# Complex Number

## 1. Logarithm of a Complex Number :

$x=0,\ y=1$

If $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log_e z = \log_e r + i\theta$$

$$\log_e z = \log_e |z| + i(\arg z)$$

$$\log_e(x+iy) = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\left|\frac{y}{x}\right| + 2n\pi i$$

~~$i\tan^{-1}(y/x)$~~

$\log(0+i) = \frac{1}{2}\times 0$

$\Rightarrow i\tan^{-1}\tan\frac{\pi}{2} + 2n\pi i$

$\Rightarrow i\frac{\pi}{2} + 2n\pi i$

$\Rightarrow i\pi\left(\frac{4n+1}{2}\right)$

**General value of the argument, we shall get the general value of $\log_e Z$**

$$\log_e z = \log_e |z| + (2n\pi)i \qquad n = 0, 1, 2, \ldots$$

$$\log_e Z = \log |z| + i\tan^{-1}\left|\frac{y}{x}\right| + (2n\pi)i$$

# Logarithm of a Complex Number:

**Example.** (i) $\log(2+2i)$

(ii) $\log_2(1+i\sqrt{3})$

**Note:** (i) $\log \bar{z} = \overline{(\log z)}$

(ii) $\log(-z) = \log z + \pi i$

# Exponential Functions of Complex Number:

$i^i \qquad e^{i\log i}$

(i) $e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta) \Rightarrow e^{\alpha+i\beta} = e^{\alpha}\cdot \alpha^{i\beta}$

(ii) $a^{\alpha+i\beta} = \cancel{a}^{\alpha}\cdot e^{i\beta\log a}, (a \in \mathrm{R})$

$= a^{\alpha}\{\cos(\beta\log a) + i\sin(\beta\log a)\}$

(iii) $(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$

$e^{p+iq}$

$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$

$e^{p}\cdot e^{iq}$

$= e^{p+iq}$

$= e^{p(\cos q+i\sin q)}$

where $p = \frac{\alpha}{2}\log(x^2 + y^2) - \beta\tan^{-1}\frac{y}{x}$

$q = \frac{\beta}{2}\log(x^2 + y^2) + \alpha\tan^{-1}\frac{y}{x}$

$p = \frac{\alpha}{2}\log(x^2 + y^2) - \beta\tan^{-1}\left|\frac{y}{x}\right|$

# Exponential Functions of Complex Number:

**Example**

(i) $4^{2+3i}$

(ii) $(1+i)^{1-i}$

$$a^{\alpha+i\beta} = a^{\alpha} \cdot e^{i\beta \log a}, (a \in \mathrm{R})$$

$$= a^{\alpha}\{\cos(\beta \log a) + i\sin(\beta \log a)\}$$

$$(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$$

$$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$$

$$= e^{p+iq}$$

$$= e^{p}(\cos q + i\sin q)$$

$\boxed{\log z = \log|z| + i\tan^{-1}\left|\frac{y}{x}\right| + 2n\pi i}$

## Values of $i^i$

$\log i \Rightarrow \log\sqrt{x^2+y^2} + i\tan^{-1}\left|\frac{1}{0}\right| + 2n\pi i$

$\Rightarrow 0 + \frac{\pi}{2} i + 2n\pi i$

$\Rightarrow i\pi\left(\frac{1}{2} + 2n\right)$

$\Rightarrow i\pi\left(\frac{4n+1}{2}\right)$

$i^i = e^{i\log i}$

$= e^{i \cdot i\pi\left(\frac{4n+1}{2}\right)}$

$\Rightarrow e^{-\pi\left[\frac{4n+1}{2}\right]}$ जहाँ $n = 0, 1, 2, 3, \ldots$

$= e^{-\frac{\pi}{2}}, e^{-\frac{5\pi}{2}}, e^{-\frac{9\pi}{2}}, e^{-\frac{13\pi}{2}}$

**Observations:**

(i) $i^i$ is a real number.

**(ii) Principal value of $i^i = e^{-\pi/2}$ (iii) $0 < i^i < 1$.**

(iv) Values of $i^i$ are in GP with common ratio $\underline{e^{-2\pi}}$

$e^{-\frac{5\pi}{2}+\frac{\pi}{2}} = e^{-2\pi}$

$e^{-\frac{13\pi}{2}+\frac{9\pi}{2}}$

$e^{-\frac{4\pi}{2}}$

$\Rightarrow e^{-2\pi}$

# Square Root of a Complex Number

$z = x + iy$

$\sqrt{z} = \sqrt{x+iy}$

$|z| = \sqrt{x^2+y^2}$

$$\sqrt{z} = \sqrt{x + iy}$$

$$= \pm\left[\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{1/2} + i\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{1/2}\right]$$

or

$$\pm\left[\left(\frac{|z|+x}{2}\right)^{1/2} + i\left(\frac{|z|-x}{2}\right)^{1/2}\right]$$

**Note:**

(i) To find $\sqrt{x-iy}$ replace $i$ by $-i$ in above results.

(ii) $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

**Short Method:** To obtain $\sqrt{x+iy}$ find two numbers $a, b$ such that

$ab=\frac{y}{2}$, $a^2-b^2=x$

$$ab = y/2 \text{ and } a^2 - b^2 = x$$
$$\sqrt{x+iy} = \pm(a+ib)$$
$$\sqrt{x-iy} = \pm(a-ib)$$

$7+24i$

**Example.** Find the square root of $7+24i, 7-24i, -7+24i, -7-24i$

$ab=\frac{24}{2}=12$

$a^2-b^2=7$

$4^2-3^2=7$

$4+3i$

$7-24i$

$ab=-\frac{24}{2}=-12$

$a^2-b^2=7$

3. The square root of $5 + 12i$ is:

$5 + 12i$ का वर्गमूल है:

(a) $(3 + 2i)$

(b) $(2 + 3i)$

(c) $(2 - 3i)$

(d) (3-2i)

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$3 \times 2 = 6$

$ab = \frac{12}{2} = 6$

$a^2 - b^2 = 5$

$3^2 - 2^2 = 5$

$$\sqrt{5+12i} = \pm\left[\left(\frac{13+5}{2}\right)^{\frac{1}{2}} + i\left(\frac{13-5}{2}\right)^{\frac{1}{2}}\right]$$

$\Rightarrow 3 + i2$

$= 3+2i$