DSSSB TGT PART(A+B)

MATHS

COMPLEX ANALYSIS

PART-04

LIVE 13/09/2024 04:00 PM

Complex Number

1. Logarithm of a Complex Number :

$z = r(\cos\theta + i\sin\theta)$

If $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\log z = \log[r(\cos\theta + i\sin\theta)]$

$\Rightarrow \log_e z = \log_e r + i\theta$

$\Rightarrow \log r + \log(\cos\theta + i\sin\theta)$

$\log_e z = \log_e |z| + i(\arg z)$

$\log_e(x + iy) = \frac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}\left|\frac{y}{x}\right|$

$\Rightarrow \log r + \log e^{(i\theta)}$

$\log r + i\theta \log e$

$\log(mn) = \log m + \log n$

$|z| = \sqrt{x^2 + y^2}$

$(x^2 + y^2)^{1/2}$

$\log m^n = n \log m$

$\log z = \log r + i\theta$

General value of the argument, we shall get the general value of $\log_e Z$

$$\log_e z = \log_e z + (2n\pi)i$$

$\log_4 Z$

$\Rightarrow \dfrac{\log Z}{\log 4}$

$\boxed{\log e = 1}$

Logarithm of a Complex Number:

Example. (i) $\log(2+2i)$ $= \frac{1}{2}\log(2^2+2^2) + i\tan^{-1}\left|\frac{2}{2}\right|$

$\Rightarrow \frac{1}{2}\log 8 + i\frac{\pi}{4}$

$\frac{1}{2}\log 2^3$

$\Rightarrow \frac{3}{2}\log 2 + i\frac{\pi}{4}$

(ii) $\log_2(1+i\sqrt{3})$

$\log_2(1+i\sqrt{3})$

$\Rightarrow \dfrac{\frac{1}{2}\log(1^2+\sqrt{3}^2)+i\tan^{-1}\left|\frac{\sqrt{3}}{1}\right|}{\log 2}$

$\Rightarrow \dfrac{\frac{1}{2}\log 2^2 + i\frac{\pi}{3}}{\log 2}$

$\Rightarrow \dfrac{\log 2 + i\frac{\pi}{3}}{\log 2}$ Ans

Note: (i) $\log \bar{z} = \overline{(\log z)}$

(ii) $\log(-z) = \log z + \pi i$

$e^{i\theta} = \cos\theta + i\sin\theta$

Exponential Functions of Complex Number:

$e^{\alpha} \cdot e^{i\beta}$

$= e^{\alpha}(\cos\beta + i\sin\beta)$

(i) $e^{\alpha+i\beta} = e^{\alpha}(\cos\ \beta + i\sin\ \beta)$

(ii) $a^{\alpha+i\beta} = a^{\alpha} \cdot e^{i\beta\log a}, (a \in \mathbf{R})$

$= a^{\alpha}\{\cos(\beta\log\ a) + i\sin(\beta\log\ a)\}$

$a^{\alpha+i\beta} \Rightarrow a^{\alpha} \cdot a^{i\beta}$

$\Rightarrow a^{\alpha} \cdot e^{i\beta\log a}$

$a^{\alpha+i\beta} \Rightarrow a^{\alpha} \cdot (\cos(\beta\log a) + i\sin(\beta\log a))$

(iii) $(x + iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\underbrace{\log(x+iy)}}$

$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right)}$

$= e^{p+iq}$

$= e^{p}(\cos\ q + i\sin\ q)$

$e^{p} \cdot e^{iq} \Rightarrow e^{p}(\cos q + i\sin q)$

where $p = \frac{\alpha}{2}\log\left(x^2 + y^2\right) - \beta\tan^{-1}\frac{y}{x}$

$$q = \frac{\beta}{2}\log\left(x^2 + y^2\right) + \alpha\tan^{-1}\frac{y}{x}$$

Exponential Functions of Complex Number:

$\alpha^{\alpha+i\beta}$

$(x+iy)^{\alpha+i\beta}$

Example

(i) 4^{2+3i} $\Rightarrow 4^{2+3i} \Rightarrow e^{(2+3i)\log 4}$

$= e^{(2+3i)\log 2^2}$

$e^{(4+6i)\log 2}$

(ii) $\underbrace{(1+i)^{1-i}}$

$(x+iy)^{a+i\beta} \Rightarrow e^{(a+i\beta)\log(x+iy)}$

$(1+i)^{1-i} \Rightarrow e^{(1-i)\log(1+i)} \Rightarrow$

$\Rightarrow e^{(1-i)\left[\frac{1}{2}\log(1^2+1^2)+i\tan^{-1}\left(\frac{1}{1}\right)\right]}$

$\Rightarrow e^{(1-i)\left[\frac{1}{2}\log 2+i\frac{\pi}{4}\right]}$

$$a^{\alpha+i\beta} = a^{\alpha}\cdot e^{i\beta\log a}, (a \in \mathbf{R})$$

$$= a^{\alpha}\{\cos(\beta\log a) + i\sin(\beta\log a)\}$$

$$(x+iy)^{\alpha+i\beta} = e^{(\alpha+i\beta)\log(x+iy)}$$

$$= e^{(\alpha+i\beta)\left(\frac{1}{2}\log(x^2+y^2)+i\tan^{-1} y/x\right\}}$$

$$= e^{p+iq}$$

$$= e^{p}(\cos q + i\sin q)$$

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Values of i^i

Observations:

(i) i^i is a real number.

(ii) Principal value of $i^i = e^{-\pi/2}$ (iii) $0 < i^i < 1$.

(iv) Values of i^i are in GP with common ratio $e^{-2\pi}$