eqn. (*i*) and (*iii*), we have

$$\frac{16}{r}(1 + r + r^2) = 56$$

$$\Rightarrow \quad 2r^2 - 5r + 2 = 0$$

$$\Rightarrow \quad (r - 2)(2r - 1) = 0$$

$$\Rightarrow \quad r = 2, \frac{1}{2} \qquad \Rightarrow \quad a = 8 \text{ or } 32$$

So, the Numbers are 8, 16, 32 or 32, 16, 8.

Or

$$S_n = 1 + 5 + 14 + 30 + 55 + \ldots + T_n$$

$$\frac{S_n = 1 + 5 + 14 + 30 + \ldots + T_n}{0 = 1 + 4 + 9 + 16 + 25 + \ldots - T_n}$$

$$T_n = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots \text{ upto } n \text{ term} = \frac{n(n+1)(2n+1)}{6}$$

$$T_n = \frac{1}{6}\left[2n^3 + 3n + 1\right]n = \frac{1}{6}[2n^3 + 3n^2 + n]$$

$$S_n = \frac{1}{6}\left[2\Sigma n^3 + 3\Sigma n^2 + \Sigma n\right]$$

$$= \frac{1}{6}\left[\frac{2n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\frac{n(n+1)}{2}\left[n(n+1) + 2n + 1 + 1\right] = \frac{1}{12}n(n+1)(n^2 + 3n + 2)$$

Solution 14:

	f_i	x_i	d_i	$f_i d_i$	$d_i'^2$	$f_i d_i'^2$
1 – 2	1	1.5	– 6	– 6	36	36
2 – 3	3	2.5	– 5	– 15	25	75
3 – 4	6	3.5	– 4	– 24	16	96
4 – 5	8	4.5	– 3	– 24	9	72
5 – 6	11	5.5	– 2	– 22	4	44
6 – 7	13	6.5	– 1	– 13	1	13
7 – 8	13	7.5	0	0	0	0
8 – 9	14	8.5	1	14	1	14
9 – 10	10	9.5	2	20	4	40
10 – 11	12	10.5	3	36	9	108
11 – 12	8	11.5	4	32	16	128
12 – 13	5	12.5	5	25	25	125
13 – 14	2	13.5	6	12	36	72
	$\Sigma f_i = 106$			$\Sigma f_i d_i = 35$		$\Sigma f_i d_i'^2 = 823$

$$\sigma^2 = \frac{823}{106} - \left(\frac{35}{106}\right)^2 = \frac{823 \times 106 - 35^2}{106^2} = \frac{86013}{106}$$

$$\sigma = 2.766 \approx 2.77.$$

SECTION B

Solution 15: Truth table of $[(\sim p) \wedge q] \Rightarrow (p \vee q)$

(*i*)

p	q	$\sim p$	$\sim p \wedge q$	$p \vee q$	$(\sim p \wedge q) \Rightarrow (p \vee q)$
F	F	T	F	F	T
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	F	T	T

(*ii*) Required equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, as per question

Distance between two foci = $2ae = 8$

$\Rightarrow \quad ae = 4$ ... (*i*)

Distance between directrix = $2\frac{a}{e} = 18$

$\Rightarrow \quad \frac{a}{e} = 9$ ... (*ii*)

Eqn. (*i*) × (*ii*) gives $a^2 = 36 \quad \Rightarrow \quad b^2 = a^2 - a^2e^2 = 36 - 16 = 20$

Required equation is $\frac{x^2}{36} + \frac{y^2}{20} = 1$

(*iii*) **Converse:** A +ve integer is prime only if it has no divisors other than 1 and itself.

Contrapositive: If a +ve integer has divisors other than 1 and itself then the +ve integer is not prime.

Solution 16:

General equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow \quad y - mx = \pm\sqrt{a^2m^2 - b^2}$

$\Rightarrow \quad y^2 + m^2x^2 - 2mxy = a^2m^2 - b^2$

$\Rightarrow \quad m^2(x^2 - a^2) + m(-2xy) + (y^2 + b^2) = 0$

Which is a quadratic equation in terms of m

Perpendicular of the roots $m_1m_2 = \frac{y^2 + b^2}{x^2 - a^2} = -1$ [$\because$ tangents are perpendicular]

$$y^2 + b^2 = -x^2 + a^2$$

$\Rightarrow \quad \boxed{x^2 + y^2 = a^2 - b^2}$ which is the required locus.

Or

Given equation of parabola $y^2 - 2y = -8x + 23$

$\Rightarrow \quad y^2 - 2y + 1 = -8x + 23 + 1$

$\Rightarrow \quad (y-1)^2 = -8(x-3)$

$\Rightarrow \quad (y-1)^2 = -4.2(x-3) \quad \equiv \quad Y^2 = -4.2X$ (II type)

Vertex (3, 1), $\boxed{a = 2}$

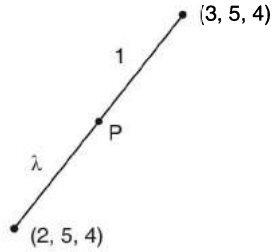
Focus $(-2, 0) \Rightarrow x - 3 = -2, \Rightarrow x = 1, y = 1 \Rightarrow$ Focus (1, 1)

Directrix: $x - 3 = 2 \Rightarrow x = 5$

Solution 17:

Let the required ratio be $\lambda : 1$ and the point of division be P.

Then co-ordinates of P are $\left(\frac{3\lambda+2}{\lambda+1}, \frac{5\lambda+5}{\lambda+1}, \frac{4\lambda+4}{x+1}\right)$ ... (*i*)



Since P lies on yz plane, the x co-ordinate of $P = 0$

$\therefore \quad \frac{3\lambda+2}{\lambda+1} = 0 \Rightarrow 3\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2}{3}$

Therefore the line joining (2, 5, 4) and (3, 5, 4) is divided by the yz plane externally in the ratio 2 : 3.

Putting the value of λ in (*i*)

Coordinate of point of the section = (0, 5, 4)

Or

Let the point $P(\alpha, 0, \gamma)$ on the zx plane equidistant from the points $A(1, -1, 0)$, $B(2, 1, 2)$ and $C(3, 2, -1)$. Then,

$$PA^2 = (\alpha - 1)^2 + 1 + \gamma^2$$
$$PB^2 = (\alpha - 2)^2 + 1 + (\gamma - 2)^2$$
$$PC^2 = (\alpha - 3)^2 + 4 + (\gamma + 1)^2$$

Since P is equidistant from A, B, C $\Rightarrow PA^2 = PB^2 = PC^2$. Using $PA^2 = PB^2$, we have

$$\alpha^2 - 2\alpha + 1 + 1 + y^2 = \alpha^2 - 4\alpha + 4 + 1 + \gamma^2 - 4\gamma + 4$$

$\Rightarrow \quad 2\alpha + 4\gamma = 7$ ... (*i*)

Using $PB^2 = PC^2$, we have

$$\alpha^2 - 4\alpha + 4 + 1 + \gamma^2 - 4\gamma + 4 = \alpha^2 - 6\alpha + 9 + 4 + \gamma^2 + 2\gamma + 1$$

$\Rightarrow \quad 2\alpha - 6\gamma = 5$ ...(*ii*)

Solving (*i*) and (*ii*), we get $\alpha = \frac{31}{10}, \gamma = \frac{1}{5}$

$\therefore$ Point $P\left(\frac{31}{10}, 0, \frac{1}{5}\right)$

Solution 18: Given axis of a parabola is $3x + 4y + 6 = 0$

Required equation of parabola is the locus of P such that, $PM^2 = 4a\ PN$ ...(1)

PM = perpendicular distance of $P(x, y)$ from x-axis $= \frac{3x+4y+16}{\sqrt{9+16}}$

Equation of $PN = 4x - 3y + \lambda = 0$ as $PN \perp PM$

As it passes through (–2, 0)

$$-8 + \lambda = 0 \Rightarrow \lambda = 8$$

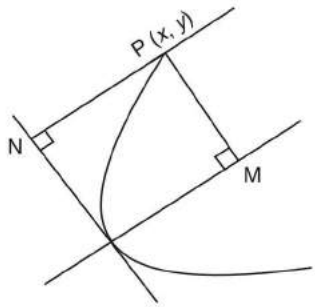
$\therefore$ Equation of PN : $4x - 3y + 8 = 0$

PN = perpendicular distance of PN from y-axis

$$= \frac{4x - 3y + 8}{\sqrt{16+9}}$$

Given, $\frac{(3x+4y+6)^2}{25} = \frac{4(4x-3y+8)}{5}$

$\Rightarrow 9x^2 + 24xy + 16y^2 - 44x + 108y - 124 = 0.$



SECTION C

Solution 19:

(*i*) $r = 0.8$, $\Sigma dxdy = 60$

$\sigma_y = \frac{5}{2}$, $\Sigma dx^2 = 90$

$\frac{\Sigma dy^2}{n} = \frac{5}{2} \Rightarrow \Sigma dy^2 = \frac{5}{2}n$

$$\frac{4}{5} = \frac{n\Sigma dxdy}{\sqrt{\Sigma dx^2 \Sigma dy^2}}$$

$$\frac{4}{5} = \frac{n^2 \times 60 \times 60 \times 2}{90 \times 5n}$$

Solving, $n = 10$.

(*ii*) $\because$ 4 students failed, therefore, their marks are less than 35. Hence arranging the list of marks $*_1, *_2, *_3, *_4$, 35, 38, 39, 45, 48, 57, 63, 78, 81, 92, 99

Median = $\frac{15+1}{2}$th obser. = 8th obser. = 45

(*iii*)

Commodity	*Price in* ₹		$I = \frac{P_1}{p_0} \times 100$
	P_0(2010)	P_1 (2014)	
A	100	140	140
B	80	120	150
C	160	180	112.5
D	220	240	109.1
E	40	40	100
			$\Sigma I = 611.6$

Index number (using simple average of price relatives)

$$= \frac{\Sigma I}{n} = \frac{611.5}{5}$$

$= 122.32.$

Solution 20:

Marks in Mathematics (X)	*Marks in Statistics (Y)*	$dx = x_i - \bar{x}$	$dy = y_i - \bar{y}$	dx^2	dy^2	$dxdy$
15	25	–15	–8	225	64	120
18	25	–12	–8	144	64	96
21	27	–9	–6	81	36	54
24	27	–6	–6	36	36	36
27	31	–3	–2	9	4	6
30	33	0	0	0	0	0
36	35	6	2	36	4	12
39	41	9	8	81	64	72
42	41	12	8	144	64	96
48	45	18	12	324	144	216
				1080	480	708

$\bar{x} = 30, \ \bar{y} = 33$

$$\sigma = \frac{\Sigma dxdy}{\sqrt{\Sigma dx^2 \Sigma dy^2}} = \frac{708}{\sqrt{480 \times 1080}} = 0.98$$

Or

Heights of Father (X)	*Heights of Son (Y)*	R_X	R_Y	$d = R_X - R_Y$	d^2
165	173	9	5.5	3.5	12.25
160	168	11	9.5	1.5	2.25
170	173	6.5	5.5	1	1
163	165	10	11.5	–1.5	2.25
173	175	4.5	3	1.5	2.25
158	168	12	9.5	2.5	6.25
178	173	2	5.5	–3.5	12.25
168	165	8	11.5	–3.5	12.25
173	180	4.5	1	3.5	12.25
170	170	6.5	8	–1.5	2.25
175	173	3	5.5	–2.5	6.25
180	178	1	2	–1	1
					$\Sigma d^2 = 72.5$

$$d^2 = \frac{1}{12}(2^3-2)+\frac{1}{12}(2^3-2)+\frac{1}{12}(4^3-4)+\frac{1}{12}(2^3-2)+\frac{1}{12}(2^3-2)$$

$$= 7$$

$$r = 1-\frac{6[\Sigma d^2 + \text{correction}]}{n(n^2-1)}$$

$$= 1-\frac{6[72.5+7]}{12(12^2-1)} = 0.72$$

Solution 21:

Given $\Sigma fi = 47 \quad \Rightarrow a + b + 27 = 47$

$$\Rightarrow a + b = 20 \qquad \ldots (i)$$

Given Mode = 37, therefore modal class = 30 – 40

Using mode $= l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h$

$$37 = 30 + \frac{15-a}{30-(a+b)} \times 10$$

$$7 = \frac{15-a}{10} \times 10 \qquad \text{[Using 1]}$$

$$a = 8 \Rightarrow b = 12.$$

Or

	Ist Sample	2nd Sample
No. of Observations	35	65
Mean	80	70
Standard Deviation	4	3

Combined Mean $(\bar{x}) = \frac{35.80 + 65.70}{35+65} = 73.5$

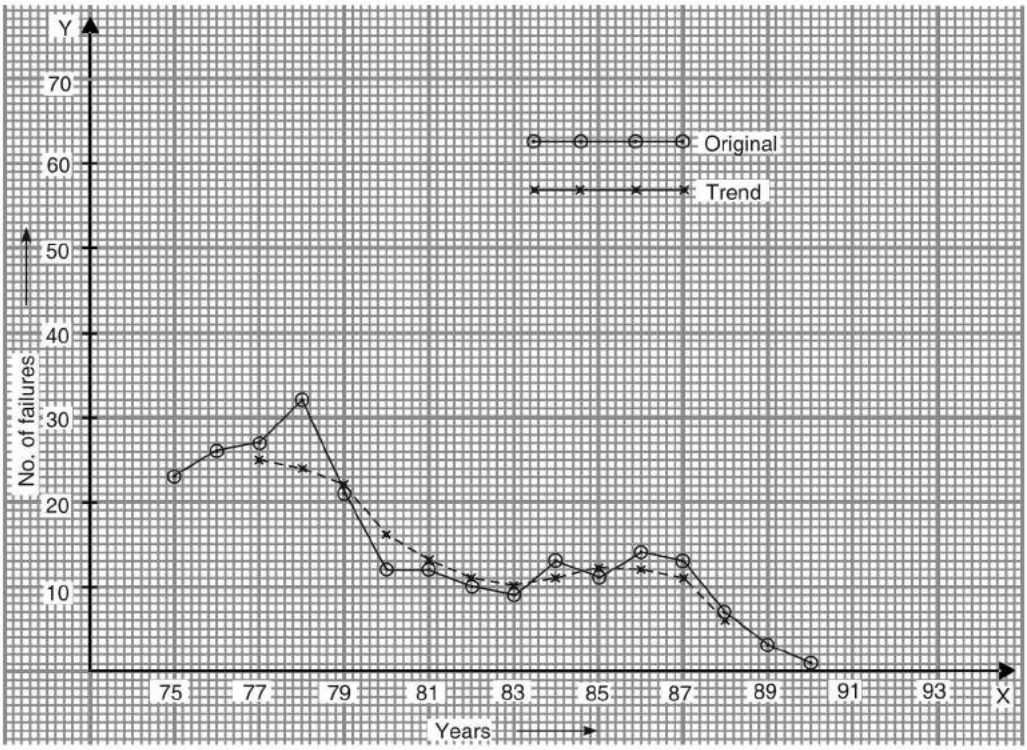
$\therefore \quad d_1 = x_1 - \bar{x} = 80 - 73.5 = 6.5; \quad d_2 = x_2 - \bar{x} = 70 - 73.5 = -3.5$

Using $\sigma^2 = \frac{n_1 d_1^2 + n_2 d_2^2 + n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} = \frac{35.6.5^2 + 35.(-3.5)^2 + 35.16 + 65.9}{35+65} = 34.2$

$\sigma = \sqrt{34.2} = 5.85$

Solution 22:

Year	*No. of failures*	*4-year moving total*	*4-year moving average*	*4-year moving average centered*
1975	23			
1976	26			
		109	27.25	
1977	28			26.88
		106	26.5	
1978	32			24.75
		92	23	
1979	20			21
		76	19	
1980	12			16.2
		54	13.5	
1981	12			12.13
		43	10.75	
1982	10			10.88
		44	11	
1983	9			10.88
		43	10.75	
1984	13			11.25
		47	11.75	
1985	11			12.13
		50	12.5	
1986	14			12
		46	11.5	
1987	12			10.5
		38	9.5	
1988	9			7.88
		25	6.25	
1989	3			
1990	1			


Y
70
60
50
40
30
20
10
No. of failures
Original
Trend
75
77
79
81
83
85
87
89
91
93
X
Years

PAPER 2

The Question Paper consists of three sections A, B and C.

Candidates are required to attempt all the questions from ***Section A*** *and all the questions* ***EITHER*** *from* ***Section B Or Section C.***

Section A: *Internal choice has been provided in three questions of four marks each and two questions of six marks each.*

Section B: *Internal choice has been provided in two questions of four marks each.*

Section C: *Internal choice has been provided in two questions of four marks each.*

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

Slide rule may be used.

SECTION A

Question 1: **[10 × 2 = 20]**

(*i*) A set B is given as $B = \{1, 2\}$. Some elements of $A \times B$ are (3, 1), (5, 1) and (7, 2). Find the remaining elements of $A \times B$ such that $n(A \times B)$ is least.

(*ii*) An arc 15 ft long describes an angle of 5 radians at the centre of a circle. Find the radius of the circle.

(*iii*) If A, B, C, D are angles of a cyclic quadrilateral, then prove that $\cos A + \cos B + \cos C + \cos D = 0$.

(*iv*) Find the modulus and argument of $2\sqrt{2}\,(\cos 225° + i \sin 225°)$.

(*v*) Evaluate: $\lim\limits_{x \to 5^+} (x - [x])$.

(*vi*) If $f(x) = \sqrt[3]{(1+x^2)^4}$, find $f'(1)$.

(*vii*) Evaluate: $\sin^2 24° - \sin^2 6°$.

(*viii*) Find the condition for the equation $ax^2 + bx + c = 0$, so that one root is m times the other.

(*ix*) How many 5-digit numbers can be formed using the digits 2, 4, 7, 9, 0 if no digit is repeated?

(*x*) A card is drawn randomly from a pack of 52 playing cards. What is the probability that the card drawn is neither a spade nor a queen.

Question 2: A real valued function is given by $f(x) = x^2 + 4$, find its domain and range. **[4]**

Or

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one but not onto.

Question 3: A polygon has 35 diagonals. Find the number of sides. **[4]**

Question 4: Use principle of mathematical induction, to prove that **[4]**

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots \ldots \ldots + n \cdot n! = (n+1)! - 1, \text{ for all } n \in N.$$

Question 5: Find the square root of $a^2 - 1 + 2ai$. **[4]**

Question 6: Solve: $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$, $(-180° \leq \theta \leq 180°)$. **[4]**

Or

Prove that, $\cos^2 \frac{\pi}{12} + \cos^2 \frac{3\pi}{4} + \cos^2 \frac{5\pi}{12} = \frac{3}{2}$.

Question 7: Given that $y = (3x-1)^2 + (2x-1)^3$, find $\frac{dy}{dx}$. Hence find the point (s) on the curve at which $\frac{dy}{dx} = 0$. **[4]**

Or

Evaluate: $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$.

Question 8: Find the image of the point (1, 2) about the line $x + y + 1 = 0$. **[4]**

Or

Find the equations of straight lines through the point A (3, –2) and inclined at 60° to the line $\sqrt{3}x + y = 1$.

Question 9: Find the 6th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^n$, if

$$C_0 + C_1 + C_2 + \ldots + C_n = 1024.$$ **[4]**

Question 10: Show that the point (0, 3), (–6, 0), (–1, 5) and (–4, –1) are concylic. **[4]**

Question 11: If $\tan \frac{\alpha}{2}$ and $\tan \frac{\beta}{2}$ are the roots of the equation **[6]**

$8x^2 - 26x + 15 = 0$, then find the value of $\cos(\alpha + \beta)$.

Or

If $\frac{\cos x}{\cos(x-2y)} = \lambda$, then show that $\tan(x-y) = \left(\frac{1-\lambda}{1+\lambda}\right)\cot y$.

Question 12: If sum of the roots of $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then show that $2a^2c = ab^2 + bc^2$. **[6]**

Question 13: If a, b, c are in AP; b, c, d are in GP and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP; prove that a, c, e are in GP. **[6]**

Or

Find, five terms are in GP, such that their sum is $\frac{11}{2}$ and product is 32.

Question 14: Find the values of f_1, f_2 and f_3 from the following data: **[6]**

Class Interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	4	16	f_1	f_2	40	f_3	4

Given that median = 33.5, mode = 34 and total frequency is 230.

SECTION B

Question 15: **[3 × 2 = 6]**

(*i*) Construct truth table for ~ [$p \wedge (\sim q)$] and find which *implication* has same truth value.

(*ii*) Find the equation of ellipse with centre at origin, length of latus rectum is 5 and eccentricity is $\frac{2}{3}$.

(*iii*) Write the converse of the contrapositive of $p \Rightarrow q$.

Question 16: Find the equation of hyperbola with foci at the points (–3, 5) and (5, 5) and length of latus rectum is $2\sqrt{8}$. **[4]**

Or

Reduce the following equation of parabola to a standard form, hence find the vertex, focus and the equations of the directrix and equation of latus rectum.

$4x - y^2 + 2y - 13 = 0$

Question 17: Using section formula, prove that three points $A(-2, 3, 5)$, $B(1, 2, 3)$ and $(7, 0, -1)$ are collinear. **[4]**

Or

The midpoints of the sides of a triangle are $\left(3, 2, \frac{3}{2}\right), \left(1, \frac{3}{2}, 3\right)$ and $\left(2, \frac{5}{2}, \frac{5}{2}\right)$. Find the coordinates of centroid.

Question 18: Does the straight line $\frac{x}{a}+\frac{y}{b}=2$ touch the ellipse $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=2$? If it touches; find the coordinates of the point of contact. **[6]**

SECTION C

Question 19: **[3 × 2 = 6]**

(*i*) Calculate standard deviation of the numbers 11, 12, 13, ..., 20.

(*ii*) Find the Cov (*X*, *Y*) of the data given below:

(1, 5), (2, 7), (3, 9), (4, 11), (5, 10), (6, 9), (7, 8), (8, 7), (9, 6), (10, 5)

(*iii*) A small industrial concern used three raw materials *A*, *B* and *C* in its manufacturing process. The prices of the materials are as shown below:

	2006	2016
A	4	5
B	60	57
C	36	42

Using 2006 as the base year, calculate for 2016 a simple aggregate price index.

Question 20: Calculate Spearman's coefficient of rank correlation from the following data and interpret the result. **[4]**

X	16	19	22	28	25	31	37	40	43	49
Y	25	25	27	31	27	33	35	41	45	41

Or

The mathematical aptitude score of 10 computer programmers with their job performance is given below:

Mathematics score	7	5	1	4	3	0	2	6	8	9
Job performance rating	8	16	8	9	5	4	3	8	17	12

Calculate the Karl Pearson's correlation coefficient and interpret the result.

Question 21: From a frequency distribution consisting of 18 observations, the mean and standard deviation were found to be 7 and 4 respectively. But on comparison with original data, it was found that a figure 12 was miscopied as 21 in calculations. Calculate the correct mean and standard deviation. **[4]**

Or

Find the inter-quartile range, semi inter-quartile range and coefficient of quartile deviation from the following frequency distribution.

Marks	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90
No. of students	60	45	120	25	90	80	120	60

Question 22: The following table gives the quarterly death rate per thousand of a city for the years 1953 to 1955. **[6]**

Quarter Ending

Year	*March*	*June*	*September*	*December*
1953	13.9	10.3	8.1	10.6
1954	13.8	9.8	7.8	10.8
1955	14.2	10.1	7.8	10.0

Plot these figures on a graph. Calculate the suitable moving averages and plot them on the same graph.

ANSWERS

Q.1. (*i*) Remaining elements of $A \times B$ are (3, 2), (5, 2), (7, 1) (*ii*) 3 ft.

(*iii*) Modulus = $2\sqrt{2}$, Argument = $-\frac{3\pi}{4}$ (*v*) 0 (*vi*) $\frac{2^{\frac{10}{3}}}{3}$ (*vii*) $\frac{\sqrt{5}-1}{8}$ (*iii*) $mb^2 = ac^2 (m + 1)^2$

(*ix*) 96 (*x*) $\frac{9}{13} \approx 0.69$

Q.2. $D_f = R$, $R_f = [4, \infty)$

Q.3. 10

Q.5. $a + i$ or $-a - i$

Q.6. $\left\{-\pi, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\right\}$

Q.7. $\frac{dy}{dx} = 6(4x^2 - x)$, Points on the curve are (0, 0), $\left(\frac{1}{4}, \frac{3}{16}\right)$ Or $\frac{1}{2}$

Q.8. (–3, –2) Or $y + 2 = 0$; $\sqrt{3}x - y - (3\sqrt{3} + 2) = 0$

Q.9. $-\frac{8064}{x^5}$

Q.11. $-\frac{627}{725}$

Q.13. $\frac{1}{2}, -1, -2, -4, 8$ or $8, -4, -2, -1, \frac{1}{2}$

Q.14. $f_1 = 60, f_2 = 100, f_3 = 6$

Q.15. (*i*)

p	q	$\sim q$	$p \wedge \sim q$	$\sim [P \wedge (\sim q)]$	$(p \Rightarrow q)$
F	F	T	F	T	T
F	T	F	F	T	T
T	F	T	T	F	F
T	T	F	F	T	T

(*ii*) $\frac{4x^2}{81}+\frac{4y^2}{45}=1,$ (*iii*) $\sim p \Rightarrow \sim q$

Q.16. $(x-1)^2 - (y-5)^2 = 8$

Or $(y-1)^2 = 4(x-3)$, Vertex (3, 1), Focus (4, 1), equation of directrix: $x = 2$; equation of latus rectum: $x = 4$

Q.17. Or Centroid $\left(2, 2, \frac{7}{3}\right)$

Q.18. Yes; Point of contact (a, b)

Q.19. (*i*) 2.87 (*ii*) Cov $(x, y) = -1.95$ (*iii*) Price index = 104

Q.20. Spearman's rank correlation coefficient $(r) = 0.98$

Or Karl Pearson's coefficient of correlation $(\rho) = 0.0005$

Q.21. Correct mean = 6.5, correct standard deviation = 2.5 Or 38.75, 19.38, 0.37

Q.22. 4 quarterly moving average is most suitable

4 quarterly moving averages are 10.71, 10.64, 10.54, 10.53, 10.6, 10.7, 10.7, 10.63

PAPER 3

The Question Paper consists of three sections A, B and C.

Candidates are required to attempt all the questions from ***Section A*** *and all the questions* ***EITHER*** *from* ***Section B Or Section C.***

Section A: *Internal choice has been provided in three questions of four marks each and two questions of six marks each.*

Section B: *Internal choice has been provided in two questions of four marks each.*

Section C: *Internal choice has been provided in two questions of four marks each.*

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

Slide rule may be used.

SECTION A

Question 1: **[10 × 2 = 20]**

(*i*) Prove that, $U - (U - A) = A$, where U is the universal set.

(*ii*) If the perimeter of a certain sector of a circle is equal to the length of the arc of the semicircle having the same radius, find the angle of the sector in degrees, minutes and seconds.

(*iii*) Find the value of θ, if $m^2 \sin\frac{\pi}{2} - n^2 \sin\frac{3\pi}{2} + 2mn \sec\theta = (m-n)^2$, $0 \le \theta \le \pi$.

(*iv*) Find the least positive value of n, if $\left(\frac{1+i}{1-i}\right)^n = 1$.

(*v*) If $\lim\limits_{x \to -a} \frac{x^9 + a^9}{x + a} = 9$, find the value of a.

(*vi*) If $f(x) = \alpha x^n$, prove that $\alpha = \frac{f'(1)}{n}$.

(*vii*) Prove that, $\sin 100° - \sin 20° - \sin 40° = 0$.

(*viii*) Find the value of λ and μ if both roots of the equation $(3\lambda + 1)x^2 = (2\lambda + 3\mu)x - 3$ are infinite.

(*ix*) There are 12 points in a plane, of which 5 are collinear. Find the number of straight lines obtained by joining these points in pairs.

(*x*) The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and electric contract is $\frac{4}{9}$. If the probability of getting at least one contract is $\frac{4}{5}$, find the probability that he will get both.

Question 2: Define modulus function. Write its domain and range. Draw the graph of it. **[4]**

Or

Z is the set of integers. Describe the following relation in set builder form, given its domain and range. {(0, –7), (2, –5), (4, –3), (–13, –20), ...}

Question 3: Find the number of (*i*) combinations (*ii*) permutations of four letters each that can made from the letters of the word "TENNESSEE". **[4]**

Question 4: Use principle of mathematical induction, to prove that **[4]**

$7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25 for all $n \in N$.

Question 5: If ω is a cube root of unity, show that $\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}+\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=-1$. **[4]**

Question 6: Solve: $2\cos^2\theta = 3\sin\theta$ $(0° \le \theta \le 360°)$. **[4]**

Or

Prove that, cosec $10° - \sqrt{3}$ sec $10° = 4$.

Question 7: Differentiate w.r.t. 'x' $f(x) = \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$. **[4]**

Or

Evaluate: $\lim_{x\to\infty} \sqrt{x^2+x+1}-\sqrt{x^2+1}$.

Question 8: A rod of length 1 m slides between two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2. **[4]**

Or

Find the equations of the bisectors of the angles between the lines $12x + 5y - 4 = 0$ and $3x + 4y + 7 = 0$. Prove that bisectors are at right angles to each other.

Question 9: If the coefficient of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^{2n}$ are in AP, prove that $2n^2 - 9n + 7 = 0$. **[4]**

Question 10: Find the equations of the tangents to the circle $x^2 + y^2 = 25$ which are parallel to the line $2x - y + 4 = 0$. **[4]**

Question 11: If tan $\frac{x-y}{2}$, tan z, tan $\frac{x+y}{2}$ are in GP, then show the cos x = cos y . cos $2z$. **[6]**

Or

Prove that tan 6° tan 42° tan 66° tan 78° = 1.

Question 12: If x is real, prove that the values of $\frac{x^2-3x+4}{x^2+3x+4}$ lies between $\frac{1}{7}$ and 7. **[6]**

Question 13: The sum of four numbers in GP is 60 and the arithmetic mean of the first and the last numbers is 18. Find the numbers. **[6]**

Or

If S be the sum, P be the product and R be the sum of reciprocals of n terms in GP prove that $P^2 = \left(\frac{S}{R}\right)^n$.

Question 14: An original frequency table with mean 10.5 and variance 9.9 was lost but the following table derived from it was found. Construct the original table. **[6]**

u_i	–2	–1	0	1	2
f_i	1	6	7	4	2

SECTION B

Question 15: **[3 × 2 = 6]**

(*i*) Construct truth table for $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

(*ii*) Find the values of k so that the line $2x + y + k = 0$ may touch the hyperbola $3x^2 - y^2 = 3$.

(*iii*) Negation of disjunction of p and q is equivalent to the negation of both p and q. Verify.

Question 16: Find the equation of parabola with its axis parallel to x-axis and passing through the points (–2, 1), (1, 2) and (–1, 3). **[4]**

Or

Find the centre, foci and the equations of the directrices of the ellipse

$$8x^2 + 9y^2 - 16x + 18y - 55 = 0.$$

Question 17: What is the geometrical significance of x-coordinate of a point in space. Hence find the ratio in which the line segement, joining the points $P(2, 3, 4)$ and $Q(-3, 5, -4)$ is divided by the yz plane. Also find the point of contact. **[4]**

Or

Find the equation of locus of a point whose distance from y-axis is equal to its distance from (2, 1, –1).

Question 18: Find the equation of the parabola whose focus and vertex are (5, 3) and (3, 1) respectively. **[6]**

SECTION C

Question 19: **[3 × 2 = 6]**

(*i*) The standard deviation of the numbers 2, 3, 11, x is $3\frac{1}{2}$, find the value of x.

(*ii*) Find $\rho(x, y)$ if Cov $(x, y) = -16.5$, *var* $(x) = 2.25$ and $\sigma_y = 12$.

(*iii*) Index number for the total cost of raw materials used for the manufacturing of the commodity in 2015, using 2001 as the base year calculated as 179.94. If the commodity was sold for ₹ 1055.75 in 2010, calculate the selling price in 2015 on assumption that selling prices are directly proportional to the cost of raw materials.

Question 20: Calculate Karl Pearson's coefficient of correlation between the heights of husbands and wives based on the following data (given in inches) and interpret the result. Take assumed means of husbands and wives are 70 and 66 respectively. **[4]**

Couple	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Height of husband	76	75	75	72	72	71	71	70	68	68	68	68	67	67	62
Height of wife	71	70	70	67	71	65	65	67	64	65	65	66	63	65	61

Or

Marks in Mathematics	15	18	21	24	27	30	36	39	42	48
Marks in Statistics	25	25	27	27	31	33	35	41	41	45

Find spearman's rank correlation coefficient.

Question 21: Find the mode of the following distribution. **[4]**

Marks	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90
No. of students	60	45	120	25	90	80	100	60

Or

An analysis of daily wages of casual labourers in two firms A and B belonging to the same industry, gives the following result:

	Firm A	*Firm B*
No. of workers	50	60
Average daily wages (in ₹)	113	120
SD	6.5	8.2

Find the mean and SD of wages of all casual labourers in the two firms taken together.

Question 22: The table given below shows the number of visitors (in hundreds) to a certain exhibition over a period of two weeks: **[6]**

Week 1	52	48	64	68	52	70	72
Week 2	55	47	51	65	58	75	81

Calculate the 7 days moving averages and illustrate these and the original information on the same graph, using the same scales.

ANSWERS

Q.1. (*ii*) 65°27′16″ (*iii*) $\theta = \pi$ (*iv*) Least value of $n = 4$ (*v*) $a = \pm 1$ (*viii*) $\lambda = -\frac{1}{3}, \mu = \frac{2}{9}$

(*ix*) 57 (*x*) $\frac{14}{45} \approx 0.31$

Q.2. Or $\{(x, y) \mid x \in Z, x - y = 7\}$, Domain = {0, 2, 4, –13, ...}, Range = {–7, –5, –3, –20, ...}

Q.3. (*i*) Number of combinations = 17, (*ii*) Number of permutations = 163

Q.6. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

Q.7. $f'(x) = 2\left[x + \frac{x^3}{\sqrt{x^4 - 1}}\right]$ Or $\frac{1}{2}$

Q.8. $\frac{9x^2}{4} + 9y^2 = l^2$ Or $7x - 9y - 37 = 0$; $99x + 77y + 71 = 0$

Q.9. $y = 2x \pm 5\sqrt{5}$

Q.13. 4, 8, 16, 32 or 32, 16, 8, 4

Q.14.

Class Interval	3–6	6–9	9–12	12–15	15–18
Frequency	1	6	7	4	2

Q.15. (*i*)

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

(*ii*) $k = \pm 1$

Q.16. $y^2 - y + 2x + 4 = 0$

Or Centre (1, –1), Foci (2, –1); (0, –1), Equation of directrices: $x = -8$; $x = 10$

Q.17. Directed distance from yz plane; Ratio: 2 : 3; Point of contact $\left(0, \frac{1}{5}, \frac{4}{5}\right)$

Or $y^2 - 4x - 2y + 2z + 6 = 0$

Q.18. $x^2 - 2xy + y^2 - 20x - 12y + 68 = 0$

Q.19. (*i*) 6 or $4\frac{2}{3}$ (*ii*) $\rho(x, y) = -0.92$ (*iii*) Selling price in 2015 = 1899.72

Q.20. Karl Pearson's coefficient of correlation (ρ) = 0.9

Or Spearman's rank correlation coefficient (r) = 0.97

Q.21. Mode = 34.41 Or $\bar{x} = 116.82$, $\sigma = 8.25$

Q.22. 7 days moving averages are: 60.86, 61.29, 61.14, 59.29, 58.86, 59.71, 60.43, 61.71