DSSSB TGT PART(A+B)

MATHS

COMPLEX ANALYSIS

PART-02

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1. Comples Number

A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. is called a complex number

Also x, y are called respectively the real part of z and imaginary part of z and they are expressed

$$\mathrm{Re}\ (z) \text{ and } \mathrm{Im}\ (z)$$

Example -

$$z = -5 + i \quad \Rightarrow \mathbf{Re}\,(z) \Rightarrow -5, \mathbf{Im}\,(z) \neq 1$$

$$z = 8 \quad \Rightarrow \mathbf{Re}\,(z) \Rightarrow 8, \mathbf{Im}\,(z) \neq 0$$

$$z = \sqrt{-9} \quad \Rightarrow \mathbf{Re}\,(z) \Rightarrow 0,$$

$$z = 0 \quad \Rightarrow \mathbf{Re}\,(z) \neq 0, \mathbf{Im}\,(z) \Rightarrow 3$$

2. Integral Powers of i

Since $i = \sqrt{-1}$

In General for any Integar k,

$i^{4k} = 1,$

$i^{4k+1} = i,$

$i^{4k+2} = -1,$

$i^{4k+3} = -i$

Example-

(i) i^{123}

(ii) i^{978}

(iii) l^{-147}

(iv) $(-i)^{8n+3}, n \in N$

Q. $i^n + i^{n+1} + i^{n+2} + i^{n+3}; \forall n \in \mathrm{N}$ **is equal to.**

(i) i

(ii) 0

(iii) $(i)^2$

(iv) $(i)^3$

Note-

The sum of four consecutive integral powers of i is always Zero.

3. Complex Number as an Ordered Pair of two Real Numbers

Every complex number $z = x + iy$ may be considered as an ordered pair (x, y) of its two parts. The first number of the ordered pair is its real part, and the second number of the ordered pair is its imaginary part. Thus

$$\mathbb{Z} \equiv x + \mathrm{i}y \equiv (\mathrm{x}, y)$$

Basic Algebraic Operations on Complex Numbers

If $z = x + iy$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are any complex numbers, then

(i) $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$ (equality)

(ii) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ (addition)

(iii) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$ (subtraction)

(iv) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ (multiplication)

(v) $-z = (-x) + i(-y)$ (negative)

(vi) $z \neq 0, \frac{1}{z} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$ (reciprocal)

(vii) $z \neq 0, \frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$ (division)

Geometrical Representation of a Complex Number

Geometrically every complex number $z = x + iy = (x, y)$ can be represented in xy-plane by a point whose cartesian coordinates are (x, y)

So, such a plane is named as complex plane or Argand plane or z-plane.

Example- Represent the following complex numbers in Argand diagram

$$i, \sqrt{-3}, -5 + i, 1 - \sqrt{-9}$$

Complex Conjugate

$$z = x + iy \Rightarrow \bar{z} = x - iy$$
$$z = (x, y) \Rightarrow \bar{z} = (x, -y)$$

Example. (i) If $z = 5 + i$, then $\bar{z} =$

(ii) If $z = 9$, then $\bar{z} =$

Propertics of Conjugate: If z, z_1, z_2 are any complex numbers, then

(i) $\bar{z} = z \Leftrightarrow z \in R$

(ii) $\bar{z} = -z \Leftrightarrow z = 0$ or purely imaginary

(iii) $\overline{(\bar{z})} = z$

(iv) $z + \bar{z} = 2\text{Re}\,(z)$

(v) $z - \bar{z} = 2i\text{Im}\,(z)$

(vi) $z\bar{z} = x^2 + y^2$

(vii) $\frac{z}{\bar{z}} = \frac{z^2}{|z|^2}$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ix) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(x) $\overline{z_1 Z_2} = \overline{z_1 z_2}$

(xi) $\overline{(z^n)} = (\bar{z})^n, n \in N$

(xii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

(xiii) $\left((\bar{z})^{-1}\right) = \left(\overline{z^{-1}}\right)$

Modulus of a Complex Number:

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

$$||z| = \sqrt{\{\mathrm{Re}\ ((z)\}^2 + \{\mathrm{lom}\ (z)\}^2}$$

Unimodular: A complex number z is said to be unimodular if $|z| = 1$

Q. $\sum_{n=1}^{40}(i^n + i^{-n}) = ?$

$$(\underbrace{i+i^2+i^3+i^4}_{0}+\underbrace{\cdots}_{0}\underbrace{\cdots}_{0}\underbrace{\cdots+i^{40}}_{0}) + (\underbrace{i^{-1}+i^{-2}+i^{-3}+\cdots}_{0}\underbrace{+i^{-40}}_{0})$$

$$0 \quad + \quad 0$$

$$0$$

Q. $\sum_{n=1}^{206} (i^n + i^{-n}) = ?$

$i^1 + i^{-1} + i^2 + i^{-2} + i^3 + i^{-3} \ldots$ ——

$\frac{205}{4}$ $R=1$

$\Big(\underbrace{i^1 + i^2 + i^3 + i^4}_{0} + \underbrace{\ldots}_{204} i^{206}\Big) + \Big(i^{-1} + i^{-2} + i^{-3} + i^{-4} + \ldots i^{-206}\Big)$

$i + i^2 + i^{-1} + i^{-2}$

$\cancel{i} - 1 - \cancel{i} - 1$

-2

$i^{205} + i^{206} + i^{-1} + i^{-2}$

$i + i^2 + (-i) + \frac{1}{i^2}$

$\cancel{i} - 1 - \cancel{i} - 1$

-2

Q. If $\left(1+\frac{1}{i}\right) = A + i\,B$ then the value of B ?

(a) 1

$\boxed{\text{(b) -1}}$

(c) 0

(d) None of these

$1 + \frac{1 \cdot i}{i \cdot i}$

$1 - i = A + iB$

$A = 1$

$B = -1$

➢ Argument or amplitude of a complex No

$$z = x + iy$$

arg (z) **or** amp (z)

$$\theta = \mathrm{Tan}^{-1}\left|\frac{y}{x}\right|$$

<u>Case-1</u> → If z Lies in first quadrant

$$\arg(z) = \theta = \mathrm{Tan}^{-1}\left|\frac{y}{x}\right|$$

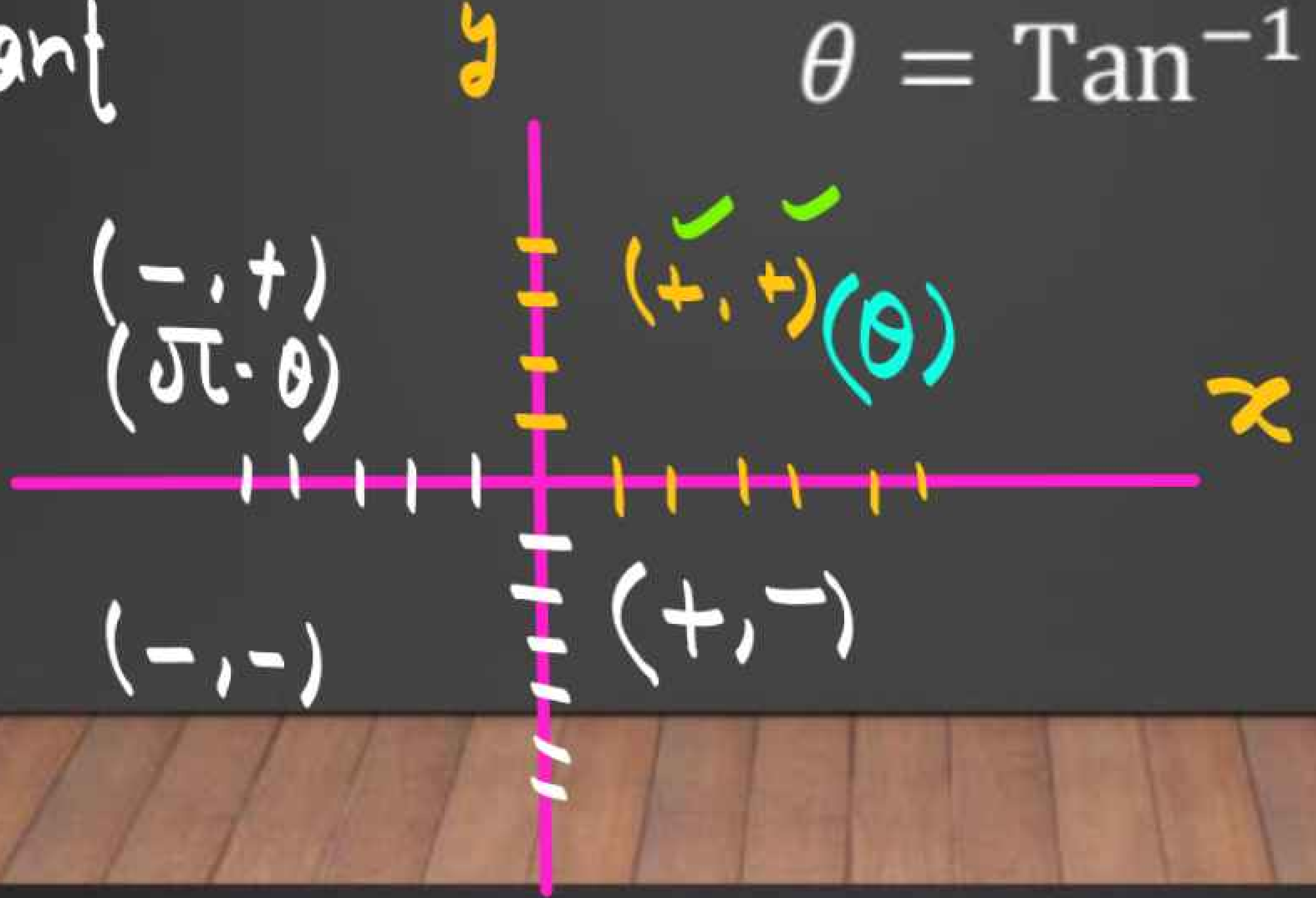
$$Z = x + iy$$

<u>Case-II</u> → If z Lies in Second quadrant

$$Z = -x + iy$$

$$\theta = \mathrm{Tan}^{-1}\left|\frac{y}{x}\right|$$

$$\boxed{\mathrm{amp}(z) \text{ or } \mathrm{Arg}(z) = \pi - \theta}$$



Case-3 → If z lies in Third quadrant

$$Z = -x - iy \quad , \quad \theta = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\boxed{Arg(z) = \theta - \pi}$$

Case-4 → If z Lies in fourth quadrant -

$$Z = x - iy \quad , \quad \theta = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\boxed{Arg(z) \text{ or } amp(z) = -\theta}$$

Q. The principal argument of the complex number $i-1$.

(a) π

(b) 2π

(c) $\boxed{\frac{3\pi}{4}}$

(d) $\frac{2\pi}{3}$

In IInd quadrant

$\arg(z) = \pi - \theta$

$\Rightarrow \pi - \frac{\pi}{4}$

$\arg \Rightarrow \frac{3\pi}{4}$

$z = -1 + i$ it is in IInd quadrant

$x = -1, \; y = +1$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

$\tan^{-1}\tan\frac{\pi}{4}$

$\boxed{\theta \Rightarrow \frac{\pi}{4}}$

$\pi = 180$

$\frac{\pi}{4} = \frac{180}{4} = 45^\circ$

2

Q. The amplitude of the complex no. $-\sqrt{3}i - 1$?

(a) $\frac{2\pi}{3}$

(b) $-\frac{2\pi}{3}$

(c) $\frac{3\pi}{4}$

(d) $\frac{-3\pi}{4}$

$z = -1 - \sqrt{3}i$ it is in III^{rd} quadrant

$x = -1,\ y = -\sqrt{3}$

$\theta = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right|$

$\theta = \tan^{-1}\tan\frac{\pi}{3}$

$\boxed{\theta = \frac{\pi}{3}}$

$\tan 60^\circ$
$\tan\frac{\pi}{3}$ $\rightarrow \sqrt{3}$

In III^{rd} quadrant

$\arg(z) = \theta - \pi$

$\Rightarrow \frac{\pi}{3} - \pi$

$\Rightarrow \frac{\pi - 3\pi}{3}$

$\text{Arg} \Rightarrow \frac{-2\pi}{3}$

Q.3. The argument of the complex Number $-6i + 6$ is?

(a) π

(b) $-\pi$

(c) $\frac{\pi}{4}$

(d) $\boxed{\frac{-\pi}{4}}$

In IVth quadrant

$\arg(z) = -\theta$

$= -\frac{\pi}{4}$

$z = 6 - 6i$ it is in IVth quadrant

$x = 6, y = -6$

$\theta = \tan^{-1}\left|\frac{-6}{6}\right|$

$\frac{6}{6} = 1 = \tan 45^\circ$

$\boxed{\theta = \frac{\pi}{4}}$

Q. Find the amplitude of $i\left(\frac{3-i}{2+i}+\frac{3+i}{2-i}\right)$?

(a) $\boxed{\frac{\pi}{2}}$ ✓

(b) $\frac{-\pi}{2}$

(c) 2π

(d) ~~0~~ ∞

$$\theta = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\Rightarrow \tan^{-1}\left|\frac{2}{0}\right|$$

$$\Rightarrow \tan^{-1}\tan\frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{2}}$$

$$i\left(\frac{(3-i)}{(2+i)}+\frac{3+i}{(2-i)}\right)$$

$$i\left[\frac{(3-i)(2-i)+(3+i)(2+i)}{4-i^2}\right]$$

$$i\left[\frac{6-\cancel{5i}-1+6+\cancel{5i}-1}{5}\right] \Rightarrow \frac{i\,\cancel{10}^{2}}{\cancel{5}} \Rightarrow 2i$$

$$Z = 0+2i$$

$x = +0,\ y = +2$

$\overset{Im(y)}{}\ \overset{Re(x)}{}$

Q. Argument of $-2i - 2\sqrt{3}$ **is?**

(a) -2π

(b) $\frac{-2\pi}{3}$

(c) 3π

(b) $\frac{-3\pi}{2}$

(e) $-\frac{5\pi}{6}$

In III_{rd} quadrant

$arg = \theta - \pi$

$\Rightarrow \frac{\pi}{6} - \pi$

$Arg \Rightarrow \frac{-5\pi}{6}$

$\theta = Tan^{-1}\left|\frac{-\cancel{2}}{+\cancel{2}\sqrt{3}}\right|$

$Tan^{-1}\, Tan\frac{\pi}{6}$

$\boxed{\theta = \frac{\pi}{6}}$

$\underline{Q} \Rightarrow -2 - 2\sqrt{3}\, i$

$\theta = Tan^{-1}\left|\frac{-\cancel{2}\sqrt{3}}{-\cancel{2}}\right|$

$Tan^{-1}\, Tan\frac{\pi}{3}$

$\theta = \frac{\pi}{3}$

$arg \Rightarrow \frac{\pi}{3} - \pi$

$\Rightarrow \frac{-2\pi}{3}$

Q. Argument of $-i - 1$ is ?

(a) $\frac{3\pi}{4}$

(b) $\boxed{\frac{-3\pi}{4}}$

(c) 2π

(d) 3π

* Properties of argument →

(1) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(2) $\arg\left|\frac{z_1}{z_2}\right| = \arg z_1 - \arg z_2$.

(3) $\arg z^n = n\arg z$

(4) $\arg(\bar{z}) = \underbrace{-\arg z}$

(5) $\arg(z_1 \cdot \bar{z}_2) = \arg z_1 + \arg \bar{z}_2$

$\Rightarrow \arg z_1 - \arg z_2$

(6) $|z|^2 = z.\bar{z}$

Q. If z is a complex no. of unit Modulus and argment is θ then arg $\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to?

(a) $-\theta$

(b) $\frac{\pi}{2}-\theta$

(c) θ

(d) $\pi-\theta$

Polar form of Complex No

$$z = x + iy,\ x = r\cos\,\theta + y = r\sin\,\theta$$

$$z = r\cos\theta + ir\sin\,\theta\ \boxed{r^2 = x^2 + y^2}$$

$$z = r(\cos\,\theta + i\sin\,\theta)$$

$$\boxed{r = \sqrt{x^2+y^2}}$$

$$\theta = \tan^{-1}\left|\frac{y}{x}\right|$$

Q. The polar form of $-1-\sqrt{-3}$ is?

$Z = -1-\sqrt{3}\,i$

$x=-1,\ y=-\sqrt{3}$

$r=\sqrt{x^2+y^2}$

$\sqrt{1+3} \Rightarrow 2$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

$\boxed{\theta = \frac{\pi}{3}}$

$Z = r(\cos\theta + i\sin\theta)$

$Z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$