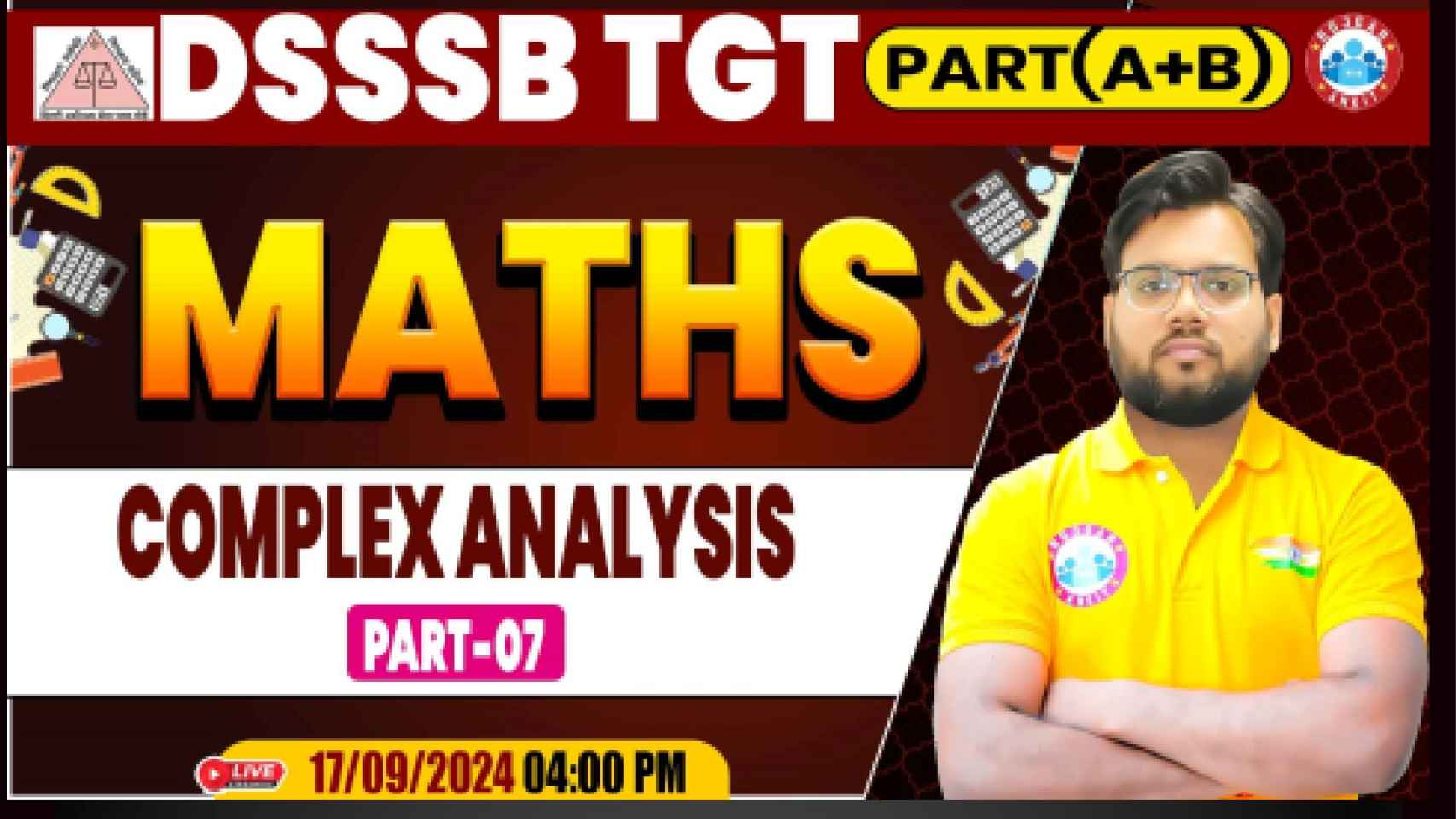
DSSSB TGT
PART(A+B)
MATHS
COMPLEX ANALYSIS
PART-07
LIVE
17/09/2024 04:00 PM

Complex Number

1. Logarithm of a Complex Number :

If $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log_e z = \log_e r + i\theta$$

$$\log_e z = \log_e |z| + i(\arg z)$$

$$\log_e (x + iy) = \frac{1}{2}\log_e \left(x^2 + y^2\right) + i\tan^{-1}(y/x)$$

General value of the argument, we shall get the general value of $\log_e Z$

$$\log_e z = \log_e z + (2n\pi)i$$

Logarithm of a Complex Number:

Example. (i) $\log(2+2i)$

(ii) $\log_2(1+i\sqrt{3})$

Note: (i) $\log \bar{z} = (\log z)$

(ii) $\log(-z) = \log z + \pi i$

Exponential Functions of Complex Number:

(i) $e^{\alpha+i\beta} = e^{\alpha}(\cos\ \beta + i\sin\ \beta)$

(ii) $a^{\alpha+i\beta} = a^{\alpha} \cdot e^{i\beta\log a}, (a \in \mathrm{R})$

$= a^{\alpha}\{\cos(\underline{\beta}\log\ a) + i\sin(\beta\log\ a)\}$

(iii) $(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$

$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$

$= e^{p+iq}$

$= e^{p(\cos\ q+i\sin\ q)}$

where $p = \frac{\alpha}{2}\log(x^2 + y^2) - \beta\tan^{-1}\frac{y}{x}$

$$q = \frac{\beta}{2}\log(x^2 + y^2) + \alpha\tan^{-1}\frac{y}{x}$$

Exponential Functions of Complex Number:

Example

(i) 4^{2+3i}

(ii) $(1+i)^{1-i}$

$$a^{\alpha+i\beta} = a^{\alpha}\cdot e^{i\beta\log a}, (a \in \mathrm{R})$$

$$= a^{\alpha}\{\cos(\beta\log a) + i\sin(\beta\log a)\}$$

$$(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$$

$$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$$

$$= e^{p+iq}$$

$$= e^{p}(\cos q + i\sin q)$$

Values of i^i

Observations:

(i) i^i is a real number.

(ii) Principal value of $i^i = e^{-\pi/2}$ (iii) $0 < i^i < 1$.

(iv) Values of i^i are in GP with common ratio $e^{-2\pi}$

Square Root of a Complex Number

$$\sqrt{z} = \sqrt{x + iy}$$

$$\pm\left[\left(\frac{\sqrt{x^2 + y^2} + x}{2}\right)^{1/2} + i\left(\frac{\sqrt{x^2 + y^2} - x}{2}\right)^{1/2}\right]$$

or

$$\pm\left[\left(\frac{|z| + x}{2}\right)^{1/2} + i\left(\frac{|z| - x}{2}\right)^{1/2}\right]$$

Note:

(i) To find $\sqrt{x-iy}$ replace i by $-i$ in above results.

(ii) $\sqrt{i}=\pm\frac{1}{\sqrt{2}}(1+i)$

Short Method: To obtain $\sqrt{x+iy}$ find two numbers a, b such that

$$ab = y/2 \text{ and } a^2 - b^2 = x$$

$$\sqrt{x+iy} = \pm(a+ib)$$

$$\sqrt{x-iy} = \pm(a-ib)$$

Example. Find the square root of $7+24i, 7-24i, -7+24i, -7-24i$

Locus of z : To find locus of z, we replace z by $x + iy$ in the given relation

Example. If $|z| = a$, then

Example. If $|z - c| = a$ where $a, c \in \mathrm{R}$, then

Example. If $|z| \leq a$, then

De Moivre's Theorem

If $n \in Q$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

or

$$(\cos\theta, \sin\theta)^n = (\cos n\theta, \sin n\theta)$$

Note: This theorem is not valid when n is not a rational number or the complex number is not in the form $\cos\theta + i\sin\theta$. For example.

$$(\cos\theta + i\sin\theta)^{\sqrt{2}} \neq \cos(\sqrt{2}\theta) + i\sin\left(\sqrt{2}\theta\right) \quad [\because \sqrt{2} \notin \mathrm{Q}]$$

$$(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$$

$$(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$$

Example of De Moivre's Theorem

Example $\cdot\ (\cos\ \pi/4, \sin\ \pi/4)^3$

Example $\cdot\ (\sin\ \pi/3 + i\cos\ \pi/3)^4$

Powers of Complex Numbers

$$z^n = (r^n, n\theta)$$

Example. Evaluate: $(1+i)^5 + (1-i)^5$

Roots of Complex Numbers

$$z^{1/n} = \left(r^{1/n}, \frac{2m\pi+\theta}{n}\right) = r^{1/n} e^{i\frac{2m\pi+\theta}{n}}, m = 0, 1, 2, \ldots, (n-1)$$

The value corresponding to $m = 0$ is called the principal value of $z^{1/n}$

Example. Find the value of $(1 + i\sqrt{3})^{1/3}$

Roots of Unity

$$1^{1/n} = (1, 2m\pi/n)$$
$$= \cos(2m\pi/n) + i\sin(2m\pi/n)$$
$$= e^{i(2m\pi/n)}, m = 0, 1, 2, \dots, (n-1)$$
$$= 1, e^{i(2\pi/n)}, e^{i(4\pi/n)}, \dots, e^{i[2(n-1)\pi/n]}$$
$$= 1, \alpha, \alpha^2, \dots, \alpha^{n-1}$$

Particular Cases :

(A) When $n = 3$ (cube roots of unity):

$$1^{\frac{1}{3}} = \left(1, \frac{2m\pi}{3}\right)$$

$$\underbrace{1+\omega+\omega^2=0}$$

$$\boxed{\omega^3 = 1}$$

$$1^{\frac{1}{3}} = \left(1, \frac{2m\pi}{3}\right)$$

$$\Rightarrow \cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}$$

$m=0 \Rightarrow 1+0 = 1\ (1)$

$2 \times 60 = 120$

$m=1 \Rightarrow \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$m=1 \Rightarrow -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$m=1 = \frac{-1+i\sqrt{3}}{2}\ (\omega)$

$m=2 \Rightarrow \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow \frac{-1-i\sqrt{3}}{2}\ (\omega^2)$

1. The square root of i is

i का वर्गमूल है:

(a) $\frac{1}{2}(\mathbf{1}+\mathbf{i})$

(b) $\frac{1}{2}(\mathbf{1}-\mathbf{i})$

(c) $\pm\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{i})$

(d) $\pm\frac{1}{\sqrt{2}}(\mathbf{1}-\mathbf{i})$

UKPSC GIC 2018

UP PGT 2013

2. The polar form of $-1-\sqrt{(-3)}$ is

$-1-\sqrt{(-3)}$ का ध्रुवीय रूप है

(a) $2\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$

(b) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

(c) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(d) $2\left(\cos\frac{\pi}{3}-1\sin\frac{\pi}{3}\right)$

UP TGT-2016 (08-03-2019)

3. The square root of $5 + 12i$ is:

$5 + 12i$ का वर्गमूल है:

(a) $(3 + 2i)$

(b) $(2 + 3i)$

(c) $(2 - 3i)$

(d) (3-2i)

KVS PGT 23-12-2018

4. The three cube roots of $z = -8i$ are

$z = -8i$ के तीन घनमूल हैं

(a) $2i, -\sqrt{3} - i, \sqrt{3} - i$

(b) $-2i, -\sqrt{3} - i, \sqrt{3} - i$

(c) $2i, -\sqrt{3} - i, \sqrt{3} + i$

(d) $2i, -\sqrt{3} - i, -\sqrt{3} + i$

LT 2018

$(2i)^3$

$8i^3$

$\underline{\underline{-8i}}$

$i^3 = -i$

$$(-\sqrt{3} - i)^3 = \left[-(\sqrt{3}+i)\right]^3$$

$$= -\left(3\sqrt{3} + i^3 + 3\sqrt{3}\, i(\sqrt{3}+i)\right)$$

$$= -(\cancel{3\sqrt{3}} - i + 9i - \cancel{3\sqrt{3}})$$

$$= -(8i) = -8i$$

$$(\sqrt{3} - i)^3 \Rightarrow 3\sqrt{3} - i^3 - 3\sqrt{3}\, i(\sqrt{3} - i)$$

$$\Rightarrow 3\sqrt{3} + i - 9i + 3\sqrt{3}\, i^2$$

$$= \cancel{3\sqrt{3}} - 8i - \cancel{3\sqrt{3}}$$

$$= -8i$$

5. If $A + iB = \frac{3-2i}{7+4i}$, then value of $A - B$ is-

यदि $A + iB = \frac{3-2i}{7+4i}$, तो $A - B$ का मान है

(a) $\frac{1}{5}$

(b) $\frac{2}{5}$

(c) $\frac{3}{5}$

(d) $-\frac{1}{5}$

$$A + iB = \frac{(3-2i)(7-4i)}{(7+4i)(7-4i)} = \frac{21-26i-8}{49+16}$$

$7^2 - (4i)^2$

$49 - 16i^2$

$\Rightarrow \frac{13-26i}{65}$

UP PGT-2016 (02-02-2019)

$\Rightarrow \frac{13}{65} - \frac{26i}{65}$

$\Rightarrow \frac{1}{5} - \frac{2}{5}i$

$A = \frac{1}{5}$, $B = -\frac{2}{5}$

$\frac{1}{5} - \left(-\frac{2}{5}\right)$

$B = -\frac{2}{5}$

$A - B$

$\frac{1}{5} - \left(-\frac{2}{5}\right)$

$\Rightarrow \frac{1}{5} + \frac{2}{5} \Rightarrow \frac{3}{5}$

6. The amplitude of $\frac{1+i}{1+\sqrt{3}i}$ is

$-1 \frac{1+i}{1+\sqrt{3}i}$ का आयाम-

(a) $\frac{\pi}{12}$

(b) $\frac{-\pi}{12}$

(c) $\frac{\pi}{4}$

(d) $\frac{-\pi}{4}$

JDD-75-PGT TIER II-X-15 28.06.2015

7. The modulus of $\frac{(1+2i)(2-i)}{3-4i}$ is equal –

$\frac{(1+2i)(2-i)}{3-4i}$ का मापांक इसके बराबर है-

(a) 5

(b) $\sqrt{5}$

(c) 1

(d) $\frac{1}{\sqrt{5}}$

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8. Simplify $\dfrac{(\cos 3\theta + i\sin 3\theta)^4 \cdot (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 \cdot (\cos 5\theta + i\sin 5\theta)^{-4}}$

Properties of Cube Roots of Unity:

(i) $\omega^3 = 1$

$\omega^0+\omega^1+\omega^2=0$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $\omega^n + \omega^{n+1} + \omega^{n+2} = 0 \forall n \in Z$

(i.e., the sum of three consecutive integral power of $\omega = 0$) $1+\omega^2+\omega$

(iv) $1 + \omega^{\mathrm{n}} + \omega^{2\mathrm{n}} \begin{cases} 3 \, when & n = 3k \\ 0 \, when & n \neq 3k' \quad k \in Z \end{cases}$

(v) $1 \cdot \omega \cdot \omega^2 = 1$

(vi) $(\omega)^2 = \omega^2, (\omega^2)^2 = \omega$

(vii) $1/\omega = \bar{\omega}^2, 1/\omega^2 = \omega$

(i.e., ω, ω^2 are reciprocal of each other)

(viii) $\bar{\omega} = \omega^2, \overline{\omega^2} = \omega$

(i.e., ω, ω^2 are conjugate of each other)

(ix) ω, ω^2 are roots of the equation $z^2 + z + 1 = 0$

$\omega^3=1$

$1+\omega+\omega^2=0$

$\omega^4=\omega^3.\omega$

$\Rightarrow 1.\omega$

$\Rightarrow \omega$

$\omega^5=\omega^2$

$\omega^6=\omega^3.\omega^3=1$

$\omega^1+\omega^2+\omega^3=\boxed{}$

$$\frac{1}{\omega} = \omega^2$$

$$\frac{1}{\omega} = \frac{\omega^3}{\omega} = \underline{\omega^2}$$

$$\Rightarrow 1 \cdot \omega \cdot \omega^2 = 1$$

$$1 \times \left(\frac{-1+\sqrt{3}\,i}{2}\right)\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$(a-b)(a+b) = a^2 - b^2$$

$$-\left(\frac{\sqrt{3}\,i-1}{2}\right)\left(\frac{\sqrt{3}\,i+1}{2}\right)$$

$$-\left(\frac{(\sqrt{3}\,i)^2 - 1^2}{4}\right) = -\left(\frac{-3-1}{4}\right) = -\left(\frac{-4}{4}\right) = -(-1) = 1$$

$\omega \quad 1 \quad \omega^2$

$\omega^{301} + \omega^{201} + \omega^{101} = ?$

$\omega^{300} \cdot \omega + (\omega^3)^{?} + \omega^{99} \cdot \omega^2$

$= \omega + 1 + \omega^2$

$= 1 + \omega + \omega^2$

$= 0$

1

-1

ω

ω^2

None of these

$\omega = \frac{-1+\sqrt{3}i}{2}$

(x) $1 - i\sqrt{3} = -2\omega$ or $-2\omega^2$, $\sqrt{3} + i = -2\omega i$ or $-2\omega^2 i$

$1 + i\sqrt{3} = -2\omega^2$ or -2ω, $\sqrt{3} - i = 2\omega i$ or $2\omega^2 i$.

If $\frac{1-i\sqrt{3}}{2} = -\omega$, then $\frac{1+i\sqrt{3}}{2} = -\omega^2$ and if $\frac{1+i\sqrt{3}}{2} = -\omega$, then $\frac{1-i\sqrt{3}}{2} = -\omega^2$

(xi) Arguments of imaginary cube roots $= 2\pi/3, -2\pi/3$.

(xii) $1, \omega, \omega^2$ are vertices of an equilateral triangle with

side $= \sqrt{3}$, area $= 3\sqrt{3}/4$

Properties of Cube Roots of Unity:

When $n = 4$ (Fourth roots of unity):

$$1^{1/4} = (1, 2m\pi/4)$$

0, 1, 2, 3

$$= \cos\left(\frac{2m\pi}{4}\right) + i\sin(2m\pi/4), m = 0, 1, 2, 3$$

$$= 1, (\cos \pi/2 + i\sin \pi/2), (\cos \pi + i\sin \pi)$$

$$(\cos 3\pi/2 + i\sin 3\pi/2)$$

$$= 1, e^{(\pi/2)}, e^{\pi i}, e^{(3\pi/2)i}$$

$$= 1, i, -1, -i$$

$$\therefore 1^{1/4} = \pm 1, \pm i$$

Properties of Cube Roots of Unity:

Q. If $1, \omega, \omega^2$ are cube roots of unity and | $a + b + c = 0$. then $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$ is equal to

(i) 0

(ii) 3abs.

(iii) 27abs

(iv) None

Complex Number

Roots of -1

$$(-1)^{1/2} = i, -i$$

$$(-1)^{1/3} = -1, \frac{1}{2}(1 \pm i\sqrt{3}) = -1, -\omega, -\omega^2$$

$$(-1)^{1/4} = \pm \pm \frac{1}{\sqrt{2}}(1 \pm i)$$

Example $\cdot\ (-8)^{\frac{1}{3}} =$

Geometry of Complex Numbers

(i) If z_1 and z_2 are two complex numbers representing points P and Q, then

$$PQ = |z_1 - z_2|$$

Z =

(ii) If $R(z)$ divides line segment PQ in the ratio $m_1 : m_2$, then

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

(iii) If z_1, z_2, z_3 represent vertices A, B, C of $\triangle ABC$, then

$$\angle A = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

(iv) Complex numbers z_1, z_2, z_3 are collinear, if

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0$$

ie. $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(v) If complex numbers z_1, z_2, z_3 are vertices of a triangle, then its

$$\text{centroid} = \frac{z_1 + z_2 + z_3}{3}$$

Some Particular Locus

(i) Equation of the line joining complex numbers z_1 and z_2 $z = tz_2 + (1-t)z_1, t \in \mathrm{R}$

or

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{z_2 - z_1}$$

or

$$\begin{vmatrix} z & z & 1 \\ z_1 & z_1 & 1 \\ z_2 & z_2 & 1 \end{vmatrix} = 0$$

Some Particular Locus

(ii) General equation of a line in complex plane:

$$\bar{a}z + a\bar{z} + b = 0$$

(iii) Equation of a circle in central form:

$$|z - z_0| = r$$

(where z_0 is the center and r is the radius of the circle)

(iv) General equation of a circle :

$$Z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

$$\begin{pmatrix} \text{Centre is the point } -a \\ \text{radius } = \sqrt{|a|^2 - b} \end{pmatrix}$$

Complex Number
Some Particular Locus

(v) Equation of a circle in diameter form:

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

or

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

(which z_1, z_2 are end points of a diameter of the circle)

(vi) $|z - z_1| = |z - z_2|$

is the equation of the perpendicular bisector of the line segment joining points z_1 and z_2.

(vii) $\left|\left(\frac{z-z_1}{z-z_2}\right)\right| = k$, or $|z - z_1| = k|z - z_2|, k > 0, k \neq 1$ is the equation of a circle.

Some Particular Locus

(viii) $\arg\left(\frac{z-z_1}{z-z_2}\right) = \pm\frac{\pi}{2}$

represents a circle where z_1, z_2 are end points of a diameter of the circle.

(ix) $\arg\left(\frac{z-z_1}{z-z_2}\right) = 0$ or π represents a line joining points z_1 and z_2.

(x) $|z-z_1| + |z-z_2| = |z_1-z_2|$ represents line segment joining points z_1 and z_2.

(xi) $|z-z_1| + |z-z_2| = k, k > |z_1-z_2|$ represents an ellipse whose foci are at points z_1 and z_2.

(xii) $|z-z_1| - |z-z_2| = k, k < |z_1-z_2|$ represents a hyperbola whose foci are at points z_1 and z_2.

(xiii) $|z - z_1| = \text{Re}(z)$ or $\text{Im}(z)$ represents a parabola.

Q. If α, β, γ are cube roots of a negative real number, then for any x, y, z $\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha}$ equals

(a) ω

(b) ω^2

(c) 1

(d) $-\omega$

$p = -m$ $\quad m > 0$

$p^{\frac{1}{3}} = (-m)^{\frac{1}{3}}$

$= (-m)^{\frac{1}{3}} \cdot (1)^{\frac{1}{3}}$

$\Rightarrow (-m)^{\frac{1}{3}} [1, \omega, \omega^2]$

$= \underset{\alpha}{(-m)^{\frac{1}{3}}}, \underset{\beta}{(-m)^{\frac{1}{3}}\omega}, \underset{\gamma}{(-m)^{\frac{1}{3}}\omega^2}$

$\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha}$

$\Rightarrow \frac{\cancel{(-m)^{\frac{1}{3}}}[x+y\omega+z\omega^2]}{\cancel{(-m)^{\frac{1}{3}}}[x\omega+y\omega^2+z]}$

$\Rightarrow \frac{\cancel{(x+y\omega+z\omega^2)}}{\omega\cancel{(x+y\omega+z\omega^2)}} \Rightarrow \frac{1}{\omega}$

$\Rightarrow \frac{1}{\omega} = \omega^2$

$\Rightarrow \frac{z}{\omega} = z\omega^2$

Q. If ω is an imaginary cube root of unity, then $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots.100$ factors is equal to

(a) 1

(b) -1

(c) ω

(d) ω^2

H.W.