# 1. Comples Number

**A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. is called a complex number**

**Also $x, y$ are called respectively the real part of $z$ and imaginary part of $z$ and they are expressed**

$$\text{Re}\,(z) \text{ and } \text{Im}\,(z)$$

**Example -**

$$z = -5 + i \quad \Rightarrow \text{Re}(z) \Rightarrow -5, \text{Im}(z) \neq 1$$

$$z = 8 \quad \Rightarrow \text{Re}(z) \Rightarrow 8, \text{Im}(z) \neq 0$$

$$z = \sqrt{-9} \quad \Rightarrow \text{Re}(z) \Rightarrow 0,$$

$$z = 0 \quad \Rightarrow \text{Re}(z) \neq 0, \text{Im}(z) \Rightarrow 3$$

# 2. Integral Powers of $i$

**Since** $i = \sqrt{-1}$

## In General for any Integar $k$,

$i^{4k} = 1,$

$i^{4k+1} = i,$

$i^{4k+2} = -1,$

$i^{4k+3} = -i$

**Example-**

(i) $i^{123}$

(ii) $i^{978}$

(iii) $l^{-147}$

(iv) $(-i)^{8n+3}, n \in N$

Q. $i^n + i^{n+1} + i^{n+2} + i^{n+3}; \forall n \in N$ is equal to.

(i) $i$

(ii) 0

(iii) $(i)^2$

(iv) $(i)^3$

## Note-

**The sum of four consecutive integral powers of $i$ is always Zero.**

# 3. Complex Number as an Ordered Pair of two Real Numbers

Every complex number $z = x + iy$ may be considered as an ordered pair $(x, y)$ of its two parts. The first number of the ordered pair is its real part, and the second number of the ordered pair is its imaginary part. Thus

$$\mathbb{Z} \equiv x + \mathbf{i}y \equiv (\mathbf{x}, y)$$

# Basic Algebraic Operations on Complex Numbers

If $z = x + iy$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are any complex numbers, then

(i) $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$ (equality)

(ii) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ (addition)

(iii) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$ (subtraction)

(iv) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ (multiplication)

(v) $-z = (-x) + i(-y)$ (negative)

(vi) $z \neq 0, \frac{1}{z} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$ (reciprocal)

(vii) $z \neq 0, \frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$ (division)

## Geometrical Representation of a Complex Number

Geometrically every complex number $z = x + iy = (x, y)$ can be represented in $xy$-plane by a point whose cartesian coordinates are $(x, y)$

So, such a plane is named as complex plane or Argand plane or $z$-plane.

**Example- Represent the following complex numbers in Argand diagram**

$$i, \sqrt{-3}, -5 + i, 1 - \sqrt{-9}$$

# Complex Conjugate

$$z = x + iy \Rightarrow \bar{z} = x - iy$$
$$z = (x, y) \Rightarrow \bar{z} = (x, -y)$$

**Example. (i) If $z = 5 + i$, then $\bar{z} =$**

**(ii) If $z = 9$, then $\bar{z} =$**

**Propertics of Conjugate**: If $z, z_1, z_2$ are any complex numbers, then

(i) $\bar{z} = z \Leftrightarrow z \in R$

(ii) $\bar{z} = -z \Leftrightarrow z = 0$ or purely imaginary

(iii) $\overline{(\bar{z})} = z$

(iv) $z + \bar{z} = 2\text{Re}\,(z)$

(v) $z - \bar{z} = 2i\text{Im}\,(z)$

(vi) $z\bar{z} = x^2 + y^2$

(vii) $\frac{z}{\bar{z}} = \frac{z^2}{|z|^2}$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ix) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(x) $\overline{z_1 Z_2} = \overline{z_1 z_2}$

(xi) $\overline{(z^n)} = (\bar{z})^n, n \in N$

(xii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

(xiii) $\left((\bar{z})^{-1}\right) = \left(\overline{z^{-1}}\right)$

## Modulus of a Complex Number:

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

$$||z| = \sqrt{\{\text{Re}\ ((z)\}^2 + \{\text{Iom}\ (z)\}^2}$$

**Unimodular:** A complex number $z$ is said to be unimodular if $|z| = 1$

**Q.** $\sum_{n=1}^{40}(i^n + i^{-n}) = ?$

Q. $\sum_{n=1}^{206}(i^n + i^{-n}) = ?$

Q. If $\left(1+\frac{1}{i}\right)=A+i\,B$ then the value of $B$ ?

(a) 1

(b) -1

(c) 0

(d) None of these

## ➢ Argument or amplitude of a complex No

$$z = x + iy$$

$\arg(z)$ **or** $\text{amp}(z)$

$$\theta = \text{Tan}^{-1}\left|\frac{y}{x}\right|$$

**Q. The principal argument of the complex number $i - 1$.**

(a) $\pi$

(b) $2\pi$

(c) $\frac{3\pi}{4}$

(d) $\frac{2\pi}{3}$

**Q. The amplitude of the complex no. $-\sqrt{3}i - 1$ ?**

(a) $\frac{2\pi}{3}$

(b) $-\frac{2\pi}{3}$

(c) $\frac{3\pi}{4}$

(d) $\frac{-3\pi}{4}$

Q. The argument of the complex Number $-6i + 6$ is?

(a) $\pi$

(b) $-\pi$

(c) $\frac{\pi}{4}$

(d) $\frac{-\pi}{4}$

**Q. Find the amplitude of $i\left(\frac{3-i}{2+i}+\frac{3+i}{2-i}\right)$ ?**

(a) $\frac{\pi}{2}$

(b) $\frac{-\pi}{2}$

(c) $2\pi$

(d) $\infty$

**Q. Argument of $-2i - 2\sqrt{3}$ is?**

(a) $-2\pi$

(b) $\frac{-2\pi}{3}$

(c) $3\pi$

(b) $\frac{-3\pi}{2}$

Q. Argument of $-i - i$ is ?

(a) $\frac{3\pi}{4}$

(b) $\frac{-3\pi}{4}$

(c) $2\pi$

(d) $3\pi$

## * Properties of argument →

**(1)** $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

**(2)** $\arg\left|\frac{z_1}{z_2}\right| = \arg z_1 - \arg z_2.$

**(3)** $\arg z^n = n\arg z$

**(4)** $\arg(\bar{z}) = -\arg z$

**(5)** $\arg(z_1 \cdot \bar{z}_2) = \arg z_1 + \arg \bar{z}_2$

$\Rightarrow \arg z_1 - \arg z_2$

**(6)** $|z|^2 = z.\bar{z}$

**Q. If $z$ is a complex no. of unit Modulus and argment is $\theta$ then arg $\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to?**

(a) $-\theta$

(b) $\frac{\pi}{2}-\theta$

(c) $\theta$

(d) $\pi-\theta$

## Polar form of Complex No

$z = x + iy,\ x = r\cos\ \theta$ • $y = r\sin\ \theta$

$z = r\cos\theta + ir\sin\ \theta$ $r^2 = x^2 + y^2$

$z = r(\cos\ \theta + i\sin\ \theta)$

$Z = x + iy$ , $x = r\cos\theta$ , $y = r\sin\theta$

$Z = r\cos\theta + i\,r\sin\theta$

**Q. The polar form of $-1-\sqrt{-3}$ is?**

$Z = -1-\sqrt{3}\,i$

$r = \sqrt{x^2+y^2}$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

$x = -1,\ y = -\sqrt{3}$

$r = 2$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

III rd quadrant

$\theta \Rightarrow \frac{\pi}{3}$

$\arg = \theta - \pi \Rightarrow \frac{\pi}{3} - \pi \Rightarrow -\frac{2\pi}{3} = \theta$

polar form $\Rightarrow r(\cos\theta + i\sin\theta)$

$\Rightarrow 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

(i) $1+i$ $r=\sqrt{2}$

$x=1, y=1$

$\theta=\frac{\pi}{4}=\theta$ ✓

$z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(ii) $3+4i$

$r=\sqrt{9+16}$

$r=5$

$\theta=\tan^{-1}\left|\frac{y}{3}\right|$

$z=5\left(\cos\tan^{-1}\frac{y}{3}+i\sin\tan^{-1}\frac{y}{3}\right)$

(iv) $-i$

$z=0-i$

$x=0, y=-1$

$r=1$

IV th quadrant

$\theta=\frac{\pi}{2}$

$\theta=-\frac{\pi}{2}$

(iii) $i-\sqrt{3} \Rightarrow -\sqrt{3}+i$ $r=2$

$x=-\sqrt{3}, y=1$

$\theta=\frac{\pi}{6}$

IInd quadrant $\Rightarrow \pi-\theta \Rightarrow \pi-\frac{\pi}{6} \Rightarrow \frac{5\pi}{6}$

$z=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

# Modulus of a Complex Number :

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

$$|z| = \sqrt{\{\mathrm{Re}\ (z)\}^2 + \{\mathrm{Im}\ (z)\}^2}$$

**Example.** $|3 - 4i| =$

$|3 - 4i|$

$\Rightarrow \sqrt{3^2 + (-4)^2} = 5$

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

**unimodular:** A complex number $z$ is said to be unumodular if $|z| = 1$

$z = 1/2 + i\sqrt{3}/2$ $\Rightarrow |z| = \sqrt{\frac{1}{4} + \frac{3}{4}} \Rightarrow 1$

Note: $z \neq 0. z/\bar{z}$ and $z|z|$ are alwoys unimodular complex numbers

# 9. Properties of Modulus:

(i) $|z| \in R$

(ii) $|z| \geq 0$

(iii) $|z| \geq |Re\ (z)|$

(iv) $|z| \geq |Im\ (z)|$

(v) $|z| = |-z| = |\bar{z}| = |zi|$

(vi) $|z|^2 = z\bar{z}$

(vii) $|z_1 z_2| = |z_1||z_2|$

(viii) $|z^n| = |z|^n, n \in N$

(ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(x) $|z| = 1 \Leftrightarrow \bar{z} = 1/z$

$|\bar{z}| = 5$

$\bar{z} = 3-4i$

$z = 3+4i$

$|z| = 5$

$-z = -3-4i$

$|-z| = 5$

$zi = 3i-4$

$zi = -4+3i$

$|zi| = 5$

$6+8i \Rightarrow |z| = 10$

$6-8i \quad |z| = 10$

$Re(z) = 6, \quad Im(z) = -8$

$|Re(z)| = 6, \quad |Im(z)| = 8$

$z = 5+12i \qquad |z|^2 = 13^2 = 169$

$\bar{z} = 5-12i$

$z.\bar{z} = (5+12i)(5-12i)$

$\Rightarrow 5^2 - (12i)^2$

$\Rightarrow 25 - 144(-1) \Rightarrow 169$

**(xi)** $z^{-1} = \bar{z}/|z|^2$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$\frac{3-4i}{|z|^2}$$

$$|z| = 5$$

$z = 3+4i$ तो $\bar{z} = 3-4i$

$$z^{-1} \Rightarrow \frac{1}{z} = \frac{1}{3+4i} \times \frac{3-4i}{3-4i}$$

$$z^{-1} \Rightarrow \frac{3-4i}{9+16} \Rightarrow \frac{3-4i}{25}$$

## Triangle Inequalities: For any complex numbers $z_1, z_2$

$z_1 = 3+4i,\ |z| = 5$

$z_2 = 6+8i,\ |z| = 10$

(i) $|z_1 + z_2| \le |z_1| + |z_2|$

(ii) $|z_1 - z_2| \le |z_1| + |z_2|$

(iii) $|z_1 + z_2| \ge ||z_1| - |z_2||$

(iv) $|z_1 - z_2| \ge ||z_1| - |z_2||$

$5 - 10 \qquad |5 - 10|$

$|-5| \qquad |-5|$

$5 \ge 5$

1. In a complex plane $|z_1 - z_2|$ is equal to the distance between two points representing $z_1$ and $z_2$. It is so because

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2. The above inequalities will be equalities only when $O, A, B$ are collinear.

3. In the set $\underline{C}$ of complex numbers, the order relations ' $<$ ' and ' $>$ ' are not defined. Hence for any $z_1, z_2 \in C, z_1 < z_2$ or $z_1 > z_2$ are meaningless unless $z_1, z_2$ are real numbers

# Some Important Results for Modulus and Argument:

**If $z_1, z_2$ are two complex numbers with arguments $\theta_1, \theta_2$ respectively, then**

**(i)** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

**(ii)** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow z_1 \mid z_2$ **is purely imaginary**

**(iii)** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

**(iv)** $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi/2$

$\Leftrightarrow z_1/z_2$ **is purely imaginary**

**(v)** $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

$\Leftrightarrow \arg(z_1) - \arg(z_2) = 2n\pi, n \in Z$

**(vi)** $|z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

**Hence** $\text{Max}|z_1 + z_2| = |z_1| + |z_2|$

$MIn|z_1 + z_2| = ||z_2| - |z_2||$

# Exponential form of Complex No. →

$z = x + iy$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

$$z = re^{i\theta}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ is known as Eular's form

$z \Rightarrow 1 + i$

$r = \sqrt{2}$

$\theta = \frac{\pi}{4}$

in the Ist quadrant $\theta = \theta$

$z = re^{i\theta}$

$z \Rightarrow \sqrt{2}\, e^{i\frac{\pi}{4}}$ → Exponential form

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ → Polar form

# $-1+\sqrt{-3}$ का Exponential form?

$\theta=\tan^{-1}\left|\frac{y}{x}\right|$

$\theta=\tan^{-1}\left|\frac{\sqrt{3}}{-1}\right|$

$\boxed{\theta=\frac{\pi}{3}}$

$z=-1+\sqrt{3}\,i$

$r=2$

$\theta=\frac{\pi}{3}$

IInd quadrant $\Rightarrow \pi-\theta$

$\Rightarrow \pi-\frac{\pi}{3}$

$\Rightarrow \frac{2\pi}{3}$

$z=re^{i\theta}$

$\Rightarrow 2e^{i\frac{2\pi}{3}}$ $\longrightarrow$ Expo. form

$\Rightarrow 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$