Coset :- Let G be a group and H is Subgroup of G and $a \in G$. The set $a.H = \{a.h / h \in H\}$ is called Left coset of H in G.

ऐसे ही $H.a = \{h.a / h \in H\}$, is Called Right Coset of H is G.

multiplication

⇓

Addition → $a+H = \{a+h \mid h \in H\}$ Left coset of H in G

→ $H+a = \{h+a / h \in H\}$ Right coset of H in G

Q. If $G = \mathbb{Z}$, find all coset of H in G ? → And $H = 4\mathbb{Z}$

A. 4 Distinct coset

B. 3 Distinct coset

C. Indentical coset

D. None of these.

* Only in Case of $\mathbb{Z}$

No. of Coset $= \frac{4\mathbb{Z}}{\mathbb{Z}}$

$(\mathbb{Z},+)$ $\quad \mathbb{Z}=\{-\infty \ldots -5,-4,-3,-2,-1,0,1,2,3,4,5 \ldots \infty\}$

$H=4\mathbb{Z}'=\{-\infty \ldots -20,-16,-12,-8,-4,0,4,8,12,16,20 \ldots \infty\}$

$0+H=H$

$1+H=\{-\infty \ldots -19,-15,-11,-7,-3,1,5,9,13,17 \ldots\}$

$2+H=\{-\infty \ldots -18,-14,-10,-6,-2,2,6,10,14,18 \ldots\}$

$3+H=\{-\infty \ldots -17,-13,-9,-5,-1,3,7,11,15,19,23 \ldots \infty\}$

$4+H=\{-\infty \ldots -16,-12,-8,-4,0,4,8,12,16,20 \ldots$

$5+H=\{-\infty \ldots -15,-11,-7,-3,1,5,9,13,17 \ldots$

$6+H=$

If $G = Z$ and $H = 2z$ then no. of coset?

$Z = \{-\infty \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \infty\}$

No. of coset $\Rightarrow 2$

$H = 2z = \{-\infty \ldots -8, -6, -4, -2, 0, 2, 4, 6, 8 \ldots \infty\}$

$0 + H \Rightarrow H$ ✓

$1 + H = \{-\infty \ldots -7, -5, -3, -1, 1, 3, 5, 7, 9 \ldots \infty\}$ ✓

$2 + H = \{-\infty \ldots -6, -4, -2, 0, 2, 4, 6, 8 \ldots \infty\}$

Same

Properties on Corsets: Let H be a subgroup of G and a, b $\in$ G.

Then,

(i) $a \in aH$ (i.e. coset of every subgroup always non-empty)

(ii) $aH = H$ iff $a \in H$.

(iii) $aH = bH$ or $aH \cap bH = \emptyset$ (null set) cosets are either identical or disjoint.

(iv) $aH = bH$ iff $a^{-1}be \in H$ or Ha Hb iff $ab^{-1} \in H$.

(v) $|ah| = |bH|$

(vi) $aH = Ha$ iff $H = aHa^{-1}$.

(vii) aH (coset of G) is need not be subgroup of G.

(viii) aH is a subgroup of G iff $a \in H$.

(ix) H is subgroup of G, then, H itself is right as well as left coset of G by e (identity element) as $He = eH = H$.

(x) If H is subgroup of G. Then, $H H = H$.

Note :

1. If G is abelian group. Then, every left coset of H in G is right coset of H in G.

$a \cdot H = H \cdot a$ ✓

$a + H = H + a$ ✓

$G \to$ Abelian

2. If G is cyclic group. Then, every left coset of H in G is right coset of H in G.

$G \to$ Cyclic

Index of a Subgroup: The index of a subgroup H of a group G is defined as the number of distinct right (or left) coset of H in G.

3. If G be a $\boxed{\text{Finite}}$ Group, Then $i_G(H)$

$$i_G(H) \Rightarrow \frac{O(G)}{O(H)}$$

$Z_n = \{0, 1, 2, 3, \ldots (n-1)\}$

$O(Z_n) = n$

$Z_{20} = \{0, 1, 2, 4, \ldots 17, 18, 19\}$

$H = \langle 4 \rangle$

$\Rightarrow \{0, 4, 8, 12, 16\}$

Q. If G = Z_{20} and H = <4> then $i_G(H)$?

A. 1

B. 2

C. 3

D. 4

$i_G(H) = \dfrac{O(G)}{O(H)}$

$\Rightarrow \dfrac{20}{5} = 4$

Q. $G = Q_4$, $H = \{\pm 1, \pm i\}$, find all Distinct cosets of H in Q_4

$O(Q_4) = 8 \rightarrow$ Finite

$O(H) = 4$

$G = Q_4 = \{\pm 1, \pm i, \pm j, \pm k$; $i^2 = j^2 = k^2 = -1$

$i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$

$j \cdot i = -k$, $k \cdot j = -i$, $i \cdot k = -j$

A. 3

B. 2 ✓

C. 0

D. None of these

$H = \{\pm 1, \pm i\}$

Q. $G = \mathbb{Z}$ and $H = 6\mathbb{Z}$ then $i_G(H) = ?$

$\{\text{infinite}\}$

$O(H) \to$ infinite

A. 4

B. 6

C. 1

D. infinite

Q. $G = 2\mathbb{Z}$ and $H = 10\mathbb{Z}$ then $i_G(H) = ?$

A. 0

B. 2

C. 5

D. infinite

Q. $G = 3\mathbb{Z}$ and $H = 15\mathbb{Z}$ then $i_G(H) = ?$

A. 0

B. 2

C. 5

D. infinite

Q. How many Distinct cosets of H = {I, (123) ,(132)} in S_3 ?

$S_3 = \{I, (12), (23), (13), (123), (132)\} \longrightarrow$ finite $O(S_3) = 6$

A 1

B. 3

C. 2

D. Infinite

$H = \{I, (123), (132)\}$

$O(H) = 3$

$\Rightarrow \frac{O(S_3)}{O(H)} = \frac{6}{3}$

$O(S_n) = n!$ $\quad 3! = 3 \times 2 \times 1 = 6$