DSSSB TGT PART(A+B)

MATHS

COMPLEX ANALYSIS

PART-06

LIVE 16/09/2024 04:00 PM

Complex Number

1. Logarithm of a Complex Number :

If $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log_e z = \log_e r + i\theta$$

$$\log_e z = \log_e |z| + i(\arg z)$$

$\log_e (x + iy) = \frac{1}{2}\log_e (x^2 + y^2) + i\tan^{-1}(y/x)$

General value of the argument, we shall get the general value of $\log_e Z$

$$\log_e z = \log_e z + (2n\pi)i$$

Logarithm of a Complex Number:

Example. (i) $\log(2+2i)$

(ii) $\log_2(1+i\sqrt{3})$

Note: (i) $\log \bar{z} = (\log z)$

(ii) $\log(-z) = \log z + \pi i$

Exponential Functions of Complex Number:

(i) $e^{\alpha+i\beta} = e^{\alpha}(\cos\ \beta + i\sin\ \beta)$

(ii) $a^{\alpha+i\beta} = a^{\alpha} \cdot e^{i\beta\log\ a}, (a \in \mathrm{R})$

$= a^{\alpha}\{\cos\ (\beta\log\ a) + i\sin\ (\beta\log\ a)\}$

(iii) $(x + iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log\ (x+iy)}$

$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log\ (x^2+y^2)+i\tan^{-1}\ y/x\right\}}$

$= e^{p+iq}$

$= e^{p(\cos\ q+i\sin\ q)}$

where $p = \frac{\alpha}{2}\log(x^2 + y^2) - \beta\tan^{-1}\frac{y}{x}$

$$q = \frac{\beta}{2}\log(x^2 + y^2) + \alpha\tan^{-1}\frac{y}{x}$$

Exponential Functions of Complex Number:

Example

(i) 4^{2+3i}

(ii) $(1+i)^{1-i}$

$$a^{\alpha+i\beta} = a^{\alpha}\cdot e^{i\beta\log a}, (a \in \mathrm{R})$$
$$= a^{\alpha}\{\cos(\beta\log a) + i\sin(\beta\log a)\}$$

$$(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$$
$$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$$
$$= e^{p+iq}$$
$$= e^{p}(\cos q + i\sin q)$$

Values of i^i

Observations:

(i) i^i is a real number.

(ii) Principal value of $i^i = e^{-\pi/2}$ (iii) $0 < i^i < 1$.

(iv) Values of i^i are in GP with common ratio $e^{-2\pi}$

Square Root of a Complex Number

$$\sqrt{z} = \sqrt{x + iy}$$

$$\pm\left[\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{1/2} + i\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{1/2}\right]$$

or

$$\pm\left[\left(\frac{|z|+x}{2}\right)^{1/2} + i\left(\frac{|z|-x}{2}\right)^{1/2}\right]$$

$$\pm\left[\left(\frac{|z|+x}{2}\right)^{\frac{1}{2}} + i\left(\frac{|z|-x}{2}\right)^{\frac{1}{2}}\right]$$

Note:

(i) To find $\sqrt{x-iy}$ replace i by $-i$ in above results.

(ii) $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

$\sqrt{0+i}$

$x=0,\ y=1\ ,\ |z|=1$

$\pm\left[\left(\frac{1+0}{2}\right)^{\frac{1}{2}} + i\left(\frac{1-0}{2}\right)^{\frac{1}{2}}\right]$

$\Rightarrow \pm\left[\left(\frac{1}{2}\right)^{\frac{1}{2}} + i\left(\frac{1}{2}\right)^{\frac{1}{2}}\right]$

$= \pm\left[\frac{1}{\sqrt{2}}(1+i)\right]$

$ab = \frac{y}{2}$

$a^2 - b^2 = x$

$\sqrt{7+24i} = \pm(4+3i)$

$ab = \frac{24}{2} = 12$

$a^2 - b^2 = 7$

$16 - 9 = 7$

Short Method: To obtain $\sqrt{x+iy}$ find two numbers a, b such that

$$ab = y/2 \text{ and } a^2 - b^2 = x$$

$$\sqrt{x+iy} = \pm(a+ib)$$

$$\sqrt{x-iy} = \pm(a-ib)$$

Example. Find the square root of $7+24i, 7-24i, -7+24i, -7-24i$

$\sqrt{7-24i}$

$\pm(-4+3i)$

$\pm(4-3i)$

$ab = -\frac{24}{2} = -12$ ✓

$a^2-b^2 = 7$

$a=-4$ $\quad a^2=16$

$b=3$ $\quad b^2=9$

$ab=-12$ $\quad a^2-b^2=7$

$\sqrt{-7+24i}$

$ab = \frac{24}{2} \Rightarrow 12$

$a^2-b^2 = -7$

$a=3,\ b=4$

$\pm(3+4i)$

$\sqrt{-7-24i}$

$\underset{3\ (-4)}{ab} = -\frac{24}{2} = -12$

$a^2-b^2 = -7$

$9-16=-7$ ✓

$\pm(3-4i)$

Locus of z : To find locus of z, we replace z by $x+iy$ in the given relation

$z = x + iy$

$(x-x')^2 + (y-y')^2 = a^2$

Example. If $|z| = a$, then

$|x+iy|$

$\sqrt{x^2+y^2} = a$

$\boxed{x^2+y^2 = a^2}$

केन्द्र $=(0,0)$
त्रिज्या $= a$

$|z-c| = a$

$|x+iy-c| = a$

$|(x-c)+iy| = a$

$\sqrt{(x-c)^2+y^2} = a$

$(x-c)^2+(y-0)^2 = a^2$

Example. If $|z-c| = a$ where $a, c \in R$, then

केन्द्र $\Rightarrow (-c, 0)$
त्रिज्या $= a$

Example. If $|z| \le a$, then

De Moivre's Theorem

If $n \in Q$, then

$(\cos\theta, \sin\theta)^n = (\cos\theta + i\sin\theta)^n$

परिमेय

$$(\cos\ \theta + i\sin\ \theta)^n = \cos\ n\theta + i\sin\ n\theta$$

or

$$(\cos\ \theta, \sin\ \theta)^n = (\cos\ n\theta, \sin\ n\theta)$$

Note: This theorem is not valid when n is not a rational number or the complex number is not in the form $\cos\ \theta + i\sin\ \theta$. For example.

अपरिमेय

$$(\cos\ \theta + i\sin\ \theta)^{\sqrt{2}} \neq \cos\left(\sqrt{2}\theta\right) + i\sin\left(\sqrt{2}\theta\right) \quad [\because \sqrt{2} \notin Q]$$

$$(\cos\ \theta + i\sin\ \phi)^n \neq \cos\ n\theta + i\sin\ n\phi$$

$$(\sin\ \theta + i\cos\ \theta)^n \neq \sin\ n\theta + i\cos\ n\theta$$

Example of De Moivre's Theorem

Example · $(\cos \pi/4, \sin \pi/4)^3$

Example · $(\sin \pi/3 + i\cos \pi/3)^4$

$\frac{3 \times \cancel{180}^{45}}{\cancel{4}}$

$\Rightarrow 135$

$\Rightarrow (90+45)$

$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3$

$\Rightarrow \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\Rightarrow -\cos 45^\circ + i\sin 45^\circ$

$\Rightarrow -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \Rightarrow \frac{-1+i}{\sqrt{2}}$

$\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^4$

$\left[i\left[\frac{1}{i}\sin\frac{\pi}{3} + \cos\frac{\pi}{3}\right]\right]^4$

$= i^4\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]^4$

$= \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}$

$\Rightarrow -\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow \frac{-1+\sqrt{3}i}{2}$

$\frac{1}{i} \times \frac{i}{i} \Rightarrow \frac{i}{i^2}$

$\Rightarrow \frac{i}{-1}$

$i^4 = 1$

$i^3 = -i$

$\frac{1}{i} = -i$

$\frac{4 \times \cancel{180}^{60}}{\cancel{3}}$

240

$180+60$

Powers of Complex Numbers

$$z^n = (r^n, n\theta)$$

Example. Evaluate: $(1+i)^5 + (1-i)^5$

$$\boxed{Z = x + iy}$$

$$Z^n = r^n, n\theta$$

$$(x+iy)^n = r^n(\cos n\theta + i \sin n\theta)$$

$$r = \sqrt{x^2+y^2}$$

$$\Uparrow$$

$$(x+iy)^n = r^n(\cos n\theta + i\sin n\theta)$$

$n=5$

$(1+i)^5 + (1-i)^5$

$x=1, y=1$

$r = \sqrt{1+1} = \sqrt{2}$

$r = \sqrt{1+1} = \sqrt{2}$

$$\left((2)^{\frac{1}{2}}\right)^5 \times \left[\cos 5\theta + i\sin 5\theta\right] + \left(2^{\frac{1}{2}}\right)^5 \left[\cos 5\theta - i\sin 5\theta\right]$$

$2^{\frac{5}{2}} \cdot 2^1$

$$2^{\frac{5}{2}}\left[2\cos 5\theta\right]$$

$$\Rightarrow 2^{7/2}\left[\cos 5\theta\right]$$

Roots of Complex Numbers

$(x+iy)^{\frac{1}{n}}$

$$z^{1/n} = \left(r^{1/n}, \frac{2m\pi+\theta}{n}\right) = r^{1/n} e^{i\frac{2m\pi+\theta}{n}}, m = 0, 1, 2, \dots, (n-1)$$

$$r^{\frac{1}{n}}\left(\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n}\right)$$

The value corresponding to $m = 0$ is called the principal value of $z^{1/n}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$m = 0, 1, 2$

Example. Find the value of $(1+i\sqrt{3})^{1/3}$ $n = 3$

$$r = \sqrt{1+3}$$
$$r = 2$$

$$\theta = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\boxed{\theta = \frac{\pi}{3}}$$

$$\Rightarrow (1 + \sqrt{3}\, i)^{\frac{1}{3}}$$

$m = 0, 1, 2$

$n - 1$

$3 - 1 = 2$

$$\Rightarrow 2^{\frac{1}{3}}\left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9} + \cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9} + \cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right]$$

$$(x+iy)^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n}\right]$$

$$m=1 \quad \frac{2\pi + \frac{\pi}{3}}{3} \Rightarrow \frac{7\pi}{9}$$

$$m = 2 = \frac{4\pi + \frac{\pi}{3}}{3} \Rightarrow \frac{13\pi}{9}$$

Roots of Unity

$1, \omega, \omega^2$

$$1^{1/n} = (1, 2m\pi/n)$$
$$= \cos(2m\pi/n) + i\sin(2m\pi/n)$$
$$= e^{i(2m\pi/n)}, m = 0, 1, 2, \ldots, (n-1)$$
$$= 1, e^{i(2\pi/n)}, e^{i(4\pi/n)}, \ldots, e^{i[2(n-1)\pi/n]}$$
$$= 1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$$

Particular Cases :

(A) When $n = 3$ (cube roots of unity):

1. The square root of i is

i का वर्गमूल है:

(a) $\frac{1}{2}(1+i)$

(b) $\frac{1}{2}(1-i)$

(c) $\pm\frac{1}{\sqrt{2}}(1+i)$

(d) $\pm\frac{1}{\sqrt{2}}(1-i)$

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UP PGT 2013

2. The polar form of $-1-\sqrt{(-3)}$ is

$-1-\sqrt{(-3)}$ का ध्रुवीय रूप है

(a) $2\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$

(b) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

(c) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(d) $2\left(\cos\frac{\pi}{3}-1\sin\frac{\pi}{3}\right)$

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3. The square root of $5 + 12i$ is:

$5 + 12i$ का वर्गमूल है:

(a) $(3 + 2i)$

(b) $(2 + 3i)$

(c) $(2 - 3i)$

(d) (3-2i)

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3×2 $\quad \sqrt{5+12i}$ $\quad (3+2i)$

$3^2 - 2^2$ $\quad ab = \frac{12}{2} = 6$

$\Rightarrow 9-4 \Rightarrow 5$ $\quad a^2 - b^2 = 5$

$a \times b = \frac{12}{2} = 6$

2×3

$a^2 - b^2 = 5$

$4 - 9 = -5$

4. The three cube roots of $z = -8i$ are

$z = -8i$ के तीन घनमूल हैं

(a) $2i, -\sqrt{3} - i, \sqrt{3} - i$

(b) $-2i, -\sqrt{3} - i, \sqrt{3} - i$

(c) $2i, -\sqrt{3} - i, \sqrt{3} + i$

(d) $2i, -\sqrt{3} - i, -\sqrt{3} + i$

LT 2018

5. If $A + iB = \frac{3-2i}{7+4i}$, then value of $A - B$ is-

यदि $A + iB = \frac{3-2i}{7+4i}$, तो $A - B$ का मान है

(a) $\frac{1}{5}$

(b) $\frac{2}{5}$

(c) $\frac{3}{5}$

(d) $-\frac{1}{5}$

UP PGT-2016 (02-02-2019)

6. The amplitude of $\frac{1+i}{1+\sqrt{3}i}$ is

[redacted] $\frac{1+i}{1+\sqrt{3}i}$ का आयाम-

(a) $\frac{\pi}{12}$

(b) $\frac{-\pi}{12}$

(c) $\frac{\pi}{4}$

(d) $\frac{-\pi}{4}$

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7. The modulus of $\frac{(1+2i)(2-i)}{3-4i}$ is equal –

$\frac{(1+2i)(2-i)}{3-4i}$ का मापांक इसके बराबर है-

(a) 5

(b) $\sqrt{5}$

(c) 1

(d) $\frac{1}{\sqrt{5}}$

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8. Simplify $\dfrac{(\cos 3\theta + i\sin 3\theta)^4 \cdot (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 \cdot (\cos 5\theta + i\sin 5\theta)^{-4}}$

$$\Rightarrow \frac{\cancel{(\cos 12\theta + i\sin 12\theta)} \times (\cos 20\theta - i\sin 20\theta)}{\cancel{(\cos 12\theta + i\sin 12\theta)} \cdot (\cos(-20\theta) + i\sin(-20\theta))}$$

$$\Rightarrow \frac{\cos 20\theta - i\sin 20\theta}{\cos 20\theta - i\sin 20\theta}$$

$$\Rightarrow 1$$

Properties of Cube Roots of Unity:

(i) $\omega^3 = 1$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $\omega^n + \omega^{n+1} + \omega^{n+2} = 0 \forall n \in Z$

(i.e., the sum of three consecutive integral power of $\omega = 0$)

(iv) $1 + \omega^{\mathrm{n}} + \omega^{2\mathrm{n}} \begin{cases} 3\, when & n = 3k \\ 0\, when & n \neq 3k' \quad k \in Z \end{cases}$

(v) $1 \cdot \omega \cdot \omega^2 = 1$

(vi) $(\omega)^2 = \omega^2, (\omega^2)^2 = \omega$

(vii) $1/\omega = \bar{\omega}^2, 1/\omega^2 - \omega$

(i.e., ω, ω^2 are reciprocal of each other)

(viii) $\bar{\omega} = \omega^2, \overline{\omega^2} = \omega$

(i.e., ω, ω^2 are conjugate of each other)

(ix) ω, ω^2 are roots of the equation $z^2 + z + 1 = 0$

(x) $1 - i\sqrt{3} = -2\omega$ or $-2\omega^2, \sqrt{3} + i = -2\omega i$
or $-2\omega^2 i$
$1 + i\sqrt{3} = -2\omega^2$ or $-2\omega, \sqrt{3} - i = 2\omega i$
or $2\omega^2 i$.
If $\frac{1-i\sqrt{3}}{2} = -\omega$, then $\frac{1+i\sqrt{3}}{2} = -\omega^2$ and
if $\frac{1+i\sqrt{3}}{2} = -\omega$, then $\frac{1-i\sqrt{3}}{2} = -\omega^2$

(xi) Arguments of imaginary cube roots $= 2\pi/3, -2\pi/3$.

(xii) $1, \omega, \omega^2$ are vertices of an equilateral triangle with

side $= \sqrt{3}$, area $= 3\sqrt{3}/4$

Properties of Cube Roots of Unity:

When $n = 4$ (Fourth roots of unity):

$$1^{1/4} = (1, 2m\pi/4)$$

$$= \cos\left(\frac{2m\pi}{4}\right) + i\sin\,(2m\pi/4), m = 0, 1, 2, 3$$

$$= 1, (\cos\,\pi/2 + i\sin\,\pi/2), (\cos\,\pi + i\sin\,\pi)$$

$$(\cos\,3\pi/2 + i\sin\,3\pi/2)$$

$$= 1, e^{(\pi/2)}, e^{\pi i}, e^{(3\pi/2)i}$$

$$= 1, i, -1, -i$$

$$\therefore 1^{1/4} = \pm 1, \pm i$$