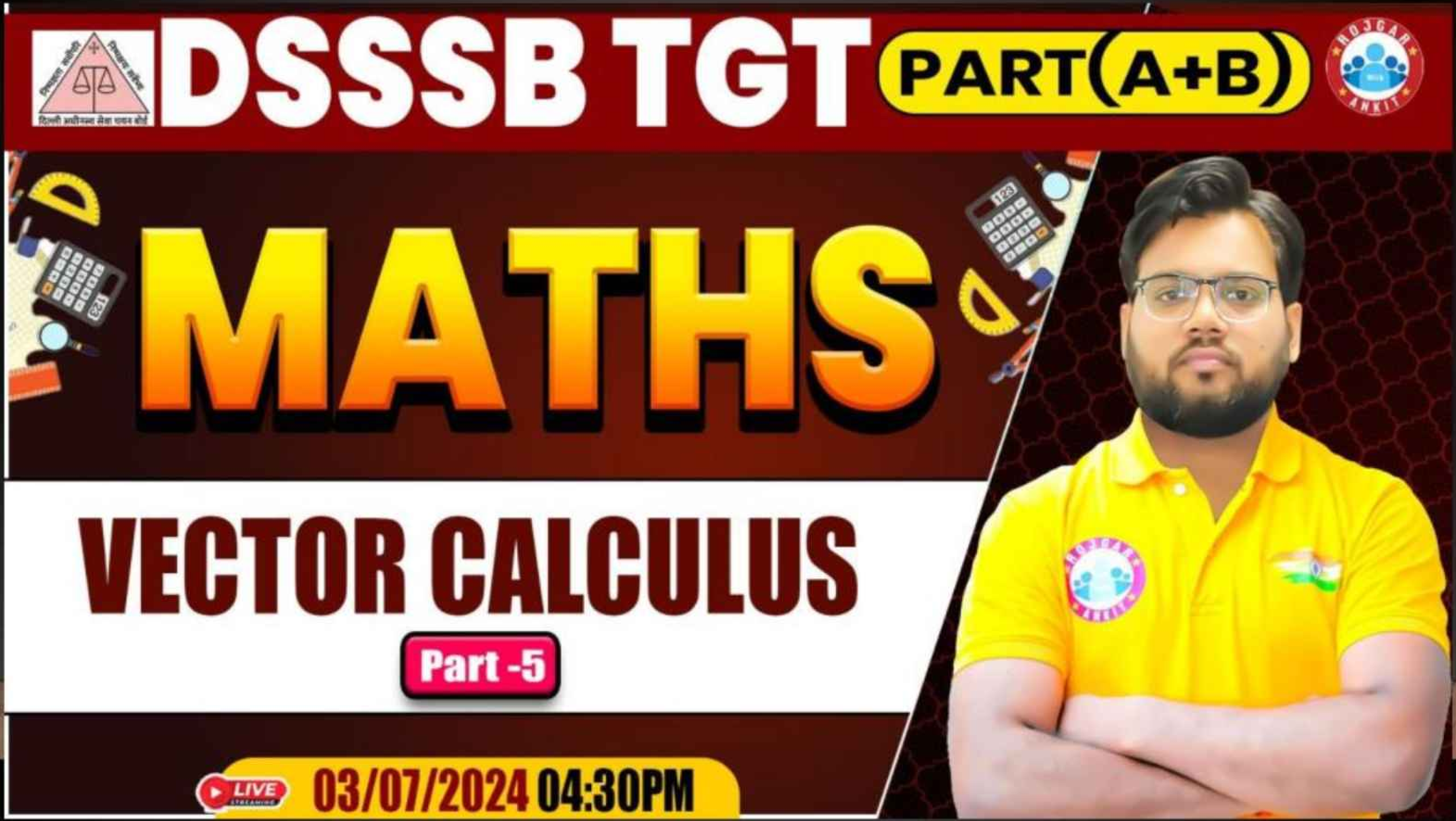
DSSSB TGT
PART (A+B)
MATHS
VECTOR CALCULUS
Part -5
LIVE
03/07/2024 04:30PM

grad Div Curl

Identity I: $\mathbf{div}\ (\boldsymbol{ua}) = \boldsymbol{u} \cdot \mathbf{div}\ \boldsymbol{a} + \boldsymbol{a} \cdot \mathbf{grad}\ \boldsymbol{u}$

या

$$\nabla \cdot (\boldsymbol{ua}) = \boldsymbol{u}(\nabla \cdot \boldsymbol{a}) + \boldsymbol{a} \cdot (\nabla \boldsymbol{u})$$

$$\mathbf{curl}(\boldsymbol{ua}) = (\mathbf{grad}\,\boldsymbol{u}) \times \boldsymbol{a} + \boldsymbol{u}\,\mathbf{curl\,a}$$

या

$$\boldsymbol{\nabla} \times (\boldsymbol{ua}) = (\boldsymbol{\nabla u}) \times \boldsymbol{a} + \boldsymbol{u}(\boldsymbol{\nabla} \times \boldsymbol{a})$$

$\mathrm{div}(\mathrm{grad}\,\phi)$, $\mathrm{curl}(\mathrm{grad}\,\phi)$, $\mathrm{div}(\mathrm{curl}\,f)$, ~~curl~~ Curl (curl f)

तथा $\mathrm{grad}(\mathrm{div}\,f)$

ज्ञात कर सकते हैं जो द्वि कोटि अवकलनीय फलन कहलाते हैं।

Second order

$$\mathbf{grad}\,(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b} + \boldsymbol{b}\times \boldsymbol{curl\,a} + \boldsymbol{a}\times \boldsymbol{curl\,b}$$

या

$$\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b} + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) + \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b})$$

[Property I : div grad $\phi = \nabla^2\phi$]

Laplacian operator (लाप्लासिय ~~संख्याओं~~)
संकारक

∇^2.

संत्कारक $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ को लाप्लासियन संकारक कहते हैं।

Property 2: curl gradu $= 0 = \nabla \times (\nabla u)$

i.e. the curl of the gradient of a scalar function u is zero.

(अर्थात् किसी अदिश फलन u की प्रवणता का कुन्तल शून्य होता है)

1. The directional derivative of $\phi = xy + yz + zx$ at $(1,1,1)$ in the direction of the vector $\hat{\imath} - \hat{\jmath} + 2\hat{k}$ is.

बिन्दु $(1,1,1)$ पर $\phi = \mathrm{xy} + \mathrm{yz} + \mathrm{zx}$ का दिक् अवकलज सदिश $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ की दिशा में है

(a) $\frac{1}{3}$

(b) $\frac{2}{3}$

(c) $\frac{5}{3}$

(d) $\frac{7}{3}$

2. A vector, which is normal to the surface $x^2 - xy + yz = 5$ at the pint $(1, 2, 3)$ is

एक सदिश, जो पृष्ठ $x^2 - xy + yz = 5$ के बिन्दु $(1, 2, 3)$ पर अभिलम्ब हो, है

(a) $\hat{\imath} + 2\hat{\jmath}$

(b) $2\hat{\jmath} + 2\hat{k}$

(c) $2\hat{\mathrm{j}} + \hat{\mathrm{k}}$

(d) $\hat{\jmath} + 2\hat{k}$

3. If $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\bar{a}$ is a constant vector, then curl $(\bar{r} \times \bar{a})$

यदि $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ और $\vec{a}$ एक अचर सदिश है, तो curl $(\vec{r} \times \vec{a})$ होगा-

(a) $-\vec{a}$

(b) $-2a$

(c) $-3\vec{a}$

(d) उपर्युक्त में से कोई नहीं

Property 3: div curl $a = 0 = \nabla \cdot (\nabla \times a)$

i.e the divergerice of curl of any vector a is zero.

(अर्थांत् किसी सदिश फलन a के कुन्तल का अपसरण शून्य होता हैं)

4. Divergence of the vector $x^2z\hat{\imath} + xy\hat{\jmath} - yz^2\hat{k}$ at $-(1, -1, 1)$ is-

$x^2z\hat{\imath} + xy\hat{\jmath} - yz^2\hat{k}$ वेक्टर क्षेत्र का $(1, -1, 1)$ पर अपसरण है-

(a) 0

(b) 3

(c) 5

(d) 6

5. पृष्ठ $xy^3z^2 = 4$ के बिन्दु $(-1, -1, 2)$ पर इकाई सदिश अभिलम्ब है:

(a) $-4\hat{i} - 12\hat{j} + 4\widehat{K}$

(b) $-\frac{1}{\sqrt{11}}(\hat{i} + \hat{j} - \hat{k})$

(c) $\frac{-1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$

(d) $\hat{i} + \hat{j} - \hat{k}$

$\hat{f} = xy^3z^2 - 4$

unit scactor normal

$\Rightarrow \frac{\vec{F}}{|\vec{F}|}$

6. यदि $\bar{F} = (x^2 + ay^2 + x)\hat{i} - (2xy + y)\hat{j}$ अघूर्णी सदिश है, तो a का मान होगा?

irrotational vector

(a) 2

(b) -1

(c) 0

(d) -2

$Curl\ \bar{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+ay^2+x & -2xy-y & 0 \end{vmatrix}$$

$i(0-0) - j(0-0) + k(-2y-2ay) = 0$

$-2y-2ay=0$

$-2y(1+a)=0$

$1+a=0$

$a=-1$

7. If $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$, then the value of div curl $\vec{F}$ is?

यदि $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$ तो div curl $\vec{F}$ का मान होगा-

(a) 3

(b) 2

(c) 1

(d) 0

$\nabla\cdot(\nabla\times\vec{F})$

$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(i(2z-x)-0\,j+k(z-x^2)\right)$

$\frac{\partial}{\partial x}(2z-x)+0+\frac{\partial}{\partial z}(z-x^2)$

$-1+1$

$=0$

div.$(\nabla\times\vec{F})$

$\nabla\cdot(\nabla\times\vec{F})$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= i(2z-x) - j(0-0) + k(z-x^2)$$

8. If the vector $\vec{F}(x, y. z) = (x + 2y + az)\hat{\imath} + x\hat{k}$ is irrotational then the value of a is.

यदि सदिश $\vec{F}(x, y. z) = (x + 2y + az)\hat{\imath} + x\hat{k}$ आघूर्णी है तो a का मान है

(a) 2

(b) 1

(c) 0

(d) None of these/ इनमें से कोई नहीं

9. सदिश $\hat{i} + 2\hat{j} + 2\hat{k}$ की दिशा में $f(x, y, z) = xy^2 + yz^3$ का बिन्दु $(2, -1, 1)$ पर दिक् - अवकलन होगा?

directional derivative

(a) $\frac{-11}{2}$

(b) $\frac{8}{3}$

(c) 5

(d) $-\frac{11}{3}$

10. If the vector $(2x+3y)\hat{\imath}+(2y-z)\hat{\jmath}+(3x+az)\hat{k}$ is solenoidal, then the value of the constant ' a ' will be.

यदि सदिश $(2x+3y)\hat{\imath}+(2y-z)\hat{\jmath}+(3x+az)\hat{k}$ परनालिक हो, तो नियतांक ' a ' का मान होगा

$\mathrm{div}\,\vec{F}=0$

(a) -2

(b) -4

(c) 2

(d) 4

11. If the vector $(x+3y)i+(y-2z)j+(x+az)k$ is solenoidal then the value of the constant ' a ' will be

(a) -1

(b) -2

(c) 2

(d) None of the above

12. The derivative of a constant vector is

(a) A scalar

(b) a constant vector

(c) Zero vectors

(d) None of these

13. **Find** : $\int f(t)dt$, **where** $f(t) = (t - t^2)i + 2t^3 j - 3k]$

Integration of Vectors

Theorem. If $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, then

$$\int \mathbf{f}(t)dt = \mathbf{i}\int f_1(t)dt + \mathbf{j}\int f_2(t)dt + \mathbf{k}\int f_3(t)dt.$$

$\int 1\, dt = t$

$\int x\, dx \Rightarrow \frac{x^2}{2} + c$

SOME STANDARD RESULTS.

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

14. We have $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}$.

Therefore $\int \left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\right) dt = \mathbf{r}\cdot\mathbf{s} + c$, where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar.

15. We have $\frac{d}{dt}(\mathbf{r}^2) = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}$.

Therefore $\int \left(2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \mathbf{r}^2 + c$.

Here the constant of integration c is a scalar quantity.

16. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2 = 2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}$.

Therefore we have $\int\left(2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\right)dt = \left(\frac{d\mathbf{r}}{dt}\right)^2 + c$.

Here the constant of integration c is a scalar quantity.

Also $\left(\frac{d\mathbf{r}}{dt}\right)^2 = \frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}$.

17. We have $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$.

$$\therefore \int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}.$$

18. If $f(t) = (t - t^2)i + 2t^3 j - 3k$, find

(i) $\int f(t)dt$ and

(ii) $\int_1^2 f(t)dt$

(i) $\Rightarrow i\int(t-t^2)dt + j\int 2t^3 dt - k\int 3dt$

$\Rightarrow i\left(\frac{t^2}{2} - \frac{t^3}{3}\right) + j\left(\frac{2t^4}{4}\right) - k3t$

$\left[\left(\frac{t^2}{2} - \frac{t^3}{3}\right)i + \frac{t^4}{2}j - 3t\,k\right]_1^2 + C$

(ii)

$\left[\left(2 - \frac{8}{3}\right)i + 8j - 6k - \left(\frac{1}{2} - \frac{1}{3}\right)i - \frac{1}{2}j + 3k\right]$

$-\frac{2}{3}i - \frac{1}{6}i + 8j - \frac{1}{2}j - 6k + 3k$

$-\frac{5}{6}i + \frac{15}{2}j - 3k$

$$= \left(\frac{t^2}{2} + \frac{t^3}{3}\right)\mathbf{i} + \frac{t^4}{2}\mathbf{j} - 3t\mathbf{k} + c$$

$$= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

19. Evaluate $\int_0^1 (e^t\mathbf{i} + e^{-2t}\mathbf{j} + t\mathbf{k})\,dt.$

$$i\int_0^1 e^t + j\int_0^1 e^{-2t} + k\int_0^1 t$$

$$i\left[e^t\right]_0^1 + j\left[\frac{e^{-2t}}{-2}\right]_0^1 + k\left[\frac{t^2}{2}\right]_0^1$$

$$i\left[e-1\right] + j\left[\frac{e^{-2}}{-2} - \frac{1}{-2}\right] + k\left[\frac{1}{2} - 0\right]$$

$$(e-1)i + \left[\frac{1}{2} - \frac{e^{-2}}{2}\right]j + \frac{1}{2}k$$

$$\boxed{(e-1)i - \frac{1}{2}\left(e^{-2}-1\right)j + \frac{1}{2}k}$$

ANSWER $= (e - 1)\mathbf{i} - \frac{1}{2}(e^{-2} - 1)\mathbf{j} + \frac{1}{2}\mathbf{k}.$

$$\int e^{-2t}\,dt$$

$$\frac{e^{-2t}}{-2}$$

$$\frac{d}{dt}\left(e^{-2t}\right)$$

$$= e^{-2t} \cdot (-2)$$

20. If $\mathbf{f}(t) = t\mathbf{i} + (t^2 - 2t)\mathbf{j} + (3t^2 + 3t^3)\mathbf{k}$, **find** $\int_0^1 \mathbf{f}(t)dt$.

$$i\left[\frac{t^2}{2}\right]_0^1 + j\left[\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}}\right]_0^1 + k\left[\frac{\cancel{3}t^3}{\cancel{3}} + \frac{3t^4}{4}\right]_0^1$$

$$\frac{1}{2}i + \left(\frac{1}{3} - 1\right)j + \left(1 + \frac{3}{4}\right)k$$

$$\frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$$

$$= \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{4}\mathbf{k}.$$

21. If $\mathbf{r} = t\mathbf{i} - t^2\mathbf{j} + (t-1)\mathbf{k}$ and $\mathbf{s} = 2t^2\mathbf{i} + 6t\mathbf{k}$, evaluate

(i) $\int_0^2 \mathbf{r}\cdot\mathbf{s}\,dt$,

(ii) $\boxed{\int_0^2 \mathbf{r}\times\mathbf{s}\,dt}$

↓

Answer

↓

Third page पर

$r\cdot s \Rightarrow 2t^3 + 6t(t-1)$

$= 2t^3 + 6t^2 - 6t$

(i) $\int_0^2 r\cdot s\,dt$

$\Rightarrow \left[\cancel{2}\frac{t^4}{\cancel{4}_{2}} + \cancel{6}^{2}\frac{t^3}{\cancel{3}} - \cancel{6}^{3}\frac{t^2}{\cancel{2}}\right]_0^2$

$8 + 16 - 12$

$8 + 4 = 12$

$= 12.$

$$\underbrace{= -24\mathbf{i} - \frac{40}{3}\mathbf{j} + \frac{64}{5}\mathbf{k}.}_{\text{ANSWER}}$$

22. If $r(t) = ti - 3j + 2tk$ है तथा $S(t) = i - 2j + 2k$ है तथा $V(t) = 3i + tj - k$ है तो $\int_1^2 r \cdot (s \times v) dt$ का मान क्या होगा?

(a) 0

(b) 49

(c) 36

(d) -5

23. Evaluate $\int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\,dt$, where $\mathbf{r} = 2t^2\mathbf{i} + t\mathbf{j} - 3t^3\mathbf{k}$.

$$= -42i + 90j - 6k$$